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Editorial

"Differential Equations of Mathematical Physics and Related Problems of Mechanics"— Editorial 2021–2023

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Based on the published papers in this Special Issue of the elite scientific journal *Mathematics*, we herein present the Editorial for "Differential Equations of Mathematical Physics and Related Problems of Mechanics", the main topics of which are fundamental and applied research on differential equations in mathematical physics and mechanics. Namely, the main topics of the Special Issue are as follows:

A. Mathematical Physics and PDE: wave equations and Boltzmann equations; solvability and regularity; spectral theory, scattering, and inverse problems; and variational methods.

B. Related Problems of Analysis and Continuum Mechanics and Stochastic Models and Probabilistic Methods, including random matrices and stochastic PDEs: variational formulations of gradient theories of elasticity and micropolar and micromorphic models of solids and liquids; the identification of the material parameters of generalized continuum theories; size-dependent models of thin structures and composite materials; and electroelastic processes in the nano- and microfields of mechanical engineering and related applied problems.

The privilege of our Special Issue, "Differential Equations of Mathematical Physics and Related Problems of Mechanics", within the framework of *Mathematics* is that we could cover both the applied and experimental, as well as fundamental, sections of modern science in mathematics, physics, and their applications in space, techniques, and medicine.

It is well known that the section "Functional interpolation" is one of the sections of *Mathematics*. In this regard, we would like to give a brief overview of the activities of our Special Issue within the framework of this section, conducted during the period of 2021–2023, and there is no better picture than to provide a detailed description of the published papers. These articles, to some extent, cover the scope and topics that we set out in the description of the Special Issue.

1. Contributions to the Special Issue

Contribution 1 is devoted to the study of some solutions of the three-dimensional Laplace equation in terms of linear combinations of generalized hypergeometric functions in prolate elliptic geometry modeling modern tokamak shapes. In this article, the authors transform Laplace's equation into Heun's equation and give an analytical solution to Laplace's equation for some parameter values. Since the Laplace equation and the Grad–Shafranov equation differ by one sign, the procedure presented in the paper can be repeated to find an analytical solution to the Grad–Shafranov equation.

Next, the authors introduce cap-cyclide geometry and, using some transformations of variables, move from the Grad–Shafranov equation to the Heun equation. After this, the authors define standard toroidal geometry as a special case of cap-cyclide geometry and find the analytical solution to Laplace's equation in cap-cyclide coordinates for some parameter values.

Note that such solutions are valid for specific parameter values, and the solutions obtained in the presented paper are compared with the solutions obtained in standard toroidal geometry.

As is known, the problem of the asymptotic behavior of solutions of various differential operators of mathematical physics in domains with smooth or non-smooth boundaries is one of the topical areas of modern science. In Contribution 2, the author considers the problem of the asymptotic behavior of solutions in a multidimensional integral equation with homogeneous kernels. At the same time, proposing a special research methodology, the author defines a special class of continuous functions with a given asymptotic behavior in the vicinity of zero and proves that if the free term of an integral equation belongs to this class and the equation itself is solvable, then its solution also belongs to the same class.

There is always an interest in studying the qualitative behavior of solutions of second-order neutral differential equations and in the use of this equation in modeling many problems in engineering and physics, particularly problems associated with loss-less transmission lines, population dynamics, automatic control, the mixing of fluids, and vibrating masses attached to an elastic bar. In Contribution 3, the oscillatory properties of a neutral differential equation are studied, and new monotonic properties of solutions to this equation, which are characterized by an iterative nature, are presented. Using these properties, the authors obtain new oscillation conditions that guarantee the oscillation of all solutions.

As is known, boundary value problems with singularities play an important role in applied problems of mathematics and in fracture mechanics. In Contribution 4, for the Stokes problem with a homogeneous Dirichlet boundary condition in a polygonal domain with one re-entrant corner on its boundary, the existence and uniqueness of an R_ν-generalized solution in weighted spaces is proved. In Contribution 4, an R_ν-generalized solution to this problem in an asymmetric variational formulation is defined, and such a defined solution allows one to construct numerical methods for finding an approximate solution without a loss of accuracy. Note that the singularity can be caused both by the degeneracy of the coefficients and the right-hand sides of the equation and by the presence of re-entrant corners on the boundary of a polygonal domain.

For some areas of theoretical physics, such as wave mechanics, the theory of oscillations, etc., the solution of problems is reduced to the problem of eigenvalues. In addition, the question of the unambiguous definition of a mechanical system, i.e., the Hamilton function, through the spectrum of eigenvalues of the linear differential equation associated with it is important. In the case when the string is vibrating and the boundary conditions are natural, it is shown in [1] that the spectrum of eigenvalues uniquely determines the differential equation, which in Schrödinger's theory is called the "amplitude equation". Paper [2] deals with the problem of determining the Hill Equation (or the one-dimensional Schrödinger equation) from its spectrum, as well as with deriving the Hill equation from specific properties of its discriminant.

Contribution 5 is devoted to the study of the asymptotic behavior of solutions of the Cauchy problem for a second-order hyperbolic equation with periodic coefficients as $t \to \infty$ in the case when the Hill operator is positive. To obtain this asymptotic expansion for $t \to \infty$, methods of the spectral theory of differential operators are used, as well as the properties of the spectrum of the Hill operator with periodic coefficients. We also note article [3], in which the asymptotic behavior as $t \to \infty$ of solutions of the initial boundary value problem for a second-order hyperbolic equation with periodic coefficients on the semi-axis is obtained.

In Contribution 6, which deals with an applied problem where a material has bimodular, functionally graded properties compared with the traditional materials typically used in classical Föppl–von Kármán equations. At the same time, when considering deformation, not only a large deflection is taken into account, as in classical Föppl–von Kármán equations, but also the larger rotation angle, which is incorporated by adopting the precise curvature formulas but not the simple second-order derivative term of the reflection.

The authors here use the biparametric perturbation method to solve the improved Föppl–von Kármán equation, in which improvements to the equations occur in both materials and deformations. To fully demonstrate the effectiveness of the proposed biparametric perturbation method, two sets of parameter combinations, one being a material parameter with central defection and the other being a material parameter with load, are used for the solutions of the improved Föppl–von Kármán equations.

The results obtained in this paper show that not only are two sets of solutions from different parameter combinations consistent, but they can also be reduced to a single-parameter solution to the disturbance. According to the authors, the successful application of the biparametric perturbation method provides new ideas for solving such nonlinear differential equations.

Contribution 7 considers the problem of the propagation of non-stationary longitudinal waves in an infinite viscoelastic layer of functionally graded materials with plane-parallel boundaries. It is worth noting that transient wave processes in viscoelastic structures made from functionally graded materials remain poorly studied. In addition, the physical and mechanical parameters of functionally graded materials continuously depend on the transverse coordinate, and the wave process propagates along this same coordinate. In Contribution 7, when studying the viscoelastic properties of the material, these properties are taken into account using linear Boltzmann–Volterra integral relations, and the viscoelastic layer of functionally graded materials is replaced by a piecewise homogeneous layer consisting of a large number of sublayers.

The authors of this paper also construct a solution to a non-stationary dynamic problem for a piecewise homogeneous layer and use an example to show the convergence of the results with an increase in the number of sublayers in the approximating piecewise homogeneous layer. Further, they consider the problem of the propagation of non-stationary longitudinal waves in the cross-section of an infinitely long viscoelastic hollow cylinder of functionally graded materials, where the properties of the material are continuously changing along the radius.

In Contribution 8, the authors consider a problem concerning discontinuities in the solutions of mathematical physics, which describe real processes and are not observed in experiments. At the same time, the appearance of discontinuities is associated with differential calculus, based on the analysis of infinitesimal quantities.

The authors also introduce nonlocal functions and nonlocal derivatives, which, unlike the traditional approach to a point, are not given, but are the results of averaging over small but finite intervals of the independent variable. Note that the classical equations of mathematical physics retain their form and include nonlocal functions. The approach to studying problems proposed by the authors leads to continuous solutions to singular problems in mathematical physics, and examples include the problem of string oscillation, the problem of a circular membrane loaded with concentrated forces, etc., which means that the analytical results are confirmed by experimental data.

Contribution 9 is devoted to physical and mathematical modeling and analytical prediction of the state of stress and strain in non-prismatic slender elastic continua. Non-prismatic slender continua are the prototypical models of many structural elements used in engineering applications. Structural components of wind turbine blades and towers, as well as load-bearing elements of aircraft wings and civil bridges, are a few examples. Elements with non-prismatic shapes are widely employed in both the industrial and civil engineering sectors for their superior structural efficiency compared with prismatic elements.

A paradigmatic prismatic element for which a few closed-form solutions is known as the classical de Saint-Venant's cylinder. The state of stress and strain in the cross-sections orthogonal to the axis of a non-prismatic cylinder is characterized by non-trivial terms that explicitly depend on the cross-sectional taper and that cannot be predicted via stepped-beam approaches based on the results known for de Saint-Venant's cylinder.

A variational principle provides the field equations that govern the mechanical behavior of such continua. The continua include a set of partial differential equations (PDEs) with Neumann-type boundary conditions that explicitly account for the influence of cross-sectional taper on the cross-sectional stress and strain fields, represent a generalization of the PDEs that governs the state of stress and strain in de Saint-Venant's cylinder, and reduce exactly to these equations in the prismatic case. PDEs with appropriate boundary conditions are the prototypical mathematical model of many physical problems, but they cannot be solved in closed forms in general. This is also the case for the PDE problem at hand, which can always be solved numerically, but admits closed-form solutions in only a few cases. Paradigmatic examples in which closed-form analytical solutions for the problem at hand are obtainable in terms of stresses, strains, or related mechanical quantities of interest for engineering applications are presented and discussed. As shown in papers [4,5], the analytical prediction of the state of stress and strain in non-prismatic slender continua is much more difficult than that in prismatic ones.

Fourteen manuscripts were submitted for publication in our Special Issue, all of which underwent a rigorous process of review for sustainability and scientific merit. As a result, only nine of the best papers were accepted for publication.

I hope that this Special Issue will inspire young talents with natural ambitions to make important and new discoveries in the field of Mathematics in Differential Equations of Mathematical Physics and Related Problems of Mechanics.

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