The Equilibrium Solutions for a Nonlinear Separable Population Model

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Abstract: The paper investigates a nonlinear model that describes population dynamics with an age structure. The fertility rate, which varies with age, follows a nonconstant pattern. The model exhibits a multiplicative structure for both fertility and mortality rates. Remarkably, this multiplicative structure renders the model separable. In this setting, it is shown that the number of births in unit time can be expressed using a system of nonlinear ordinary differential equations. The asymptotic behavior of solutions to this system has been established for a specific case. This result is significant because it provides a mathematical framework for understanding the dynamics of birth rates in certain settings. Furthermore, this paper explicitly identifies the steady-state solution and the equilibrium solution. As in any research paper, new directions of study remain open.

Keywords: population dynamics; equilibrium density function

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1. Introduction

Our paper investigates a population model with an age structure. This model and its variants have been studied in numerous works, including [1–12]. A quick review of these works is necessary. The work of [1] focuses on mathematical models in population biology and epidemiology. The second edition of their book [2] provides valuable insights for understanding population dynamics. It covers essential concepts and applications in the field. The work of Hoppensteadt [3] examines the mathematical theories of populations, including demographics, genetics, and epidemics. The author of [4] explores the mathematical aspects of modeling cell population dynamics. The study is relevant to computational biology. The book by Iannelli [6] provides a rigorous mathematical theory of age-structured population dynamics and insights into population modeling. In the work of [7], the authors introduce a population pyramid dynamics model with an analytical solution. In [8], the authors apply mathematics to medical problems, particularly in the context of population dynamics. In 1911, the authors of [9] examined a problem of age distribution. The work explores the variability in age distributions in populations and the concept of stable types. The study laid the groundwork for understanding population dynamics. In 2020, [10] proposed a model of population with an age structure. The study investigates a linear continuous model related to the McKendrick model, which describes the age structure in population dynamics. The book by Webb [11], published in 1985, analyzes the mathematical models for age-structured populations. It explores nonlinear effects and density-dependent birth rates.
All the above references cover a wide range of topics related to age-structured population dynamics, including predation, cannibalism, and mathematical modeling. However, a more extensive body of literature on this topic has been published so far (see also [13–15]). In what follows, we will consider only those papers which are of significant relevance to our research.

The seminal work of [8] considers the age structure in the dynamics of a one-sex population model, assuming that female population dynamics can be modeled as a function of two variables, namely age and time. This model takes as inputs an age-specific mortality intensity and an age-specific fertility function. The number of individuals at a given time whose age is lower than a certain age is given by an integral of a function $p(a, t)$ of two variables (age and time). The size of the population (the total number of individuals) can also be obtained by integrating $p(a, t)$. It appears only natural to find $p(a, t)$, which is described by a system of integral and differential equations. This system is shown to be reduced to a Volterra integral equation. The latter has a unique solution established by a fixed-point theorem, but this solution is not available in closed form, and its numerical computation based on fixed-point iterations seems complicated. The next class of population models is the nonlinear models obtained in situations where mortality and fertility are functions of age and population size (see [16] for more on this). We point out that the book [16] presents the basic approach to age-structured population dynamics, covering essential concepts and applications.

Let us analyze three seminal works in the paradigm of nonlinear population models. The first work to incorporate population-size-dependent mortality and fertility rates (rendering the model as nonlinear) is [5]. The analysis contributes to understanding population behavior.

The above paper also characterized equilibrium population and established a condition for the equilibrium to be locally asymptotically stable. A special class of models, the separable ones, is considered in that paper. Ref. [12] shows that the stability classification depends in many cases on the marginal birth and death rates as measures of the sensitivity of fertility and mortality; in other cases more information is required to determine the stability. The work of [13] investigated the local stability of an age-structured population with density-dependent fertility and mortality. The research, published in 1987, provides insights into equilibrium population configurations.

In what follows, the contributions of our paper will be considered. Our setting is a nonlinear population model, in a special case where mortality and fertility are separable functions of age and population size. The age part of the fertility is assumed as in [13], which is a model with a nonconstant (in age) fertility rate. In this paradigm, we establish the existence of the steady state for the nonlinear equations characterizing the population dynamics. Moreover, these are shown to be equivalent to a nonlinear system of differential equations. The latter is shown to have an equilibrium solution, which we find explicitly. The equilibrium stability analysis can also be established.

The remainder of this paper is organized as follows: Section 2 presents the population models, both linear and nonlinear. Section 3 presents our nonlinear model and the main results of the paper. An illustrative example is provided in Section 4. The paper continues with Section 5, which presents some findings regarding the asymptotic behavior of solutions for the considered model. Some remarks are provided in Section 6. The work ends with conclusions and new directions of study in Section 7.

2. Population Models

Firstly, the linear population model will be introduced. The exposition here follows [16]. The dynamics of the population is expressed in terms of the density of the population of age $a$, at time $t$, denoted as $p(a, t)$. The total population at $t$, denoted by $P(t)$, can be obtained by integrating its density, i.e.,

$$P(t) = \int_0^\infty p(a, t) da.$$
Next let us introduce fertility and mortality. The number of offspring, borne by individuals during the infinitesimal time interval \([t, t + dt]\), and the infinitesimal age interval \([a, a + da]\) is

\[ \beta(a, t), \]

also referred to as the age-specific fertility. The number of offspring during the infinitesimal interval \([t, t + dt]\) is then

\[ B(t) = \int_0^\infty \beta(a, t)p(a, t)da. \]

The number of deaths of individuals during the infinitesimal time interval \([t, t + dt]\) and the infinitesimal age interval \([a, a + da]\) is

\[ \mu(a, t). \]

The number of deaths during the infinitesimal interval \([t, t + dt]\) is then

\[ D(t) = \int_0^\infty \mu(a, t)p(a, t)da. \]

The probability that an individual of age \(a - x\) at time \(t - x\) will survive up to time \(t\) (with age \(a\)) is given by

\[ \pi(a, t, x) = e^{- \int_0^x \mu(a - \sigma, t - \sigma) d\sigma}. \]

In the case of time-independent mortality,

\[ \pi(a) = e^{- \int_0^a \mu(\sigma)d\sigma}, \]

is the probability for a newborn to survive to age \(a\), also known as the survival probability.

In the following, we will derive the linear Lotka–McKendrick Equation. The fertility and mortality rates \(\beta(a)\) and \(\mu(a)\) are assumed to be time-independent, as they only depend on the age \(a\). The number of individuals with age younger than \(a\) at time \(t\), denoted by \(N(a, t)\), is given by

\[ N(a, t) = \int_0^a p(\sigma, t)d\sigma. \]

Next, let us examine the number of individuals with age lower than \(a + h\) at time \(t + h\), i.e., \(N(a + h, t + h)\). This number will comprise \(N(a, t)\) and all the newborn in the time interval \([t, t + h]\) (their age will be lower than \(a + h\)), which is

\[ \int_t^{t+h} B(s)ds. \]

One then needs to adjust for the deaths of newborn, through the time interval \([t, t + h]\), and the deaths on \([t, t + h]\) of individuals older than \(a\), and this number of deaths is

\[ \int_0^h \int_0^{a+s} \mu(\sigma)p(\sigma, t + s)d\sigma ds. \]

This is the case because

\[ \int_0^{a+s} \mu(\sigma)p(\sigma, t + s)d\sigma, \]

which gives the number of individuals who die at \(t + s\) at an age younger than \(a + s\). Therefore,

\[ N(a + h, t + h) = N(a, t) + \int_t^{t+h} B(s)ds - \int_0^h \int_0^{a+s} \mu(\sigma)p(\sigma, t + s)d\sigma ds. \]
By differentiating this with respect to $h$, and then taking $h = 0$, we obtain
\[ p(a, t) + \int_0^a p_t(\sigma, t)d\sigma = B(t) - \int_0^a \mu(\sigma)p(\sigma, t)d\sigma. \]

Next, by differentiating with respect to $a$, we obtain
\[ p(a, t) + p_a(a, t) + \mu(a)p(a, t) = 0. \]

Also, setting $a = 0$ yields
\[ p(0, t) = B(t), \]

but on the other hand,
\[ B(t) = \int_0^\infty \beta(\sigma)p(\sigma, t)d\sigma. \]

Consequently, we obtained the following system:
\[
\begin{align*}
    p_a(a, t) + p_t(a, t) + \mu(a)p(a, t) &= 0 & \text{for } a, t \geq 0, \\
    p(0, t) &= \int_0^\infty \beta(\sigma)p(\sigma, t)d\sigma & \text{for } t > 0, \\
    p(a, 0) &= p_0(a) & \text{for } a > 0.
\end{align*}
\]

1. A Special Linear Population Model

Inspired by [17–19], recent work [20] considers a model with a pseudo-exponential survival probability $\pi(a)$, i.e.,
\[ \pi(a) = \sum_{i=1}^n c_i e^{-\mu_i a}, \]
for positive constants $c_i, \mu_i$ with
\[ \sum_{i=1}^n \mu_i^2 \neq 0, c_1 \neq 0 \text{ and } \sum_{i=1}^n c_i = 1. \]

Moreover, we present the case of
\[ \pi(a) = \left( \sum_{i=0}^n c_i a^i \right) e^{-\mu_1 a}, c_0 \neq 0 \text{ and } \sum_{i=1}^n c_i = 1. \]

In this setting, finding $p(a, t)$ is reduced to a linear ODE system. In special situations, (2) and (3), a closed-form solution is obtained by means of Laplace transform.

2.2. Nonlinear Population Models

These are models in which the death rate $\mu(a, p)$ and the fertility rate $\beta(a, p)$ are functions of age and population size. The age density function at time $t \geq 0$, $p(a, t)$ satisfies the following nonlinear equations:
\[
\begin{align*}
    p_t(a, t) + p_a(a, t) + \mu(a, P(t))p(a, t) &= 0 \\
    p(0, t) &= \int_0^\infty \beta(\sigma)\mu(\sigma, P(t))p(\sigma, t)d\sigma \\
    p(a, 0) &= p_0(a) \\
    P(t) &= \int_0^\infty p(\sigma, t)d\sigma,
\end{align*}
\]

where $P(t)$ is the total population at $t$. As shown in [16], this system of nonlinear equations can be reduced to integral equations. Specifically, let
\[ \pi(a, t, x, P) = e^{-\int_0^x \mu(a, t, P(\sigma))d\sigma} \]
and
\[ p(a, t) = \begin{cases} \frac{p_0(t - a)\pi(a, t, t, P)}{B(t - a)\pi(a, t, a, P)} & \text{for } t \leq a \\ \frac{B(t - a)\pi(a, t, t, P)}{p_0(t - a)\pi(a, t, a, P)} & \text{for } t > a. \end{cases} \]

Here, the function \( B \) solves the following system of equations:
\[
\begin{cases}
B(t) = \int_0^t \beta(\sigma, P(t))\pi(\sigma, t, \sigma, P)B(\sigma)d\sigma + F(t, P) \\
P(t) = \int_0^\infty \pi(\sigma, t, \sigma, P)B(\sigma)d\sigma + G(t, P)
\end{cases}
\]
(5)

where
\[
\begin{cases}
F(t, P) = \int_0^\infty \beta(\sigma, P(t))\pi(\sigma, t, \sigma, P)p_0(\sigma - t)d\sigma \\
G(t, P) = \int_0^\infty \pi(\sigma, t, \sigma, P)p_0(\sigma - t)d\sigma.
\end{cases}
\]
(6)

This system (5) can be solved through the following iterative method:
\[
\begin{cases}
B^{k+1}(t) = \int_0^t \beta(\sigma, P^k(t))\pi(\sigma, t, \sigma, P^k)B^k(\sigma)d\sigma + F(t, P^k) \\
p^{k+1}(t) = \int_0^\infty \pi(\sigma, t, \sigma, P^k)B^k(\sigma)d\sigma + G(t, P^k)
\end{cases}
\]
(7)

see [16] for more details on this. The next subsection presents a special class of nonlinear population models known as separable population models.

2.3. Separable Population Models

We will now particularize the nonlinear model by choosing the following fertility and mortality rates:
\[
\beta(a, p) = R_0\beta_0\Phi(p)e^{-\rho p}, \quad \mu(a, p) = \mu_0 + \Psi(p).
\]

By substituting this into the nonlinear system, one obtains
\[
\begin{cases}
p_0(a, t) + p(a, t) + (\mu_0 + \Psi(P(t)))p(a, t) = 0 \\
p(0, t) = \int_0^\infty R_0\Phi(P(t))e^{-\rho p}p(\sigma, t)d\sigma \\
p(a, 0) = p_0(a) \\
P(t) = \int_0^\infty p(\sigma, t)d\sigma.
\end{cases}
\]

These nonlinear equations can be reduced to the following ordinary differential equation (ODE) system:
\[
\begin{cases}
P'(t) = -(\mu_0 + \Psi(P(t)))P(t) + R_0\beta_0\Phi(P(t))Q(t) \\
Q'(t) = (R_0\beta_0\Phi(P(t)) - \rho - \mu_0 - \Psi(P(t)))Q(t)
\end{cases}
\]
(8)

where
\[ Q(t) = \int_0^\infty e^{-\rho p}p_1(a, t)da, \]

see [16] for more details.

Now let us focus on the existence of steady states. The net reproduction rate is defined as
\[
R(x) = R_0\Phi(x)\int_0^\infty \beta_0e^{-(\rho + \mu_0 + \Psi(x))a}da.
\]

This quantity was introduced first by [5]. In accordance with [5], the quantity \( R(x) \) represents the number of children expected to be born by an individual when the population is \( x \). The steady state \( P^* \) is given by the following equation:
\[
R(P^*) = 1,
\]
which in this setting reads
\[ R_0 \beta_0 \Phi(P^*) = \rho + \mu_0 + \Psi(P^*). \]

3. A Special Nonlinear Population Model

We consider the following logistic system:
\[
\begin{cases}
  p_t(a, t) + p_a(a, t) + \mu_0 p(a, t) + \Psi(P(t))p(a, t) = 0 \\
  p(0, t) = R_0 \Phi(P(t)) \int_0^\infty \sum_{i=0}^{n-1} \beta_i \sigma^i e^{-\rho \sigma} p(\sigma, t) d\sigma \\
  p(a, 0) = p_0(a) \\
  P(t) = \int_0^\infty p(\sigma, t) d\sigma
\end{cases}
\]
where \( p(a, t) \) is the age density function at time \( t \geq 0 \), \( P(t) \) is the total population at time \( t \), \( p_0(a) \) is the initial population at time \( t = 0 \), \( a \in [0, \infty) \), \( \mu_0 > 0 \) is the intrinsic mortality term, \( R_0 > 0 \) is the birth modulus, and \( \rho, \sigma \) are prescribed positive parameters. In the real world, the age profile is given by \( p(a, t) \).

The fertility is given by
\[
\beta(a, p) = R_0 \Phi(p) \sum_{i=0}^{n-1} \beta_i a^i e^{-\rho a},
\]
with \( \beta_0, \beta_1, \ldots, \beta_{n-1} \in (0, \infty) \).

The mortality (the death rate) is given by
\[
\mu(a, P) = \mu_0 + \Psi(P).
\]

The pair \( (\beta(a, p), \mu_0) \) represents an age-dependent intrinsic birth–death process, while \( \Psi(P(t)) \) models an external mortality that is uniform across all ages and depends only on the weighted sizes.

We begin by assuming that \( \Phi \) and \( \Psi \) are continuous on \( [0, \infty) \) and continuously differentiable on \( (0, \infty) \) and
\[
\Phi(x) \geq 0, \Phi'(x) < 0, \Phi(0) = 1, \Phi(+\infty) = 0,
\]
\[
\Psi(x) \geq 0, \Psi'(x) > 0, \Psi(0) = 0, \Psi(+\infty) = +\infty.
\]

We also adopt the normalization condition
\[
\int_0^\infty \sum_{i=0}^{n-1} \beta_i a^i e^{-(\rho + \mu_0) a} da = 1.
\]

By using the Gamma integral
\[
\Gamma(p) = \int_0^\infty x^{p-1} e^{-x} dx, \quad p > 0,
\]
in (12), we can obtain
\[
\sum_{i=0}^{n-1} \frac{\beta_i i!}{(\rho + \mu_0)^{i+1}} = 1.
\]
which allows the parameter $R_0$ in (9) to play the role of an intrinsic basic reproduction number denoted by $R(x)$ in what follows next. Regarding the existence of steady states, the net reproduction rate at size $x$ takes the following form:

$$
R(x) = R_0 \Phi(x) \int_0^\infty \sum_{i=0}^{n-1} \beta_i \int_0^\infty d \alpha e^{-i(\rho + \mu_0 + \Psi(x)) \alpha} da
$$

$$
= R_0 \Phi(x) \sum_{i=0}^{n-1} \beta_i \int_0^\infty d \alpha e^{-i(\rho + \mu_0 + \Psi(x)) \alpha} \frac{dt}{(\rho + \mu_0 + \Psi(x))^{i+1}}
$$

$$
= R_0 \Phi(x) \sum_{i=0}^{n-1} \beta_i \int_0^\infty \frac{dt}{(\rho + \mu_0 + \Psi(x))^{i+1}}
$$

within our setting, as described on page 154 in [16] or page 288 in [5]. One initial observation about assumptions (10) and (11) is that

$$
\lim_{x \to +\infty} R(x) = 0 \text{ and } R(x) \text{ is a decreasing function (i.e., } R'(x) < 0). \tag{14}
$$

Indeed, $R(x)$ is a decreasing function because

$$
R'(x) = \sum_{i=0}^{n-1} \beta_i i! \left[ R_0 \Phi'(x)(\rho + \mu_0 + \Psi(x)) - R_0 \Phi(x) \Psi'(x) \right] - \sum_{i=0}^{n-1} i \beta_i i! R_0 \Phi(x) \Psi'(x) < 0,
$$

for all $x \geq 0$. Moreover,

$$
\lim_{x \to +\infty} R(x) = 0,
$$

is satisfied in light of the asymptotic conditions on $\Psi, \Phi$.

As is commonly known (see page 154 in [16] or [5] [Theorem 6, pages 288-289]), since we have a single weighted size $P$, we use the fact that nontrivial stationary sizes $P^*$ must satisfy

$$
R(P^*) = 1. \tag{15}
$$

This condition is both necessary and sufficient for nontrivial stationary sizes to exist with total population $P^*$.

Next, let us focus on the task of finding $P(t)$. We will denote this by

$$
P_i(t) = \int_0^\infty \sigma^{i-1} e^{-\rho \sigma} p(\sigma, t) d\sigma, \text{ for } i = 1, 2, \ldots, n.
$$

The subsequent step is to note that the renewal condition, the total birth rate, or the fertility rate at time $t$ can be expressed in the new notations as follows:

$$
p(0, t) = R_0 \Phi(P(t)) \sum_{i=0}^{n-1} \beta_i \int_0^\infty \sigma^{i-1} e^{-\rho \sigma} p(\sigma, t) d\sigma
$$

$$
= R_0 \Phi(P(t)) \sum_{i=0}^{n-1} \beta_i P_{i+1}(t).
$$
Furthermore, the first derivative of \( P(t) \) yields

\[
P'(t) = \int_{0}^{\infty} p_1(a,t) \, da
\]

\[
= - \int_{0}^{\infty} p_0(a,t) \, da - (\mu_0 + \Psi(P(t))) \int_{0}^{\infty} p(a,t) \, da
\]

\[
= p(0,t) - (\mu_0 + \Psi(P(t))) P(t)
\]

\[
= -(\mu_0 + \Psi(P(t))) P(t) + R_0 \Phi(P(t)) \sum_{i=0}^{n-1} \beta_i P_{i+1}(t)
\]

and likewise

\[
P'_1(t) = \int_{0}^{\infty} e^{-\alpha p_1} p_1(a,t) \, da
\]

\[
= - \int_{0}^{\infty} e^{-\alpha p_0} p_0(a,t) \, da - (\mu_0 + \Psi(P(t))) \int_{0}^{\infty} e^{-\alpha p} p(a,t) \, da
\]

\[
= p(0,t) - \rho \int_{0}^{\infty} e^{-\alpha p_0} p_0(a,t) \, da - (\mu_0 + \Psi(P(t))) P_1(t)
\]

\[
= R_0 \Phi(P(t)) \sum_{i=0}^{n-1} \beta_i P_{i+1} - \rho P_1(t) - (\mu_0 + \Psi(P(t))) P_1(t)
\]

\[
= (R_0 \beta_0 \Phi(P(t)) - \rho - \mu_0 - \Psi(P(t))) P_1(t) + R_0 \Phi(P(t)) \sum_{i=0}^{n-1} \beta_i P_{i+1}.
\]

Similarly, the first derivative of \( P_{i+1}(t) (i = 1, ..., n - 1) \) can be calculated to obtain

\[
P'_{i+1}(t) = \int_{0}^{\infty} a^i e^{-\alpha a} p_i(\sigma,t) \, d\sigma
\]

\[
= - \int_{0}^{\infty} a^i e^{-\alpha a} p_0(\sigma,t) \, d\sigma - (\mu_0 + \Psi(P(t))) \int_{0}^{\infty} a^i e^{-\alpha a} p(a,t) \, d\sigma
\]

\[
= \int_{0}^{\infty} \left( i a^{i-1} e^{-\alpha a} - a^i \rho e^{-\alpha a} \right) p(\sigma,t) \, d\sigma - (\mu_0 + \Psi(P(t))) \int_{0}^{\infty} a^i e^{-\alpha a} p(a,t) \, d\sigma
\]

\[
= i \int_{0}^{\infty} \left( d^{-1} e^{-\alpha a} - a^i \rho e^{-\alpha a} \right) p(a,t) \, d\sigma - (\mu_0 + \Psi(P(t))) \int_{0}^{\infty} a^i e^{-\alpha a} p(a,t) \, d\sigma
\]

\[
= i P_i(t) - \rho P_{i+1}(t) - (\mu_0 + \Psi(P(t))) P_{i+1}(t)
\]

\[
= i P_i(t) - (\rho + \mu_0 + \Psi(P(t))) P_{i+1}(t).
\]

Our objective is to obtain the existence of solutions to model (9). To achieve this, we are led to the system of first-order differential equations

\[
\begin{align*}
P'(t) &= -(\mu_0 + \Psi(P(t))) P(t) + R_0 \Phi(P(t)) \sum_{i=0}^{n-1} \beta_i P_{i+1} \\
P'_1(t) &= (R_0 \beta_0 \Phi(P(t)) - \rho - \mu_0 - \Psi(P(t))) P_1(t) + R_0 \Phi(P(t)) \sum_{i=0}^{n-1} \beta_i P_{i+1} \\
P'_{i+1}(t) &= i P_i(t) - (\rho + \mu_0 + \Psi(P(t))) P_{i+1}(t), \quad i = 1, ..., n - 1 \text{ and } n \geq 2
\end{align*}
\]

coupled with the initial conditions

\[
P(0) = P_0, \quad P_1(0) = P_0, \quad i = 1, 2, ..., n
\]

where

\[
P(0) = \int_{0}^{\infty} p_0(a) \, da, \quad P_i(0) = \int_{0}^{\infty} a^i e^{-\alpha a} p_0(a) \, da, \quad i = 1, 2, ..., n.
\]

We are now in a position to state and prove the following four theorems, which are the main results of Section 3.

**Theorem 1.** Every solution to system (16) is also a solution to the ordinary nonlinear system (9), and vice versa.
Proof of the Theorem 1. Studying the existence of solutions for system (16) is equivalent to studying the existence of solutions to (9) because, if the pair 

\[(P(t), P_1(t), ..., P_n(t))\]
solves (16), then by setting

\[B(t) = p(0, t) = R_0 \Phi(P(t)) \sum_{i=0}^{n-1} \beta_i P_{i+1}(t)\]

we obtain the solution to (9) via the usual formula

\[p(a, t) = \begin{cases} 
  p_0(a - t) e^{-\int_0^t (\mu_0 + \Psi(P(\sigma))) d\sigma} & a \geq t, \\
  B(t - a) e^{-\int_0^{t-a} (\mu_0 + \Psi(P(\sigma))) d\sigma} & a < t.
\end{cases}\]

Thus, to prove the existence of solutions to model (9), we can focus on analyzing (16). In the following theorem, we are interested in the existence of stationary solutions to this problem (16). These solutions may not be unique for certain parameter values and may lead to complex bifurcations. In the model of the problem, any stationary solution is referred to as an equilibrium density function. In a nonlinear model for population dynamics with an age structure, the equilibrium point is a state where the population remains constant over time.

The next theorem provides the conditions for nontrivial equilibrium solutions.

Theorem 2. If \(R_0 > 1\), then a unique nontrivial equilibrium solution \((P^*, P_1^*, ..., P_n^*)\) of (16) exists.

Proof of the Theorem 2. The equilibrium solution \((P^*, P_1^*, ..., P_n^*)\) of (16) is given by

\[\begin{align*}
0 &= -(\mu_0 + \Psi(P^*)) P^* + R_0 \Phi(P^*) \sum_{i=0}^{n-1} \beta_i P_{i+1}^* \\
0 &= (R_0 \beta_0 \Phi(P^*) - \rho - \mu_0 - \Psi(P^*)) P_1^* + R_0 \Phi(P^*) \sum_{i=1}^{n-1} \beta_i P_{i+1}^* \\
0 &= i P_i^* - (\rho + \mu_0 + \Psi(P^*)) P_{i+1}^*, \quad i = 1, ..., n-1.
\end{align*}\]

(18)

To solve nonlinear algebraic system (18), we focus on the third equation \((i = 1, ..., n-1)\), from which we obtain the following successively:

\[P_{i+1}^* = \frac{i}{(\rho + \mu_0 + \Psi(P^*))} P_i^* = \frac{i}{\rho + \mu_0 + \Psi(P^*)} P_{i-1}^* = \frac{i!}{(\rho + \mu_0 + \Psi(P^*))} P_1^*.

The subsequent step is to substitute the determined quantities

\[P_{i+1}^* = \frac{i!}{(\rho + \mu_0 + \Psi(P^*))} P_{i+1}^*, \quad i = 1, ..., n-1.

(19)\]
into the first equation of (18). By equivalence, we obtain

\[ 0 = -(\mu_0 + \Psi(P^*))P^* + R_0\Phi(P^*)\beta_0P_1^* + R_0\Phi(P^*)\frac{1}{n} \sum_{i=1}^{n-1} \beta_iP_i^* \\
\iff
0 = -(\mu_0 + \Psi(P^*))P^* + R_0\Phi(P^*)\beta_0P_1^* + R_0\Phi(P^*)\frac{n-1}{\sum_{i=1}^{n} (\rho + \mu_0 + \Psi(P^*)^i)P_i^*} \\
\iff
0 = -(\mu_0 + \Psi(P^*))P^* + \left[ \beta_0 + \frac{n-1}{\sum_{i=1}^{n} (\rho + \mu_0 + \Psi(P^*)^i)} \beta_i \right] P_1^* \\
\]

Equation

\[ \frac{n-1}{\sum_{i=0}^{n-1} \frac{\beta_i}{(\rho + \mu_0 + \Psi(P^*)^i)}} = 1, \quad (20) \]

can be rewritten for nontrivial stationary sizes \( P^* \) in the following form:

\[ \frac{n-1}{\sum_{i=0}^{n-1} \frac{\beta_i}{(\rho + \mu_0 + \Psi(P^*)^i)}} = \frac{\rho + \mu_0 + \Psi(P^*)}{R_0\Phi(P^*)}. \quad (21) \]

Finally, since nontrivial stationary sizes \( P^* \) are given by (21), we can deduce from the above equation that

\[ P_1^* = \frac{(\mu_0 + \Psi(P^*))}{\left[ \beta_0 + \frac{n-1}{\sum_{i=1}^{n-1} \frac{\beta_i}{(\rho + \mu_0 + \Psi(P^*)^i)}} \right]} P^* = \frac{(\mu_0 + \Psi(P^*))}{\rho + \mu_0 + \Psi(P^*)} P^* \quad (22) \]

The existence of a nontrivial stationary solution for system (16) can be expressed as

\[ P_1^* = \frac{\mu_0 + \Psi(P^*)}{\rho + \mu_0 + \Psi(P^*)} P^* \quad \text{and} \quad P_{i+1}^* = \frac{i!}{(\rho + \mu_0 + \Psi(P^*))^i} P_i^*, \quad i = 1, \ldots, n-1, \quad (23) \]

if the second equation in (18) is checked by (23). However, this is a straightforward exercise due to the following equivalence:

\[ 0 = (R_0\beta_0\Phi(P^*) - \rho - \mu_0 - \Psi(P^*))P_1^* + R_0\Phi(P^*)\sum_{i=1}^{n-1} \beta_iP_i^* \\
\iff
0 = (R_0\beta_0\Phi(P^*) - \rho - \mu_0 - \Psi(P^*))P_1^* - R_0\Phi(P^*)\beta_0P_1^* + R_0\Phi(P^*)\sum_{i=1}^{n-1} \beta_iP_i^* \\
\iff
0 = (-\rho - \mu_0 - \Psi(P^*))P_1^* + \left[ \beta_0 + \frac{n-1}{(\rho + \mu_0 + \Psi(P^*)^i)} \beta_i \right] R_0\Phi(P^*) P_1^* \\
\iff
0 = (-\rho - \mu_0 - \Psi(P^*))P_1^* + \frac{\rho + \mu_0 + \Psi(P^*)}{R_0\Phi(P^*)^i} R_0\Phi(P^*) P_1^*. \]

We finally present an alternative proof for the positivity of the equilibrium. Since \( R(x) \) is a bijection, its inverse \( R^{-1}(x) \) exists and is a decreasing function. Indeed,

\[ R^{-1}(x) = \frac{1}{R(R^{-1}(x))} \]

and \( R'(t) < 0 \) for all \( t \geq 0 \), and we have that \( R^{-1}(x) < 0 \) for all \( x \geq 0 \), i.e., \( R^{-1} \) is a decreasing function. Finally, it is worth noting that (15) implies

\[ P^* = R^{-1}(1). \quad (24) \]
On the other hand, from (13), we have
\[ R(0) = R_0 \implies 0 = R^{-1}(R_0). \]  
Due to the fact that \( R_0 > 1 \) and \( R^{-1} \) is a decreasing function, we have
\[ 0 = R^{-1}(R_0) < R^{-1}(1) = P^* \]  
where we have used (24) and (25). Clearly, \( P^* > 0 \) implies that \( P^*_i > 0 \) for all \( i = 1, ..., n \). \( \square \)

We state the third main theorem.

**Theorem 3.** If \( R_0 < 1 \), then the only trivial equilibrium exists.

**Proof of Theorem 3.** Clearly, system (16) has at least the trivial solution. Also, (26) proved that if \( R_0 < 1 \), then
\[ 0 = R^{-1}(R_0) > R^{-1}(1) = P^* \]  
and so only the trivial equilibrium exists. \( \square \)

Let us note that the stability of the equilibrium point \((P^*, P^*_1, ..., P^*_n)\) is determined by the sign of the real part of the eigenvalues of the Jacobian matrix of the system. Depending on the parameter combinations chosen, the model can show that stability as well as instability of the nontrivial equilibrium. The stability of the equilibrium point is also important, as it determines whether the population will return to the equilibrium point after a disturbance or continue to move away from it. Furthermore, the existence of periodic solutions can occur when passing from one case to the other. Hence, nonlinear models can have multiple equilibrium points, and the stability of each point can vary depending on the parameters of the model.

The final theorem in this section presents the necessary and sufficient conditions for the existence of a nontrivial equilibrium.

**Theorem 4.** One unique positive nontrivial equilibrium exists if and only if \( R_0 > 1 \).

**Proof of Theorem 4.** In this case, (15) becomes (20). Due to (14), this equation (20) has one, and only one, nontrivial solution if, and only if, \( R_0 > 1 \). The fact that the condition \( R_0 > 1 \) is necessary and sufficient for the existence of a nontrivial equilibrium implies that \( R_0 \) acts as a bifurcation parameter. Under the assumptions of the model, it is obvious that we have a forward bifurcation at the point \( R_0 \). The proof of positivity in **Theorem 2** can also confirm the result using the technique employed. \( \square \)

4. An Illustrative Example

We consider an illustrative example to validate the theoretical findings in **Theorem 2**. The specific data provided include the following parameters and functions:

\[
\begin{align*}
n & = 2, \quad \Phi(x) = e^{-x}, \quad \Psi(x) = x^2, \quad R_0 = 3 \\
\mu_0 & = 1, \quad \rho = 2, \quad \beta_0 = 1, \quad \beta_1 = 6.
\end{align*}
\]

In this scenario, algebraic equation (20) transforms into
\[
\frac{R_0 e^{-\rho' \beta_0}}{\rho + \mu_0 + P^*} + \frac{R_0 e^{-\rho' \beta_1}}{\rho + \mu_0 + P^*^2} - 1 = 0.
\]
Alternatively, after substituting our specific data,

\[
\frac{3e^{-P^*}}{(3 + P^{*2})} + \frac{18e^{-P^*}}{(3 + P^{*2})^2} - 1 = 0.
\]

To find the numerical solution for this equation, we can use an online equation solver such as Wolfram Alpha. The result is approximately

\[ P^* \simeq 0.788636. \]

Returning to (23), we obtain

\[
P_1^* = \frac{\mu_0 + \Psi(P^*)}{\rho \mu_0 + \Psi(P^*)} P^* = \frac{1 + (0.788636)^2}{3 + (0.788636)^2} \cdot 0.788636 \simeq 0.353159
\]

and

\[
P_2^* = \frac{1}{(\rho \mu_0 + \Psi(P^*))} P_1^* = \frac{1}{3 + (0.788636)^2} \cdot 0.353159 \simeq 0.097505.
\]

Finally, the equilibrium point is given by

\[ (P^*, P_1^*, P_2^*) = (0.788636, 0.353159, 0.097505). \]

The Jacobian of the system at this point \((P^*, P_1^*, P_2^*)\) is given by

\[
J = \begin{vmatrix}
-1 - 3P^{*2} - e^{-P^*} (3P_1^* + 18P_2^*) & 3e^{-P^*} & 18e^{-P^*} \\
(2P^* - 3e^{-P^*)} P_1^* - 18e^{-P^*} P_2^* & 3e^{-P^*} - 3 - (P^*)^2 & 18e^{-P^*} \\
-2P^* P_2^* & 1 & -\left(3 + (P^*)^2\right)
\end{vmatrix}.
\]

Evaluating the Jacobian at the equilibrium point, we obtain

\[
J = \begin{vmatrix}
-3.06679 & 1.36339278 & 8.18035671 \\
-1.83615 & -2.25855395 & 8.18035671 \\
-0.153791 & 1 & -3.621946
\end{vmatrix}
\]

and computing its eigenvalues, we observe that they are

\[ -1.3907 + 1.6467i, -1.3907 - 1.6467i, -6.1658. \]

Since all eigenvalues have a real part strictly smaller than zero, the nonlinear equilibrium point is asymptotically stable. This also confirms the following:

\[ \text{trace}\, J = -1 - 3P^{*2} - e^{-P^*} (3P_1^* + 18P_2^*) + 3\left(e^{-P^*} - 1\right) - (P^*)^2 - \left(3 + (P^*)^2\right) < 0. \]

In the next section, for the special case \(n = 1\) considered by [16], we study the asymptotic behavior at infinity of the solution.

5. The Asymptotic Behavior of the Solution for the System (16) as Time Tends to Infinity

We examine the special case where \(n = 1\) under the same assumptions used in Section 3. The ODE system (16) is as follows:

\[
\begin{align*}
P'(t) &= -(\mu_0 + \Psi(P(t)))P(t) + R_0 \beta_0 \Phi(P(t))P_1(t), \\
P_1'(t) &= (R_0 \beta_0 \Phi(P(t)) - \rho - \mu_0 - \Psi(P(t)))P_1(t).
\end{align*}
\]

(28)
According to Proposition 8.5, page 237 in [16], if
\[ P(0) > P_1(0) > 0, \]
then it follows that
\[ P(t) > P_1(t) > 0 \text{ for all } t \geq 0. \] (29)
From this observation, we can deduce the following results about the behavior of populations as time tends to +infinity.

**Theorem 5.** If \( R_0 < 1 \), then
\[ \lim_{t \to +\infty} P(t) = 0. \] (30)

**Proof of Theorem 5.** In light of \( P(t) > P_1(t) > 0 \), it follows that
\[ P'(t) < (R_0 \beta_0 \Phi(P(t)) - \rho - \mu_0 - \Psi(P(t)))P(t). \]
Moreover,
\[ R_0 \beta_0 \Phi(P(t)) - \rho - \mu_0 - \Psi(P(t)) < 0, \]
in light of
\[ R(P(t)) \leq R_0 < 1. \]
Thus, \( P(t) \) is decreasing. Let
\[ \lim_{t \to +\infty} P(t) = L, \]
where \( L \geq 0 \). Next, we show that \( L = 0 \). The condition \( R_0 < 1 \) implies that
\[ R_0 \beta_0 \Phi(P(t)) - \rho - \mu_0 - \Psi(P(t)) \leq (R_0 - 1)(\rho + \mu_0 + \Psi(P(t))). \]
Moreover,
\[ \rho + \mu_0 + \Psi(P(t)) \geq \rho + \mu_0 + \Psi(L). \]
Therefore,
\[ R_0 \beta_0 \Phi(P(t)) - \rho - \mu_0 - \Psi(P(t)) \leq (R_0 - 1)(\rho + \mu_0 + \Psi(L)) < 0 \]
as a result of a simple application of Gronwall inequality. In summary, the given result establishes the behavior of populations under specific conditions, and **Theorem 5 provides insights into the long-term behavior when \( R_0 < 1 \).**

**Theorem 6.** If \( R_0 > 1 \), then
\[ \lim_{t \to +\infty} P(t) \neq 0. \] (31)

**Proof of Theorem 6.** We consider two cases:

**Case 1.** If \( R(P(t)) > 1 \) for all \( t \geq 0 \), then the condition \( R(P(t)) > 1 \) implies that
\[ R_0 \beta_0 \Phi(P(t)) - \rho - \mu_0 - \Psi(P(t)) > 0, \]
and consequently, \( P'(t) > 0 \). This leads to the conclusion that \( t \to P_1(t) \) is strictly increasing, which, when combined with \( P(t) > P_1(t) > 0 \), yields the desired result.

**Case 2.** If \( R(P(t)) \leq 1 \) for all \( t \geq 0 \), this implies that
\[ P(t) > R^{-1}(1) = P^* > 0 \text{ for all } t \geq 0, \]
where we use the fact that \( R^{-1} \) is a decreasing function.

To summarize, the result presented in **Theorem 6** illuminates population behavior under specific conditions and provides valuable insights into long-term dynamics when
$R_0 > 1$. This is the case when $\lim_{t \to +\infty} P(t) \neq 0$, and then the population does not vanish at infinity.

6. Some Remarks

Regarding all the obtained results, we can make the following two observations:

**Remark 1.** Our model analysis can be expanded to encompass more general mortality patterns. Specifically, we consider the following mortality:

$$\mu(a, p) = \mu_0(t) + \Psi(p).$$

Here, $\mu_0(t)$ represents a deterministic function, which follows the Gompertz function

$$\mu_0(t) = a + be^{-ct},$$

for some constants $a, b$ and $c$. However, in such cases, the trivial equilibrium becomes the sole equilibrium point. The trivial equilibrium corresponds to a constant mortality rate that does not vary with age. Essentially, the system reaches a stable state where no further changes occur. In summary, our model allows us to explore mortality patterns beyond simple scenarios by considering both deterministic factors and additional influences. However, when using the Gompertz function, the equilibrium point simplifies to this stable, unchanging state.

Since in the article of [18] the fitting of combinations of exponentials to probability distributions is investigated, we are in position to give the following.

**Remark 2.** Our approach is versatile and can be extended to accommodate a broader class of fertility functions. Specifically, consider the following fertility:

$$\beta(a, p) = R_0\beta_0\Phi(p)F(a),$$

where $R_0$ represents a constant, $\beta_0$ is a baseline fertility rate, $\Phi(p)$ captures the influence of external factors, and $F(a)$ is a continuous function of age. Remarkably, we can approximate $F(a)$ using the expression

$$\frac{n-1}{\sum_{i=0}^{n-1} \beta_i e^{-\rho a}},$$

as a result of [18]. In summary, Dufresne’s research validates the utility of this adaptable approximation for modeling fertility patterns, allowing us to explore a broader spectrum of demographic scenarios.

Finally, we point out that in [17] a new polynomial exponential family is introduced. The study investigates its density, hazard rate functions, numerical properties, and applications to real-world datasets.

7. Conclusions and New Directions of Study

Our work primarily presents an equilibrium solution for a nonlinear ordinary differential equation system. As a result, it is demonstrated that the equilibrium solution solves a separable population model. The results are summarized in Theorems 1–4. Furthermore, in a specific case, Theorems 5 and 6 describe the asymptotic behavior of the solution at infinity for the resulting nonlinear ordinary differential equation system. New directions of study remain open. For instance, it is worth investigating whether Theorems 5 and 6 (specifically Case 1) continue to hold true in the broader context outlined in this research paper. In this context, our task is to demonstrate that for all $t \geq 0$, the following holds: $P(t) > P_1(t) > \ldots > P_n(t)$. This result was suggested by (27) and (29). Currently, we are not aware of any algorithm that can compute the equilibrium point $P^*$ under the general conditions specified in this paper, even though it may be possible to obtain it in exact form for some special cases.

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