

Article

The Augmented Weak Sharpness of Solution Sets in Equilibrium Problems

Ruyu Wang ¹, Wenling Zhao ^{1,*}, Daojin Song ¹ and Yaozhong Hu ²

¹ School of Mathematics and Statistics, Shandong University of Technology, Zibo 255049, China; 13121204020@stumail.sdut.edu.cn (R.W.); djs@sdut.edu.cn (D.S.)

² Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB T6G 2G1, Canada; yaozhong@ualberta.ca

* Correspondence: wenlingzhao@sdut.edu.cn

Abstract: This study considers equilibrium problems, focusing on identifying finite solutions for feasible solution sequences. We introduce an innovative extension of the weak sharp minimum concept from convex programming to equilibrium problems, coining this as weak sharpness for solution sets. Recognizing situations where the solution set may not exhibit weak sharpness, we propose an augmented mapping approach to mitigate this limitation. The core of our research is the formulation of augmented weak sharpness for the solution set. This comprehensive concept encapsulates both weak sharpness and strong non-degeneracy within feasible solution sequences. Crucially, we identify a necessary and sufficient condition for the finite termination of these sequences under the premise of augmented weak sharpness for the solution set in equilibrium problems. This condition significantly broadens the scope of the existing literature, which often assumes the solution set to be weakly sharp or strongly non-degenerate, especially in mathematical programming and variational inequality problems. Our findings not only shed light on the termination conditions in equilibrium problems but also introduce a less stringent sufficient condition for the finite termination of various optimization algorithms. This research, therefore, makes a substantial contribution to the field by enhancing our understanding of termination conditions in equilibrium problems and expanding the applicability of established theories to a wider range of optimization scenarios.



Citation: Wang, R.; Zhao, W.; Song, D.; Hu, Y. The Augmented Weak Sharpness of Solution Sets in Equilibrium Problems. *Mathematics* **2024**, *12*, 352. <https://doi.org/10.3390/math12020352>

Academic Editor: Marjan Mernik

Received: 20 December 2023

Revised: 18 January 2024

Accepted: 20 January 2024

Published: 22 January 2024



Copyright: © 2024 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

Keywords: equilibrium problems; feasible solution sequence; regular normal cone; weak sharpness; augmented weak sharpness; finite termination

MSC: 90C31; 49J40; 49K40; 91A10; 49M37

1. Introduction

In this paper, we explore the equilibrium problem denoted as $EP(\phi, S)$:

$$\text{Find } \bar{x} \in S \text{ such that } \phi(\bar{x}, y) \geq 0, \quad \forall y \in S,$$

where $S \subset \mathbb{R}^n$ is a closed convex set, and $\phi : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a function such that

$$\phi(x, x) = 0, \quad \forall x \in S. \quad (1)$$

Let $\bar{S} = \{x \in S \mid \phi(x, y) \geq 0, \quad \forall y \in S\} \neq \emptyset$ be the solution set of $EP(\phi, S)$ and

$$\tilde{S} = \{x \in S \mid \exists u \in \partial_y \phi(x, x), \text{ such that } \langle u, y - x \rangle \geq 0, \quad \forall y \in S\}$$

is the stationary points set of $EP(\phi, S)$, where $\partial_y \phi(x, x)$ is the generalized sub-differential at x of $\phi(x, \cdot)$ (Definition 8.3 [1]), or the sub-differential for short.

The relation between $\bar{S}$ and $\tilde{S}$ is discussed, where it is noted that while the general relation

$$\bar{S} \subseteq \tilde{S} \quad (2)$$

does not always hold, under certain conditions, such as lower semi-continuity of $\phi(x, \cdot)$ and the satisfaction of the (BCQ) constraint qualification in $\bar{S}$, this inclusion is established. This condition is particularly valid when $\phi(x, \cdot)$ is locally Lipschitz or convex on $\mathbb{R}^n$.

The model $EP(\phi, S)$ serves as a unified model encompassing various optimization problems, including mathematical programming, variational inequality, the Nash equilibrium, and saddle-point problems, for example, [2–5]. The study extends to vector optimization problems, demonstrating the versatility of $EP(\phi, S)$. Previous research efforts have expanded the model to include generalized quasi-variational inequality problems. The concept of an equilibrium problem plays a central role in various applied sciences, such as physics, economics, engineering, transportation, sociology, chemistry, biology, and other fields [6–8]. The theory of gap functions, developed in the variational inequalities, is extended to a general equilibrium problem in [9]. Van et al. [10] provide sufficient conditions and characterizations for linearly conditioned bifunction associated with an equilibrium problem. The problem $EP(\phi, S)$ also has significant implications in practice, especially in healthcare [11–13], the financial economy [14], multi-agent games [15,16], and so on. Milasi et al. [14] focus on the analysis of an economic equilibrium model under time and uncertainty by using a stochastic variational inequality approach. Nagurney et al. [13] construct a supply chain game theory network framework that captures labor constraints under three different scenarios. For different scenarios, appropriate equilibrium constructs are defined, along with their variational inequality formulations. Li et al. [11] propose a new non-smooth, two-stage stochastic equilibrium model of medical supplies in epidemic management. The effectiveness of the model is validated based on actual data from Wuhan, China, which suffered from the COVID-19 pandemic. Fargetta et al. [12] model the competition among hospitals as a Nash equilibrium problem and introduce a stochastic programming model to design and evaluate the behavior of each demand location.

Regarding the finite termination of algorithms for $EP(\phi, S)$, particularly in mathematical programming and variational inequality problems, existing research has predominantly concentrated on concepts such as the weak sharp minimum and strong non-degeneracy of the solution set. Pioneering studies by scholars like Rockafellar [17], Polyak [18], Ferris [19], and others have laid down conditions for finite termination utilizing specific algorithms. Nonetheless, the reliance on algorithmic frameworks has highlighted the necessity for more expansive research into conditions that ensure finite termination, irrespective of the algorithms employed. Early significant contributions in this area were made by Burke and Moré [20], who established the necessary and sufficient conditions for the finite termination of feasible solution sequences in smooth programming problems that converge to strongly non-degenerate points. Subsequently, Burke and Ferris [21] broadened these findings to encompass differentiable convex programming. In a further extension, Marcotte and Zhu [22] generalized these principles to continuous variational inequality problems characterized by pseudo-monotonicity⁺. Al-Homidan et al. [23] consider weak sharp solutions for the generalized variational inequality problem, in which the underlying mapping is set-valued. Huang et al. [24] give several characterizations of the weak sharpness in terms of the primal gap function associated with the mixed variational inequality. Nguyen [25] presents the concept of weak sharpness in variational inequality problems, specifically within Hadamard spaces. Subsequently, numerous researchers have explored the concept of weak sharpness of the solution set, particularly focusing on its implications for the finite convergence of diverse algorithms applied to variational inequality problems. This topic has been extensively investigated in various studies, as detailed in references [26–29], among others. In contrast to the aforementioned literature, this paper focuses on augmented weak sharpness in equilibrium problems, without being confined to mathematical programming problems or variational inequality problems. We establish both the necessary

and sufficient conditions for the finite termination of feasible solution sequences under the criterion of augmented weak sharpness in equilibrium problems. Additionally, we showcase how these results extend to mathematical programming problems, variational inequality problems, Nash equilibrium problems, and global saddle-point problems. The results corresponding to the latter two scenarios have not been studied in other literature.

This paper introduces and elaborates on the concepts of weak sharpness and strong non-degeneracy of the solution set within the framework of $EP(\phi, S)$. To tackle scenarios where these characteristics are not present, we propose an augmented mapping on the solution set. This leads to the definition of augmented weak sharpness of the solution set for feasible solution sequences. This novel concept not only generalizes weak sharpness and non-degeneracy but is also employed in establishing the necessary and sufficient conditions for the finite termination of feasible solution sequences under the premise of augmented weak sharpness.

The remainder of the paper is organized as follows. Section 2 provides preliminary information and discusses several special cases of $EP(\phi, S)$. In Section 3, the notion of augmented weak sharpness is introduced for the solution set of $EP(\phi, S)$ under general conditions. Section 4 presents many examples illustrating situations where weak sharpness or non-degeneracy is not satisfied, but augmented weak sharpness holds. Section 5 establishes the finite identification of feasible solution sequences under the condition of augmented weak sharpness and presents consequences, generalizing results from conditions of weak sharpness or strong non-degeneracy. We provide a conclusion in Section 6.

2. Preliminary

This section delineates the foundational concepts and exemplifies specific instances pertinent to $EP(\phi, S)$, which serves as a cornerstone for the ensuing discussions.

Consider a boundless sequence $N \subseteq \{1, 2, \dots\}$ and a series of sets $C^k \subset \mathbb{R}^n$, where $k = 1, 2, \dots$. The upper limit and lower limit of this sequence of sets are defined as follows:

$$\begin{aligned}\limsup_{k \rightarrow \infty} C^k &= \{x \in \mathbb{R}^n \mid \exists N, x^k \in C^k, \text{ such that } \lim_{k \in N, k \rightarrow \infty} x^k = x\}, \\ \liminf_{k \rightarrow \infty} C^k &= \{x \in \mathbb{R}^n \mid \exists x^k \in C^k, \text{ such that } \lim_{k \rightarrow \infty} x_k = x\}.\end{aligned}$$

Consequently, it can be deduced that $\liminf_{k \rightarrow \infty} C^k \subseteq \limsup_{k \rightarrow \infty} C^k$.

The tangent cone to a set $C \subset \mathbb{R}^n$ at the point $\bar{x} \in C$ is defined as follows:

$$T_C(\bar{x}) = \{d \in \mathbb{R}^n \mid \exists x^k \in C, x^k \rightarrow \bar{x}, \tau^k \downarrow 0, (k \rightarrow \infty), \text{ with } \lim_{k \rightarrow \infty} \frac{x^k - \bar{x}}{\tau^k} = d\}.$$

The regular normal cone to C at $\bar{x}$ is defined as:

$$\hat{N}_C(\bar{x}) = \{d \in \mathbb{R}^n \mid \langle d, x - \bar{x} \rangle \leq o(\|x - \bar{x}\|), \forall x \in C\}.$$

In general, the normal cone to C at $\bar{x}$ is defined as: $N_C(\bar{x}) = \limsup_{x^k \in C, x^k \rightarrow \bar{x}} \hat{N}_C(x^k)$. The polar cone of C is defined as: $C^\circ = \{y \in \mathbb{R}^n \mid \langle y, x \rangle \leq 0, \forall x \in C\}$. According to (Proposition 6.5 [1]), it is established that $T_C(x)^\circ = \hat{N}_C(x)$. Furthermore, in the case where C is convex, as per (Theorem 6.9 [1]), it holds that: $N_C(\bar{x}) = \hat{N}_C(\bar{x}) = \{d \mid \langle d, x - \bar{x} \rangle \leq 0, \forall x \in C\}$. The projection of a point $x \in \mathbb{R}^n$ onto a closed set C is defined as: $P_C(x) = \arg \min_{y \in C} \|y - x\|$, and the distance from a point $x \in \mathbb{R}^n$ to the set C is denoted as $\text{dist}(x, C) = \inf_{y \in C} \|y - x\|$. When C is a closed set, then $\text{dist}(x, C) = \|P_C(x) - x\|$.

Assuming that the subdifferential of $\psi(\cdot)$ at a point $x \in C$ satisfies $\partial\psi(x) \neq \emptyset$, the projected subdifferential of $\psi(\cdot)$ at x is defined as:

$$P_{T_C(x)}(-\partial\psi(x)) = \{P_{T_C(x)}(-u) \mid u \in \partial\psi(x)\}.$$

If $\psi(\cdot)$ exhibits continuous differentiability within a neighborhood of the point $x \in C$, as per (Exercise 8.8 [1]), it is established that $\partial\psi(x) = \{\nabla\psi(x)\}$. Consequently, this implies that the projected subdifferential is equivalent to the projected gradient $P_{T_C(x)}(-\nabla\psi(x))$.

We define a sequence $\{x^k\} \subset \mathbb{R}^n$ as terminating finitely to C if there exists a k_0 such that $x^k \in C$ for all $k \geq k_0$. In $EP(\phi, S)$, the function ϕ is monotonic on $S \times S$ if it satisfies $\phi(x, y) + \phi(y, x) \leq 0$, for $\forall (x, y) \in S \times S$. Furthermore, a function ϕ is pseudo-monotone on $S \times S$ if the condition $\phi(x, y) \geq 0$ necessarily implies $\phi(y, x) \leq 0$, for $\forall (x, y) \in S \times S$.

The model $EP(\phi, S)$ serves as a unified model encompassing various optimization problems, including mathematical programming, variational inequality, the Nash equilibrium, and saddle-point problems, and so on. In the following sections, we demonstrate through several examples how other problems are special cases of equilibrium problems $EP(\phi, S)$ (see [3]).

Example 1. Consider the function $\phi : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$, defined as

$$\phi(x, y) = f(y) - f(x), \quad \text{for } \forall (x, y) \in \mathbb{R}^n \times \mathbb{R}^n, \quad (3)$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a given function. Clearly, ϕ adheres to the condition in (1). Therefore, the problem $EP(\phi, S)$ translates into the subsequent mathematical programming challenge:

$$(MP) \quad \text{Find } \bar{x} \in S \subseteq \mathbb{R}^n, \text{ such that } f(\bar{x}) \leq f(y), \text{ for } \forall y \in S.$$

Based on (3), it follows that the function ϕ exhibits monotonicity across $\mathbb{R}^n \times \mathbb{R}^n$.

Example 2. Consider the function $\phi : S \times S \rightarrow \mathbb{R}$, where $S \subseteq \mathbb{R}^n$, defined as

$$\phi(x, y) = \langle F(x), y - x \rangle, \quad \text{for } \forall (x, y) \in S \times S, \quad (4)$$

and $F : S \rightarrow \mathbb{R}^n$ is a given mapping. It is evident that ϕ fulfills the condition in (1). Consequently, the problem $EP(\phi, S)$ is equivalent to the following variational inequality problem:

$$(VIP) \quad \text{Find } \bar{x} \in S, \text{ such that } \langle F(\bar{x}), y - \bar{x} \rangle \geq 0, \text{ for } \forall y \in S.$$

Based on (4), it can be deduced that ϕ is monotonic on $S \times S \iff F$ is monotonic on S , and ϕ is pseudo-monotonic on $S \times S \iff F$ is pseudo-monotonic over S .

Example 3. Suppose $S = S_1 \times S_2$, where $S_1 \subseteq \mathbb{R}^{n_1}$, $S_2 \subseteq \mathbb{R}^{n_2}$, and $(n_1 + n_2 = n)$ are all non-empty closed convex sets. Let $x = (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and $y = (y_1, y_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Define $\phi : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ as

$$\phi(x, y) = \varphi(y_1, x_2) - \varphi(x_1, y_2), \quad \text{for } \forall (x, y) \in \mathbb{R}^n \times \mathbb{R}^n, \quad (5)$$

where $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$. Clearly, ϕ adheres to (1). Therefore, $EP(\phi, S)$ is transformed into the following global saddle-point problem:

$$(SPP) \quad \text{Find } \bar{x} = (\bar{x}_1, \bar{x}_2) \in S_1 \times S_2, \text{ such that } \varphi(\bar{x}_1, y_2) \leq \varphi(\bar{x}_1, \bar{x}_2) \leq \varphi(y_1, \bar{x}_2), \\ \text{for } \forall (y_1, y_2) \in S_1 \times S_2.$$

It follows from (5) that ϕ is monotonic over $\mathbb{R}^n \times \mathbb{R}^n$.

Example 4. Let us consider a set $I = \{1, 2, \dots, n\}$ and a Cartesian product $S = \prod_{i \in I} S_i$, where each $S_i \subseteq \mathbb{R}$, for $i \in I$, represents a non-empty closed convex set. For a vector $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, we define

$$x^i = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n),$$

with necessary adjustments for the scenarios when $i = 1$ or $i = n$. Suppose the function $\phi : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ is given by

$$\phi(x, y) = \sum_{i \in I} (f_i(x^i, y_i) - f_i(x)), \quad \text{for } \forall (x, y) \in \mathbb{R}^n \times \mathbb{R}^n, \quad (6)$$

where each $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ is a specified function. Clearly, ϕ satisfies the condition outlined in (1). Therefore, the problem $EP(\phi, S)$ translates into the following Nash equilibrium problem:

$$(NEP) \quad \text{Find } \bar{x} \in S, \text{ such that } f_i(\bar{x}) \leq f_i(\bar{x}^i, y_i), \text{ for } \forall y_i \in S_i, \text{ and } i \in I.$$

3. The Augmented Weak Sharpness in Equilibrium Problem

In this section, we explore the concepts of weak sharpness and strong non-degeneracy within the solution set $\bar{S} \subset S$ of $EP(\phi, S)$ defined by ϕ and S . Additionally, to provide more lenient conditions for the finite identification of a feasible solution sequence, we introduce an augmented mapping on the solution set $\bar{S}$. This leads to the establishment of the concept of augmented weak sharpness for the solution set $\bar{S}$ in relation to feasible solution sequences. For various cases and under broad assumptions, we demonstrate that this novel concept extends the traditional notions of weak sharpness and strong non-degeneracy.

Initially, we present the definition of weak sharp minima in (MP), as discussed in [30,31].

Definition 1. In (MP), a solution set $\bar{S} \subset S$ is weak sharp minimal if there exists a constant $\alpha > 0$ such that, for $\forall x \in \bar{S}$, the following inequality is satisfied:

$$f(y) - f(x) \geq \alpha \cdot \text{dist}(y, \bar{S}), \quad \forall y \in S. \quad (7)$$

Here, the constant α and the set $\bar{S}$ are referred to as the modulus and domain of sharpness for the function f over the set S , respectively. It is evident that $\bar{S}$ constitutes a set of global minima for the function f over the set S .

If (MP) is characterized as a non-smooth convex program, then the solution set $\bar{S}$ is identified as a weak sharp minimal set with modulus α if, and only if, the following condition is met:

$$\alpha B \subset \partial f(x) + [T_S(x) \cap N_{\bar{S}}(x)]^\circ, \quad \forall x \in \bar{S}, \quad (8)$$

where B represents a unit ball. This relationship is elaborated in (Theorem 2.6 (c), [30]).

In the scenario where (MP) is a smooth convex programming problem, the set $\bar{S}$ is deemed a weak sharp minimal set with modulus α if, and only if, the condition expressed below is satisfied:

$$-\nabla f(\bar{x}) \in \text{int} \bigcap_{x \in \bar{S}} [T_S(x) \cap N_{\bar{S}}(x)]^\circ, \quad \forall \bar{x} \in \bar{S}, \quad (9)$$

as discussed in (Corollary 2.7 (c), [30]). Notably, in instances where f is smooth, the conditions (9) and (8) are equivalent since $\nabla f(\cdot)$ remains constant over the set $\bar{S}$. This equivalence is further explicated in (Corollary 6 [21]).

To extend these characteristics to variational inequalities and smooth non-convex programming problems, several studies ([22,32–35]) have employed (9) to define the weak sharpness of the solution set in these contexts. We now propose to use (8) to define the weak sharpness of the solution set $\bar{S}$ for $EP(\phi, S)$.

Definition 2. In $EP(\phi, S)$, $\partial_y \phi(x, x) \neq \emptyset$ for $\forall x \in S$, the solution set $\bar{S} \subset S$ is considered to be a weak sharp set with modulus α , if there exists a constant $\alpha > 0$ such that

$$\alpha B \subset \partial_y \phi(x, x) + [T_S(x) \cap \hat{N}_{\bar{S}}(x)]^\circ, \quad \forall x \in \bar{S}. \quad (10)$$

Remark 1. In (10), the notation $\hat{N}_{\bar{S}}(\cdot)$ is utilized instead of $N_{\bar{S}}(\cdot)$, acknowledging that $\bar{S}$ may not be convex. In cases where $\bar{S}$ is non-convex, according to (Proposition 6.5 [1]), $\hat{N}_{\bar{S}}(\cdot)$ forms a closed convex cone, whereas $N_{\bar{S}}(\cdot)$ is merely a closed cone, and it is established that $\hat{N}_{\bar{S}}(\cdot) \subseteq N_{\bar{S}}(\cdot)$. When $\bar{S}$ is convex, as per (Theorem 6.9 [1]), the aforementioned relationship holds with equality. Hence, it is deduced that $[T_{\bar{S}}(x) \cap N_{\bar{S}}(x)]^\circ \subseteq [T_{\bar{S}}(x) \cap \hat{N}_{\bar{S}}(x)]^\circ$. Therefore, using $\hat{N}_{\bar{S}}(\cdot)$ in Definition 2 provides more relaxed conditions.

We now proceed to introduce the concept of strong non-degeneracy in $EP(\phi, S)$.

Definition 3. In $EP(\phi, S)$, assuming that $\phi(x, \cdot)$ is differentiable at each point x in S , we define the solution set $\bar{S} \subset S$ as strongly non-degenerate if

$$-\nabla_y \phi(\bar{x}, \bar{x}) \in \text{int} N_{\bar{S}}(\bar{x}), \quad \forall \bar{x} \in \bar{S}, \quad (11)$$

and each point $\bar{x}$ in this context is referred to as a strongly non-degenerate point.

Weak sharpness and strong non-degeneracy are two sufficient conditions for the finite termination of an algorithm, but in many cases, these conditions are difficult to satisfy. Therefore, this paper proposes a less stringent condition for finite termination, i.e., the augmented weak sharpness of the solution set. The augmented weak sharpness is an extension of the concepts of weak sharpness and strong non-degeneracy.

Definition 4. In $EP(\phi, S)$, let $\bar{S} \subset S$ be a closed set. For any $x \in S$, it is required that $\partial_y \phi(x, x) \neq \emptyset$, and consider a sequence $\{x^k\} \subset S$. The set $\bar{S}$ is said to be augmented weak sharp with respect to the sequence $\{x^k\}$. Given an infinite sequence $K = \{k \mid x^k \notin \bar{S}\}$, there exists an augmented mapping (set-valued mapping) $H : \bar{S} \rightarrow 2^{\mathbb{R}^n}$ satisfying the following conditions:

(a) there exists a constant $\alpha > 0$, such that

$$\alpha B \subset H(z) + [T_{\bar{S}}(z) \cap \hat{N}_{\bar{S}}(z)]^\circ, \quad \forall z \in \bar{S},$$

(b) for $\forall u^k \in \partial_y \phi(x^k, x^k)$ and every $v^k \in H(P_{\bar{S}}(x^k))$, the following inequality is observed:

$$\limsup_{k \in K, k \rightarrow \infty} \psi_k = \frac{1}{\|x^k - P_{\bar{S}}(x^k)\|} \langle u^k - v^k, x^k - P_{\bar{S}}(x^k) \rangle \geq 0.$$

Now, we will discuss the inclusion relation between the augmented weak sharpness of the solution set $\bar{S}$ and the weak sharpness as well as the strong non-degeneracy in non-smooth and smooth cases. Proposition 1 demonstrates the relationship between the augmented weak sharpness and the weak sharpness in the non-smooth case. Propositions 4 and 6, respectively, showcase the relationships of the augmented weak sharpness with weak sharpness and strong non-degeneracy in the smooth case.

3.1. The Non-Smooth Case

The following proposition addresses the relationship between the weak sharpness and the augmented weak sharpness in this context.

Proposition 1. In $EP(\phi, S)$, suppose $\bar{S} \subset S$ is a closed set, $\partial_y \phi(x, x) \neq \emptyset$ for any $x \in S$, and $\partial_y \phi(x, x)$ is monotonic over S . If $\bar{S}$ is a weakly sharp set in $EP(\phi, S)$, then, for every sequence $\{x^k\} \subset S$, the set $\bar{S}$ is also augmented weakly sharp.

Proof. Consider an infinite sequence $K = \{k \mid x^k \notin \bar{S}\}$. Define the mapping $H(z)$ for all $z \in \bar{S}$ as $H(z) = \partial_y \phi(z, z)$. Condition (a) in Definition 4 is satisfied by (10). The monotonicity of $\partial_y \phi(x, x)$ ensures that condition (b) is also fulfilled. $\square$

The next proposition provides a sufficient condition for the monotonicity of $\partial_y \phi(x, x)$.

Proposition 2. In $EP(\phi, S)$, if ϕ satisfies the following conditions:

- (i) For all $x \in S$, $\phi(x, \cdot)$ is a convex function over $\mathbb{R}^n$,
- (ii) The function ϕ is monotonic (or pseudo-monotonic) over $S \times S$.

Then, $\partial_y \phi(x, x)$ is monotonic (or pseudo-monotonic) over S .

Proof. By (i), $\partial_y \phi(x, x)$ is non-empty for all $x \in S$. Let $u_x \in \partial_y \phi(x, x)$ and $u_z \in \partial_y \phi(z, z)$. If ϕ is monotonic over $S \times S$, the inequality

$$\begin{aligned} 0 &\geq \phi(x, z) + \phi(z, x) \\ &= \phi(x, z) - \phi(x, x) + \phi(z, x) - \phi(z, z) \quad [\text{by (1)}] \\ &\geq \langle u_x, z - x \rangle + \langle u_z, x - z \rangle \quad [\text{by (i)}] \\ &= \langle u_x - u_z, z - x \rangle, \end{aligned}$$

holds, indicating the monotonicity of $\partial_y \phi(x, x)$. Similarly, for pseudo-monotonicity, $\langle u_x, z - x \rangle \geq 0$ implies $\langle u_z, x - z \rangle \leq 0$, fulfilling the pseudo-monotonic condition. $\square$

Remark 2. The following examples demonstrate that the conditions in Proposition 2 are sufficient but not necessary for the monotonicity of $\partial_y \phi(x, x)$ over S .

Example 5. In $EP(\phi, S)$ with $\phi(x, y) = e^{x-y} - e^{y-x}$ and $S = [0, 1]$, $\nabla_y \phi(x, x)$ is monotonic over S , even though $\phi(x, \cdot)$ is non-convex over S for all $x \in [0, 1]$.

Example 6. In $EP(\phi, S)$ with $\phi(x, y) = e^{y^2 - x^2} - 1$ and $S = \mathbb{R}$, $\nabla_y \phi(x, x)$ is monotonic over S , even though ϕ is not monotonic over $S \times S$.

The research in (Theorem 5 [21]) provides a characteristic description for the solution set of non-smooth convex programming. This result not only deepens the understanding of the nature of the solution set in convex programming but is also pivotal in analyzing the weak sharp minimality of such a set. The next step is to extend this result to the solution set of $EP(\phi, S)$ and apply it in analyzing the weak sharpness of the solution set.

Theorem 1. In $EP(\phi, S)$, suppose that ϕ satisfies the following conditions:

- (i) For $\forall x \in S$, $\phi(x, \cdot)$ is a convex function over $\mathbb{R}^n$.
- (ii) For $\forall x, z \in \bar{S}$, there is $\phi(x, y) = \phi(z, y)$, $\forall y \in \mathbb{R}^n$.

Furthermore, suppose $\bar{x} \in \bar{S}$, and A is a convex set satisfying $\bar{S} \subseteq A \subseteq S$. Then, we have

$$\bar{S} = \{x \in S \mid \partial_y \phi(x, x) \cap (-N_S(x)) = \partial_y \phi(\bar{x}, \bar{x}) \cap (-N_S(\bar{x}))\} \quad (12)$$

$$= \{x \in A \mid \partial_y \phi(x, x) \cap (-N_S(x)) = \partial_y \phi(\bar{x}, \bar{x}) \cap (-N_S(\bar{x}))\}. \quad (13)$$

Proof. Denote $\hat{S} = \{x \in S \mid \partial_y \phi(x, x) \cap (-N_S(x)) = \partial_y \phi(\bar{x}, \bar{x}) \cap (-N_S(\bar{x}))\}$. We aim to show that $\bar{S} = \hat{S}$, as indicated in (12). Since A satisfies $\bar{S} \subseteq A \subseteq S$, replacing S with A in (12) immediately yields (13). First, we prove $\bar{S} \subseteq \hat{S}$. Let $\hat{x} \in \bar{S}$, and consider $\hat{u} \in \partial_y \phi(\hat{x}, \hat{x}) \cap (-N_S(\hat{x}))$, we have $\hat{u} \in \partial_y \phi(\hat{x}, \hat{x})$, and

$$\langle \hat{u}, x - \hat{x} \rangle \geq 0, \quad \forall x \in S. \quad (14)$$

Using the convexity of $\phi(x, \cdot)$, $\phi(x, x) = 0$, and the condition $\phi(x, y) = \phi(z, y)$ for $x, z \in \bar{S}$, it can be shown that $\phi(\hat{x}, \hat{x}) \cap (-N_S(\hat{x})) \neq \emptyset$, and $\phi(\hat{x}, \bar{x}) = \phi(\bar{x}, \bar{x}) = 0$. Thus, by a combination of (i) and (14), we obtain that $0 = \phi(\hat{x}, \bar{x}) - \phi(\hat{x}, \hat{x}) \geq \langle \hat{u}, \bar{x} - \hat{x} \rangle \geq 0$, i.e.,

$$\langle \hat{u}, \bar{x} - \hat{x} \rangle = 0. \quad (15)$$

In view of (i), (ii) and (15), we immediately obtain that for $\forall y \in \mathbb{R}^n$

$$\begin{aligned}\phi(\bar{x}, y) - \phi(\bar{x}, \bar{x}) &= \phi(\hat{x}, y) - \phi(\hat{x}, \hat{x}) \\ &\geq \langle \hat{u}, y - \hat{x} \rangle \\ &= \langle \hat{u}, y - \bar{x} \rangle + \langle \hat{u}, \bar{x} - \hat{x} \rangle \\ &= \langle \hat{u}, y - \bar{x} \rangle.\end{aligned}$$

Then, $\hat{u} \in \partial_y \phi(\bar{x}, \bar{x})$ holds, according to (14) and (15), we obtain that

$$\begin{aligned}\langle \hat{u}, x - \bar{x} \rangle &= \langle \hat{u}, x - \hat{x} \rangle + \langle \hat{u}, \hat{x} - \bar{x} \rangle \\ &= \langle \hat{u}, x - \hat{x} \rangle \geq 0\end{aligned}$$

holds for $\forall x \in S$, i.e., $\hat{u} \in -N_S(\bar{x})$. Thus,

$$\partial_y \phi(\hat{x}, \hat{x}) \cap (-N_S(\hat{x})) \subseteq \partial_y \phi(\bar{x}, \bar{x}) \cap (-N_S(\bar{x})).$$

Since $\bar{x}, \hat{x}$ are selected from $\bar{S}$ arbitrarily, the inverse inclusion relation of the above relationship also holds. Therefore, we have $\bar{S} \subseteq \hat{S}$.

Now, we prove $\hat{S} \subseteq \bar{S}$. Assume $\hat{x} \in \hat{S}$. Since $\phi(\bar{x}, \bar{x}) \cap (-N_S(\bar{x})) \neq \emptyset$, then we have $\partial_y \phi(\hat{x}, \hat{x}) \cap (-N_S(\hat{x})) \neq \emptyset$. Taking $\hat{u} \in \partial_y \phi(\hat{x}, \hat{x}) \cap (-N_S(\hat{x}))$, for $\forall y \in S$, we have

$$\begin{aligned}\phi(\hat{x}, y) &= \phi(\hat{x}, y) - \phi(\hat{x}, \hat{x}) \\ &\geq \langle \hat{u}, y - \hat{x} \rangle \geq 0,\end{aligned}$$

i.e., $\hat{x} \in \bar{S}$, $\hat{S} \subseteq \bar{S}$. $\square$

Remark 3. In the case of the convex programming problem, which is a specific instance of $EP(\phi, S)$, conditions (i) and (ii) are readily satisfied, as illustrated in Example 1. However, it is important to note that these conditions are not limited to convex programming within the scope of $EP(\phi, S)$. For instance, consider $\phi(x, y) = (1 - x^2)(y - x)$ with $S = [-1, 1]$, where both (i) and (ii) are evidently fulfilled.

Denote the set

$$G := \bigcap_{x \in \bar{S}} [T_S(x) \cap \hat{N}_{\bar{S}}(x)]^\circ,$$

and consider the following proposition within the context of $EP(\phi, S)$.

Proposition 3. Under the assumptions of Theorem 1, and additionally assuming that $\bar{S} \subset S$ is a closed set and $\partial_y \phi(x, x)$ is monotonic over S , if the condition

$$-\partial_y \phi(x, x) \cap (-N_S(x)) \subset \text{int} G \quad \text{for all } x \in \bar{S}, \quad (16)$$

is met, then the set $\bar{S}$ is augmented weakly sharp for all sequences $x^k \subset S$.

Proof. First, by the assumption (i) in Theorem 1 and (Theorem 23.4, [36]), we obtain that $\partial_y \phi(x, x)$ is a non-empty compact set for $\forall x \in \mathbb{R}^n$. Furthermore, by Theorem 1, we obtain that $\partial_y \phi(x, x) \cap (-N_S(x))$ is a non-empty compact constant set over $\bar{S}$. In view of (16), there exists a constant $\alpha > 0$ such that for $\forall x \in \bar{S}$,

$$\alpha B - \partial_y \phi(x, x) \cap (-N_S(x)) \subset [T_S(x) \cap \hat{N}_{\bar{S}}(x)]^\circ.$$

Based on the above formula, for $\forall x \in \bar{S}$, we obtain

$$\begin{aligned}\alpha B &\subset \partial_y \phi(x, x) \cap (-N_S(x)) + [T_S(x) \cap [\hat{N}_{\bar{S}}(x)]^\circ] \\ &\subset \partial_y \phi(x, x) + [T_S(x) \cap [\hat{N}_{\bar{S}}(x)]^\circ].\end{aligned}$$

Therefore, $\bar{S}$ is a weak sharp set by Definition 2. In view of the monotonicity of $\partial_y \phi(x, x)$ and Proposition 1, the proof has been completed. $\square$

3.2. The Smooth Case

In this subsection, we operate under the assumption that the function $\phi(x, \cdot)$ is continuously differentiable on $\mathbb{R}^n$ for all $x \in S$. Consequently, this implies that $\partial_y \phi(x, \cdot) = \nabla_y \phi(x, \cdot)$.

Proposition 4. In $EP(\phi, S)$, let us consider a closed subset $\bar{S} \subset S$ and a sequence $x^k \subset S$ that satisfies the following condition:

$$\lim_{k \rightarrow \infty} |\nabla_y \phi(x^k, x^k) - \nabla_y \phi(P_{\bar{S}}(x^k), P_{\bar{S}}(x^k))| = 0. \quad (17)$$

If the set $\bar{S}$ is identified as weak sharp, then it can also be considered as augmented weakly sharp with respect to the sequence x^k .

Proof. Let $K = \{k \mid x^k \notin \bar{S}\}$ be an infinite sequence, and set

$$H(z) = \nabla_y \phi(z, z), \quad \forall z \in \bar{S}.$$

By (10), we know that (a) holds in Definition 4, and by (17), we immediately obtain that

$$\limsup_{k \in K, k \rightarrow \infty} \psi_k = \limsup_{k \in K, k \rightarrow \infty} \frac{1}{\|x^k - P_{\bar{S}}(x^k)\|} \langle \nabla_y \phi(x^k, x^k) - H(P_{\bar{S}}(x^k)), x^k - P_{\bar{S}}(x^k) \rangle = 0,$$

i.e., Definition 4 (b) holds. $\square$

Proposition 5. In $EP(\phi, S)$, suppose $\bar{S} \subset S$ is a closed set, $\{x^k\} \subset S$, $\{\nabla_y \phi(x^k, x^k)\}$ is bounded and any one of its accumulations $\bar{p}$ satisfies $-\bar{p} \in \text{int}G$. Then, $\bar{S}$ is augmented weakly sharp with respect to $\{x^k\}$.

Proof. Let $K = \{k \mid x^k \notin \bar{S}\}$ be an infinite sequence. According to the hypotheses, there must be an accumulation $\bar{p}$ of $\{\nabla_y \phi(x^k, x^k)\}_{k \in K}$ satisfying $-\bar{p} \in \text{int}G$, i.e., there exists a constant $\alpha > 0$ such that

$$\alpha B \subset \bar{p} + [T_S(z) \cap \hat{N}_{\bar{S}}(z)]^\circ, \quad \forall z \in \bar{S}. \quad (18)$$

Letting $H(z) = \bar{p}$ for $\forall z \in \bar{S}$, by (18), we know that (a) holds in Definition 4. Furthermore let $K_0 \subseteq K$ such that

$$\lim_{k \in K_0, k \rightarrow \infty} \nabla_y \phi(x^k, x^k) = \bar{p}. \quad (19)$$

By (19), we immediately obtain that

$$\begin{aligned}\limsup_{k \in K, k \rightarrow \infty} \psi_k &\geq \limsup_{k \in K_0, k \rightarrow \infty} \psi_k \\ &= \lim_{k \in K_0, k \rightarrow \infty} \frac{1}{\|x^k - P_{\bar{S}}(x^k)\|} \langle \nabla_y \phi(x^k, x^k) - \bar{p}, x^k - P_{\bar{S}}(x^k) \rangle \\ &= 0,\end{aligned}$$

i.e., Definition 4 (b) holds. $\square$

In the final analysis of $EP(\phi, S)$, we establish a crucial connection between the notions of strong non-degeneracy and augmented weak sharpness within the framework of solution sets.

Proposition 6. In $EP(\phi, S)$, suppose $\nabla_y \phi(x, x)$ is continuous over S , $\{x^k\} \subset S$ is bounded and any of its accumulations is strongly non-degenerate. Then, $\bar{S}$ is augmented weakly sharp with respect to $\{x^k\}$.

Proof. Let $K = \{k \mid x^k \notin \bar{S}\}$ be an infinite sequence. Then, there must be an accumulation point $\bar{x}$ of $\{x^k\}_{k \in K}$. Suppose $K_0 \subseteq K$ such that

$$\lim_{k \in K_0, k \rightarrow \infty} x^k = \bar{x}. \quad (20)$$

By the assumptions of the strong non-degeneracy of $\bar{x}$ and the continuity of $\nabla_y \phi(x, x)$, and (Proposition 5.1 [37]), we know that $\bar{x}$ is an isolated point of $\bar{S}$. Thus, we have $T_{\bar{S}}(\bar{x}) = \{0\}$, $\hat{N}_{\bar{S}}(\bar{x}) = T_{\bar{S}}(\bar{x})^\circ = \mathbb{R}^n$. Therefore, we obtain that

$$[T_S(\bar{x}) \cap \hat{N}_{\bar{S}}(\bar{x})]^\circ = T_S(\bar{x})^\circ = N_S(\bar{x}). \quad (21)$$

According to (11) and (21), we know that there exists a constant $\alpha > 0$ such that

$$\alpha B \subset \nabla_y \phi(\bar{x}, \bar{x}) + N_S(\bar{x}) = \nabla_y \phi(\bar{x}, \bar{x}) + [T_S(\bar{x}) \cap \hat{N}_{\bar{S}}(\bar{x})]^\circ. \quad (22)$$

Now, we define the augmented mapping

$$H(z) = \begin{cases} \nabla_y \phi(\bar{x}, \bar{x}), & z = \bar{x}, \\ \mathbb{R}^n, & z \neq \bar{x}. \end{cases} \quad (23)$$

According to (22) and (23), we can immediately obtain that

$$\alpha B \subset H(z) + [T_S(z) \cap \hat{N}_{\bar{S}}(z)]^\circ, \forall z \in \bar{S},$$

i.e., (a) holds in Definition 4. Since x is an isolated point of $\bar{S}$, by (20), for all sufficiently large $k \in K_0$, it holds that $P_{\bar{S}}(x^k) = \bar{x}$. Therefore, according to (20) and (23), and the continuity of $\nabla_y \phi(x, x)$, we obtain that

$$\begin{aligned} \limsup_{k \in K, k \rightarrow \infty} \psi_k &\geq \limsup_{k \in K_0, k \rightarrow \infty} \psi_k \\ &= \lim_{k \in K_0, k \rightarrow \infty} \frac{1}{\|x^k - \bar{x}\|} \langle \nabla_y \phi(x^k, x^k) - \nabla_y \phi(\bar{x}, \bar{x}), x^k - \bar{x} \rangle \\ &= 0 \end{aligned}$$

i.e., Definition 4 (b) holds. $\square$

4. Some Examples

In this section, we present illustrative examples within the framework of $EP(\phi, S)$ to demonstrate instances where the solution set S does not maintain weak sharpness but instead satisfies the condition of augmented weak sharpness.

Example 7. Consider the following mathematical programming problems (MP) as a special case of $EP(\phi, S)$:

$$\min_{x \in S} f(x) = \max\{0, x_1 x_2\} + \sum_{i=1}^2 \max\{0, x_i\},$$

where

$$S = \{x \in \mathbb{R}^2 \mid x_1 \leq 1, x_2 \leq 1\},$$

$$\bar{S} = \{x \in \mathbb{R}^2 \mid x_1 = 0, x_2 \leq 0\} \cup \{x \in \mathbb{R}^2 \mid x_1 \leq 0, x_2 = 0\}.$$

This is a non-smooth and non-convex programming problem, and the solution set $\bar{S}$ of it is non-convex. By Example 1, we know that

$$\phi(x, y) = f(y) - f(x), \quad (x, y) \in \mathbb{R}^2 \times \mathbb{R}^2.$$

Therefore, $\partial_y \phi(x, \cdot) = \partial f(\cdot)$.

When $x \in S \setminus \bar{S}$, we have

$$\partial f(x_1, x_2) = \begin{cases} (x_2 + (\frac{x_1}{|x_1|})^+, x_1 + (\frac{x_2}{|x_2|})^+), & \text{if } x_1 x_2 > 0 \\ ((\frac{x_1}{|x_1|})^+, (\frac{x_2}{|x_2|})^+), & \text{if } x_1 x_2 < 0 \\ \{u \in \mathbb{R}^2 \mid u_1 \in [\frac{x_1}{\|x\|}, x_2 + 1], \\ \quad u_2 \in [\frac{x_2}{\|x\|}, x_1 + 1]\}, & \text{if } x_1 x_2 = 0 \text{ and } x_1 + x_2 > 0, \end{cases} \quad (24)$$

where $(t)^+ = \max\{0, t\}$.

When $x \in \bar{S}$, we have

$$\partial f(x_1, x_2) = \begin{cases} \{u \in \mathbb{R}^2 \mid u_1 \in [0, 1], u_2 \in [0, 1]\}, & \text{if } x = (0, 0) \\ \{u \in \mathbb{R}^2 \mid u_1 \in [x_2, 1], u_2 = 0\}, & \text{if } x_1 = 0, x_2 < 0. \\ \{u \in \mathbb{R}^2 \mid u_1 = 0, u_2 \in [x_1, 1]\}, & \text{if } x_1 < 0, x_2 = 0. \end{cases} \quad (25)$$

Note that $\bar{S} \subset \text{int}S$, for all $x \in \bar{S}$, we have $T_S(x) = \mathbb{R}^2$. Then, for $\forall x \in \bar{S}$, we obtain

$$\begin{aligned} [T_S(x) \cap \hat{N}_{\bar{S}}(x)]^\circ &= \hat{N}_{\bar{S}}(x)^\circ \\ &= \begin{cases} \{\xi \in \mathbb{R}^2 \mid \xi_1 \leq 0, \xi_2 \leq 0\}, & \text{if } x = (0, 0) \\ \{\xi \in \mathbb{R}^2 \mid \xi_1 = 0, \xi_2 \in (-\infty, +\infty)\}, & \text{if } x_1 = 0, x_2 < 0 \\ \{\xi \in \mathbb{R}^2 \mid \xi_1 \in (-\infty, +\infty), \xi_2 = 0\}, & \text{if } x_1 < 0, x_2 = 0 \end{cases} \end{aligned} \quad (26)$$

By (25), the second and third formula of (26), we obtain that

$$\partial f(x) + \hat{N}_{\bar{S}}(x)^\circ = \begin{cases} \{\xi \in \mathbb{R}^2 \mid \xi_1 \in [x_2, 1], \xi_2 \in (-\infty, +\infty)\}, & \text{if } x_1 = 0, x_2 < 0 \\ \{\xi \in \mathbb{R}^2 \mid \xi_1 \in (-\infty, +\infty), \xi_2 \in [x_1, 1]\}, & \text{if } x_1 < 0, x_2 = 0. \end{cases} \quad (27)$$

By (27), we know that the model $\alpha > 0$ is not a constant in Definition 2, which is related to $x \in \bar{S}$, and when $\|x\| \rightarrow 0$, $\alpha \rightarrow 0^+$, i.e., the constant $\alpha > 0$ does not exist. Therefore, $\bar{S}$ is not a weak sharp set.

Now, take a small enough $\varepsilon > 0$, and let

$$\Omega_\varepsilon = \{x \in \mathbb{R}^2 \mid -\varepsilon < x_1 < 0, -\varepsilon < x_2 < 0\}.$$

Below, we will prove that $\bar{S}$ is an augmented weak sharp set with respect to arbitrary sequences $\{x^k\} \subset S \setminus \Omega_\varepsilon$. Let $K = \{k \mid x^k \notin \bar{S}\}$ be an infinite sequence. We introduce the augmented mapping $H : \bar{S} \Rightarrow \mathbb{R}^2$ as follows:

$$H(x_1, x_2) = \begin{cases} (\varepsilon, \varepsilon), & \text{if } x = (0, 0) \\ \{v \in \mathbb{R}^2 \mid |v_1| \leq \varepsilon, v_2 = 0\}, & \text{if } x_1 = 0, x_2 < 0 \\ \{v \in \mathbb{R}^2 \mid v_1 = 0, |v_2| \leq \varepsilon\}, & \text{if } x_1 < 0, x_2 = 0. \end{cases} \quad (28)$$

According to (26), (28), and the condition (a) in Definition 4 holds, i.e., there exists a constant $\alpha > 0$ such that for all $x \in \bar{S}$, $\alpha B \subset H(x) + \hat{N}_{\bar{S}}(x)^\circ$.

For $k \in K$, let

$$\psi_k = \frac{1}{\|x^k - P_{\bar{S}}(x^k)\|} \langle u^k - v^k, x^k - P_{\bar{S}}(x^k) \rangle,$$

where $u^k \in \partial f(x^k)$, $v^k \in H(P_{\bar{S}}(x^k))$, and

$$P_{\bar{S}}(x^k) = \begin{cases} (0, 0), & \text{if } x_1^k \geq 0, x_2^k \geq 0, \text{ and } x_1^k + x_2^k > 0 \\ (0, x_2^k), & \text{if } x_1^k > 0, x_2^k < 0, \text{ or } x_2^k \leq x_1^k < 0 \\ (x_1^k, 0), & \text{if } x_1^k < 0, x_2^k > 0, \text{ or } x_1^k \leq x_2^k < 0. \end{cases} \quad (29)$$

According to (24), (28), and (29), we can easily prove that $\{x^k\}_{k \in K} \subset S \setminus \Omega_\varepsilon$ satisfies the condition (b) in Definition 4. For simplicity, we only prove the case that $\{x^k\}_{k \in K}$ lies in the third quadrant. At the same time, considering $x^k \notin \Omega_\varepsilon$, we have

(i) when $x_2^k \leq x_1^k < 0$ and $x_2^k \leq -\varepsilon$, by the first item of (24), (29), and the second item of (28), we have

$$\psi_k = \frac{1}{|x_1^k|} (x_2^k - v_1^k) x_1^k = |x_2^k| + v_1^k \geq \varepsilon - |v_1^k| \geq 0,$$

(ii) when $x_1^k \leq x_2^k < 0$ and $x_1^k \leq -\varepsilon$, by the first item of (24), (29), and the third item of (28), we have

$$\psi_k = \frac{1}{|x_2^k|} (x_1^k - v_2^k) x_2^k = |x_1^k| + v_2^k \geq \varepsilon - |v_2^k| \geq 0.$$

By (i) and (ii), we obtain that

$$\limsup_{k \in K, k \rightarrow \infty} \psi_k \geq 0,$$

i.e., (b) in Definition 4 holds.

In addition, we note that Remark 1 is verified through Example 7. As in this example, the regular normal cone of $\bar{S}$ at $(0, 0)$ is

$$\hat{N}_{\bar{S}}(0, 0) = \{\xi \in \mathbb{R}^2 \mid \xi_1 \geq 0, \xi_2 \geq 0\},$$

while the normal cone under the general meaning is

$$N_{\bar{S}}(0, 0) = \hat{N}_{\bar{S}}(0, 0) \cup \{\xi \in \mathbb{R}^2 \mid \xi_1 \in (-\infty, 0), \xi_2 = 0\} \cup \{\xi \in \mathbb{R}^2 \mid \xi_1 = 0, \xi_2 \in (-\infty, 0)\}.$$

Here, $N_{\bar{S}}(0, 0)$ is a closed cone, but not a convex cone. Furthermore, according to $T_{\bar{S}}(0, 0) = \mathbb{R}^2$, it follows that

$$[T_{\bar{S}}(0, 0) \cap \hat{N}_{\bar{S}}(0, 0)]^\circ = \hat{N}_{\bar{S}}(0, 0)^\circ = \{\xi \in \mathbb{R}^2 \mid \xi_1 \leq 0, \xi_2 \leq 0\},$$

$$[T_{\bar{S}}(0, 0) \cap N_{\bar{S}}(0, 0)]^\circ = N_{\bar{S}}(0, 0)^\circ = \{(0, 0)\}.$$

The advantage of the regular cone has been shown here compared with the normal cone under the general meaning.

Example 8. Consider the following VIP(F, S) as a special case of EP(ϕ, S):

$$F(x) = (\cos x_1, e^{x_2}),$$

where $S = \{x \in \mathbb{R}^2 \mid 0 \leq x_1 \leq \frac{\pi}{2}, x_2 \geq 0\}$, $\bar{S} = \{(0, 0), (\frac{\pi}{2}, 0)\}$. By Example 2, we have $\phi(x, y) = \langle F(x), y - x \rangle$, $(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2$. Then, we have $\partial_y \phi(x, \cdot) = F(x)$.

When $x \in \bar{S}$, we have

$$F(x) = \begin{cases} (1, 1), & x = (0, 0) \\ (0, 1), & x = (\frac{\pi}{2}, 0), \end{cases} \quad (30)$$

and

$$[T_{\bar{S}}(x) \cap N_{\bar{S}}(x)]^\circ = N_{\bar{S}}(x) = \begin{cases} \{\xi \in \mathbb{R}^2 \mid \xi_1 \leq 0, \xi_2 \leq 0\}, & x = (0, 0) \\ \{\xi \in \mathbb{R}^2 \mid \xi_1 \geq 0, \xi_2 \leq 0\}, & x = (\frac{\pi}{2}, 0). \end{cases} \quad (31)$$

This is a non-monotonic variational inequality problem. By (30) and (31), one can see that $\bar{S}$ is not a weak sharp set.

Next, we will prove that $\bar{S}$ is an augmented weak sharp set with respect to the sequence $\{x^k\} \subset S$, which satisfies the following conditions:

- (i) $\{x^k\} \subset \{x \in \mathbb{R}^2 \mid x_1 \leq \frac{\pi}{4}\} \cup \{x \in \mathbb{R}^2 \mid x_1 \geq \frac{\pi}{4}, x_1 + x_2 \geq \frac{\pi}{2}\};$
- (ii) $\lim_{k \rightarrow \infty} \text{dist}(x^k, \bar{S}) = 0.$

For this purpose, let $K = \{k \mid x^k \notin \bar{S}\}$ be an infinite sequence. Taking $\lambda \in (0, \frac{1}{2})$, we introduce the augmented mapping $H : \bar{S} \rightarrow \mathbb{R}^2$ as follows:

$$H(x) = \begin{cases} (\lambda, \lambda), & \text{if } x = (0, 0) \\ (-\lambda, \lambda), & \text{if } x = (\frac{\pi}{2}, 0). \end{cases} \quad (32)$$

By (31) and (32), we obtain that the condition (a) in Definition 4 holds. Now, for $k \in K$, let

$$\psi_k = \frac{1}{\|x^k - P_{\bar{S}}(x^k)\|} \langle F(x^k) - H(P_{\bar{S}}(x^k)), x^k - P_{\bar{S}}(x^k) \rangle,$$

where

$$P_{\bar{S}}(x^k) = \begin{cases} (0, 0), & \text{if } x_1^k \leq \frac{\pi}{4} \\ (\frac{\pi}{2}, 0), & \text{if } x_1^k \geq \frac{\pi}{4}. \end{cases} \quad (33)$$

By the condition (ii), the accumulations of the bounded sequence $\{x^k\}_{k \in K}$ are only possibly $\bar{x} = (0, 0)$ or $\bar{x} = (\frac{\pi}{2}, 0)$. Without loss of generality, let $\bar{x} = (\frac{\pi}{2}, 0)$ be one of its accumulations. Then, there exists an infinite subsequence $K_0 \subseteq K$ such that

$$\lim_{k \in K_0, k \rightarrow \infty} x^k = (\frac{\pi}{2}, 0). \quad (34)$$

According to (32)–(34), when $k \in K_0$ is big enough, we obtain that

$$\begin{aligned} \psi_k &= \frac{1}{\|x^k - \bar{x}\|} [(\cos x_1^k + \lambda)(x_1^k - \frac{\pi}{2}) + (e^{x_2^k} - \lambda)x_2^k] \\ &\geq \frac{1}{\|x^k - \bar{x}\|} [\cos x_1^k(x_1^k - \frac{\pi}{2}) + (e^{x_2^k} - 2\lambda)x_2^k] \quad [\text{by (i)}] \\ &\geq \frac{1}{\|x^k - \bar{x}\|} (x_1^k - \frac{\pi}{2}) \cos x_1^k \quad [\text{by } \lambda \in (0, \frac{1}{2})] \end{aligned}$$

Furthermore, by (34), we immediately obtain that

$$\limsup_{k \in K, k \rightarrow \infty} \psi_k \geq \lim_{k \in K_0, k \rightarrow \infty} \frac{1}{\|x^k - \bar{x}\|} (x_1^k - \frac{\pi}{2}) \cos x_1^k = 0,$$

i.e., (b) in Definition 4 holds.

Example 9. Consider $EP(\phi, S)$, where

$$\phi(x, y) = (e^{y_1^2} + x_2^2) - (e^{x_1^2} + y_2^2),$$

$S = S_1 \times S_2 = [0, 1] \times [-1, 1]$, and $\bar{S} = \{(0, 1), (0, -1)\}$.

By Example 3, one can see that this is a saddle-point problem (SPP), i.e., find $\bar{x} = (\bar{x}_1, \bar{x}_2) \in S_1 \times S_2$ such that

$$\phi(\bar{x}_1, y_2) \leq \phi(\bar{x}_1, \bar{x}_2) \leq \phi(y_1, \bar{x}_2), \quad \forall (y_1, y_2) \in S_1 \times S_2,$$

where $\varphi(x_1, x_2) = e^{x_1^2} + x_2^2$. It can be easily seen that

$$\nabla_y \phi(x, x) = (2x_1 e^{x_1^2}, -2x_2), \quad (35)$$

and

$$\nabla_y \phi(x, x) = \begin{cases} (0, -2), & \text{if } x = (0, 1) \\ (0, 2), & \text{if } x = (0, -1). \end{cases} \quad (36)$$

When $x \in \bar{S}$, we have

$$[T_{\bar{S}}(x) \cap N_{\bar{S}}(x)]^\circ = N_{\bar{S}}(x) = \begin{cases} \{\xi \in \mathbb{R}^2 \mid \xi_1 \leq 0, \xi_2 \geq 0\}, & \text{if } x = (0, 1) \\ \{\xi \in \mathbb{R}^2 \mid \xi_1 \leq 0, \xi_2 \leq 0\}. & \text{if } x = (0, -1). \end{cases} \quad (37)$$

By (36) and (37), one can see that $\bar{S}$ is not a weak sharp set and that $\bar{S}$ is not a strongly non-degenerate set.

Next, we will prove that $\bar{S}$ is an augmented weak sharp set with respect to the sequence $\{x^k\} \subset S$, which satisfies the following conditions:

- (i) $\{x^k\} \subset \{x \in \mathbb{R}^2 \mid x_1 + |x_2| \leq 1\}$;
- (ii) $\lim_{k \rightarrow \infty} \text{dist}(x^k, \bar{S}) = 0$.

For this purpose, let $K = \{k \mid x^k \notin \bar{S}\}$ be an infinite sequence. Taking $\lambda \in (0, 1)$, we introduce the augmented mapping $H : \bar{S} \rightarrow \mathbb{R}^2$ as follows:

$$H(x) = \begin{cases} (\lambda, -\lambda), & \text{if } x = (0, 1) \\ (\lambda, \lambda), & \text{if } x = (0, -1). \end{cases} \quad (38)$$

By (37) and (38), we obtain that the condition (a) in Definition 4 holds. Now we will prove the condition (b) in Definition 4 also holds. By (ii), one can see that the accumulations of $\{x^k\}$ are $\bar{x} = (0, 1)$ or $\bar{x} = (0, -1)$. Without loss of generality, assume that there exists a sequence $k_0 \subseteq K$ such that

$$\lim_{K \in K_0, k \rightarrow \infty} x^k = (0, 1). \quad (39)$$

For $k \in K$, let

$$\psi_k = \frac{1}{\|x^k - \bar{x}\|} \langle \nabla_y \phi(x^k, x^k) - H(P_{\bar{S}}(x^k)), x^k - P_{\bar{S}}(x^k) \rangle,$$

where

$$P_{\bar{S}}(x^k) = \begin{cases} (0, 1), & \text{if } x^k \geq 0 \\ (0, -1), & \text{if } x^k \leq 0. \end{cases} \quad (40)$$

According to (35) and (38)–(40), when $k \in K_0$ is big enough, we obtain that

$$\begin{aligned} \psi_k &= \frac{1}{\|x^k - \bar{x}\|} [(2x_1^k e^{(x_1^k)^2} - \lambda)x_1^k + (\lambda - 2x_2^k)(x_2^k - 1)] \\ &\geq \frac{1}{\|x^k - \bar{x}\|} [(2x_2^k - \lambda)(1 - x_2^k) - \lambda x_1^k] \\ &\geq \frac{1}{\|x^k - \bar{x}\|} [(2x_2^k - \lambda)(1 - x_2^k) - \lambda(1 - x_2^k)] \quad (\text{by (i)}) \\ &= \frac{1}{\|x^k - \bar{x}\|} 2(x_2^k - \lambda)(1 - x_2^k) \geq 0 \quad (\text{by } \lambda \in (0, 1)) \end{aligned}$$

Thus, we have

$$\limsup_{k \in K, k \rightarrow \infty} \psi_k \geq \lim_{k \in K_0, k \rightarrow \infty} \frac{1}{\|x^k - \bar{x}\|} 2(x_2^k - \lambda)(1 - x_2^k) \geq 0,$$

i.e., (b) in Definition 4 holds.

Example 10. Considering EP(ϕ, S), where

$$\phi(x, y) = (e^{y_1 - x_2} - e^{x_2 - y_1} - e^{x_1 - x_2} + e^{x_2 - x_1}) + (e^{y_2^2 - x_1^2} - e^{x_2^2 - x_1^2}),$$

$S = S_1 \times S_2 = [0, 1] \times [0, 1]$, $\bar{S} = \{(0, 0)\}$, $\bar{x} = (0, 0)$. By Example 4, one can see that this is a Nash equilibrium problem (NEP), i.e., find $\bar{x} \in S$ such that

$$f_1(\bar{x}) \leq f_1(y_1, \bar{x}_2), \quad \forall y_1 \in S_1,$$

$$f_2(\bar{x}) \leq f_2(\bar{x}_1, y_2), \quad \forall y_2 \in S_2,$$

where $f_1(x) = e^{x_1 - x_2} - e^{x_2 - x_1}$, $f_2(x) = e^{x_2^2 - x_1^2} - 1$. It is easy to see that

$$\nabla_y \phi(x, x) = (e^{x_1 - x_2} + e^{x_2 - x_1}, 2x_2 e^{x_2^2 - x_1^2}), \quad (41)$$

$$\nabla_y \phi(0, 0) = (2, 0), \quad (42)$$

$$[T_S(0, 0) \cap N_S(0, 0)]^\circ = N_S(0, 0) = \{\xi \in \mathbb{R}^2 \mid \xi_1 \leq 0, \xi_2 \leq 0\}. \quad (43)$$

By (42) and (43), we obtain that $\bar{S}$ is not a weak sharp set, i.e., the point $(0, 0)$ is not a strongly non-degenerate point.

Now, we will prove that $\bar{S}$ is an augmented weak sharp set for arbitrary sequences $\{x^k\} \subset S \cap \{x \in \mathbb{R}^2 \mid x_2 \leq x_1\}$. For this purpose, let $K = \{k \mid x^k \notin \bar{S}\}$ be an infinite sequence. Take $\lambda \in (0, \frac{1}{2})$, we introduce the augmented mapping $H : \bar{S} \rightarrow \mathbb{R}^2$ as follows:

$$H(0, 0) = (\lambda, \lambda). \quad (44)$$

By (43) and (44), one can see that the condition (a) in Definition 4 holds. Furthermore, according to (41) and (44), for $k \in K$, we obtain that

$$\begin{aligned} \psi_k &:= \frac{1}{\|x^k\|} \langle \nabla_y \phi(x^k, x^k) - H(0, 0), x^k \rangle \\ &= \frac{1}{\|x^k\|} [(e^{x_1^k - x_2^k} + e^{x_2^k - x_1^k} - \lambda)x_1^k + (2x_1^k e^{(x_2^k)^2 - (x_1^k)^2} - \lambda)x_2^k] \\ &\geq \frac{1}{\|x^k\|} [(e^{x_1^k - x_2^k} + e^{x_2^k - x_1^k} - \lambda)x_1^k - \lambda x_2^k] \\ &\geq \frac{1}{\|x^k\|} (e^{x_1^k - x_2^k} - 2\lambda)x_1^k \quad [by \quad x_2^k \leq x_1^k] \\ &\geq \frac{1}{\|x^k\|} (1 - 2\lambda)x_1^k \geq 0. \quad [by \quad \lambda \in (0, \frac{1}{2})] \end{aligned}$$

Thus, the condition (b) in Definition 4 also holds.

Example 11. Considering the following non-convex programming problem (MP):

$$\min_{x \in S} f(x) = \sin x_1 + \max\{0, x_2\},$$

where $S = \{x \in \mathbb{R}^2 \mid 0 \leq x_1 \leq \frac{\pi}{3}, -1 \leq x_2\}$, $\bar{S} = \{x \in \mathbb{R}^2 \mid x_1 = 0, -1 \leq x_2 \leq 0\}$, and

$$\partial f(x_1, x_2) = \begin{cases} (\cos x_1, 1), & \text{if } x_2 > 0 \\ \{u \in \mathbb{R}^2 \mid u_1 = \cos x_1, u_2 \in [0, 1]\}, & \text{if } x_2 = 0 \\ (\cos x_1, 0), & \text{if } x_2 < 0. \end{cases} \quad (45)$$

Noting that $\bar{S}$ is convex, we have

$$[T_S(x) \cap N_{\bar{S}}(x)]^\circ = \begin{cases} \{\xi \in \mathbb{R}^2 \mid \xi_1 \leq 0, \xi_2 \leq 0\}, & \text{if } x = (0, 0) \\ \{\xi \in \mathbb{R}^2 \mid \xi_1 \leq 0, \xi_2 \in (-\infty, \infty)\}, & \text{if } x_1 = 0, -1 \leq x_2 < 0. \end{cases} \quad (46)$$

When $x \in \bar{S}$, from (45), one can see

$$\partial f(x_1, x_2) = \begin{cases} \{u \in \mathbb{R}^2 \mid u_1 = 1, u_2 \in [0, 1]\}, & \text{if } x = (0, 0) \\ (1, 0), & \text{if } x_1 = 0, -1 \leq x_2 < 0. \end{cases} \quad (47)$$

According to (46) and (47), we obtain that $\bar{S}$ is a weak sharp set. Furthermore, if we assume $\{x^k\} \subset S$, then,

$$P_S(x^k) = \begin{cases} (0, 0), & \text{if } x_2^k > 0 \\ (0, x_2^k), & \text{if } x_2^k \leq 0. \end{cases} \quad (48)$$

Denote the mapping $H(x) = \partial f(x)$ over $\bar{S}$, and then it is easy to prove that $\bar{S}$ is an augmented weak sharp set with respect to the sequence $\{x^k\}$, which satisfies the condition $\lim_{k \rightarrow \infty} \text{dist}(x^k, \bar{S}) = 0$.

Note that by modifying the mapping $\partial f(\cdot)$, we can prove $\bar{S}$ is augmented weak sharp with respect to arbitrary $\{x^k\} \subset S$. For this purpose, let $K = \{k \mid x^k \notin \bar{S}\}$ be an infinite sequence. Taking $\lambda \in (0, \cos \frac{\pi}{3}]$, we introduce the augmented mapping $H : \bar{S} \rightarrow \mathbb{R}^2$ as follows:

$$H(x) = \begin{cases} (\lambda, \lambda), & \text{if } x = (0, 0) \\ (\lambda, 0), & \text{if } x_1 = 0, -1 \leq x_2 < 0. \end{cases} \quad (49)$$

According to (46) and (49), one can see that the condition (a) in Definition 4 holds. Furthermore, for $k \in K$, let

$$\psi_k = \frac{1}{\|x^k - P_{\bar{S}}(x^k)\|} \langle u^k - v^k, x^k - P_{\bar{S}}(x^k) \rangle,$$

where $u^k \in \partial f(x^k)$, $v^k \in H(P_{\bar{S}}(x^k))$. By (45), (48), and (49), we immediately obtain that

$$\psi_k = \begin{cases} \frac{1}{\|x^k\|} [(\cos x_1^k - \lambda)x_1^k + (1 - \lambda)x_2^k] \geq 0, & \text{if } x_1^k \geq 0, x_2^k > 0 \\ \frac{1}{|x_1^k|} (\cos x_1^k - \lambda)x_1^k \geq 0, & \text{if } x_1^k > 0, x_2^k \leq 0 \end{cases} \quad (\text{by the definition of } \lambda).$$

Thus, the condition (b) in Definition 4 also holds.

Here are some additional examples that are analogous to the previous ones:

1.

$$\begin{aligned} EP(\phi, S) \quad & \phi(x, y) = x_1^2 y_1^2 + x_2^2 y_2^2 - x_1^4 - y_2^4, \\ & S = \{x \in \mathbb{R}^2 \mid 0 \leq x_1 \leq 1, -1 \leq x_2 \leq 1\}, \\ & \bar{S} = \{(0, 1), (0, -1)\}. \end{aligned}$$

2.

$$\begin{aligned} EP(\phi, S) \quad & \phi(x, y) = e^{y_1^2 - x_1^2} + y_2 - x_2 - 1, \\ & S = \{x \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 \geq 0\}, \\ & \bar{S} = \{(0, 0)\}. \end{aligned}$$

3.

$$(MP) \quad \begin{aligned} \min_{x \in S} \quad & \left\{ f(x) = x_1 \log(1 + x_2) + \frac{1}{2}(x_1 - 1)^2(x_2 - 1)^2 \right\}, \\ S = \quad & \{x \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 \geq 0, x_1^2 + x_2^2 \leq 1\}, \\ \bar{S} = \quad & \{(1, 0), (0, 1)\}. \end{aligned}$$

4.

$$(MP) \quad \begin{aligned} \min_{x \in S} \quad & \{f(x) = \sin x_1 \cos x_2\}, \\ S = \quad & \{x \in \mathbb{R}^2 \mid 0 \leq x_1 \leq \frac{\pi}{2}, 0 \leq x_2 \leq \frac{\pi}{2}\}, \\ \bar{S} = \quad & \{x \in \mathbb{R}^2 \mid x_1 = 0, 0 \leq x_2 \leq \frac{\pi}{2}\} \cup \{x \in \mathbb{R}^2 \mid 0 \leq x_1 \leq \frac{\pi}{2}, x_2 = \frac{\pi}{2}\}. \end{aligned}$$

5.

$$VIP(F, S) \quad \begin{aligned} F(x) = \quad & (-x_1 e^{1-x_1^2}, x_2(x_2^2 - 1)), \\ S = \quad & \{x \in \mathbb{R}^2 \mid -1 \leq x_1 \leq 1, -1 \leq x_2 \leq 1\}, \\ \bar{S} = \quad & \{((-1)^i, (-1)^j) \mid i = 1, 2, j = 1, 2\}. \end{aligned}$$

5. Finite Termination of Feasible Solution Sequence

In this section, under the condition that the solution set $\bar{S}$ of $EP(\phi, S)$ is augmented weak sharp with respect to $\{x^k\} \subset S$, we present the condition of finite identification of $\{x^k\}$ (see Theorem 2). Applying the result to four special cases of $EP(\phi, S)$ (see Examples 1–4), we derive a series of results for the finite identification of feasible solution sequences in these cases. The first two cases, respectively, refer to the mathematical programming and variational inequalities problems. These two results are generalizations of the corresponding results in the literature under the condition that $\bar{S}$ is weakly sharp or strongly non-degenerate. The last two cases mean the saddle-point and Nash equilibrium problems; the finite identification problems of feasible solution sequences have not been reported by other authors in the literature.

Theorem 2. In $EP(\phi, S)$, let $\bar{S} \subset S$ be a closed set, and for every $x \in S$, assume $\partial_y \phi(x, x) \neq \emptyset$. If $\bar{S}$ is augmented weak sharp with respect to a sequence $x^k \subset S$, then the following statements hold:

(i) Assuming condition (2) is satisfied. If the sequence x^k terminates finitely to $\bar{S}$, then,

$$0 \in \liminf_{k \rightarrow \infty} P_{T_S(x^k)}(-\partial_y \phi(x^k, x^k)). \quad (50)$$

(ii) If condition (50) is met, then the sequence x^k terminates finitely to $\bar{S}$.

Proof. (i) If $x^k \in \bar{S}$, then by (2), we know that there exist $u^k \in \partial_y \phi(x^k, x^k)$ such that $-u^k \in N_{\bar{S}}(x^k)$. Therefore, by the convexity of S and projective decomposition, we have that $P_{T_S(x^k)}(-u^k) = 0$, i.e., (50) holds.

(ii) Suppose (50) holds. Now, we prove that $\{x^k\}$ terminates finitely to $\bar{S}$. If not, then $K = \{k \mid x^k \notin \bar{S}\}$ is an infinite sequence. According to this, $\bar{S}$ is augmented weakly sharp with respect to $\{x^k\} \subset S$, there exist a mapping $H : \bar{S} \rightarrow 2^{\mathbb{R}^2}$ and a constant $\alpha > 0$ such that

$$\alpha B \subset H(z) + [T_S(z) \cap \hat{N}_{\bar{S}}(z)]^\circ, \forall z \in \bar{S}, \quad (51)$$

and for $\forall u^k \in \partial_y \phi(x^k, x^k)$, $v^k \in H(P_{\bar{S}}(x^k))$, we have

$$\limsup_{k \in K, k \rightarrow \infty} \frac{1}{\|x^k - P_{\bar{S}}(x^k)\|} \langle u^k - v^k, x^k - P_{\bar{S}}(x^k) \rangle \geq 0. \quad (52)$$

Let $z^k = P_{\bar{S}}(x^k)$. Then, for $\forall z \in \bar{S}$, we have $\|z^k - x^k\|^2 \leq \|z - x^k\|^2$, and

$$\langle x^k - z^k, z - z^k \rangle \leq \frac{1}{2} \|z - z^k\|^2 = o(\|z - z^k\|).$$

Therefore, by the definition of $\hat{N}_{\bar{S}}(\cdot)$, we obtain that

$$x^k - z^k \in \hat{N}_{\bar{S}}(z^k). \quad (53)$$

Furthermore, by the convexity of S , we have

$$x^k - z^k \in T_S(z^k), \quad z^k - x^k \in T_S(x^k). \quad (54)$$

According to (53) and (54), we immediately obtain that

$$x^k - z^k \in T_S(z^k) \cap \hat{N}_{\bar{S}}(z^k). \quad (55)$$

Let $g_k = \frac{x^k - z^k}{\|x^k - z^k\|}$ ($k \in K$). By (51), one can see that there exists $\bar{v}^k \in H(z^k)$, $\bar{\xi}^k \in [T_S(z^k) \cap \hat{N}_{\bar{S}}(z^k)]^\circ$ such that for $\forall k \in K$. We have

$$\alpha g_k = \bar{v}^k + \bar{\xi}^k. \quad (56)$$

By (55) and (56), for $\forall k \in K$, we obtain that

$$\alpha = \langle \bar{v}^k, g_k \rangle + \langle \bar{\xi}^k, g_k \rangle \leq \langle \bar{v}^k, g_k \rangle. \quad (57)$$

On the other hand, from (50), one can see

$$0 \in \liminf_{k \in K, k \rightarrow \infty} P_{T_S(x^k)}(-\partial_y \phi(x^k, x^k)).$$

Therefore, there exists $\bar{u}^k \in \partial_y \phi(x^k, x^k)$ such that

$$\lim_{k \in K, k \rightarrow \infty} P_{T_S(x^k)}(-\bar{u}^k) = 0. \quad (58)$$

Using (54), (57), and the properties of the projected gradient (Lemma 3.1 [38]), we immediately obtain that

$$\begin{aligned} \alpha &\leq \langle \bar{v}^k, g_k \rangle \\ &= \langle -\bar{u}^k, -g_k \rangle - \langle \bar{u}^k - \bar{v}^k, g_k \rangle \\ &\leq \max\{\langle -\bar{u}^k, d \rangle \mid d \in T_S(x^k), \|d\| \leq 1\} - \langle \bar{u}^k - \bar{v}^k, g_k \rangle \\ &= \|P_{T_S(x^k)}(-\bar{u}^k)\| - \langle \bar{u}^k - \bar{v}^k, g_k \rangle. \end{aligned}$$

According to (52) and (58),

$$\begin{aligned} \alpha &\leq \liminf_{k \in K, k \rightarrow \infty} \{\|P_{T_S(x^k)}(-\bar{u}^k)\| - \langle \bar{u}^k - \bar{v}^k, g_k \rangle\} \\ &= - \limsup_{k \in K, k \rightarrow \infty} \langle \bar{u}^k - \bar{v}^k, g_k \rangle \leq 0, \end{aligned}$$

which leads to a contradiction. The proof has been completed. $\square$

Applying Theorem 2 to the special cases (Examples 1–4) of $EP(\phi, S)$, we can obtain the following corollaries.

Corollary 1. *The following conclusions hold:*

- (1) In (MP), suppose that $\bar{S} \subset S$ is a closed set, $\partial f(x) \neq \emptyset$ for $\forall x \in S$, and $\bar{S}$ is augmented weak sharp with respect to $\{x^k\} \subset S$. So the following conclusions are established:

(i) Suppose that (2) holds. If $\{x^k\}$ terminates finitely to $\bar{S}$, then we have

$$0 \in \liminf_{k \rightarrow \infty} P_{T_S(x^k)}(-\partial f(x^k)). \quad (59)$$

(ii) If (59) holds, then $\{x^k\}$ terminates finitely to $\bar{S}$.

- (2) In VIP(F, S), suppose that $\bar{S} \subset S$ is a closed set, $\bar{S}$ is augmented weak sharp with respect to $\{x^k\} \subset S$. Then, $\{x^k\}$ terminates finitely to $\bar{S}$ if, and only if,

$$\lim_{k \rightarrow \infty} P_{T_S(x^k)}(-F(x^k)) = 0. \quad (60)$$

- (3) In (SPP), suppose that $\bar{S} \subset S$ is a closed set, for $\forall x \in S$, $\partial \varphi(x) \neq \emptyset$, $\bar{S}$ is augmented weakly sharp with respect to $\{x^k\} \subset S$. So the following conclusions are established:

(i) Suppose that (2) holds. If $\{x^k\}$ terminates finitely to $\bar{S}$, then we have

$$0 \in \liminf_{k \rightarrow \infty} P_{T_S(x^k)}((-\partial_{y_1} \varphi(x^k), \partial_{y_2} \varphi(x^k))). \quad (61)$$

(ii) If (61) holds, then $\{x^k\}$ terminates finitely to $\bar{S}$.

- (4) In (NEP), suppose that $\bar{S} \subset S$ is a closed set, $\partial_{y_i} f_i(x) \neq \emptyset$ for $\forall x \in S$ and $i \in I = \{1, 2, \dots, n\}$, $\bar{S}$ is augmented weak sharp with respect to $\{x^k\} \subset S$. So, the following conclusions are established:

(i) Suppose that (2) holds. If $\{x^k\}$ terminates finitely to $\bar{S}$, then we have

$$0 \in \liminf_{k \rightarrow \infty} P_{T_S(x^k)}(-(\partial_{y_i} f_i(x^k), i \in I)). \quad (62)$$

(ii) If (62) holds, then $\{x^k\}$ terminates finitely to $\bar{S}$.

Next, we apply Theorem 2 to a kind of important function, i.e., for $\forall x \in S$, $\phi(x, \cdot)$ is a locally Lipschitz function on $\mathbb{R}^n$. For this function, by (Theorems 9.13 and 8.15 [1]), we know (2) is established, and $\partial_y \phi(x, x) \neq \emptyset$. Therefore, by Theorem 2, we have the following corollary:

Corollary 2. In $EP(\phi, S)$, suppose that $\bar{S} \subset S$ is a closed set, for $\forall x \in S$, $\phi(x, \cdot)$ is locally Lipschitz, $\bar{S}$ is augmented weak sharp with respect to $\{x^k\} \subset S$, then $\{x^k\} \subset S$ terminates finitely to $\bar{S}$, if and only if (50) holds.

By Corollary 2 and Proposition 1, we immediately get the following corollary.

Corollary 3. In $EP(\phi, S)$, suppose $\bar{S} \subset S$ is a closed set, $\phi(x, \cdot)$ is a locally Lipschitz function on $\mathbb{R}^n$ for $\forall x \in S$, and $\partial_y \phi(x, x)$ is monotone over S . If $\bar{S}$ is a weak sharp set, then $\{x^k\} \subset S$ terminates finitely to $\bar{S}$, if and only if (50) holds.

Remark 4. By Remark 2, one can see that the monotonicity of $\partial_y \phi(x, x)$ does not imply the convexity of $\phi(x, \cdot)$, and the reverse is also true. For example, $\phi(x, y) = x^2 y - x^3$, $(x, y) \in \mathbb{R} \times \mathbb{R}$.

Notice that a finite convex function on $\mathbb{R}^n$ is locally Lipschitz; therefore, by Corollary 3 and Proposition 2, we immediately obtain the following corollary:

Corollary 4. In $EP(\phi, S)$, suppose $\bar{S} \subset S$ is a closed set, for $\forall x \in S$, $\phi(x, \cdot)$ is a convex function on $\mathbb{R}^n$, and $\phi(\cdot, \cdot)$ is monotonic over $S \times S$. If $\bar{S}$ is a weak sharp set, then $\{x^k\} \subset S$ terminates finitely to $\bar{S}$, if and only if (50) holds.

For the special cases (Examples 1–3) of $EP(\phi, S)$, we have the following corollaries.

Corollary 5. *The following conclusions are established.*

- (1) *In (MP), suppose $\bar{S} \subset S$ is a weak sharp minimal set. Then, $\{x^k\} \subset S$ terminates finitely to $\bar{S}$, if and only if (59) holds.*
- (2) *In $VIP(F, S)$, suppose $F(\cdot)$ is monotonic over S , $\bar{S} \subset S$ is a closed and weak sharp set. Then, $\{x^k\} \subset S$ terminates finitely to $\bar{S}$, if, and only if, (60) holds.*
- (3) *In (SPP), for $\forall x_2 \in S_2$, $\varphi(\cdot, x_2)$ is a convex function over $\mathbb{R}^{n_1}$; for $\forall x_1 \in S_1$, $\varphi(x_1, \cdot)$ is a concave function over $\mathbb{R}^{n_2}$, and $\bar{S} \subset S$ is a weak sharp set. Then, $\{x^k\} \subset S$ terminates finitely to $\bar{S}$, if and only if (61) holds.*

Proof. According to the hypotheses in (1) and (2), one can see that they all meet the hypotheses conditions of Corollary 3. For (3), by its hypotheses and (5), one can see that the hypotheses conditions of Corollary 4 are satisfied. So, we immediately obtain that these conclusions (1)–(3) hold. $\square$

Remark 5. *Corollary 5 (1) is equivalent to (Theorem 3.1 [35]).*

By Corollary 2 and Proposition 3, we can obtain the following corollary:

Corollary 6. *Under the hypothesis in Theorem 1, and supposing $\bar{S} \subset S$ is closed and $\partial_y \phi(x, x)$ is monotonic over S . If for $\forall x \in \bar{S}$,*

$$-\partial_y \phi(x, x) \cap (-N_S(x)) \subset \text{int}G,$$

then $\{x^k\} \subset S$ terminates finitely to $\bar{S}$, if and only if (50) holds.

Since the convex programming (MP), as a special case of $EP(\phi, S)$, satisfies the hypotheses in Corollary 6, we can obtain the following corollary:

Corollary 7. *Suppose that in (MP), it holds that for $\forall x \in \bar{S}$,*

$$-\partial f(x) \cap (-N_S(x)) \subset \text{int}G.$$

Then, $\{x^k\} \subset S$ terminates finitely to $\bar{S}$, if and only if (59) holds.

Remark 6. *Corollary 7 is a generalization of (Theorem 4.7 [30]) in the smooth convex programming, and in Corollary 7, the two hypotheses about $\{x_k\}$ and $\nabla f(\cdot)$ in (Theorem 4.7 [30]) are removed.*

Next, we consider the smooth situation, the case that $\phi(x, \cdot)$ is locally Lipschitz. Therefore, by Proposition 4 and Corollary 2, we obtain the following corollary:

Corollary 8. *In $EP(\phi, S)$, suppose $\bar{S} \subset S$ is a closed set, and $\{x^k\} \subset S$ satisfies (17). If $\bar{S}$ is weak sharp, then $\{x^k\}$ terminates finitely to $\bar{S}$, if and only if*

$$\lim_{k \rightarrow \infty} P_{T_S(x^k)}(-\nabla_y \phi(x^k, x^k)) = 0. \quad (63)$$

By Proposition 5 and Corollary 2, we obtain the following corollary.

Corollary 9. *In $EP(\phi, S)$, suppose $\bar{S} \subset S$ is a closed set, and $\{x^k\} \subset S$, $\{\nabla_y \phi(x^k, x^k)\}$ is bounded and any of its accumulation $\bar{p}$ satisfies $-\bar{p} \in \text{int}G$. Then, $\{x^k\}$ terminates finitely to $\bar{S}$, if and only if (63) holds.*

Remark 7. *In the special cases $VIP(F, S)$ of $EP(\phi, S)$, Corollary 9 is a generalization and improvement of (Theorem 3.3 [35]), i.e., $[T_S(x) \cap N_S(x)]^\circ$ in (Theorem 3.3 [35]) is replaced with*

$[T_S(x) \cap \hat{N}_{\bar{S}}(x)]^\circ$, and the assumption of the continuity of $F(\cdot)$ is removed. It is worthwhile to note that (Theorem 3.3 [35]) has even improved (Theorem 3.2 [34]).

By Proposition 6 and Corollary 2, we have the following corollary.

Corollary 10. *In $EP(\phi, S)$, suppose $\nabla_y \phi(x, x)$ is continuous over S , $\{x^k\} \subset S$ is bounded, and any of its accumulation is strongly non-degenerate. Then, $\{x^k\}$ terminates finitely to $\bar{S}$, if and only if the (63) holds.*

Remark 8. *In smooth problems, a special case of $EP(\phi, S)$, Corollary 10 is just (Theorem 5.3 [33]), and the latter is an extension of (Corollary 3.5 [20]).*

By Theorem 2 and a series of its corollaries, one can see that, under normal conditions, the weak sharpness or strong non-degeneracy of the solution set is a special case of the augmented weak sharpness with respect to the feasible solution sequence. On the other hand, for some algorithms in mathematical programming and variational inequalities, for example, the proximal point algorithm, the gradient projection algorithm and the SQP algorithm, and so on (see [18,20,32–34,38,39]), the projected gradient of the point sequence generated by them all converge to zero, i.e., (50) holds. Therefore, the notion of augmented weak sharpness of the solution set presented by us provides weaker sufficient conditions than the weak sharpness or strong non-degeneracy for the finite termination of these algorithms.

6. Conclusions

In this paper, we introduce a novel concept related to the solution set of equilibrium problems, namely, the augmented weak sharpness of the solution set. This concept extends the ideas of weak sharpness and strong non-degeneracy to sequences of feasible solutions. We elucidate the relationship between the augmented weak sharpness and the traditional notions of weak sharpness and strong non-degeneracy, providing sufficient conditions for the presence of augmented weak sharpness in both non-smooth and smooth cases. Through examples, we show that the requirements for the augmented weak sharpness of the solution set are less stringent than those for weak sharpness. We establish both the necessary and sufficient conditions for the finite termination of feasible solution sequences under the criterion of augmented weak sharpness in equilibrium problems. Additionally, we demonstrate how these results can be applied to mathematical programming problems, variational inequality problems, Nash equilibrium problems, and global saddle-point problems. Based on these conditions, we conclude that the criteria for augmented weak sharpness are more relaxed compared to those for weak sharpness and strong non-degeneracy.

Author Contributions: Conceptualization, R.W., W.Z., D.S. and Y.H.; methodology, R.W., W.Z., D.S. and Y.H.; writing—original draft preparation, R.W., W.Z. and D.S.; writing—review and editing, W.Z. and Y.H.; supervision, W.Z. and D.S.; funding acquisition, W.Z., D.S. and Y.H. All authors have read and agreed to the published version of the manuscript.

Funding: This research was supported by the National Natural Science Foundations of China (12371305), the Natural Science Foundation of Shandong Province (ZR2021MA066, ZR2023MA020), the Natural Sciences and Engineering Research Council of Canada (RGPIN-2018-05687), and a centennial fund of the University of Alberta.

Data Availability Statement: Data sharing is not applicable to this article.

Conflicts of Interest: The authors declare no conflicts of interest.

References

- Rockafellar, R.; Wets, R. *Variational Analysis*; Springer: New York, NY, USA, 1998.
- Bigi, G.; Castellani, M.; Pappalardo, M.; Passacantando, M. Existence and solution methods for equilibria. *Eur. J. Oper. Res.* **2013**, *227*, 1–11. [\[CrossRef\]](#)
- Blum, E. From optimization and variational inequalities to equilibrium problems. *Math. Stud.* **1994**, *63*, 123–145.
- Combettes, P.L.; Hirstoaga, S.A. Equilibrium programming in Hilbert spaces. *J. Nonlinear Convex Anal.* **2005**, *6*, 117–136.
- Fan, K. A minimax inequality and applications. *Inequalities* **1972**, *3*, 103–113.
- Flâm, S.D.; Antipin, A.S. Equilibrium programming using proximal-like algorithms. *Math. Program.* **1996**, *78*, 29–41. [\[CrossRef\]](#)
- Konnov, I. *Equilibrium Models and Variational Inequalities*; Elsevier: Amsterdam, The Netherlands, 2007.
- Moudafi, A. On finite and strong convergence of a proximal method for equilibrium problems. *Numer. Funct. Anal. Optim.* **2007**, *28*, 1347–1354. [\[CrossRef\]](#)
- Mastroeni, G. Gap functions for equilibrium problems. *J. Glob. Optim.* **2003**, *27*, 411–426. [\[CrossRef\]](#)
- Van Nguyen, L.; Ansari, Q.H.; Qin, X. Linear conditioning, weak sharpness and finite convergence for equilibrium problems. *J. Glob. Optim.* **2020**, *77*, 405–424. [\[CrossRef\]](#)
- Li, M.; Zhang, C.; Ding, M.; Lv, R. A two-stage stochastic variational inequality model for storage and dynamic distribution of medical supplies in epidemic management. *Appl. Math. Model.* **2022**, *102*, 35–61. [\[CrossRef\]](#)
- Fargetta, G.; Maugeri, A.; Scrimali, L. A stochastic Nash equilibrium problem for medical supply competition. *J. Optim. Theory Appl.* **2022**, *193*, 354–380. [\[CrossRef\]](#)
- Nagurney, A. Supply chain game theory network modeling under labor constraints: Applications to the COVID-19 pandemic. *Eur. J. Oper. Res.* **2021**, *293*, 880–891. [\[CrossRef\]](#) [\[PubMed\]](#)
- Milasi, M.; Scopelliti, D. A stochastic variational approach to study economic equilibrium problems under uncertainty. *J. Math. Anal. Appl.* **2021**, *502*, 125243. [\[CrossRef\]](#)
- Zangenehmehr, P.; Farajzadeh, A. On Solutions of Generalized Implicit Equilibrium Problems with Application in Game Theory. *Adv. Math. Financ. Appl.* **2022**, *7*, 391–404.
- Meng, Q.; Nian, X.; Chen, Y.; Chen, Z. Neuro-adaptive control for searching generalized Nash equilibrium of multi-agent games: A two-stage design approach. *Neurocomputing* **2023**, *530*, 69–80. [\[CrossRef\]](#)
- Rockafellar, R.T. Monotone operators and the proximal point algorithm. *SIAM J. Control Optim.* **1976**, *14*, 877–898. [\[CrossRef\]](#)
- Polyak, B.T. *Introduction to Optimization*; Optimization Software, Publications Division: New York, NY, USA, 1987.
- Ferris, M.C. Finite termination of the proximal point algorithm. *Math. Program.* **1991**, *50*, 359–366. [\[CrossRef\]](#)
- Burke, J.V.; Moré, J.J. On the identification of active constraints. *SIAM J. Numer. Anal.* **1988**, *25*, 1197–1211. [\[CrossRef\]](#)
- Burke, J.V.; Ferris, M.C. Characterization of solution sets of convex programs. *Oper. Res. Lett.* **1991**, *10*, 57–60. [\[CrossRef\]](#)
- Marcotte, P.; Zhu, D. Weak sharp solutions of variational inequalities. *SIAM J. Optim.* **1998**, *9*, 179–189. [\[CrossRef\]](#)
- Al-Homidan, S.; Ansari, Q.H.; Burachik, R.S. Weak sharp solutions for generalized variational inequalities. *Positivity* **2017**, *21*, 1067–1088. [\[CrossRef\]](#)
- Huang, H.; He, M. Weak sharp solutions of mixed variational inequalities in Banach spaces. *Optim. Lett.* **2018**, *12*, 287–299. [\[CrossRef\]](#)
- Nguyen, L.V. Weak sharpness and finite termination for variational inequalities on Hadamard manifolds. *Optimization* **2021**, *70*, 1443–1458. [\[CrossRef\]](#)
- Al-Homidan, S.; Ansari, Q.H.; Van Nguyen, L. Finite convergence analysis and weak sharp solutions for variational inequalities. *Optim. Lett.* **2017**, *11*, 1647–1662. [\[CrossRef\]](#)
- Liu, Y. Weakly sharp solutions and finite convergence of algorithms for a variational inequality problem. *Optimization* **2018**, *67*, 329–340. [\[CrossRef\]](#)
- Wu, Z. Characterizations of weakly sharp solutions for a variational inequality with a pseudomonotone mapping. *Eur. J. Oper. Res.* **2018**, *265*, 448–453. [\[CrossRef\]](#)
- Wu, Z.; Wu, S.Y. Weak sharp solutions of variational inequalities in Hilbert spaces. *SIAM J. Optim.* **2004**, *14*, 1011–1027. [\[CrossRef\]](#)
- Burke, J.V.; Ferris, M.C. Weak sharp minima in mathematical programming. *SIAM J. Control Optim.* **1993**, *31*, 1340–1359. [\[CrossRef\]](#)
- Ferris, M.C. Weak Sharp Minima and Penalty Functions in Mathematical Programming. Ph.D. Thesis, University of Cambridge, Cambridge, UK, 1988.
- Wang, C.; Liu, Q.; Yang, X. Convergence properties of nonmonotone spectral projected gradient methods. *J. Comput. Appl. Math.* **2005**, *182*, 51–66. [\[CrossRef\]](#)
- Wang, C.; Zhao, W.; Zhou, J.; Lian, S. Global convergence and finite termination of a class of smooth penalty function algorithms. *Optim. Methods Softw.* **2013**, *28*, 1–25. [\[CrossRef\]](#)
- Xiu, N.; Zhang, J. On finite convergence of proximal point algorithms for variational inequalities. *J. Math. Anal. Appl.* **2005**, *312*, 148–158. [\[CrossRef\]](#)
- Zhou, J.; Wang, C. New characterizations of weak sharp minima. *Optim. Lett.* **2012**, *6*, 1773–1785. [\[CrossRef\]](#)
- Tyrrell Rockafellar, R. *Convex Analysis*; Cambridge University Press: Cambridge, UK, 1970.
- Wang, C.Y.; Zhang, J.Z.; Zhao, W.L. Two error bounds for constrained optimization problems and their applications. *Appl. Math. Optim.* **2008**, *57*, 307–328. [\[CrossRef\]](#)

38. Calamai, P.H.; Moré, J.J. Projected gradient methods for linearly constrained problems. *Math. Program.* **1987**, *39*, 93–116. [\[CrossRef\]](#)
39. Xiu, N.; Zhang, J. Some recent advances in projection-type methods for variational inequalities. *J. Comput. Appl. Math.* **2003**, *152*, 559–585. [\[CrossRef\]](#)

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.