Article

Three New Proofs of the Theorem

\[ \text{rank } f(M) + \text{rank } g(M) = \text{rank} (f, g)(M) + \text{rank}[f, g](M) \]

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Abstract: It is well known that in \(\mathbb{C}[X]\), the product of two polynomials is equal to the product of their greatest common divisor and their least common multiple. In a recent paper, we proved a similar relation between the ranks of matrix polynomials. More precisely, the sum of the ranks of two matrix polynomials is equal to the sum of the rank of the greatest common divisor of the polynomials applied to the respective matrix and the rank of the least common multiple of the polynomials applied to the respective matrix. In this paper, we present three new proofs for this result. In addition to these, we present two more applications.

Keywords: rank; Jordan canonical form; matrix polynomials; Sylvester inequality; Frobenius inequality

MSC: 15A03; 15A20; 15A24

1. Introduction

In an euclidean ring \((A, +, \cdot)\), for any elements \(a, b \in A\), there exists the greatest common divisor \(d = (a, b) \in A\) and there exists the least common multiple \(m = [a, b] \in A\). For the rings \((\mathbb{Z}, +, \cdot)\) and \((\mathbb{C}[X], +, \cdot)\), the relation \(a \cdot b = d \cdot m\) is well known (see [1] p. 150). By \(\mathbb{C}[X]\), we denote the set of all polynomials in the indeterminate \(X\) with complex coefficients. In a recent paper (see [2]), we proved a similar relation between the ranks of matrix polynomials.

Theorem 1. For any two polynomials \(f, g \in \mathbb{C}[X]\) and for any matrix \(M \in M_n(\mathbb{C})\), the following relation holds:

\[ \text{rank } f(M) + \text{rank } g(M) = \text{rank} (f, g)(M) + \text{rank}[f, g](M), \]

where \(d := (f, g)\) denotes the greatest common divisor and \(m := [f, g]\) denotes the lowest common multiple of the polynomials \(f, g\).

In [2], the proof is presented using the method of elementary transformations in block-partitioned matrices. In this paper, we present three more proofs of the previous theorem. The first proof uses the reduction of a matrix to Jordan’s canonical form. The following two proofs reduce the equality to be proved to the study of the case of equality from Frobenius inequality.

The rank of a matrix is a fundamental concept in numerical linear algebra and engineering. For example, in control theory, the rank of a matrix can be used to determine whether a linear system is controllable or observable.

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2. Main Results

First Proof of Theorem 1 (Vasile Pop). For the first proof, we use the reduction of the matrices which appear in Theorem 1 to Jordan’s canonical form. More precisely, we show how the relation of this theorem can be reduced to the case in which $M$ is a Jordan cell:

$$J_{\lambda} = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}.$$  

For this, we recall some basic facts (see, for example, [3], pp. 164–169; [4], pp. 95–100).

Theorem 2. Let $M \in \mathcal{M}_n(\mathbb{C})$ be a matrix and $f \in \mathbb{C}[X]$ a polynomial. Then,

(i) There is an invertible matrix $P \in \text{GL}_n(\mathbb{C}) = \{ A \in \mathcal{M}_n(\mathbb{C}) \mid \det A \neq 0 \}$ such that the matrix $J_M = P^{-1} \cdot M \cdot P$ has a diagonal-block form

$$J_M = J_{\lambda_1} \oplus J_{\lambda_2} \oplus \cdots \oplus J_{\lambda_k},$$

where $\lambda_1$, $\lambda_2$, ..., $\lambda_k$ are the distinct eigenvalues of $M$ and $J_{\lambda_1}$, $J_{\lambda_2}$, ..., $J_{\lambda_k}$ are the corresponding Jordan cells;

(ii) $\text{rank } M = \text{rank } J_M = \text{rank } J_{\lambda_1} + \text{rank } J_{\lambda_2} + \cdots + \text{rank } J_{\lambda_k};$

(iii) $f(M) = P \cdot f(J_M) \cdot P^{-1};$

(iv) $f(J_M) = f(J_{\lambda_1}) \oplus f(J_{\lambda_2}) \oplus \cdots \oplus f(J_{\lambda_k});$

(v) for any Jordan cell $J_{\lambda} \in \mathcal{M}_m(\mathbb{C})$, we have

$$f(J_{\lambda}) = \begin{bmatrix} f(\lambda) & f'(\lambda) & \frac{f''(\lambda)}{2!} & \cdots & \frac{f^{(m-2)}(\lambda)}{(m-2)!} & \frac{f^{(m-1)}(\lambda)}{(m-1)!} \\ 0 & f(\lambda) & f'(\lambda) & \cdots & \frac{f^{(m-3)}(\lambda)}{(m-3)!} & \frac{f^{(m-2)}(\lambda)}{(m-2)!} \\ 0 & 0 & f(\lambda) & \cdots & \frac{f^{(m-4)}(\lambda)}{(m-4)!} & \frac{f^{(m-3)}(\lambda)}{(m-3)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & f(\lambda) & f'(\lambda) \\ 0 & 0 & 0 & \cdots & 0 & f(\lambda) \end{bmatrix};$$

(vi) for any Jordan cell $J_{\lambda} \in \mathcal{M}_m(\mathbb{C})$, we have

$$\text{rank } f(J_{\lambda}) = 0$$

if $\lambda$ is a root of $f$ with multiplicity at least $m$ (i.e., $f(\lambda) = f'(\lambda) = \cdots = f^{(m-1)}(\lambda) = 0$). For the other cases, we have

$$\text{rank } f(J_{\lambda}) = m - \sigma_f(\lambda),$$

where $\sigma_f(\lambda)$ is the order of multiplicity of the $\lambda$ for the polynomial $f$ (i.e., $f(\lambda) = f'(\lambda) = \cdots = f^{(i)}(\lambda) = 0$, but $f^{(i+1)}(\lambda) \neq 0$, with $i + 1 = \sigma_f(\lambda)$).

Using Theorem 2 (i)–(iv), proving equality (1) is reduced to proving the following relation:

$$\text{rank } f(J_{\lambda}) + \text{rank } g(J_{\lambda}) = \text{rank } (f \circ g)(J_{\lambda}) + \text{rank } [f, g](J_{\lambda}),$$

for any $J_{\lambda}$ Jordan cell of $M$. 

Let $\lambda$ be an eigenvalue of the matrix $M$ and $J_\lambda$ the corresponding Jordan cell. With the above notations, 
\[(f, g) = d \text{ and } [f, g] = m,
\]
we have
\[\sigma_d(\lambda) = \min(\sigma_f(\lambda), \sigma_g(\lambda)) = \sigma_f(\lambda) \land \sigma_g(\lambda)\]
and
\[\sigma_m(\lambda) = \max(\sigma_f(\lambda), \sigma_g(\lambda)) = \sigma_f(\lambda) \lor \sigma_g(\lambda).
\]

By Theorem 2 (vi), to prove equality (2), we discuss the following three cases.
(a) If $\sigma_f(\lambda) \geq k$ and $\sigma_g(\lambda) \geq k$, then $\sigma_d(\lambda) \geq k$ and $\sigma_m(\lambda) \geq k$. Then, relation (2) becomes $0 + 0 = 0 + 0$, which, evidently, is true.
(b) If $\sigma_f(\lambda) \geq k$ and $\sigma_g(\lambda) = p < k$, then $\sigma_d(\lambda) = p < k$ and $\sigma_m(\lambda) \geq k$, and relation (2) becomes $p + 0 = p + 0$, which, evidently, is true.
(c) If $\sigma_f(\lambda) = p < k$ and $\sigma_g(\lambda) = q < k$, then $\sigma_d(\lambda) = p \land q$ and $\sigma_m(\lambda) = p \lor q$, and relation (2) becomes $p + q = p \land q + p \lor q$, which, evidently, is true. 
\[\square\]

**Second Proof of Theorem 1 (Alexandru Negrescu).** Consider $f_1, g_1 \in \mathbb{C}[X]$, two polynomials, such that
\[f = d \cdot f_1, \quad g = d \cdot g_1, \quad \text{and} \quad (f_1, g_1) = 1.
\]
As a consequence, we have
\[m = d \cdot f_1 \cdot g_1.
\]
We also consider the matrices $A, B, C \in \mathcal{M}_n(\mathbb{C})$ such that
\[A = f_1(M), \quad B = d(M), \quad \text{and} \quad C = g_1(M).
\]
We deduce that
\[f(M) = (f_1 \cdot d)(M) = f_1(M) \cdot d(M) = AB,
\]
\[g(M) = (d \cdot g_1)(M) = d(M) \cdot g_1(M) = BC,
\]
and
\[m(M) = (f_1 \cdot d \cdot g_1)(M) = f_1(M) \cdot d(M) \cdot g_1(M) = ABC.
\]
Therefore, the equality to be proved is equivalent to
\[\text{rank}(AB) + \text{rank}(BC) = \text{rank}(B) + \text{rank}(ABC),
\]
which we recognize as the equality case of Frobenius inequality.

In 2002, Tian and Styan (see [5] (Theorem 1)) proved the following result.

**Theorem 3.** Let $A \in \mathcal{M}_{m,n}(\mathbb{C}), B \in \mathcal{M}_{n,k}(\mathbb{C}), \text{ and } C \in \mathcal{M}_{k,l}(\mathbb{C})$ be given. Then, the following three statements are equivalent:

(a) $\text{rank} \begin{bmatrix} O & AB \\ BC & B \end{bmatrix} = \text{rank}(AB) + \text{rank}(BC)$;

(b) $\text{rank}(ABC) = \text{rank}(AB) + \text{rank}(BC) - \text{rank}(B)$;

(c) there are $X$ and $Y$ such that $BCX + YAB = B$.

Thus, the equality to be demonstrated is reduced to showing that there are two matrices, $X, Y \in \mathcal{M}_n(\mathbb{C})$, such that
\[g(M) \cdot X + Y \cdot f(M) = d(M),
\]
which is an immediate consequence of the following theorem (see [1] (Theorem 3.31, p. 135)).
**Theorem 4.** If \( \mathbb{K} \) is a field, \( f, g \in \mathbb{K}[X] \), and \( d \) is their greatest common divisor, then there are \( \varphi_1, \varphi_2 \in \mathbb{K}[X] \) such that
\[
g \cdot \varphi_1 + \varphi_2 \cdot f = d.
\]

Indeed, we can choose \( X = \varphi_1(M) \) and \( Y = \varphi_2(M) \), whereby the proof is finished. \( \square \)

**Third Proof of Theorem 1 (Mihai Opincariu).** As put forward in the previous proof, the equality to be demonstrated is equivalent to
\[
\text{rank}(AB) + \text{rank}(BC) = \text{rank}(B) + \text{rank}(ABC),
\]
where the notations are those of the second proof \( (A = f_1(M), B = d(M), C = g_1(M)) \).

To prove relation (3), we use Sylvester’s Theorem (see [6] (Theorem 2.6)), which presents the equality case in Sylvester’s inequality.

**Theorem 5.** For any matrices \( A, B \in \mathcal{M}_n(\mathbb{C}) \), the following relation holds:
\[
\text{rank } B - \text{rank } (AB) = \dim(\text{Ker } A \cap \text{Im } B). \tag{4}
\]

If, in relation (4), we replace \( B \) with \( BC \), we obtain
\[
\text{rank } (BC) - \text{rank } (ABC) = \dim(\text{Ker } A \cap \text{Im } (BC)). \tag{5}
\]

Subtracting relation (5) from (4), we obtain
\[
\text{rank } B + \text{rank } (ABC) = \text{rank } (AB) + \text{rank } (BC) + \dim(\text{Ker } A \cap \text{Im } B) - \dim(\text{Ker } A \cap \text{Im } (BC)). \tag{6}
\]

Since \( \text{Im } (BC) \subseteq \text{Im } B \), we obtain the well-known Frobenius inequality
\[
\text{rank } (AB) + \text{rank } (BC) \leq \text{rank } B + \text{rank } (ABC), \tag{7}
\]
and by (6), it follows that equality case (3) is true if and only if
\[
\text{Ker } A \cap \text{Im } B = \text{Ker } A \cap \text{Im } (BC).
\]

So, taking into account the notations at the beginning of this proof, it remains to show the relation
\[
\text{Ker } (f_1(M)) \cap \text{Im } (d(M)) = \text{Ker } (f_1(M)) \cap \text{Im } (d(M)g_1(M)).
\]

It is enough to prove the inclusion “\( \subset \)”, since the other is obvious.

We have:
\[
x \in \text{Ker } (f_1(M)) \cap \text{Im } (d(M)) \quad \text{iff} \quad f_1(M) \cdot x = 0 \text{ and } x = d(M) \cdot y.
\]

According to Theorem 4, since \( (f_1, g_1) = 1 \), there exist two polynomials, \( f_2, g_2 \in \mathbb{C}[X] \), such that
\[
f_2 \cdot f_1 + g_2 \cdot g_1 = 1,
\]

hence
\[
f_2(M) \cdot f_1(M) + g_2(M) \cdot g_1(M) = I_n.
\]

By multiplying in the last relation by \( x \) and taking into account that \( x \in \text{Ker } (f_1(M)) \), we obtain
\[
g_2(M) \cdot g_1(M) \cdot x = x.
\]

Since \( x = d(M) \cdot y \), we have
\[
g_2(M) \cdot g_1(M) \cdot d(M) \cdot y = x.
equivalent to
\[ d(M) \cdot g_1(M)(g_2(M) \cdot y) = x, \]
or
\[ x = d(M) \cdot g_1(M) \cdot z, \]
where \( z = g_2(M) \cdot y, \)
that is,
\[ x \in \text{Im}(d(M) \cdot g_1(M)) \]
and the proof is finished. \( \square \)

Besides the applications highlighted in [2], Section 3, we present two others. The first of these is closely related to a problem given in the 2004 edition of the Undergraduate Mathematics Competition at the Mechanics and Mathematics Faculty of Taras Shevchenko National University of Kyiv, Ukraine (see [7], p. 27).

**Proposition 1.** Let \( A, B, C, \) and \( D \) be \( n \times n \) real matrices such that
\[ A^T = BCD, \quad B^T = CDA, \quad C^T = DAB, \quad \text{and} \quad D^T = ABC. \]
We denote \( S = ABCD. \) Then,
\[ \text{rank} \ S + \text{rank} (S - I_n) + \text{rank} (S + I_n) = 2n. \]

**Proof.** First, we show that \( S \) is a tripotent matrix. Indeed, we have
\[
\]

If we consider in Theorem 1 the polynomials \( f = x \) and \( g = x^2 - 1, \) for which \( d = 1 \) and \( m = x^3 - x, \) we obtain that \( \text{rank} \ S + \text{rank} (S^2 - I_n) = n + \text{rank} (S^3 - S); \) thereby, taking into account that the matrix \( S \) is tripotent, we deduce that \( \text{rank} \ S + \text{rank} (S^2 - I_n) = n. \)

If we consider in Theorem 1 the polynomials \( f = x - 1 \) and \( g = x + 1, \) for which \( d = 1 \) and \( m = x^2 - 1, \) we obtain that \( \text{rank} (S - I_n) + \text{rank} (S + I_n) = n + \text{rank} (S^2 - I_n), \) so \( \text{rank} (S^2 - I_n) = \text{rank} (S - I_n) + \text{rank} (S + I_n) - n \) and the conclusion follows. \( \square \)

The following result was proposed by Bogdan Sebacher in the 2019 edition of the Traian Lalescu National Mathematics Contest for University Students, Romania (see [8]).

**Proposition 2.** Let \( a \in \mathbb{C}, \) with \( a \neq 0, n \geq 2, \) and \( A \in \mathcal{M}_n(\mathbb{C}). \) Then,
\[ \text{rank} \left( aA - A^2 \right) = \text{rank} A + \text{rank} (aI_n - A) - n. \]

**Proof.** If we consider in Theorem 1 the polynomials \( f = x \) and \( g = a - x, \) for which \( d = 1 \) and \( m = ax - x^2, \) we obtain that \( \text{rank} A + \text{rank} (aI_n - A) = \text{rank} I_n + \text{rank} (aA - A^2), \) and the conclusion follows. \( \square \)

In addition to those presented in [2], we offer two more consequences of Theorem 1, which can be easily shown by the method of mathematical induction.

**Corollary 1.** Let \( f_1, f_2, \ldots, f_m \in \mathbb{K}[X] \) be polynomials and \( A \in \mathcal{M}_n(\mathbb{K}). \) Let \( \left( f_{i_1}, f_{i_2}, \ldots, f_{i_p} \right) \) denote the greatest common divisor of polynomials \( f_{i_1}, f_{i_2}, \ldots, f_{i_p} \) (the maximum-degree monic polynomial that divides all polynomials \( f_{i_1}, f_{i_2}, \ldots, f_{i_p} \)) and \( \left[ f_{i_1}, f_{i_2}, \ldots, f_{i_p} \right] \) denote the lowest common multiple of the polynomials \( f_{i_1}, f_{i_2}, \ldots, f_{i_p} \) (the minimum-degree monic polynomial that is a multiple of all polynomials \( f_{i_1}, f_{i_2}, \ldots, f_{i_p} \)). Then,
\[
\text{rank}[f_1, f_2, \ldots, f_m](A) = \sum_{i_1=1}^{m} \text{rank}(f_{i_1})(A) - \sum_{i_1 < i_2} \text{rank}(f_{i_1}, f_{i_2})(A) + \\
+ \sum_{i_1 < i_2 < i_3} \text{rank}(f_{i_1}, f_{i_2}, f_{i_3})(A) + \ldots + \\
+ (-1)^{m-1} \text{rank}(f_1, f_2, \ldots, f_m)(A)
\]

and

\[
\text{rank}(f_1, f_2, \ldots, f_m)(A) = \sum_{i_1=1}^{m} \text{rank}[f_{i_1}](A) - \sum_{i_1 < i_2} \text{rank}[f_{i_1}, f_{i_2}](A) + \\
+ \sum_{i_1 < i_2 < i_3} \text{rank}[f_{i_1}, f_{i_2}, f_{i_3}](A) + \ldots + \\
+ (-1)^{m-1} \text{rank}[f_1, f_2, \ldots, f_m](A).
\]

3. Conclusions

This paper provides three new proofs that the sum of the ranks of two matrix polynomials is equal to the sum of the rank of the greatest common divisor of the polynomials applied to the respective matrix and the rank of the least common multiple of the polynomials applied to the respective matrix. In addition to these, two more applications are provided.

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References

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