On a Class of Nonlinear Elliptic Equations with General Growth in the Gradient

M. Francesca Betta 1, Anna Mercaldo 2,* and Roberta Volpicelli 2

Abstract: In this paper, we prove an existence and uniqueness result for a class of Dirichlet boundary value problems whose model is
\[
-\Delta_p u = \beta |\nabla u|^q + c(x)|u|^{p-2}u + f \quad \text{in } \Omega,
\]
\[
u = 0 \quad \text{on } \partial \Omega,
\]
where \(\Omega\) is a bounded open subset of \(\mathbb{R}^N\), \(N \geq 2\), \(1 < p < N\), \(\Delta_p u\) is the so-called \(p\)-Laplace operator, and \(p-1 < q < p\). We assume that \(\beta\) is a positive constant, \(c\) and \(f\) are measurable functions belonging to suitable Lorentz spaces. Our approach is based on Schauder fixed point theorem.

Keywords: existence; uniqueness; nonlinear elliptic equations; fixed point

MSC: 35J25; 35J60

1. Introduction

In this paper, we consider a class of Dirichlet boundary value problems whose prototype is
\[
\begin{align*}
-\Delta_p u &= \beta |\nabla u|^q + c(x)|u|^{p-2}u + f \\
u &= 0
\end{align*}
\]
in \(\Omega\), with \(\beta\) a positive constant, \(c\) and \(f\) are functions belonging to suitable Lorentz spaces. Suitable solutions are in order. We begin by dealing with the existence for Problem (1). Existence results in the case where \(0 \leq q < p-1\) and \(c \leq 0\) are well known. Indeed, in this case, an a priori estimate for every solution \(u\) of (1) in \(W_0^{1,p}(\Omega)\) can be obtained by using \(u\) as a test function, and existence follows by classical theory of pseudomonotone operators due to J. Leray and J.-L. Lions (see, for example, [3]). The same occurs when \(q = p-1\), \(\beta\) is small enough and \(c \leq 0\), since in this case, the operator \(-\Delta_p u - \beta |\nabla u|^q - c(x)|u|^{p-2}u\) is coercive. If \(\beta\) is large, this is not the case. However, this problem has been solved in the linear case in [4], and in the nonlinear case by various authors (see, for example, [5] and the references therein); the approach in some sense allows one to reduce the problem to a finite sequence of problems with coefficient \(\beta = \beta(x)\) having a norm in a suitable Lebesgue space small enough and to again obtain a priori estimates.

The limit case \(q = p\) has been studied by many authors, by proving a priori estimates as, for example, in [6] and the references therein. In those papers, the authors prove a priori
estimates and, therefore, the existence of solutions in $W^{1,p}_0(\Omega)$, which are not bounded in general, but satisfy a further regularity; the approach used in this case is based on a change of unknown function $w = \exp(\frac{\beta}{p-1}|u|-1)\text{sign}u$, which allows one to cancel the term $\beta|\nabla u|^p$.

In this paper, we focus our attention on the case $p - 1 < q < p$. When the study of the existence of solutions to Problem (1) is faced, some necessary conditions are required on the data, as shown in [7,8] for $p = 2$ and $c = 0$. These necessary conditions, when the datum is an element of a Lebesgue space, consist of the fact that the datum $f$ belongs to the $L^m(\Omega)$ with $m > \frac{N(q-p+1)}{q}$ and the norm $\|f\|_{L^m(\Omega)}$ is sufficiently small. Moreover, it is well-known that if $p = 2, f = 0$ and $c$ is a constant, the existence and uniqueness of a weak solution are guaranteed if $c < \lambda(\Omega)$, where $\lambda(\Omega)$ is the first eigenvalue of the Laplacian operator (see, for example, [9], where the more general case with $p \neq 2$ is considered).

In order to explain the difficulties related to obtaining a priori estimates, let us use $u$ as test function in (1) when $p - 1 < q < p - 1 + \frac{2}{N}$, $c \in L^\infty(\Omega)$ and $f \in L^{(p-1)'}(\Omega)$, where $p^* = \frac{pN}{p-N}$. This leads to the following estimate:

$$
\|\nabla u\|_p^p \leq \beta C_p |\Omega|^{1+\frac{1}{p} - \frac{q-1}{p}} \|\nabla u\|^p_{p} + C_p |c| \|\nabla u\|_p + C_p \|f\|_{\|\nabla|^p \|_p},
$$

where $C_p$ is the best constant in the Sobolev embedding of $W^{1,p}_0(\Omega)$ in $L^p(\Omega)$. Unfortunately, since $q > p - 1$, this inequality does not provide any a priori bound for the gradient of solution $u$, even if a size condition on $\|c\|_{L^\infty(\Omega)}$ is required.

To overcome the difficulties in deriving an estimate from (2), a method based on a continuity argument has been introduced in [10], while a comparison result with symmetrization techniques has been used in [11] when $c = 0$. Sharp assumptions on the summability of the datum $f$ (which depend on $q$) and on its smallness are also given in [11].

In order to prove a priori estimates for weak solutions, the presence of a zero-order term of the type $c(x)|u|^{p-2}u$ does not allow us to use the continuity methods quoted above, nor to obtain a comparison result by using symmetrization techniques. A first attempt to study the existence of weak solutions to Problem (1) has been made in [12], when the datum $f$ is an element of the dual space $W^{-1,p'}(\Omega)$ of the Sobolev space $W^{1,p}_0(\Omega)$, by proving the existence (without uniqueness) of a fixed point for a suitable operator.

As far as uniqueness of weak solutions concerns, a classical counterexample (recalled in Section 4), with $p = 2$, shows that uniqueness does not hold in general in the class of weak solutions and for values of $q > 1 + \frac{2}{N}$.

Nevertheless, various uniqueness results have been proven when the datum $f$ belongs to the dual space $W^{-1,p'}(\Omega)$ in the case where the equation does not involve lower order terms of the type $\beta|\nabla u|^q$ and $c(x)|u|^{p-2}u$, or in the case where the lower order term has natural growth with respect to the gradient, i.e., when $q = p$ under further regularity assumptions on the weak solutions (see, for example, [13]).

Uniqueness for the operator with a “superlinear” term of the type $\beta|\nabla u|^q$ are contained in [14], where two different uniqueness results, according to the values of $p$, namely $2N \leq p \leq 2$ and $p \geq 2$, are proven when $c = 0$. The difference between the two cases $p \leq 2$ and $p \geq 2$ is due to the assumptions that are made on the operator in (1); that is, the “strong monotonicity”. This means that uniqueness can be proven for operators like $-\text{div}(a(x)(1 + |\nabla u|^2)^{\frac{p-2}{2}}\nabla u)$ when $p \geq 2$, and both for operators $-\text{div}(a(x)(1 + |\nabla u|^2)^{\frac{p-2}{2}}\nabla u)$ and for operators $-\text{div}(a(x)|\nabla u|^{p-2}\nabla u)$ when $p \leq 2$. In both cases, $p \leq 2$ and $p \geq 2$, the results give the uniqueness of the weak solution to Problem (1) under the assumptions that $q \leq q^*$, where the value of $q^*(N, p, r)$, is given by
The main difficulty which arises in proving uniqueness results are due to the lower order term $\beta(1 + |\nabla u|^2)^{\frac{q}{2}}$ and to the fact that, in general, $\beta$ is large so that the operator is not coercive. In this case, the idea by Bottaro and Marina [4] was used, which consists of considering a coefficient $\beta = \beta(x)$ belonging to a suitable Lebesgue space $L^q(\Omega)$ and reducing to the case where $\|\beta\|_{L^q(\Omega)}$ is small enough.

In [15], the uniqueness of solutions of (1), with $p = 2$ and $c = 0$ or $c$ constant with $c < 0$, is proven for a class of regular solutions that is consistent with the existence results proven in [10]; that is, for weak solutions $u$, such that $(1 + |u|)^{q-1}u \in H^1_0(\Omega)$ with $q = \frac{(N-2)(q-1)}{2(2-q)}$. Uniqueness is proven when $p = 2$ for the whole interval of values of $q$, i.e., $1 < q < 2$, but it is basic to use convexity of the function $F(t) = t^q$ or a linearization process, which assures a higher summability of the gradient of the solutions, for low values of $q$ given by $\frac{q}{2} < 1 + 2/N$.

These results have been extended to more general classes of nonlinear elliptic equations (see, for example, [16–18] and the references therein). In all these papers, uniqueness results have been proven for strongly monotone operators, and partial intervals of values of $p$ and of $q$ are considered. As far as we know, uniqueness results for operators having both first- and zero-order terms for general coefficient $c$ are not available in the literature.

Further questions concerning the existence, multiplicity or regularity of solutions related to Problem (1) can be found in [19–22].

The main result of this paper is given by Theorem 1, stated in Section 2. It states both the existence and uniqueness for any value of $p$, $1 < p < N$, for values of $q$ given by $p - 1 < q < p - 1 + \frac{2}{N}$, under smallness assumptions on the datum $f$ and the coefficient $c$.

As pointed out, the upper bound on the value $q$ is natural when we consider the example given in Section 4, while the smallness assumption on $f$ is due to the fact that a “superlinear term” $\beta|\nabla u|^q$ with $q > p - 1$ appears. In Section 3, we prove our existence and uniqueness result by applying the Schauder fixed point theorem. Once the operator $A$ is defined in such a way that its fixed points are weak solution to (1), the main difficulty consists of proving that $A$ maps a ball $B_R$ of $W_0^{1,p}(\Omega)$ in itself for a suitable value of the radius $R$; actually, this value is a positive zero of a suitable function $G$ given in the statement of the theorem. Actually, Theorem 1 improves the result proven in [12], where just the existence was proven (without uniqueness) under stronger assumptions on the summability of $f$ and $c$. Moreover, Theorem 1 gives the uniqueness of a weak solution in a suitable ball of the Sobolev space $W_0^{1,p}(\Omega)$ for monotone operators $a$ (and no more for strongly monotone operators), depending also on $u$. In Section 4, we explicitly show the effects of the different approaches in order to prove uniqueness for Problem (1), and we highlight the novelty of the uniqueness result given by Theorem 1. Indeed, the “linearization process”, used to prove Theorem 2, can be just applied to operators satisfying further structural assumptions, such as strong monotonicity and, therefore, it gives uniqueness for a less general class of operators and for smaller interval of values of $p$.

2. Preliminaries and Statement of Main Result

In this section, we firstly recall a few properties of Lorentz spaces, and then we state the main result of the paper.

Let us begin by recalling some properties of rearrangements. If $u$ is a measurable function defined in $\Omega$, and

$$\mu(t) = |\{x \in \Omega : |u(x)| \geq t\}|, \quad t \geq 0$$

is its distribution function, then

$$u^*(s) = \sup\{t \geq 0 : \mu(t) > s\}, \quad s \in (0,|\Omega|),$$

where $u^*(s)$ is the $s$-envelope of $u$.
is the decreasing rearrangement of \( u \), and \( u_+(s) = u^s([\Omega] - s) \) is the increasing rearrangement of \( u \).

Here, \(|E|\) denotes the \( n \)-dimensional Lebesgue measure of any measurable set \( E \).

If \( \omega_N \) is the measure of the unit ball of \( \mathbb{R}^N \), and \( \Omega^p \) is the ball of \( \mathbb{R}^N \) centered at the origin with the same measure as \( \Omega \),

\[
  u^\#(x) = u^s(\omega_N|x|^N), \quad u_+(x) = u_+(\omega_N|x|^N), \quad x \in \Omega^p,
\]

denote the spherically decreasing and increasing rearrangements of \( u \), respectively.

For any \( t \in (0, +\infty) \), the Lorentz space \( L^{t,s}(\Omega) \) is the collection of all measurable functions \( u \), such that \( \|u\|_{L^{t,s}} \) is finite, where we use the notation

\[
  \|u\|_{L^{t,s}} = \left( \int_0^{t^{s/\tau}} \frac{u^s(s)^{1/\tau}}{ds} \right)^{1/r}
\]

if \( r \in ]0, \infty[ \);

\[
  \|u\|_{L^{t,s}_\infty} = \sup_{s>0} u^s(s)^{1/\tau} = \sup_{\tau>0} \tau \mu(\tau)^{1/\tau}
\]

if \( r = \infty \).

These spaces give, in some sense, a refinement of the usual Lebesgue spaces. Indeed, \( L^{t,1}(\Omega) = L^t(\Omega) \) and \( L^{t,\infty}(\Omega) = M^t(\Omega) \) is the \( L^1 \)-weak Marcinkiewicz space. The following embeddings hold true (see, for example, [23]):

\[
  L^{t_1}(\Omega) \subset L^{t_2}(\Omega), \quad \text{if} \quad t_1 < t_2, \quad (4)
\]

\[
  L^{t_1,t_2}(\Omega) \subset L^{t_2,t_2}(\Omega), \quad \text{for} \quad t_1 > t_2, \quad 0 < r_1, r_2 \leq \infty, \quad (5)
\]

and

\[
  L^{t,1}(\Omega) \subset L^t(\Omega), \quad \text{if} \quad t < t_1. \quad (6)
\]

Moreover, the following inequalities hold:

\[
  \|u\|_{L^{t_2}} \leq \left( \frac{r_1}{r_2} \right)^{\frac{1}{t_2} - \frac{1}{t_1}} \|u\|_{L^{t_1}}, \quad \text{for} \quad 0 < r_1 < r_2 \leq \infty, \quad (7)
\]

and

\[
  \|u\|_{L^{t_2,t_2}} \leq c_L \|u\|_{L^{t_1,t_1}}, \quad \text{for} \quad t_1 > t_2, \quad 0 < r_1, r_2 < \infty, \quad (8)
\]

where

\[
  c_L = \left[ \frac{t_2}{t_1} \right]^{\frac{1}{t_2}} \left( \frac{t_1}{t_2} \right)^{\frac{1}{t_1}} \frac{\mu(\Omega)^{1/\tau}}{\tau}. \quad (9)
\]

For \( 1 < t < \infty, 1 \leq r \leq \infty \), the generalized Hölder inequality holds true

\[
  \left| \int_\Omega fg \, dx \right| \leq \|f\|_{L^t} \|g\|_{L^{t',r'}}, \quad (10)
\]

for every \( f \in L^{t,r}(\Omega) \) and \( g \in L^{t',r'}(\Omega) \).

Here, for every \( 1 \leq t \leq \infty \), \( t' \) denotes the Hölder exponent \( \frac{1}{t'} + \frac{1}{r'} = 1 \).

More generally, if \( f \in L^{t_1,r_1}(\Omega) \) and \( g \in L^{t_2,r_2}(\Omega) \) with

\[
  \frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r}, \quad \frac{1}{t_1} + \frac{1}{t_2} = \frac{1}{t},
\]

then the following inequality holds true (cf., for example, [23]):

\[
  \|fg\|_{L^t} \leq t'\|f\|_{L^{t_1,r_1}} \|g\|_{L^{t_2,r_2}}, \quad (11)
\]
As pointed out in the Introduction, the aim of this paper is to prove existence and uniqueness of weak solutions to the following more general class of Dirichlet boundary value problems:

\[
\begin{cases}
-\mathrm{div} (a(x, u, \nabla u)) = H(x, \nabla u) + g(x, u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]  \hspace{1cm} (12)

where \( \Omega \) is a bounded open subset of \( \mathbb{R}^N \), \( N \geq 2 \), \( 1 < p < N \),

\[a : (x, s, z) \in \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow a(x, s, z) \in \mathbb{R}^N,\]

\[H : (x, z) \in \Omega \times \mathbb{R}^N \rightarrow H(x, z) \in \mathbb{R},\]

\[g : (x, s) \in \Omega \times \mathbb{R} \rightarrow g(x, s) \in \mathbb{R}\]

are Carathéodory functions which satisfy the ellipticity condition

\[a(x, s, z) \cdot z \geq |z|^p, \hspace{1cm} (13)\]

the monotonicity condition

\[(a(x, s, z) - a(x, s, z')) \cdot (z - z') > 0, \quad z \neq z' \hspace{1cm} (14)\]

and the growth conditions

\[|a(x, s, z)| \leq a_0|z|^{p-1} + a_1|s|^{p-1} + a_2, \quad a_0, a_1, a_2 > 0, \hspace{1cm} (15)\]

\[|H(x, z)| \leq \beta|z|^q + f(x), \quad \beta > 0, \hspace{1cm} (16)\]

\[|g(x, s)| \leq c(x)|s|^{p-1}, \quad c(x) \geq 0 \text{ a.e. in } \Omega, \hspace{1cm} (17)\]

with \( p - 1 < q < p \), for every \( s \in \mathbb{R} \) and for every \( z, z' \in \mathbb{R}^N \). Moreover, the datum \( f \) and the coefficient \( c \) are measurable functions in suitable Lorentz spaces.

The main result of the paper is given by the following theorem, which state both existence and uniqueness for weak solutions satisfying a suitable a priori estimates in \( W_0^{1,p}(\Omega) \). Actually, such a uniqueness result is a consequence of the fact that the weak solution is obtained as a fixed point of a suitable map, and such a fixed point is unique by the Schauder fixed point theorem.

**Theorem 1.** Assume that (13)–(17) hold true with

\[L^{\text{cond}}1p - 1 < q \leq p - 1 + \frac{p}{N}, \hspace{1cm} (18)\]

and

\[c \in L^\infty(\Omega), \quad f \in L^{(p)'}(\Omega).\]

If the norm of \( c \) in \( L^\infty(\Omega) \) and the norm of \( f \) in \( L^{(p)'}(\Omega) \) are sufficiently small, that is,

\[\|c\|_{L^\infty} < \frac{1}{5p} \text{ and } \|f\|_{L^{(p)'}} < C_1\]

then...
where \( S \) denotes the best constant in the embedding \( W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega) \),

\[
C_1 = \frac{(q - p + 1) \left( 1 - S^p \|c\|_{L^{p^*}}^{\frac{q}{p}} \right)}{(qS)^{\frac{q}{p^*}}} \left( \frac{N}{N - p} \right)^{\frac{p - 1}{N - p + 1}} \times \frac{\beta}{\sqrt{S}q\Omega}^{1 + \frac{q + 1}{p} - \frac{\beta}{\sqrt{S}q\Omega}} \left( \frac{N}{N - p} \right)^{\frac{1}{p} - \frac{\beta}{\sqrt{S}q\Omega}} \times \left( \frac{p - 1}{\beta} \right)^{\frac{p - 1}{p^*}} |\Omega|^{\frac{p - 1}{p^*} - \frac{p - 1}{N(p - 1 + 1)}},
\]

(20)

then there exists a unique weak solution \( u \in W_0^{1,p}(\Omega) \) to Problem (12), such that

\[
||\nabla u||_p \leq R,
\]

(21)

where \( R > 0 \) is the first positive zero of the function

\[
G(\rho) = \left( 1 - S^p \|c\|_{L^{p^*}}^{\frac{q}{p}} \right) \frac{\rho^{p - 1} - \beta S \Omega^{1 + \frac{q + 1}{p} - \frac{\beta}{\sqrt{S}q\Omega}} \left( \frac{N}{N - p} \right)^{\frac{1}{p} - \frac{\beta}{\sqrt{S}q\Omega}} \rho^\beta - S \|f\|_{L^{p^*}}^{\frac{1}{p^*}}}{\sqrt{S}q\Omega},
\]

(22)

for any \( \rho \geq 0 \).

**Remark 1.** Some comments on the smallness conditions (19) on the coefficient \( c \) and on the datum \( f \) are in order. Under Assumption (19), the function \( G(\rho) \) defined in (22) has a positive maximum \( G(\rho_0) \), where

\[
\rho_0 = \frac{(p - 1) \left( 1 - S^p \|c\|_{L^{p^*}}^{\frac{q}{p}} \right)}{\beta S q \Omega^{1 + \frac{q + 1}{p} - \frac{\beta}{\sqrt{S}q\Omega}} \left( \frac{N}{N - p} \right)^{\frac{1}{p} - \frac{\beta}{\sqrt{S}q\Omega}}},
\]

(23)

and

\[
\text{Lgrho}(\rho_0) = \frac{(q - p + 1) \left( 1 - S^p \|c\|_{L^{p^*}}^{\frac{q}{p}} \right)}{(qS)^{\frac{q}{p^*}}} \left( \frac{N}{N - p} \right)^{\frac{p - 1}{N(p - 1 + 1)}} \times \frac{\beta}{\sqrt{S}q\Omega}^{1 + \frac{q + 1}{p} - \frac{\beta}{\sqrt{S}q\Omega}} \left( \frac{N}{N - p} \right)^{\frac{1}{p} - \frac{\beta}{\sqrt{S}q\Omega}} \times \left( \frac{p - 1}{\beta} \right)^{\frac{p - 1}{p^*}} |\Omega|^{\frac{p - 1}{p^*} - \frac{p - 1}{N(p - 1 + 1)} - S \|f\|_{L^{p^*}}^{\frac{1}{p^*}}},
\]

(24)

Therefore, the smallness assumptions (19) guarantee that \( \rho_0 \) and \( G(\rho_0) \) are positive. Moreover, \( \rho_0 \) is the unique critical point with \( G'(\rho_0) > 0 \) for \( \rho < \rho_0 \) and \( G'(\rho_0) < 0 \) for \( \rho > \rho_0 \). Since \( G(0) = -S \|f\|_{L^{p^*}}^{\frac{1}{p^*}} < 0 \) and \( \lim_{\rho \to +\infty} G(\rho) = -\infty \), this allows us to assert that \( G(\rho) \) has two positive zeros.

**Remark 2.** Let us explicitly remark that Theorem 1 gives an answer to the question of existence and uniqueness for Problem (1), since this model problem belongs to the class of operators considered in (12).

3. Proof of the Main Result

In this section, we prove our main existence and uniqueness result for a weak solution to Problem (12), as stated in Theorem 1 above.

**Proof.** Consider the ball \( B_R \subset W_0^{1,p}(\Omega) \) defined by

\[
B_R = \left\{ v \in W_0^{1,p}(\Omega) : ||\nabla v||_p \leq R \right\},
\]

where \( R > 0 \) is the first positive zero of the function \( G \). \( \square \)
Consider the mapping
\[ A : B_R \rightarrow W_0^{1,p}(\Omega) \]
defined by
\[ A(v) = u \]
where \( u \in W_0^{1,p}(\Omega) \) is the unique weak solution to problem
\[
\begin{cases}
-\text{div} (a(x,v,\nabla u)) = H(x,\nabla v) + g(x,v) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]
(25)
that is,
\[
\int_{\Omega} a(x,v,\nabla u) \nabla \phi \, dx = \int_{\Omega} H(x,\nabla v) \phi \, dx + \int_{\Omega} g(x,v) \phi \, dx,
\]
for any \( \phi \in W_0^{1,p}(\Omega) \).

The existence and uniqueness of such a solution \( u \) is followed by classical results (see, for example, [3]). Indeed, since \( |\nabla v|^q \in L^q(\Omega), \ q \leq p - 1 + \frac{p}{N} \) and \( f \in L^{(p')*,p'}(\Omega) \),
then \( H(x,\nabla v) \in L^{(p')*,p'}(\Omega) \). Moreover, since \( c \in L^{(p')*,\infty}(\Omega) \), also \( g(x,v) \in L^{(p')*,p'}(\Omega) \).
Hence, the right-hand side of the equation in (25) belongs to \( L^{(p')*,p'}(\Omega) \) and by Sobolev embedding of \( W_0^{1,p}(\Omega) \) in \( L^{p',q'}(\Omega) \), it is an element of the dual space \( W^{-1,p'}(\Omega) \). Therefore, by classical results, the solution \( u \) exists, and is unique.

Now, the proof proceeds by showing that the Schauder fixed point theorem can be applied. Therefore, in Step 1 in the following, we prove that \( A \) maps \( B_R \) into itself; in Step 2, we prove that \( A \) is continuous; and, finally, in Step 3, we prove that \( A \) is a compact operator.

We proceed by dividing the proof into steps.

**STEP 1: In this step, we prove that \( A \) maps \( B_R \) into itself.**

Let \( v \in B_R \), and let \( A(v) = u \in W_0^{1,p}(\Omega) \) be the unique weak solution to Problem (25). Let us choose \( u \) as test function in (26). Then, we obtain
\[
\int_{\Omega} a(x,v,\nabla u) \nabla u \, dx = \int_{\Omega} H(x,\nabla v) u \, dx + \int_{\Omega} g(x,v) u \, dx.
\]
(27)
By the ellipticity condition (13) and the growth conditions (16) and (17), we have
\[
\int_{\Omega} |\nabla u|^p \, dx \leq \beta \int_{\Omega} |\nabla v|^q |u| \, dx + \int_{\Omega} |c||v|^{p-1} |u| \, dx + \int_{\Omega} |f||u| \, dx.
\]
(28)
By using generalized Hölder inequality (10) and inequality (7), since \( q \leq p - 1 + \frac{p}{N} \), we obtain
\[
\|\nabla u\|_p^p \leq \beta|\Omega|^{1+\frac{p-1}{p}} \|\nabla v\|_p^q \|u\|_{L^{p'}} + \|c\|_{L^{p',\infty}} \|v\|_{L^{p',p'}} \|u\|_{L^{p',p'}} + \|f\|_{L^{(p')*,p'}} \|u\|_{L^{p',p'}} \leq \beta|\Omega|^{1+\frac{p-1}{p}} \left( \frac{N-p}{N} \right)^{\frac{p}{N}} \|\nabla v\|_p^q \|u\|_{L^{p',p'}} + \|c\|_{L^{p',\infty}} \|v\|_{L^{p',p'}} \|u\|_{L^{p',p'}} + \|f\|_{L^{(p')*,p'}} \|u\|_{L^{p',p'}}.
\]
By sharp Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p',q}(\Omega)$, denoting with $S$ the best constant in such embedding, we deduce

\begin{equation}
\|\nabla u\|_p^p \leq \beta S|\Omega|^{1+\frac{1}{N} - \frac{p+1}{p}}\left(\frac{N-p}{N}\right)^\frac{1}{N} \|\nabla v\|_p^p \left(\frac{N-p}{N}\right)^\frac{1}{N} + S^p||c||_{L^{p,q}_\infty}\|\nabla v\|_p^p + S\|f\|_{L^{(p',q'),r}}\|\nabla u\|_p^p. \tag{30}
\end{equation}

Since $v \in B_R$, we have

\begin{equation}
\|\nabla u\|_p^{p-1} \leq \beta S|\Omega|^{1+\frac{1}{N} - \frac{p+1}{p}}\left(\frac{N-p}{N}\right)^\frac{1}{N} \|\nabla v\|_p^p \left(\frac{N-p}{N}\right)^\frac{1}{N} + S^p||c||_{L^{p,q}_\infty}\|\nabla v\|_p^{p-1} + S\|f\|_{L^{(p',q'),r}} \leq \beta S|\Omega|^{1+\frac{1}{N} - \frac{p+1}{p}}\left(\frac{N-p}{N}\right)^\frac{1}{N} R^q + S^p||c||_{L^{p,q}_\infty} R^{p-1} + S\|f\|_{L^{(p',q'),r}} = R^{p-1}, \tag{31}
\end{equation}

and the assertion follows.

**Step 2.** In this step, we prove that $A$ is continuous with respect to the strong convergence in $W_0^{1,p}(\Omega)$.

Assume that $w_n \in B_R$, such that

\begin{equation}
w_n \rightarrow w \text{ strongly in } W_0^{1,p}(\Omega). \tag{32}
\end{equation}

We prove that

\begin{equation}
A(w_n) \rightarrow A(w) \text{ strongly in } W_0^{1,p}(\Omega). \tag{33}
\end{equation}

By definition of $A$, $A(w_n) = u_n \in W_0^{1,p}(\Omega)$ is the unique solution to Problem (25) with $v = w_n$; that is,

\begin{equation}
\int_\Omega a(x,w_n,\nabla u_n) \nabla \phi \, dx = \int_\Omega H(x,\nabla w_n) \phi \, dx + \int_\Omega g(x,w_n) \phi \, dx, \tag{34}
\end{equation}

for all $\phi \in W_0^{1,p}(\Omega)$. Therefore, in order to show (33), we prove that

\begin{equation}
u_n \rightarrow u \text{ strongly in } W_0^{1,p}(\Omega) \tag{35}
\end{equation}

and

\begin{equation}u = A(w). \tag{36}\end{equation}

By Step 1, $u_n \in B_R$ for any $n \in \mathbb{N}$. Hence, we can extract a subsequence, still denoted by $u_n$, such that

\begin{equation}
u_n \rightarrow u \text{ weakly in } W_0^{1,p}(\Omega), \tag{37}
\end{equation}

\begin{equation}
u_n \rightarrow u \text{ strongly in } L'(\Omega), r < p^*, \tag{38}
\end{equation}

\begin{equation}u_n \rightarrow u \text{ a.e. in } \Omega. \tag{39}\end{equation}

**Step 2A.** Proof of strong convergence in $W_0^{1,p}(\Omega)$ (35) of $u_n$. 


In this step, we prove that

$$\lim_{n \to +\infty} \int_{\Omega} [a(x, w_n, \nabla u_n) - a(x, w_n, \nabla u)] \nabla (u_n - u) \, dx = 0. \quad (38)$$

Indeed, by (38) and Lemma 5 in [24] (p. 190), we can deduce

$$u_n \to u \quad \text{strongly in } W^{1,p}_0(\Omega),$$
$$\nabla u_n \to \nabla u \quad \text{a.e. in } \Omega. \quad (39)$$

Here, we just recall that Lemma 5 in [24] is a subtle generalization of the classical result, which asserts that the strong convergence of the sequence is a consequence of the weak convergence of the sequence and the convergence of the norms of its elements.

In order to prove (38), we use \( u_n - u \) as test function in (34), and we obtain

$$\int_{\Omega} [a(x, w_n, \nabla u_n) - a(x, w_n, \nabla u)] \nabla (u_n - u) \, dx \quad (40)$$

$$= - \int_{\Omega} a(x, w_n, \nabla u) \nabla (u_n - u) \, dx + \int_{\Omega} H(x, \nabla w_n)(u_n - u) \, dx$$
$$+ \int_{\Omega} g(x, w_n)(u_n - u) \, dx.$$

We prove that each term on the right-hand side of (40) goes to zero as \( n \) goes to \( +\infty \).
Indeed, since \( a(x, w_n, \nabla u) \to a(x, w, \nabla u) \) strongly in \( L^p(\Omega) \) by Lebesgue dominate convergence, and since \( u_n \to u \) weakly in \( W^{1,p}_0(\Omega) \), then

$$\lim_{n \to +\infty} \int_{\Omega} a(x, w_n, \nabla u) \nabla (u_n - u) \, dx = 0. \quad (41)$$

Now, we prove that

$$\lim_{n \to +\infty} \int_{\Omega} H(x, \nabla w_n)(u_n - u) \, dx = 0. \quad (42)$$

Indeed, by (37) and (32), since for a subsequence, denoted again with \( w_n, \nabla w_n \to \nabla w \) a.e., and \( H(x, z) \) is a Carathéodory function, we have

$$H(x, \nabla w_n)(u_n - u) \to 0 \quad \text{a.e. in } \Omega.$$  

Moreover, such a sequence is equintegrable. Indeed, by (32), \( \nabla w_n \to \nabla w \) strongly in \( \left( L^p(\Omega) \right)^N \) and for a subsequence, denoted again as \( w_n \), there exists a function \( k \in L^p(\Omega) \) such that \( \| \nabla w_n \| \leq k \) a.e. Therefore, since the sequence \( \| u_n - u \|_{L^p} \) is bounded, and the sequence \( \| u_n - u \|_{L^{p^*}} \) is bounded when \( \left( \frac{\eta}{q} \right)' \leq p^* \), i.e., when \( q \leq p - 1 + \frac{p'}{\eta} \), by Hölder inequality, we obtain

$$\left| \int_{E} H(x, \nabla w_n)(u_n - u) \, dx \right| \leq \beta \int_{E} |\nabla w_n|^q |u_n - u| \, dx + \int_{E} |f(u_n - u)| \, dx$$
\[
\leq \beta \int_{E} k^q |u_n - u| \, dx + \int_{E} |f(u_n - u)| \, dx
\leq \beta \left( \int_{E} k^p \, dx \right)^{\frac{q}{p}} \| u_n - u \|_{L^{p^*}} \left( \int_{0}^{\| f \|_{L^p}} s^{p' - q} \, ds \right)^{\frac{1}{p}} \| u_n - u \|_{L^p}
\leq C \left( \int_{E} k^p \, dx \right)^{\frac{q}{p}} + C \left( \int_{0}^{\| f \|_{L^p}} s^{p' - q} \, ds \right)^{\frac{1}{p}},$$

for any measurable subset \( E \subset \Omega \).
Here, $C$ denotes a positive constant, which depends only on the data, does not depend on $n$, and can vary from line to line. Then, by equi-integrability and Vitali theorem, (42) is proven.

Finally, let us consider the last term on the right-hand side of (40). Since $w_n$ converges almost everywhere to $w$, by (37), we have

$$g(x, w_n)(u_n - u) \to 0 \quad \text{a.e. in } \Omega.$$  

Moreover, such a sequences is equiintegrable. Indeed, by (32), $w_n \to w$ strongly in $L^{p',p}(\Omega)$, and for a subsequence denoted $w_{n_k}$, there exists a function $h \in L^{p',p}$ such that $|w_{n_k}| \leq h$ a.e. in $\Omega$. Therefore, by Hölder inequality, since the sequence $\|u_n - u\|_{p',p}$ is bounded, we obtain

$$\left| \int_E g(x, w_n)(u_n - u) \right| dx \leq \int_E |c||w_n||u_n - u| dx$$

$$\leq \int_E |c|h||u_n - u| dx$$

$$\leq \left( \int_0^{|E|} |(ch)^*|^p \frac{p'}{s} \frac{1}{s^{p-1}} ds \right)^{\frac{1}{p}} \|u_n - u\|_{p',p}$$

$$\leq C \left( \int_0^{|E|} |(ch)^*|^p \frac{p'}{s} \frac{1}{s^{p-1}} ds \right)^{\frac{1}{p}}$$

for every measurable set $E \subset \Omega$. Then, by equi-integrability and Vitali theorem, we obtain

$$\lim_{n \to +\infty} \int_{\Omega} g(x, w_n)(u_n - u) dx = 0. \quad (43)$$

Combining (40)–(43), we conclude the proof of (38).

**Step 2b. In this step, we prove that $A(w) = u$.**

Let us consider (34) and pass to the limit for $n$, which goes to $+\infty$. By (39), since $a(x, s, z)$ is a Carathéodory function bounded in $(L^p(\Omega))^N$, we obtain

$$a(x, v, \nabla u_n) \to a(x, v, \nabla u) \text{ weakly in } (L^p(\Omega))^N.$$  

This implies

$$\int_{\Omega} a(x, w_n, \nabla u_n) \nabla \phi dx \to \int_{\Omega} a(x, w, \nabla u) \nabla \phi dx, \quad (44)$$

as $n$, which goes to $+\infty$.

Moreover, by (32), we can extract a subsequence, still denoted by $w_n$, such that

$$w_n \to w \quad \text{a.e. in } \Omega \text{ and } \nabla w_n \to \nabla w \quad \text{a.e. in } \Omega.$$  

Therefore, since $H(x, z)$ and $g(x, s)$ are Carathéodory functions, by using the Vitali theorem, we obtain

$$\int_{\Omega} H(x, \nabla w_n) \phi dx \to \int_{\Omega} H(x, \nabla w) \phi dx, \quad (45)$$

$$\int_{\Omega} g(x, w_n) \phi dx \to \int_{\Omega} g(x, w) \phi dx, \quad (46)$$

as $n$, which goes to $+\infty$.

Now, we pass to the limit in (34). By combining (44)–(46), we obtain

$$\int_{\Omega} a(x, w, \nabla u) \nabla \phi dx = \int_{\Omega} H(x, \nabla w) \phi dx + \int_{\Omega} g(x, w) \phi dx,$$
for all $\phi \in W^{1,p}_0(\Omega)$. This yields that $u$ is a weak solution to (34). Since the solution to (34)
is unique, one has $u = A(w)$.

By Step 2a and Step 2b, the conclusion that $A$ is continuous follows.

**STEP 3. In this step, we prove that $A$ is a compact operator.**

To this aim, we prove that $A(B_R)$ is precompact in $W^{1,p}_0(\Omega)$. Let $w_n \in B_R$, and let
$A(w_n) = u_n \in W^{1,p}_0(\Omega)$ be the unique weak solution to Problem (25) with $v = w_n$. In order
to prove that $A(B_R)$ is precompact in $W^{1,p}_0(\Omega)$, we prove that there exists a subsequence,
still denoted with $u_n$, which strongly converges to $A(w) = u \in W^{1,p}_0(\Omega)$. Since $w_n, u_n \in B_R$,
you are bounded in $W^{1,p}_0(\Omega)$, and we can extract subsequences still denoted by $u_n$ and $w_n$,
respectively, such that

- $w_n \rightharpoonup w$ weakly in $W^{1,p}_0(\Omega)$,
- $w_n \to w$ strongly in $L^r(\Omega), r < p^*$,
- $w_n \to w$ a.e. in $\Omega$,

and

- $u_n \rightharpoonup u$ weakly in $W^{1,p}_0(\Omega)$,
- $u_n \to u$ strongly in $L^r(\Omega), r < p^*$,
- $u_n \to u$ a.e. in $\Omega$.

Now, we proceed as in Step 2a to prove that $u_n \to u$ strongly in $W^{1,p}_0(\Omega)$, and as in
Step 2b to prove that $A(w) = u \in W^{1,p}_0(\Omega)$.

This proves that $A(B_R)$ is precompact in $W^{1,p}_0(\Omega)$, and yields the conclusion.

**STEP 4: The conclusion.** By the Schauder fixed point theorem, since $A$ maps $B_R$ into itself
and is a continuous compact operator, there exists a unique fixed point of $A$. By definition
of $A$, the unique fixed point of $A$ is the unique weak solution to Problem (12).

**Remark 3. As pointed out in the Introduction, an existence result has been proven in [12] under
the assumptions**

$$ p - 1 < q \leq p - 1 + \frac{p}{N}, \quad f \in L^{(\nu-\frac{p}{q})'}(\Omega), \quad c \in L^\frac{N}{p}(\Omega). $$

We explicitly remark that Theorem 1 completes the existence result given in [12], since it also
gives a uniqueness result, which cannot be deduced by the analogous result in [12].

4. **Further Remarks**

4.1. **A Counter-Example**

As pointed out in the Introduction, the main novelty of Theorem 1 consists of the
uniqueness of weak solutions. When dealing with the question of uniqueness, one has to
consider the fact that for the model problem with $p = 2$,

$$
\begin{align*}
\begin{cases}
-\Delta u &= |\nabla u|^q \quad \text{in } B_1(0), \\
u &= 0 \quad \text{on } \partial B_1(0),
\end{cases}
\end{align*}
$$

(47)

uniqueness does not hold for solutions in $H^1_0(B_1(0))$, where $B_1(0)$ is the unitary ball. For
instance, it is well known (see, for example [7]) that, in addition to the trivial solution $u = 0$,
the function

$$
u(x) = C_\alpha(|x|^{q-1} - 1), \quad \alpha = \frac{2-q}{q-1}, \quad C_\alpha = \frac{(N - \alpha - 2)^{\frac{1}{q-1}}}{\alpha},$$

(48)

solves Problem (47) when $N > 2, 1 + 2/N < q < 2$ and $u \in H^1_0(B_1(0))$. 

We remark that, for the model Problem (47), Theorem 1 selects only the trivial solution \( u = 0 \). Indeed, since the general computation is quite involved, here we consider only the case \( N = 4, q = \frac{7}{4}, \beta = 1, c = f = 0 \). A straightforward calculation allows us to verify that the value of the constant \( R \) in (21) reads as
\[
R = \frac{\sqrt{2}}{S^3} \omega_4^\frac{1}{p},
\]
where (see [25])
\[
S = \frac{\sqrt{2}}{\sqrt{\pi}} \quad \text{and} \quad \omega_4 = \frac{\pi^2}{2}.
\]
Moreover, a further straightforward calculation proves that the function \( u \) in (48) solves Problem (47), but does not satisfy (21).

4.2. A Further Uniqueness Result

The approach based on Schauder fixed point theorem used to prove Theorem 1 allows us to obtain uniqueness for a larger class of nonlinear elliptic operators, and improves the uniqueness results already known in the literature. In order to explain the efficacy of the approach used to prove Theorem 1, we prove Theorem 2 below obtained by using a different technique, based on the summability properties of the gradient of the solution. Such an approach has been used, for example, in [16,18], and it requires us to consider a class of nonlinear elliptic operators which satisfy further standard structural conditions. Actually, the operator \( a \) cannot depend on \( u \), and it has to satisfy the following “strong monotonicity” condition, instead of the monotonicity condition (14):
\[
(a(x,z) - a(x,z')) \cdot (z - z') \geq (\epsilon + |z| + |z'|)^{p-2} |z - z'|^2,
\]
with \( \epsilon \) nonnegative and strictly positive if \( p > 2 \).

Moreover, the following locally Lipschitz condition on \( H \) has to be required
\[
|H(x,z) - H(x,z')| \leq \beta(\eta + |z| + |z'|)^{q-1} |z - z'|,
\]
where \( \beta > 0, \eta \) is nonnegative and strictly positive if \( 1 < p \leq 2 \).

As far as the zero-order term is concerned, we assume that the function \( g(x,\cdot) \) is decreasing; that is,
\[
g(x,s) - g(x,s') \leq 0 \quad \text{if} \quad s > s'.
\]

Precisely the following result holds true:

**Theorem 2.** Let \( N \geq 2 \) and \( p \) such that
\[
\begin{cases}
\frac{2N}{N+2} \leq p < 2, \quad \text{if} \quad N = 2 \\
\frac{2N}{N+2} \leq p \leq 2, \quad \text{if} \quad N \geq 3.
\end{cases}
\]

Assume that the operator \( a = a(x,\xi) \) satisfies (13)–(17), (49)–(51) with
\[
\mathcal{L}\text{cond}1 p - 1 < q \leq p - 1 + \frac{p}{N},
\]
and
\[
c \in L^{\frac{N}{p^\prime}}(\Omega), \quad f \in L^{(p^\prime)'}(\Omega).
\]

Then, Problem (12) has at most one weak solution.
Remark 4. We explicitly observe that the existence of at least a weak solution to Problem (12) is guaranteed by the assumptions that the datum \( f \) belongs to \( L^{(p')} (\Omega) \), the coefficient \( c \) belongs to \( L^{\infty} (\Omega) \), and a suitable smallness conditions on their norms are satisfied. Moreover, such a weak solution satisfies an a priori estimate in the Sobolev space \( W_0^{1,p} (\Omega) \) of the type \( \| \nabla u \|_{L^p} \leq \text{Const.} \) (see, for example [12] or Theorem 1 above).

Remark 5. An analogous uniqueness result holds true for all values of \( p \geq 2 \), under the same assumptions on the summability of \( f \) and \( c \), the assumption of strong monotonicity on the operator \( a \) (51), and the assumption of locally Lipschitz condition on \( H \) (50). This implies, together with the previous remark and Theorem 2, that Theorem 1 gives uniqueness of weak solutions to Problem (12) for a larger class of operators. Moreover, Theorem 1 also gives uniqueness for a larger interval of value of \( p \).

Remark 6. Theorem 2 can be applied to the model Problem (1) under the further assumption the coefficient \( c \) is a nonpositive function.

Proof. Assume that Problem (12) has two weak solutions \( u, v \). We prove that \( u = v \) a.e. in \( \Omega \). Let us denote

\[ w = (u - v)_+ \]

and

\[ D = \{ x \in \Omega : w(x) > 0 \} \]

Assume that \( D \) has positive measure. Let us fix \( t \in [0, \sup w] \). We denote

\[ w_t = \begin{cases} w - t & \text{if } w > t \\ 0 & \text{otherwise} \end{cases} \]

and

\[ E_t = \{ x \in D : w(x) > t \} \]

Since \( u \) and \( v \) are weak solutions to Problem (12), by using \( w_t \) as test function, the following equalities hold true:

\[ \int_{\Omega} a(x, \nabla u) \nabla w_t \, dx = \int_{\Omega} H(x, \nabla u) w_t \, dx + \int_{\Omega} g(x, u) w_t \, dx. \quad (54) \]

\[ \int_{\Omega} a(x, \nabla v) \nabla w_t \, dx = \int_{\Omega} H(x, \nabla v) w_t \, dx + \int_{\Omega} g(x, v) w_t \, dx. \quad (55) \]

Subtracting the two equalities, we obtain

\[ \int_{E_t} [a(x, \nabla u) - a(x, \nabla v)] \nabla w \, dx = \int_{E_t} [H(x, \nabla u) - H(x, \nabla v)] w_t \, dx \]

\[ + \int_{E_t} [g(x, u) - g(x, v)] w_t \, dx \quad (56) \]

By monotonicity of \( g(x, \cdot) \) (51), we obtain

\[ \int_{E_t} [a(x, \nabla u) - a(x, \nabla v)] \nabla w \, dx \leq \int_{E_t} [H(x, \nabla u) - H(x, \nabla v)] w_t \, dx. \quad (57) \]

Starting by this inequality, we can proceed as in [16]. For sake of completeness, we repeat here those arguments. By assumptions (49) and (50), we have

\[ \int_{E_t} \frac{|\nabla w_t|^2}{(|\nabla u| + |\nabla v|)^2-p} \, dx \leq \beta \int_{E_t} (\eta + |\nabla u| + |\nabla v|)^{q-1} |\nabla w_t| w_t \, dx. \quad (58) \]
Let us estimate the integral on the right-hand side of (58) by using Hölder inequality and Sobolev inequality. Since \( p \geq \frac{2N}{N+2} \), we obtain

\[
\int_{E_t} (\eta + |\nabla u| + |\nabla v|)^{q-1} |\nabla w_1| w_1 \, dx \\
\leq \int_{E_t} \frac{|\nabla w_1|}{(|\nabla u| + |\nabla v|)^{2-p}} (\eta + |\nabla u| + |\nabla v|)^{q-2} w_1 \, dx \\
\leq \Omega^{1-\frac{q+1}{p} + \frac{1}{q}} \left( \int_{E_t} \frac{|\nabla w_1|^2}{(|\nabla u| + |\nabla v|)^{2-p}} \, dx \right)^{1/2} \\
\times \left( \int_{E_t} (\eta + |\nabla u| + |\nabla v|)^p \, dx \right)^{\frac{2-p}{p}} \left( \int_{E_t} |\nabla w_1|^p \, dx \right)^{\frac{1}{p}},
\]

where \( C_p \) is the best constant in Sobolev embedding \( W_0^{1,p}(\Omega) \subset L^p(\Omega) \). From (58) and (59), we obtain

\[
\left( \int_{E_t} \frac{|\nabla w_1|^2}{(|\nabla u| + |\nabla v|)^{2-p}} \, dx \right)^{\frac{1}{2}} \\
\leq C_p \beta |\Omega|^{1-\frac{q+1}{p} + \frac{1}{q}} \left( \int_{E_t} (\eta + |\nabla u| + |\nabla v|)^p \, dx \right)^{\frac{2-p}{p}} \left( \int_{E_t} |\nabla w_1|^p \, dx \right)^{\frac{1}{p}}.
\]

On the other hand, since \( p \leq 2 \), if \( N \geq 3 \) or \( p < 2 \) if \( N = 2 \), Hölder inequality gives

\[
\int_{E_t} |\nabla w_1|^p \, dx \leq \int_{E_t} \frac{|\nabla w_1|^p}{(|\nabla u| + |\nabla v|)^{(2-p)\frac{p}{2}}} (\eta + |\nabla u| + |\nabla v|)^{(2-p)\frac{p}{2}} \, dx \\
\leq \left( \int_{E_t} \frac{|\nabla w_1|^2}{(|\nabla u| + |\nabla v|)^{2-p}} \, dx \right)^{\frac{p}{2}} \left( \int_{E_t} (\eta + |\nabla u| + |\nabla v|)^p \, dx \right)^{\frac{2-p}{p}}.
\]

Then, by using (60), we obtain

\[
1 \leq \beta C_p |\Omega|^{1-\frac{q+1}{p} + \frac{1}{q}} \left( \int_{E_t} (\eta + |\nabla u| + |\nabla v|)^p \, dx \right)^{\frac{q-p+1}{p}}.
\]

Letting \( t \to \sup \omega \), the left-hand side goes to zero; this gives a contradiction. Therefore, we conclude that \( |D| = 0 \), and we obtain \( u \leq v \) a.e. in \( \Omega \). By exchanging the role of \( u \) and \( v \), we also obtain \( v \leq u \) a.e. in \( \Omega \). This yields the conclusion. \( \square \)

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