Asymptotic Tracking Control for Mismatched Uncertain Systems with Active Disturbance Rejection

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Abstract: By introducing a set of exact disturbance estimators, a continuously tracking controller for a class of mismatched uncertain systems with exogenous disturbances will be proposed. The most appealing superiority is that the proposed exact disturbance estimators can not only estimate the external disturbances but also achieve an asymptotic estimation performance. Furthermore, with the help of a set of first-order asymptotic filters and an auxiliary system, the developed control algorithm is able to compensate for these total disturbances feedforwardly. Consequently, the whole closed-loop stability with an asymptotic tracking performance is strictly analyzed, and meanwhile applications are conducted to indicate the effectiveness of the proposed controller.

Keywords: tracking control; nonlinear systems; disturbance estimator; disturbance observer; asymptotic estimation; asymptotic tracking; active disturbance rejection

MSC: 93B52; 93C10; 93C40; 93C73; 93D20; 93D21

1. Introduction

Disturbances extensively exist in all practical systems, which may cause critical control performance degradation and even instability in developing high-performance closed-loop controllers [1–3]. Over the past decades, many advanced control algorithms such as adaptive robust control [4], robust adaptive control [5,6], sliding mode control [7,8] and so on have been proposed for various nonlinear systems to cope with modeling uncertainties. Additionally, many studies focus on rejecting disturbances by combining with disturbance observers.

Currently, there has been a growing interest in disturbance-observer-based control strategies with an active disturbance rejection ability for uncertain nonlinear systems [9–18]. And the main concept of these control strategies is to estimate the disturbances via different disturbance observers and thus compensate for them feedforwardly in developing the closed-loop controllers. Typically, Chen et al., proposed a nonlinear disturbance observer (NDOB) for nonlinear systems with disturbances governed by the exogenous system to estimate the total disturbances in an exponentially convergent rate [19]. It is worth noting that NDOB has been successfully applied to various practical systems [20,21]. Moreover, Won et al., have proposed a high-gain-disturbance-observer-based controller for hydraulic systems to improve the output tracking performance and meanwhile constrain the output tracking error [14]. Furthermore, Han has developed an active disturbance rejection controller (ADRC) for uncertain systems [22]. And the main support for ADRC is the extended state observer (ESO) [12,23,24]. Moreover, there are still some other disturbance observers [25–27] have been proposed. Notably, the aforementioned...
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disturbance observers can only achieve a bounded estimation performance. How to develop exact disturbance estimators and meanwhile acquire an asymptotic tracking performance, especially for mismatched nonlinear systems, is extremely important and challenging in designing high-performance closed-loop controllers.

Inspired by the above discussions, we will propose an asymptotic tracking controller for systems with matched and mismatched exogenous disturbances, which is of great significance both in theory and practice. Especially, the main contributions of this paper are shown as follows:

1. A set of exact disturbance estimators (EDEs) with optimized design parameters which can acquire an asymptotic estimation performance is proposed;
2. Both mismatched and matched exogenous disturbances can be effectively compensated, and meanwhile an asymptotic tracking performance can be acquired.

Notation: \( \hat{x} \) is the estimate of \( x \) with \( x - \hat{x} \) being the estimation error; and \( \text{sgn}(\cdot) \) is the signum function. In addition, the variable \( i = 1, \ldots, n \) with \( n \) being the system order; the variable \( j = 1, \ldots, n-1 \); and the variable \( \theta = 2, \ldots, n-1 \).

2. Problem Formulation

A class of nonlinear systems is employed as

\[
\begin{align*}
\dot{\phi}_j &= \phi_{j+1} + \psi_j(\bar{\phi}_j) + f_j(t) \\
\dot{\phi}_n &= g(\bar{\phi}_n)u + \psi_n(\bar{\phi}_n) + f_n(t) \\
y &= \phi_1
\end{align*}
\]  

(1)

where \( \bar{\phi}_j = [\phi_1, \ldots, \phi_j] \in \mathbb{R}^j \) with \( \phi_i \) being the system states; \( g(\bar{\phi}_n) \) and \( \psi_j(\bar{\phi}_j) \) are known nonlinear functions, especially, \( g(\bar{\phi}_n) > 0 \); \( f_j(t) \) denote unknown time-varying functions which can describe external disturbances; moreover, \( u \) and \( y \) are the control input and output, respectively.

Given a desired trajectory \( \phi_{d1} \), the control objective is to design a continuously disturbance-compensation-based control law \( u \) for (1) so that \( y = \phi_1 \) can asymptotically track \( \phi_{d1} \).

Assumption 1. The desired trajectory \( \phi_{d1} \in \mathcal{C} \).

Assumption 2. \( \hat{f}_i(t) \) exist and meanwhile are bounded.

Remark 1. Notably, not only mismatched external disturbances \( f(t) \) but also matched external disturbances \( f(t) \) are simultaneously considered in Model (1), which means that it can be applied to many practical systems, such as motor servo systems [28,29] and so on.

3. Asymptotic Tracking Controller with Active Disturbance Rejection

3.1. Exact Disturbance Estimator

Inspired by the ESO, we extend the external disturbances \( f(t) \) as new state variables \( \phi_n \). According to Assumption 2, we can define \( \hat{f}_i(t) = d_i(t) \) and then transform the system (1) as

\[
\begin{align*}
\dot{\phi}_j &= \phi_{j+1} + \psi_j(\bar{\phi}_j) + \phi_j \\
\dot{\phi}_n &= g(\bar{\phi}_n)u + \psi_n(\bar{\phi}_n) + \phi_n \\
y &= \phi_1
\end{align*}
\]  

(2)

Stimulated by [30,31], a set of novel EDEs can be proposed as [32]
\[
\begin{align*}
\dot{\phi}_i &= \varphi_{j+1} + \psi_j(\varphi_j) + \phi_{j+2} + L_{j+1}(\varphi_{j+1} - \phi_j) \\
\dot{\phi}_j &= L_{j+2}(\varphi_{j+1} - \phi_j) + \dot{\alpha}_j \text{sgn}(\varphi_{j+1} - \phi_j) \\
\dot{\phi}_n &= g(\overline{\varphi}_n)u + \psi_n(\overline{\varphi}_n) + \dot{\phi}_m + L_n(\varphi_n - \phi_n) \\
\dot{\phi}_m &= L_n(\varphi_n - \phi_n) + \dot{\alpha}_n \text{sgn}(\varphi_n - \phi_n)
\end{align*}
\]

where \(L_1\) and \(L_2\) are adjustable positive design parameters; in addition, \(\dot{\alpha}\) are the estimates of \(\alpha\) which satisfy (23) which will be introduced later.

As conducted in [23], we can parameterize \(L_1\) and \(L_2\) as \(2_1\beta_1\) and \(2_1\beta_2\), respectively, with \(\beta_1\) being adjustable positive design parameters. Therefore, (3) can be rearranged as

\[
\begin{align*}
\dot{\phi}_i &= \varphi_{j+1} + \psi_j(\varphi_j) + \phi_2 + 2\beta_1(\varphi_1 - \phi_1) \\
\dot{\phi}_j &= \beta_1(\varphi_1 - \phi_1) + \dot{\alpha}_j \text{sgn}(\varphi_1 - \phi_1) \\
\dot{\phi}_n &= g(\overline{\varphi}_n)u + \psi_n(\overline{\varphi}_n) + \dot{\phi}_m + 2\beta_2(\varphi_n - \phi_n) \\
\dot{\phi}_m &= \beta_2(\varphi_n - \phi_n) + \dot{\alpha}_n \text{sgn}(\varphi_n - \phi_n)
\end{align*}
\]

Especially, \(\dot{\alpha}\) can be updated via

\[
\dot{\alpha}_i = \frac{\lambda_i}{\beta_2} \eta_i^T F_a C_o \text{sgn}(\eta_i)
\]

where \(\lambda_i\) are positive design parameters and \(\eta_i = [\eta_1, \eta_2]^T = [\varphi_1, \varphi_2, 2, \beta_1]^T\) are new vectors. Due to incalculable variables existing in (5), we can obtain \(\dot{\alpha}\) via the following method:

\[
\dot{\alpha}_i(t) = \dot{\alpha}_i(0) + \frac{1}{2} \lambda_i \left[ \frac{3}{\beta_2} \left| \eta_i(t) \right| - \frac{3}{\beta_2} \left| \eta_i(0) \right| + \int_0^t \left| \eta_i(u) \right| du \right]
\]

Noting (2) and (4), we can obtain

\[
\begin{align*}
\dot{\phi}_i &= \varphi_i - 2\beta_1 \phi_i \\
\dot{\phi}_n &= d_i(t) - \beta_1 \phi_i - \dot{\alpha}_i \text{sgn}(\phi_i)
\end{align*}
\]

After introducing new vectors as \(\eta_i = [\eta_1, \eta_2]^T = [\varphi_i, \varphi_i, 2, \beta_1]^T\), we can rewrite (7) as

\[
\eta_i = \beta_o B \varepsilon + \frac{1}{\beta_2} \eta_i \left[ d_i(t) - \dot{\alpha}_i \text{sgn}(\eta_i) \right]
\]

where \(B_o = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix}\) and \(C_o = [0, 1]^T\).

As \(B_o\) is Hurwitz, there is a positive definite matrix \(F_o\) guaranteeing \(B_o^T F_o + F_o B_o = -I\) [12].

3.2. Controller Design

Firstly, we introduce a set of error variables \(e_i(t)\) and compensation signals \(e_i(t)\) as [33]

\[
\begin{align*}
e_i &= \varphi_i - \varphi_{i+1}, e_{i+1} = \varphi_{i+1} - \varphi_{i+2} \\
\varepsilon_i &= e_i - \tilde{\varsigma}_i, \varepsilon_{i+1} = e_{i+1} - \tilde{\varsigma}_{i+1}
\end{align*}
\]

where \(e_i = \varphi_i - \varphi_{i+1}\) (t) is the tracking error; \(\tilde{\varsigma}\) represent the auxiliary variables; and \(\tilde{\varsigma}\) indicate the filtered values of the virtual control laws \(\psi\) to be synthesized later, which can be produced via the following filters [34].
\[ L_{ij} \dot{v}_j = -v_j + v_j - \xi_{ij} + v_j (0) = v_j (0) \]
\[ \dot{\xi}_{ij} = -\frac{L_{ij} \xi_j \sigma_j^2}{\sigma_j^2 \xi_j + \delta^2 (t)} \tag{10} \]

where \( L_{ij} \) are the positive parameters; \( \xi_{ij} = v_j - v_{ij} \) indicate the filtering errors; and \( \delta^2 (t) > 0 \) and satisfy \( \int_0^\infty \delta^2 (u) du \leq \delta, \forall t \geq 0 \), with \( \delta \) being some positive constants [34]; especially, \( \sigma_j \) represent the upper bounds of \( \dot{v}_j \), which can be updated via
\[ \dot{\sigma}_j = r_j |\dot{\xi}_j| \tag{11} \]
with \( r_j \) being the adjustable positive gains.

An auxiliary system is introduced as [33]
\[ \dot{\zeta}_j = -k_i \zeta_j + \zeta_j + (v_j - v_j) \]
\[ \dot{\zeta}_n = -k_n \zeta_n \tag{12} \]

where \( k_i \) are the positive gains.

Step 1:
Differentiating \( \varepsilon_1 \) via (1), (9) and (12), one has
\[ \dot{\varepsilon}_1 = v_1 + k_i \dot{\xi}_1 + \dot{\varepsilon}_2 + \psi_1 (\bar{\theta}) + \varphi_{1,i} - \dot{\varphi}_d \tag{13} \]
Thus, \( v_1 \) can be designed as
\[ v_1 = -k_i \dot{\varepsilon}_1 - \psi_1 (\bar{\theta}) - \dot{\varphi}_{1,i} + \dot{\varphi}_d \tag{14} \]
Substituting (14) into (13) yields
\[ \dot{\varepsilon}_1 = -k_i \dot{\varepsilon}_1 + \dot{\varepsilon}_2 + \dot{\varphi}_{1,i} \tag{15} \]

Step 0:
Differentiating \( \varepsilon_0 \) via (1), (9) and (12), one yields
\[ \dot{\varepsilon}_0 = v_0 + k_i \dot{\xi}_0 + \dot{\varepsilon}_{0,i} + \psi_0 (\bar{\theta}_0) + \varphi_{0,i} - \dot{\varphi}_{0,i} \tag{16} \]
Therefore, the virtual control law \( v_0 \) can be designed as
\[ v_0 = -k_i \dot{\varepsilon}_0 - \psi_0 (\bar{\theta}_0) - \dot{\varphi}_{0,i} - \dot{\varepsilon}_{0,i} \tag{17} \]
Substituting (17) into (16) obtains
\[ \dot{\varepsilon}_0 = -k_i \dot{\varepsilon}_0 + \dot{\varepsilon}_{0,i} + \dot{\varphi}_{0,i} \tag{18} \]

Step n:
Considering (1), (9) and (12), we can acquire the time derivative of \( \varepsilon_n \) as
\[ \dot{\varepsilon}_n = g(\bar{\theta}_n) u + \psi_n (\bar{\theta}_n) + \varphi_{n,i} - \dot{\varphi}_{n,i} + k_i \dot{\zeta}_n \tag{19} \]
Finally, the actual control law \( u \) can be designed as
\[ u = g^{-1}(\bar{\theta}_n) \left[-k_i \dot{\varepsilon}_n - \psi_n (\bar{\theta}_n) - \dot{\varphi}_{n,i} - \dot{\varepsilon}_{n,i} \right] \tag{20} \]
Substituting (20) into (19) achieves
\[ \dot{\varepsilon}_n = -k_i \dot{\varepsilon}_n - \dot{\varepsilon}_{n,i} + \dot{\varphi}_{n,i} \tag{21} \]
3.3. Main Theoretical Results

**Proposition 1.** Define a set of auxiliary functions $Q_i(t)$ as:

$$Q_i = \frac{1}{\beta_{\alpha i}} \eta_i^T F_s C_o \left[ d_i(t) - \alpha_i \text{sgn}(\eta_i) \right]$$  \hspace{1cm} (22)

If the parameters $\alpha_i$ satisfy the following sufficient condition:

$$\alpha_i > D_i + \frac{3D_2}{5\beta_{\alpha i}}$$  \hspace{1cm} (23)

where $D_i = \sup_{t \geq 0} |d_i(t)|$ and $D_2 = \sup_{t \geq 0} |d_i(t)|$ are some unknown positive constants. Hence, the following inequality holds

$$\int_0^t Q(v)dv \leq \mu_i$$  \hspace{1cm} (24)

where $\mu_i = \frac{3[\alpha_i \eta_i,0(0) - \eta_i,0(0)]}{(2\beta_{\alpha i})}$.

**Proof.** See Appendix A. □

**Theorem 1.** Consider (1), the proposed controller with the designed EDEs in (4) can guarantee that all system signals are bounded as well as $e_i \rightarrow 0$ and $\Phi_{\hat{\rho}} \rightarrow 0$ as $t \rightarrow \infty$.

**Proof.** See Appendix B. □

**Remark 2.** Notably, some advanced control strategies [35,36] have been proposed. However, compared with these controllers, we have constructed a set of novel EDEs via the traditional ESO which can estimate the states and disturbances asymptotically. Additionally, to prevent over-parameterization of $\hat{\alpha}_i$ and $\hat{\sigma}_j$, we can constrain their adaptive laws through the projection mapping function in [1].

4. Illustration Example

A one-link robot arm driven by the permanent magnet direct-current motor will be employed to verify the performance of the designed controller. Denoting state variables $\varphi_1 = y_m$, $\varphi_2 = y_m'$, and $\varphi_3 = K_{mL_m}I_m$, the considered system can be presented as follows

$$\begin{align*}
\dot{\varphi}_1 &= \varphi_2 + \psi_1(\Phi) + f_1(t) \\
\dot{\varphi}_2 &= \varphi_3 + \psi_2(\Phi_2) + f_2(t) \\
\dot{\varphi}_3 &= g(\Phi) + \psi_3(\Phi_3) + f_3(t)
\end{align*}$$  \hspace{1cm} (25)

where $y_m$ and $I_m$ are the angular displacement of the load and the electric current, respectively; $\psi_1(\Phi) = 0$, $f_1(t) = 0$, $\psi_2(\Phi_2) = -B_m q_2 I_m$, $f_2(t) = \Delta z(t) I_m$, $g(\Phi) = K_m K_E I_m$, $\psi_3(\Phi_3) = -K_m K_m q_2 I_m - R_m I_m$, $f_3(t) = K_m \Delta z(t) I_m$. The definitions and physical values of the system parameters are given in Table 1.
Table 1. The physical parameters of the system.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Implication</th>
<th>Value (Unit)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_m$</td>
<td>The rotational inertia of the load</td>
<td>$2.4 \times 10^{-3}$ (kg·m²)</td>
</tr>
<tr>
<td>$B_m$</td>
<td>The viscous friction coefficient</td>
<td>2.26 (N·m·s/rad)</td>
</tr>
<tr>
<td>$R_m$</td>
<td>The armature resistance</td>
<td>3.0 (Ω)</td>
</tr>
<tr>
<td>$L_m$</td>
<td>The armature inductance</td>
<td>0.08 (H)</td>
</tr>
<tr>
<td>$K_m$</td>
<td>The torque constant</td>
<td>1.85 (N·m/A)</td>
</tr>
<tr>
<td>$K_v$</td>
<td>The electrical gain</td>
<td>2.26</td>
</tr>
<tr>
<td>$K_E$</td>
<td>The electromotive force coefficient</td>
<td>1.25 (V·s/rad)</td>
</tr>
<tr>
<td>$\Delta_2(t)$</td>
<td>External disturbance</td>
<td>$\sin (\pi t)$</td>
</tr>
<tr>
<td>$\Delta_3(t)$</td>
<td>External disturbance</td>
<td>$15 \sin (\pi t)$</td>
</tr>
</tbody>
</table>

For (25), the following four controllers are employed to track the trajectory $q_{\phi_1} = 0.1 \sin(\pi t)[1 - \exp(-0.01t^3)]$ rad.

1) **C1**: This is the developed controller in Section 3. And its parameters are tuned as Table 2.

Table 2. The controller parameters of C1.

<table>
<thead>
<tr>
<th>Controller Parameter</th>
<th>Value</th>
<th>Controller Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_1$</td>
<td>300</td>
<td>$\lambda_2$</td>
<td>$1 \times 10^7$</td>
</tr>
<tr>
<td>$k_2$</td>
<td>100</td>
<td>$r_1$</td>
<td>150</td>
</tr>
<tr>
<td>$k_3$</td>
<td>100</td>
<td>$r_2$</td>
<td>150</td>
</tr>
<tr>
<td>$\beta_{o2}$</td>
<td>300</td>
<td>$L_{c1}$</td>
<td>$5 \times 10^{-4}$</td>
</tr>
<tr>
<td>$\beta_{o3}$</td>
<td>500</td>
<td>$L_{c2}$</td>
<td>$5 \times 10^{-4}$</td>
</tr>
<tr>
<td>$\lambda_1$</td>
<td>$1.5 \times 10^5$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2) **C2**: It is same as C1 but without compensation of the mismatched external disturbances.

3) **C3**: It is same as C1 but without compensation of the matched external disturbances.

4) **C4**: It is same as C1 but without compensation of the mismatched and matched external disturbances simultaneously.

For fairness, all design parameters of C2, C3 and C4 are chosen as same as that of C1.

The contrastive tracking errors are plotted in Figure 1. It can be clearly discovered that C1 performs the best tracking performance in terms of both transient and final tracking errors, which verifies the effectiveness of the compensation performance for mismatched and matched external disturbances. Moreover, it follows from Figure 1 that the tracking error of C1 gradually approaches zero, which demonstrates the achievable asymptotic output tracking performance. Furthermore, it also means that mismatched and matched external disturbances can be exactly estimated by the introduced observer. To support this claim, the estimation performance of the system states and modeling uncertainties with the proposed observer (5) are exhibited in Figures 2 and 3, respectively. In
addition, Figure 4 plots the control input of C1, which demonstrates that the resulting control law is smooth and meanwhile bounded.

Figure 1. Comparative tracking errors.

Figure 2. Estimation performance of $\phi_2$ and $\phi_f$ with the constructed observer.
5. Conclusions

A novel asymptotic tracking controller for a class of nonlinear systems with mismatched and matched exogenous disturbances has been proposed. Especially, a set of novel exact disturbance estimators with nonlinear robust terms to further reject disturbances has been creatively constructed to estimate the total disturbances in real time. Meanwhile, the exact disturbance estimations have been exploited in designing the resulting control scheme to eliminate the effects of disturbances. Especially, asymptotic tracking performance and asymptotic disturbance estimation performance have been demonstrated via strict theoretical analysis. In addition, the application on a one-link robotic arm driven by a direct-current servo motor has been conducted to verify the achievable results.
Author Contributions: Methodology, software, validation, formal analysis, investigation, funding acquisition, data curation, writing—original draft preparation, writing—review and editing, G.Y.; Supervision, investigation, validation, L.C. All authors have read and agreed to the published version of the manuscript.

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Appendix A

Proof of Proposition 1 [30]. After integrating both sides of (22), one has

$$\int_0^t Q(v)dv = \frac{1}{\beta_0} \int_0^5 \eta_i(v) \left[ d_i(v) - \alpha_i \text{sgn}[\eta_i(v)] \right] dv$$

$$+ \frac{1}{\beta_0} \int_0^3 \frac{1}{2} \eta_i(v) \left[ d_i(v) - \alpha_i \text{sgn}[\eta_i(v)] \right] dv$$

Therefore, we have

$$\int_0^t Q(v)dv \leq \frac{1}{\beta_0} \int_0^5 \eta_i(v) \left[ |d_i(v)| + \frac{3}{2} \beta_0 \left| \frac{dd_i(v)}{dv} - \alpha_i \right| \right] dv$$

$$+ \frac{3}{2} \beta_0 \left| \eta_i(t) \right| \left[ |d_i(t)| - \alpha_i \right] + \frac{3}{2} \beta_0 \left[ \alpha_i |\eta_i(0)| - \eta_i(0)d_i(0) \right]$$

This proves Proposition 1. □

Appendix B

Proof of Theorem 1. A set of auxiliary functions $W_i$ is defined as:

$$W_i = \mu_i - \int_0^t Q(v)dv$$

It follows from Proposition 1 that $W_i \geq 0$. Therefore, a Lyapunov candidate $V_{L1}$ can be employed as

$$V_{L1} = \frac{1}{2} \sum_{i=1}^n \epsilon_i^2 + \frac{1}{2} \sum_{i=1}^n \xi_i^2 + \frac{1}{2} \sum_{i=1}^n \eta_i^T F_i \eta_i + \sum_{i=1}^n W_i$$

$$+ \frac{1}{2} \sum_{j=1}^{n-1} \epsilon_j^2 + \frac{1}{2} \sum_{j=1}^{n-1} \xi_j^2 + \frac{1}{2} \sum_{j=1}^{n-1} \alpha_j^2$$

Based on (10), the filtering error dynamics can be arranged as

$$\dot{\epsilon}_j = v_j - \frac{1}{L_{ij}} \epsilon_j + \frac{1}{L_{ij}} \xi_j$$

After substituting (12), (15), (18), (21) and (A5) into the time derivative of $V_{L1}$, we have
\[
\dot{V}_i = -\sum_{j=1}^{n} k_{ij} e_i^2 + \sum_{j=1}^{n} \bar{e}_i \dot{\phi}_j - \sum_{j=1}^{n} k_{ij} \xi_j^2 - \sum_{j=1}^{n} \zeta_j \dot{\xi}_j - \sum_{j=1}^{n} \dot{\xi}_j \xi_j
\]

\[+ \sum_{j=1}^{n} \frac{1}{\beta_{ij}} \eta_j F_i C_i \left[ d_i(t) - \dot{\alpha}_i \text{sgn}(\eta_i) \right] - \frac{1}{2} \sum_{j=1}^{n} \beta_{ij} \left\| \theta_i \right\|^2
\]

\[+ \sum_{j=1}^{n} \frac{1}{\beta_{ij}} \xi_j \xi_l - \sum_{j=1}^{n} \frac{1}{\beta_{ij}} \eta_j F_i C_i \left[ d_i(t) - \alpha_i \text{sgn}(\eta_i) \right]
\]

\[+ \sum_{j=1}^{n} \frac{1}{\lambda_{ij}} \delta_{ij} \dot{\xi}_j - \sum_{j=1}^{n} \frac{1}{\beta_{ij}} \zeta_j \dot{\xi}_j
\]  

\[= \sum_{j=1}^{n} \left( k_{ij} - \frac{1}{2} \xi_j \right) e_i^2 - (k_{ij} - 1) \xi_j^2 - \frac{1}{\beta_{ij}} \left( \eta_j F_i C_i \right) \left[ d_i(t) - \alpha_i \text{sgn}(\eta_i) \right]
\]

\[\leq \sum_{j=1}^{n} \left( k_{ij} - \frac{1}{2} \xi_j \right) e_i^2 - (k_{ij} - 1) \xi_j^2 - \frac{1}{\beta_{ij}} \left( \eta_j F_i C_i \right) \left[ d_i(t) - \alpha_i \text{sgn}(\eta_i) \right]
\]

\[\leq \frac{1}{2} \left( \xi_j^2 + \xi_{j+1}^2 \right), \quad \left\| \xi_j \right\| \leq \frac{1}{2} \left( \xi_j^2 + \xi_{j+1}^2 \right)
\]

\[\text{where } \zeta \text{ are some positive constants.}
\]

Noting (5), (10) and (11), we have

\[\dot{V}_i \leq -\sum_{j=1}^{n} k_{ij} e_i^2 + \sum_{j=1}^{n} \beta_{ij} \left\| e_i \right\| - \sum_{j=1}^{n} k_{ij} \xi_j^2 + \sum_{j=1}^{n} \frac{1}{\beta_{ij}} \left\| \xi_j \right\|^2
\]

\[+ \sum_{j=1}^{n} \frac{1}{\beta_{ij}} \delta_{ij} \dot{\xi}_j
\]

\[\leq \frac{1}{2} \left( \xi_j^2 + \xi_{j+1}^2 \right), \quad \left\| \xi_j \right\| \leq \frac{1}{2} \left( \xi_j^2 + \xi_{j+1}^2 \right)
\]

Applying Young's inequality, one has

\[\beta_{ij} \left\| e_i \right\| \leq \frac{1}{2} \left( \xi_j^2 + \xi_{j+1}^2 \right)
\]

\[\left\| \xi_j \right\| \leq \frac{1}{2} \left( \xi_j^2 + \xi_{j+1}^2 \right)
\]

\[\text{where } \zeta \text{ are some positive constants.}
\]

Based on (A8), we can arrange (A7) as

\[\dot{V}_i \leq \sum_{j=1}^{n} \left( k_{ij} - \frac{1}{2} \xi_j \right) e_i^2 - (k_{ij} - 1) \xi_j^2 - \frac{1}{\beta_{ij}} \left( \eta_j F_i C_i \right) \left[ d_i(t) - \alpha_i \text{sgn}(\eta_i) \right]
\]

\[\leq \sum_{j=1}^{n} \left( k_{ij} - \frac{1}{2} \xi_j \right) e_i^2 - (k_{ij} - 1) \xi_j^2 - \frac{1}{\beta_{ij}} \left( \eta_j F_i C_i \right) \left[ d_i(t) - \alpha_i \text{sgn}(\eta_i) \right]
\]

\[\leq \frac{1}{2} \left( \xi_j^2 + \xi_{j+1}^2 \right), \quad \left\| \xi_j \right\| \leq \frac{1}{2} \left( \xi_j^2 + \xi_{j+1}^2 \right)
\]

Define a set of variables as follows

\[k_{ij} = k_i - \frac{1}{2} \xi_i, \quad k_{ij} = k_i - 1, \quad k_{ij} = k_i - 3
\]

\[k_{ij} = k_i, \quad k_{ij} = \frac{1}{2} \left( \overline{\beta}_{ij} - \frac{\beta_{ij}^2}{\xi_i} \right), \quad k_{ij} = \frac{1}{L_{ij}} - \frac{1}{2}
\]

Therefore, we can rewrite (A10) as

\[\dot{V}_i \leq -\sum_{j=1}^{n} k_{ij} e_i^2 - \sum_{j=1}^{n} k_{ij} \xi_j^2 - \sum_{j=1}^{n} k_{ij} \left\| \eta_j \right\|^2 - \sum_{j=1}^{n} k_{ij} \xi_j^2 + \sum_{j=1}^{n} \delta_j
\]

After integrating two sides of (A11), this yields

\[V_i(t) + \int_0^t \left[ \sum_{j=1}^{n} k_{ij} e_j^2(v) + \sum_{j=1}^{n} k_{ij} \xi_j^2(v) + \sum_{j=1}^{n} \beta_{ij} \left\| \eta_j \right\|^2 + \sum_{j=1}^{n} k_{ij} \xi_j^2(v) \right] dv + \int_0^t \sum_{j=1}^{n} \delta_j dv
\]

\[\leq V_i(0) + \sum_{j=1}^{n} \delta_j
\]

It can be seen from (9) and (A12) that \(e_i, e_i, \xi_i, \xi_i, \phi_i, \eta_i \) and \( \eta_i \in L_\infty \). Afterwards, \( \phi_i \in L_\infty \) and \( \eta_i \in L_\infty \) can be proved. Moreover, \( \phi_i \in L_\infty \) can be inferred from (11). Furthermore, \( \phi_i \in L_\infty \) can be obtained due to Assumption 2 and \( \eta_i \in L_\infty \). Thus, it can be deduced from
Assumption 1 and (9) that the whole system’s signals stay bounded. And all estimation of the developed EDEs can be concluded as bounded. Notably, \( |\xi_i| \leq L_j \sigma_j \) can be obtained from (10), which means \( \xi_i \in E_\infty \). Based on the above analysis, we can acquire \( \nu \in E_\infty \). In addition, it can be inferred from (8), (9), (15), (18), (21) and (A5) that \( \tilde{\eta}_1, \tilde{\eta}_2, \tilde{\epsilon}(t) \) and \( \dot{\tilde{e}}(t) \in E_\infty \). As a result, \( \nu \to 0, \eta_1 \to 0 \) and \( \eta_2 \to 0 \) as \( t \to \infty \) can be obtained by exploiting Barbalat’s lemma, which proves Theorem 1 [29].

References


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