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# To the Question of the Solvability of the Ionkin Problem for Partial Differential Equations

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**Abstract:** We study the solvability of the Ionkin problem for some differential equations with one space variable. These equations include parabolic and quasiparabolic, hyperbolic and quasihyperbolic, pseudoparabolic and pseudohyperbolic, elliptic and quasielliptic equations and equations of many other types. For the above equations, the following theorems are proved with the use of the splitting method: the existence of regular solutions—solutions that all have weak derivatives in the sense of S. L. Sobolev and occur in the corresponding equation.

**Keywords:** spatial nonlocal problems; Ionkin condition; splitting method; regular solutions; existence; uniqueness

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## 1. Introduction

The Ionkin problem (N. I. Ionkin [1]) has been studied by many authors and for many classes of partial differential equations, and at the same time, the original method, proposed by N. I. Ionkin himself, has almost always been used. This is the method of decomposing the solution in some special biorthogonal systems of functions. In 2006, in A. M. Nakhushev's book [2] (see also [3–5]) and in the recent works [6,7] of the author of this paper, new approaches were applied to studying the Ionkin problem and close nonlocal problems—in A. M. Nakhushev's work, for second-order parabolic equations, and in the works of the author of this paper, for quasiparabolic equations, parabolic equations with an arbitrary evolution direction and elliptic equations.

In the present article, A. M. Nakhushev's approach is further developed: we show that this approach is applicable to a wide class of differential equations and that with its help, one can obtain a number of substantially new results on the solvability of the Ionkin problem and some other nonlocal problems close to it.

In 1986 in [8], N. I. Yurchuk proposed his approach to studying the solvability of the Ionkin problem for second-order parabolic equations. This approach was based on a priori estimates but it gave the existence of solutions belonging to some weighted Sobolev space. Let us clarify that in contrast to N. I. Yurchuk's approach, the approach of [6,7] gives the existence of solutions belonging to classical Sobolev spaces. The splitting method proposed below also gives the existence of regular solutions.

## 2. Statement of the Problems

Let $\Omega$ be the interval $(0,1)$ of the $Ox$ axis, $Q$ be the rectangle $\Omega\times(0,T)$ of the variables $x$ and $t$ be finite height $T$. Denote by $D_x^k$ and $D_t^k$ the derivatives $\frac{\partial^k}{\partial x^k}$ and $\frac{\partial^k}{\partial t^k}$, respectively. Furthermore, let

$$P_k(t,D_t)=\sum_{j=1}^{p_k}\alpha_{kj}(t)D_k^j,\quad k=1,\dots,m,$$

be operators with real coefficients and $L$ be the differential operator

$$L = \sum_{k=0}^{m} P_k(t, D_t) D_x^{2k} \quad .$$

For the operator $L$, define the conditions

$$U_j(x, 0) = 0, \quad j = 1, \dots m_1, \quad x \in \Omega, \tag{1}$$

$$U_j(x, T) = 0, \quad j = m_1 + 1, \dots, p_0 = \max(p_1, \dots, p_m), \quad x \in \Omega. \tag{2}$$

Nonlocal Problem I: *Find a function $u(x, t)$ that is a solution in $Q$ to the equation*

$$Lu = f(x, t)$$

*and satisfies conditions (1) and (2) and also the conditions*

$$D_x^{2k} u(x, t)\Big|_{x=0} = 0, \quad k = 0, \dots, m-1, \quad t \in (0, T), \tag{3}$$

$$D_x^{2k+1} u(x, t)\Big|_{x=0} - D_x^{2k+1} u(x, t)\Big|_{x=1} = 0, \quad k = 0, \dots, m-1, \quad t \in (0, T). \tag{4}$$

Nonlocal Problem II: *Find a function $u(x, t)$ that is a solution in $Q$ to the equation*

$$Lu = f(x, t)$$

*and satisfies conditions (1) and (2) and also the conditions*

$$D_x^{2k+1} u(x, t)\Big|_{x=0} = 0, \quad k = 0, \dots, m-1, \quad t \in (0, T), \tag{5}$$

$$D_x^{2k} u(x, t)\Big|_{x=0} - D_x^{2k} u(x, t)\Big|_{x=1} = 0, \quad k = 0, \dots, m-1, \quad t \in (0, T). \tag{6}$$

Nonlocal Problem III: *Find a function $u(x, t)$ that is a solution in $Q$ to the equation*

$$Lu = f(x, t)$$

*and satisfies conditions (1), (2), and (3) and also the condition*

$$D_x^{2k+1} u(x, t)\Big|_{x=0} + D_x^{2k+1} u(x, t)\Big|_{x=1} = 0, \quad k = 0, \dots, m-1, \quad t \in (0, T). \tag{7}$$

Nonlocal Problem IV: *Find a function $u(x, t)$ that is a solution in $Q$ to the equation*

$$Lu = f(x, t)$$

*and satisfies conditions (1), (2) and (5), and also the condition*

$$D_x^{2k} u(x, t)\Big|_{x=0} + D_x^{2k} u(x, t)\Big|_{x=1} = 0, \quad k = 0, \dots, m-1, \quad t \in (0, T). \tag{8}$$

For $m = 1$, $P_0 = D_t$, $P_1 = -I$, Nonlocal Problem I is the Ionkin problem [1] (see also [9]). Nonlocal Problem II for the same operators $P_0$ and $P_1$ can be called the problem conjugate to the Ionkin problem. If in Problems I and II, the operators $P_0$ and $P_1$ are not the same as in the Ionkin problem, then these problems can be called a generalization of the Ionkin problem and the conjugate Ionkin problem.

Nonlocal Problems III and IV for the equation $Lu = f$ have not been studied previously.

Define the linear space $H$:

$$H = \{v(x, t) : v(x, t) \in L_2(Q),\ D_t^{p_k} D_x^{2k} v(x, t) \in L_2(Q), \quad k = 0, \dots, m\}$$

(here the derivatives are understood as weak derivatives in the sense of S. L. Sobolev).

Obviously, $H$ is a Banach space with respect to the norm

$$\|v\|_H = \left( \int_Q \left[ v^2 + \sum_{k=0}^{m} \left( D_t^{p_k} D_x^{2k} v \right)^2 \right] dx\, dt \right)^{1/2}.$$

The aim of this article is to prove the existence of solutions to Nonlocal Problems I–IV belonging to $H$.

## 3. Main Results

We put $F(x,t) = f(x,t) + f(1-x,t)$.
Consider two auxiliary problems.

**Problem 1.** *Find a function $v(x,t)$ that is a solution in $Q$ to the equation*

$$Lv = F(x,t)$$

*and satisfies conditions (1) and (2) and also the condition*

$$D_x^{2k+1} v(x,t)\Big|_{x=0} = D_x^{2k+1} v(x,t)\Big|_{x=1} = 0, \quad k = 0, \dots, m-1, \quad t \in (0,T). \tag{9}$$

**Problem 2.** *Find a function $w(x,t)$ that is a solution in $Q$ to the equation*

$$Lw = f(x,t)$$

*and satisfies the conditions*

$$D_x^{2k} w(x,t)\Big|_{x=0} = 0, \quad k = 0, \dots, m-1, \quad t \in (0,T),$$

$$D_x^{2k} w(x,t)\Big|_{x=1} = D_x^{2k} v(x,t)\Big|_{x=0}, \quad k = 0, \dots, m-1, \quad t \in (0,T) \tag{10}$$

*($v(x,t)$ is a solution to Problem 1).*

<u>The Main Condition</u>: *The operators $P_k$, $k = 0, \dots, m$, the function $f(x,t)$, and conditions (1) and (2) are such that boundary value Problems A and B are uniquely solvable in $H$.*

**Theorem 1.** *Suppose the fulfillment of the Main Condition. Then the solution $w(x,t)$ to Problem 2 is a solution to Nonlocal Problem I in $H$.*

**Proof.** Alongside $v(x,t)$, the function $v(1-x,t)$ is also a solution to Problem 1. Since a solution to Problem 1 is unique, for $(x,t) \in Q$ we have

$$v(x,t) = v(1-x,t). \tag{11}$$

Next, using the solution $w(x,t)$ to Problem 2, define the function $V(x,t)$:

$$V(x,t) = w(x,t) + w(1-x,t).$$

This function satisfies the equalities

$$D_x^{2k} V(x,t)\Big|_{x=0} = D_x^{2k} w(x,t)\Big|_{x=1} = D_x^{2k} v(x,t)\Big|_{x=0}, \tag{12}$$

$$D_x^{2k} V(x,t)\Big|_{x=1} = D_x^{2k} w(x,t)\Big|_{x=1} = D_x^{2k} v(x,t)\Big|_{x=0} = D_x^{2k} v(x,t)\Big|_{x=1} \tag{13}$$

(the last equality follows from (11)). □

These equalities imply that the functions $v(x,t)$ and $V(x,t)$ satisfy identical boundary conditions, and they are both solutions to the same equation. Due to the uniqueness of solutions, we have

$$V(x,t) = v(x,t).$$

However, then

$$D_x^{2k+1} w(x,t)\Big|_{x=0} - D_x^{2k+1} w(x,t)\Big|_{x=1}$$
$$= D_x^{2k+1} v(x,t)\Big|_{x=0} + D_x^{2k+1} v(x,t)\Big|_{x=1} = 0, \quad t \in (0,T).$$

In other words, the function $w(x,t)$ satisfies the desired boundary conditions of Nonlocal Problem I.

The fulfillment of conditions (1) and (2) for $w(x,t)$, the validity of the equation $Lw = f$ and the membership $w(x,t) \in H$ are obvious.

Therefore, $w(x,t)$ is a desired solution to Nonlocal Problem I.

The theorem is proved.

We put $f_1(x,t) = \int\limits_0^x f(y,t)\,dy$.

**Theorem 2.** *Suppose that the function $f_1(x,t)$ satisfies the Main Condition. Then, Nonlocal Problem II has a solution belonging to $H$.*

**Proof.** Let $\underline{u}(x,t)$ be a solution to Nonlocal Problem I for $f_1(x,t)$. We put $u(x,t) = \underline{u}_x(x,t)$. The function $u(x,t)$ will be the desired solution to Nonlocal Problem II.

The theorem is proved. □

For proving the solvability of Nonlocal Problems III and IV, we need a modified Main Condition.

We put $F_1(x,t) = f(x,t) - f(1-x,t)$.

Consider two auxiliary problems:

**Problem 3.** *Find a function $v(x,t)$ that is a solution in $Q$ to the equation*

$$Lv = F_1(x,t)$$

*and satisfies conditions (1) and (2) and also condition (9).*

**Problem 4.** *Find a function $w(x,t)$ that is a solution in $Q$ to the equation*

$$Lw = f(x,t)$$

*and satisfies the conditions*

$$D_x^{2k} w(x,t)\Big|_{x=0} = 0, \quad D_x^{2k} w(x,t)\Big|_{x=1} = -D_x^{2k} v(x,t)\Big|_{x=0},$$
$$k = 0, \ldots, m-1, \quad t \in (0,T)$$

*($v(x,t)$ is a solution to Problem 3).*

The Modified Main Condition: *The operators $P_k$, $k = 0, \ldots, m$, the function $f(x,t)$, and conditions (1) and (2) are such that boundary value Problems $A_1$ and $B_1$ are uniquely solvable in $H$.*

**Theorem 3.** *Suppose the fulfillment of the Modified Main Condition. Then the solution $w(x,t)$ to Problem 4 is a solution to Nonlocal Problem III from $H$.*

**Proof.** Like $v(x,t)$, the function $-v(1-x,t)$ is a solution to Problem 3. Since the solution to Problem 3 is unique, then for $(x,t) \in Q$, we have $v(x,t) = -v(1-x,t)$. We put

$V_1(x,t) = w(x,t) - w(1-x,t)$. Obviously, $w(x,t)$ satisfies all conditions to Nonlocal Problem III. The membership of $w(x,t)$ in $H$ follows from the Modified Main Condition.

The theorem is proved. □

**Theorem 4.** *Suppose that $f_1(x,t)$ satisfies the Modified Main Condition. Then, Nonlocal Problem IV has a solution belonging to $H$.*

The proof of this theorem is obvious.

## 4. Examples

Theorems 1–4 imply that for proving the solvability of Nonlocal Problems I–IV (and, in particular, the solvability of the Ionkin problem), it suffices to check the fulfillment of the Main Condition or the Modified Main Condition. Let us give several examples when these conditions either hold or are easy to be seen to hold.

**Example 1.** *Quasiparabolic Equations of Arbitrary Order.*

Let $P_0$ and $P_m$ be the operators

$$P_0 = (-1)^{p+1} D_t^{2p+1}, \quad P_m = (-1)^{m+1} I.$$

In the rectangle $Q$, consider the equation

$$P_0 u + P_m D_x^{2m} u = f(x,t). \tag{14}$$

For $p = 0$, $m = 1$, this equation is the heat equation; the Ionkin problem (Nonlocal Problem I) was studied in this case (by expanding the solution in the series in special biorthogonal function systems) in [1,9]. In the more general case of second-order parabolic equations with arbitrary coefficients, the solvability of Nonlocal Problems I and II was established in [6,8]. Next, the solvability of Nonlocal Problem I in the special case of $p = 0$, $m = 2$, was studied in [10] (also with the use of expanding the solution in special biorthogonal systems).

In the general case of $p \geq 0$, $m \geq 1$, Nonlocal Problems I–IV have not been studied before.

As was shown in Section 2, for proving the solvability of Nonlocal Problems I–IV in $H$, it suffices to prove that they satisfy the Main Condition or the Modified Main Condition.

As with conditions (1) and (2), for Equation (14), choose either the conditions

$$D_t^k u(x,t)\Big|_{t=0} = 0, \quad k = 0,\dots,p, \quad x \in \Omega,$$

$$D_t^k u(x,t)\Big|_{t=T} = 0, \quad k = 0,\dots,p-1, \quad x \in \Omega \tag{15}$$

or the conditions

$$D_t^k u(x,t)\Big|_{t=0} = 0, \quad k = 0,\dots,p, \quad x \in \Omega,$$

$$D_t^k u(x,t)\Big|_{t=T} = 0, \quad k = p+1,\dots,2p-1, \quad x \in \Omega. \tag{16}$$

The solvability of boundary value Problem 1 in $H$ for Equation (14) (with conditions (15) or (16)) is not hard to prove by the classical Fourier method. Obviously, this solution is unique, and $f(x,t) \in L_2(Q)$ is a sufficient condition for the solvability (existence and uniqueness) of Problem 1.

We show that under some additional assumptions on $f(x,t)$, Problem 2 is also uniquely solvable in $H$.

Transform Problem 2: turn it into a problem with zero boundary conditions by setting

$$w(x,t) = W(x,t) - \varphi(x,t),$$

$$\varphi(x,t) = a_{2(m-1)}(x)D_x^{2(m-1)}v(0,t) + a_{2(m-2)}(x)D_x^{2(m-2)}v(0,t) + \ldots + a_0(x)v(0,t).$$

Here, the coefficients $a_k(x)$ are polynomials of degree of at most $2m-1$, and they are chosen so that the conditions

$$D_x^{2k}\varphi(x,t)\Big|_{x=0} = 0, \quad D_x^{2k}\varphi(x,t)\Big|_{x=1} = D_x^{2k}v(x,t)\Big|_{x=0}$$

$$k = 0,\ldots,m-1, \quad t \in (0,T),$$

hold. Obviously, the function $W(x,t)$ must satisfy the equation

$$LW = f(x,t) - P_0\varphi(x,t) = \widetilde{f}(x,t)$$

in $Q$.

The boundary value problem for this equation with zero boundary data for Problem 2 has a solution belonging to $H$ if $\widetilde{f}(x,t) \in L_2(Q)$. Since $f(x,t) \in L_2(Q)$, for the validity of the desired inclusion for the function $\widetilde{f}(x,t)$, it suffices that the equations

$$D_t^{2p+1}D_x^{2k}v(0,t) \in L_2([0,T]), \quad k = 0,\ldots,m-1, \tag{17}$$

hold. We show that under some additional conditions for $f(x,t)$, the solution $v(x,t)$ to boundary value Problem 1 (with conditions (15) and (16)) is such that the equations in (17) hold.

**Proposition 1.** *Suppose that the functions $D_x^k f(x,t)$, $k = 0,\ldots,2m-1$, belong to $L_2(Q)$ and for $m \geq 2$ we have*

$$D_x^{2k-1}f(x,t)\Big|_{x=0} - D_x^{2k-1}f(x,t)\Big|_{x=1} = 0, \quad k = 1,\ldots,m-1, \quad t \in (0,T). \tag{18}$$

*Then, boundary value Problem 1 with conditions (15) or (16) for Equation (14) has a solution $v(x,t)$ such that $v(x,t) \in H$, $D_t^{2p+1}D_x^{2m-1}v(x,t) \in L_2(Q)$, $D_x^{3m-1}v(x,t) \in L_2(Q)$.*

**Proof.** Consider the auxiliary problem: Find a function $v(x,t)$ that is a solution to $Q$ to the equation

$$Lv + \varepsilon(-1)^p D_t^{2p+1}D_x^{4m-2}v = F(x,t) \tag{19}$$

($\varepsilon > 0$) and satisfies conditions (15) and (9) and also the conditions

$$D_x^{2k+1}v(x,t)\Big|_{x=0} = D_x^{2k+1}v(x,t)\Big|_{x=1}, \quad k = m,\ldots,2(m-1), \quad t \in (0,T). \tag{20}$$

Define the space $H_1$:

$$H_1 = \{v(x,t) : v(x,t) \in H, \; D_t^{2p+1}D_x^{4m-2}v(x,t) \in L_2(Q)\}.$$

Boundary Value Problem (9), (15), (19), (20) has a solution $v(x,t)$ belonging to $H_1$; this is not hard to prove by using by the classical Fourier method or by the Galerkin method with the choice of a special basis (see, for example, [11]).

Multiply (19) by the function $(-1)^{p+1}(T_0-t)D_t^{2p+1}D_x^{4m-2}v(x,t)$, $T_0 > 0$, and integrate it by using $Q$. After easy calculations, we infer that solutions $v(x,t)$ to the boundary value problem (9), (15), (19), (20) satisfy the a priori estimate

$$\begin{aligned} &\int_Q \left\{ \left[D_t^{2p+1}D_x^{2m-1}v(x,t)\right]^2 + \left[D_t^p D_x^{3m-1}v(x,t)\right]^2 \right\} dx\,dt \\ &+\varepsilon \int_Q \left[D_t^{2p+1}D_x^{4m-2}v(x,t)\right]^2 dx\,dt \leq C\sum_{k=0}^{2m-1}\int_Q \left[D_x^k f(x,t)\right]^2 dx\,dt, \end{aligned} \tag{21}$$

in which the number $C$ is defined only by $T$.

Estimate (21) and the reflexivity of a Hilbert space (see [12,13]) implies that there exist sequences $\{\varepsilon_l\}_{l=1}^{\infty}$ of positive numbers and $\{v_l(x,t)\}_{l=1}^{\infty}$ of solutions to problem (9), (15), (19), (20) with $\varepsilon = \varepsilon_l$ such that as $l \to \infty$, the convergences

$$\varepsilon_l \to 0,$$

$$v_l(x,t) \to v(x,t) \quad \textit{weakly in } H,$$

$$\varepsilon_l D_t^{2p+1} D_x^{4m-2} v_l(x,t) \to 0 \quad \textit{weakly in } L_2(Q),$$

hold. Obviously, the limit function $v(x,t)$ is a desired solution to Problem 1 under condition (15).

For condition (16), all the arguments are analogous to those given above.

The proposition is proved. □

The proposition implies that, for solutions to Problem 1, under the above conditions on $f(x,t)$, equations (17) hold. As we said above, Nonlocal Problem I satisfies the Main Condition under conditions (15) or (16).

Summing up what was said above, we obtain the following theorem:

**Theorem 5.** *For any function $f(x,t)$ such that $D_x^k f(x,t)$, $k = 0,\dots,2m-1$, belong to $L_2(Q)$ and satisfying* (18) *for $m \geq 2$, Nonlocal Problem I with conditions* (15) *and* (16) *has a solution $u(x,t)$ belonging to $H$.*

The solvability of Nonlocal Problem II with conditions (15) or (16) with respect to $t$ is not hard to prove with the use of Theorem 2.

The solvability of nonlocal Problem III with conditions (15) or (16) is not hard to prove with the use of Theorem 3. We only specify that here we must also use the assertion about the presence of the additional equations in (17) for solutions $v(x,t)$ to Problem 3 and that condition (18) must be replaced by the condition *for $m \geq 2$,*

$$D_x^{2k-1} f(x,t)\Big|_{x=0} + D_x^{2k-1} f(x,t)\Big|_{x=1} = 0, \quad k = 0,\dots,2(m-1), \quad t \in (0,T). \tag{22}$$

The solvability of Nonlocal Problem IV is not hard to prove with the use of Theorem 4.

We do not give the exact statements of Nonlocal Problems II–IV due to their obviousness.

**Example 2.** *Hyperbolic and Quasihyperbolic Equations.*

We confine ourselves to the case $m = 1$.

In the rectangle $Q$, consider the equation

$$Lv \equiv (-1)^{p+1} D_t^{2p} u - u_{xx} = f(x,t) \tag{23}$$

For $p = 1$, this equation is the usual wave equation; the nonlocal Ionkin problem for this equation was studied in [14]. For $p > 1$, Equation (23) is not hyperbolic (and, in particular, the classical initial boundary value problem for it is ill-posed); Nonlocal Problems I–IV have not been studied for it before.

As with conditions (1) and (2), we use either the conditions

$$D_t^k u(x,t)\Big|_{t=0} = 0, \quad k = 0,\dots,p, \quad x \in \Omega,$$

$$D_t^k u(x,t)\Big|_{t=T} = 0, \quad k = 1,\dots,p-1, \quad x \in \Omega, \tag{24}$$

or the conditions

$$D_t^k u(x,t)\Big|_{t=0} = 0, \quad k = 0,\dots,p, \quad x \in \Omega,$$

$$D_t^k u(x,t)\Big|_{t=T} = 0, \quad k = p+1,\dots,2p-1, \quad x \in \Omega. \tag{25}$$

The solvability of Problem 1 for Equation (23) with conditions (24) with respect to $t$ in $H$ was established in [15,16], and the solvability of Problem 1 for Equation (23) with conditions (25) with respect to $t$ in $H$ was shown in [17]; in both cases, the solution $v(x,t)$ is unique, and in both cases, the following memberships for $f(x,t)$ are required: $f(x,t) \in L_2(Q)$, $f_t(x,t) \in L_2(Q)$.

We show that the solution $v(x,t)$ to Problem 1 with conditions (24) or (25) under some additional constraints on $f(x,t)$ is such that $D_t^{2p} D_x v(x,t) \in L_2(Q)$.

**Proposition 2.** *Suppose that the functions $f(x,t)$, $f_t(x,t)$, $f_{xt}(x,t)$ belong to $L_2(Q)$. Then Boundary Problem 1 with conditions (24) or (25) for Equation (23) has a solution $v(x,t)$ such that $D_t^{2p} D_x v(x,t) \in L_2(Q)$, $D_x^3 v(x,t) \in L_2(Q)$.*

**Proof.** Consider the auxiliary problem: *Find a function $v(x,t)$ that is a solution in $Q$ to the equation*

$$Lv + \varepsilon(-1)^{p+1} D_t^{2p} D_x^4 v = F(x,t) \tag{26}$$

$(\varepsilon > 0)$ and satisfies (24) and the condition

$$D_x^{2k+1} v(x,t)\Big|_{x=0} = D_x^{2k+1} v(x,t)\Big|_{x=1} = 0, \quad k = 0,1, \quad t \in (0,T). \tag{27}$$

Using the Fourier method or the Galerkin method with the choice of a special basis, it is not hard to see that problem (24), (26), (27) has a solution $v(x,t)$ such that $v(x,t) \in H$, $D_t^{2p} D_x^4 v(x,t) \in L_2(Q)$. Demonstrate that $v(x,t)$ satisfies a priori estimates uniform in $\varepsilon$.

Multiply Equation (26) by the function $(T_0 - t) D_t D_x^4 v(x,t)$, $T_0 > T$, and integrate the result over $Q$. After easy calculations, we infer that solutions $v(x,t)$ to problem (24), (26), (27) satisfy the estimate

$$\begin{aligned} &\int_Q \left\{ \left[ D_t^p D_x^2 v(x,t) \right]^2 + \left[ D_x^3 v(x,t) \right]^2 \right\} dx\,dt \\ &+\varepsilon \int_Q \left[ D_t^p D_x^4 v(x,t) \right]^2 dx\,dt + \int_\Omega \left[ D_x^3 v(x,T) \right]^2 \\ &\leq C_1 \int_Q \left[ f^2(x,t) + f_t^2(x,t) + f_{xt}^2(x,t) \right] dx\,dt, \end{aligned} \tag{28}$$

where the constant $C_1$ is determined only by $T$.

At the next step, multiply (26) by $D_t^{2p} D_x^2 v(x,t)$ and integrate the result over $Q$. Using Young's inequality and estimate (28), we conclude that solutions $v(x,t)$ to problem (24), (26), (27) admit the estimate

$$\begin{aligned} &\int_Q \left[ D_t^{2p} D_x v(x,t) \right]^2 dx\,dt + \varepsilon \int_Q \left[ D_t^{2p} D_x^3 v(x,t) \right]^2 dx\,dt \\ &\leq C_2 \int_Q \left[ f^2(x,t) + f_t^2(x,t) + f_x^2(x,t) + f_{xt}^2(x,t) \right] dx\,dt, \end{aligned} \tag{29}$$

where the constant $C_2$ is determined only by $T$.

Estimates (28) and (29) are quite enough for passing to the limit in problem (24), (26), (27). Using (28) and (29) and the reflexivity of a Hilbert space, passing to the limit in the corresponding subsequence, we conclude that Problem 1 with condition (24) for Equation (23) has a desired solution $v(x,t)$.

If in Problem 1 condition (25) is given for Equation (23), then completely analogous arguments again yield the existence of a desired solution $v(x,t)$.

The proposition is proved. □

Proposition 2 means that Nonlocal Problem I satisfies the Main Condition. Therefore, we have the following theorem:

**Theorem 6.** *For any function $f(x,t)$ such that $f(x,t) \in W_2^1(Q)$, $f_{xt}(x,t) \in L_2(Q)$, Nonlocal Problem I with conditions (24) or (25) has a solution $u(x,t) \in H$.*

It is not hard to prove the solvability of Nonlocal Problems II–IV with conditions (24) or (25) with respect to $t$ using the algorithm of Section 2 and the technique of obtaining a priori estimates presented in the proof of Proposition 2.

**Example 3.** *Elliptic and Quasielliptic Equations.*

We again confine ourselves to the case $m = 1$.

In the rectangle $Q$, consider the equation

$$(-1)^{p+1} D_t^{2p} u + u_{xx} = f(x,t) \tag{30}$$

The Ionkin problem (in its generalized statement) for Equation (30) in the case $p = 1$ (i.e, for an elliptic equation) was studied in [7], whereas for $p > 1$, Nonlocal Problems I–IV for (30) have not been studied before.

Let us consider two versions of conditions (1) and (2) again: the condition

$$D_t^k u(x,t)\Big|_{t=0} = D_t^k u(x,t)\Big|_{t=T} = 0, \quad k = 0,\dots,p-1, \quad x \in \Omega, \tag{31}$$

or the condition

$$D_t^{2k} u(x,t)\Big|_{t=0} = D_t^{2k} u(x,t)\Big|_{t=T} = 0, \quad k = 0,\dots,p-1, \quad x \in \Omega. \tag{32}$$

The Main Condition will be fulfilled for Nonlocal Problem I if the solution $v(x,t)$ to Problem 1 for Equation (30) with conditions (31) or (32) satisfies the membership $D_t^{2p} D_x v(x,t) \in L_2(Q)$; the proof of what is required is similar to the proofs of Propositions 1 and 2 (i.e., involves regularizations and a priori estimates).

**Theorem 7.** *For any function $f(x,t)$ such that $f(x,t) \in L_2(Q)$, $f_x(x,t) \in L_2(Q)$, Nonlocal Problem I with conditions (31) and (32) has a solution $u(x,t)$ belonging to $L_2(Q)$.*

The proof of this theorem is obvious.

The solvability of Nonlocal Problems II–IV for Equation (30) is also easy to prove by using Theorems 2–4.

**Example 4.** *Equations of Sobolev Type.*

Nonlocal Problems I–IV for equations of Sobolev type have rarely been studied—we can only mention the works [6,18], in which the solvability of Problems I and II was investigated for equations called pseudohyperbolic [19,20] and pseudoparabolic [21,22] in the literature. Let us show that the technique presented in Section 2 makes it possible to obtain existence theorems for solutions to Nonlocal Problems I–IV and for some other classes of Sobolev-type equations.

In the rectangle $Q$, consider the equation

$$u_{tt} - \alpha u_{xx} - u_{xxtt} = f(x,t). \tag{33}$$

The equation arises in the mathematical modeling of processes of plasma physics, in describing the dynamics of long waves on water, in electrodynamics and in elasticity theory (see [20–26]).

As with conditions (1), (2), we use either the Cauchy conditions

$$u(x,0) = u_t(x,0) = 0, \quad x \in \Omega, \tag{34}$$

or the Dirichlet conditions

$$u(x,0) = u(x,T) = 0, \quad x \in \Omega, \tag{35}$$

The solvability of Problem 1 for Equation (33) with conditions (34) or (35) in $H$ for $f(x,t) \in L_2(Q)$ is obvious. Moreover, a solution $v(x,t)$ to Problem 1 for Equation (33) with conditions (34) for $f(x,t) \in L_2(Q)$ and arbitrary $\alpha$, and with conditions (35) for $f(x,t) \in L_2(Q)$, $\alpha \leq 0$, satisfies the membership $v_{xtt}(x,t) \in L_2(Q)$. Consequently, Problem 2 for Equation (33) is also solvable in $H$. Thus, the Main Condition for Nonlocal Problem I is fulfilled both for condition (34) and for condition (35). This means that the following theorem holds:

**Theorem 8.** *For any function $f(x,t) \in L_2(Q)$, Nonlocal Problem I with condition (34) is solvable in $H$ for any $\alpha$. If condition (35) is defined in Nonlocal Problem I, then a solution from $H$ exists provided that $f(x,t) \in L_2(Q)$, $\alpha \leq 0$.*

The solvability of Nonlocal Problems II–IV is easily proved with the use of Theorems 2–4.

**Example 5.** *Degenerating Equations.*

In all the above examples, the equations under consideration were equations with constant coefficients. At the same time, all equations could have coefficients depending on $t$. Moreover, the corresponding equations could degenerate, i.e., some of the coefficients defining the type of the equation could vanish.

Now, consider the degenerating elliptic equation

$$u_{tt} + h(t)u_{xx} + \mu(t) = f(x,t), \tag{36}$$

in $Q$, in which $h(t)$ is a nonnegative function on $[0,T]$.

The Ionkin problem for Equation (36) was studied in [27,28] for $h(t) = t^m$ by the method based on representing the solution as a series in special biorthogonal function systems, where the function $\mu(t)$ also had a model form (subordinate to $h(t)$). Let us demonstrate that both for the Ionkin problem (i.e., Nonlocal Problem I) and for Nonlocal Problems II–IV, it is not hard to also obtain results on solvability in $H$ for more general equations.

As with conditions (1), (2), we use a Dirichlet condition; namely, the condition

$$u(x,0) = u(x,T) = 0, \quad x \in \Omega, \tag{37}$$

**Proposition 3.** *Suppose the fulfillment of the conditions*

$$h(t) \in C([0,T]), \quad h(t) \geq 0 \quad \textit{for } t \in [0,T]; \tag{38}$$

$$\mu(t) \in C([0,T]), \quad \mu(t) \leq 0 \quad \textit{for } t \in [0,T]. \tag{39}$$

*Then, for any function $f(x,t)$ for which one of the conditions*

$$f(x,t) \in L_2(Q), \quad f_x(x,t) \in L_2(Q), \quad f_{xx}(x,t) \in L_2(Q),$$

$$f_x(0,t) - f_x(1,t) = 0, \quad t \in (0,T), \tag{40}$$

*or*

$$f(x,t) \in L_2(Q), \quad f_x(x,t) \in L_2(Q), \quad h^{-\frac{1}{2}}(t) f_x(x,t) \in L_2(Q), \tag{41}$$

*holds, Problem 1 for Equation (36) with condition (37) has a solution $v(x,t)$ such that $v(x,t) \in H$, $v_{xtt}(x,t) \in L_2(Q)$, $h^{\frac{1}{2}}(t) v_{xxx}(x,t) \in L_2(Q)$.*

**Proof.** Consider the auxiliary problem: Find a function $v(x,t)$ that is a solution in $Q$ to the equation

$$v_{tt} + h(t) v_{xx} - \varepsilon v_{xxtt} + \mu v = f(x,t) \tag{42}$$

$(\varepsilon > 0)$ and satisfies (37) and also the condition

$$v_x(0,t) = v_{xxx}(0,t) = v_x(1,t) = v_{xxx}(0,t) = 0, \quad t \in (0,T). \tag{43}$$

The existence of a regular solution (of a solution having all square-integrable derivatives occurring in the equation) to this problem under the conditions of the theorem is obvious. Multiplying (42) first by $-v_{xxxx}(x,t)$, then by $-v_{xxtt}(x,t)$, integrating over $Q$ and using the hypotheses of the theorem, it is not hard to see that solutions $v(x,t)$ to the boundary value problem (36), (42), (43) satisfy the estimate

$$\int_Q \left[ v_{xtt}^2 + h(t) v_{xxx}^2 \right] dx\, dt + \varepsilon \int_Q \left[ v_{xxtt}^2 + v_{xxxx}^2 \right] dx\, dt \leq C_0(f), \tag{44}$$

where the constant $C_0(f)$ does not depend on $\varepsilon$. This estimate and the reflexivity of a Hilbert space imply the possibility of choosing a sequence converging to a desired solution to Problem 1 for Equation (36) with condition (37).

The proposition is proved. □

Proposition 3 means that the Main Condition is fulfilled for Nonlocal Problem I. Therefore, the following Theorem holds:

**Theorem 9.** *Suppose the fulfillment of conditions (38) and (39) and also of one of conditions (40) or (41). Then, Nonlocal Problem I has a solution $u(x,t) \in H$.*

Using Theorems 2–4, it is not hard to obtain theorems on the solvability of Equation (36) with condition (37) to Nonlocal Problems II–IV.

## 5. Comments and Supplements

5.1. The splitting method proposed in this article makes it possible to study further properties of solutions to Nonlocal Problems I–IV without investigating the properties of the corresponding function series. For example, knowing the properties of solutions to Problems A and B or to Problems $A_1$ and $B_1$, it is not hard to obtain theorems on increasing the smoothness, the boundedness of the solutions, the behavior of the solutions, etc.

5.2. In the examples presented in Section 3, some specific conditions (1) and (2) are used. Obviously, other conditions can be used—for example, for quasihyperbolic equations (23), one can use the conditions from [17]; for quasielliptic equations (30), along with conditions (31) or (32), we can use mixed conditions, etc.

5.3. The examples of Section 3 do not exhaust all classes of equations for which the splitting method is applicable. Observe first of all that in Examples 1–4, we consider equations with constant coefficients but in fact all equations can have variable coefficients (with the unconditional type preserved). In Examples 2–5, instead of the case $m = 1$, it is quite possible to consider the case $m > 1$ (in this case, conditions on the function $f(x,t)$ of the type of conditions (18) or (22) can appear). Equations of Sobolev type are certainly not limited to the simplest pseudoparabolic and pseudohyperbolic equations discussed in Example 4; for example, Nonlocal Problems I–IV can also be effectively studied for the general pseudohyperbolic equations of [25,29], etc.

5.4. It seems that the splitting method can be effectively used for studying the solvability of Nonlocal Problems I–IV with fractional derivatives.

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