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Frobenius Modules Associated to Algebra Automorphisms

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Abstract: Here, we study Frobenius bimodules associated with a pair of automorphisms of an algebra and discuss their basic properties. In particular, some equivalent conditions for a finite-dimensional bimodule are proved to be Frobenius and some isomorphisms between Ext-groups and Tor-groups of Frobenius modules over finite dimensional algebras are established.

Keywords: Frobenius module; ext-group; tor-group

MSC: 16D20; 16E30

1. Introduction

The study of Frobenius algebras and Frobenius extensions has a long history. It has long been well known that Frobenius algebras and extensions receive extensive applications; for instance, they are related to Hopf algebras [1,2], topological quantum field theory [3], Yang–Baxter equations [4], representation and homology theory [5–9], Lie theory [10,11], etc. Recently, Frobenius extensions have found applications in matrix theory and invariant theory [12–14].

Let R and S be rings and let ${}_RM_S$ be a bimodule. Assume that M is projective both as a left R-module and as a right S-module. If M satisfies certain self-dual properties (Definition 2.1 in [15]), then it is call a Frobenius bimodule. Now, assume R is a ring and that S is a subring of R. It was proved in [15] that the ring extension R/S is a Frobenius extension if and only if R, viewed as an R-S-bimodule, is a Frobenius module. Many other properties of ring extensions may be determined by Frobenius bimodules; for example, separable Frobenius extensions are determined by Frobenius bimodules, and two rings are separable equivalent if and only if they are linked by a Frobenius biseparable bimodule [16]. More properties and applications of Frobenius bimodules may be found in [14–17]. Note that Frobenius bimodules are assumed to be projective as both left modules and right modules; however, many examples show that if we drop the assumption of the projectiveness in the definition of a Frobenius bimodule and keep the self-dual property, the resulting bimodules continue to possess many properties similar to those of Frobenius bimodules.

In this paper, we provide a modified definition of Frobenius bimodules over a single algebra. Let A be an algebra and let ${}_AM_A$ be a finite-dimensional bimodule. If M admits a nondegenerate bilinear form which is balanced associated to a pair of automorphisms of A, then we say that M is a Frobenius module (more precisely, see Definition 2). Note that we drop the assumption that M is projective as a left or right A-module. Such Frobenius modules exist extensively; indeed, as is shown in Theorem 3, every finite-dimensional A-bimodule is a direct summand of a Frobenius module. Because a Frobenius module is not necessary projective, it has many nontrivial homological properties.

The rest of this paper is organized as follows.

In Section 2, we provide a precise definition of Frobenius modules associated with a pair of automorphisms of a given algebra, then discuss the basic properties of Frobenius modules. In particular, we provide a criterion condition for a bimodule to be Frobenius

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(Theorem 4) and prove that the Nakayama automorphism of a Frobenius module is a bimodule homomorphism (Proposition 4 and Corollary 3).

In Section 3, we focus on the homological properties of Frobenius modules over finite-dimensional algebras. The main results of this paper are as follows.

Theorem 1 (=Corollary 4)**.** *Let A be a finite dimensional algebra and let (ς, σ) be a pair of automorphisms of A. Assume that M is a (ς, σ)-Frobenius module; then, we have isomorphisms*

$$\mathrm{Ext}^n_A(M, M) \cong \mathrm{Ext}^n_{A^\circ}(M, M)$$

for all $n \geq 0$, where $\mathrm{Ext}^n_A(M, M)$ is the Ext-group of the left A-module ${}_AM$ and $\mathrm{Ext}^n_{A^\circ}(M, M)$ is the Ext-group of the right A-module M_A.

The above theorem shows that the Ext-group of a Frobenius module is left–right symmetric, which is a consequence of a more general result (Theorem 5).

Theorem 2 (=Theorems 6 and 8)**.** *Let A and M be the same as in Theorem A and let ${}_AX$ and Y_A be finitely generated A-modules; then, we have the following isomorphisms:*

(i) $\mathrm{Tor}^A_n(X, M)^* \cong \mathrm{Ext}^n_{A^\circ}(X, M^\sigma)$ *for $n \geq 0$;*
(ii) $\underline{\mathrm{Hom}}_A(Y, M) \cong \mathrm{Tor}^A_1(Tr(Y), M)$.

In the above theorem, $\underline{\mathrm{Hom}}_A(Y, M)$ is the stable Hom-set and $Tr(Y)$ is the Auslander–Retein translation of Y_A (see the main text above Theorem 6). Isomorphism (ii) in the above theorem may be viewed as a new explanation of Auslander–Reiten duality for Frobenius modules.

Throughout this paper, $\Bbbk$ is a field with characteristic zero and all algebras and modules considered are over the field $\Bbbk$. Letting V be a vector space, we write $V^* = \mathrm{Hom}_\Bbbk(V, \Bbbk)$.

2. Frobenius Modules Associated with Algebra Automorphisms

Let A be an algebra and let M_A be a right A-module. For $\sigma \in \mathrm{Aut}(A)$, we write M^σ for the A-module whose right A-action is twisted by σ. Below, to avoid possible confusion, we use $\diamond$ to denote the right A-action twisted by σ, that is,

$$x \diamond a = x\sigma(a)$$

for $x \in M$ and $a \in A$. Similarly, if ${}_AN$ is a left A-module, then ${}^\sigma N$ denotes the left A-module obtained from ${}_AN$ with the left A-action twisted by the automorphism σ.

Definition 1. *Let σ be an automorphism of A and let ${}_AM_A$ be an A-bimodule.*

(i) *A bilinear form $\langle -, - \rangle : M \times M \longrightarrow \Bbbk$ is said to be σ-inner-balanced if it satisfies the following condition: for all $a \in A$, $x, y \in M$,*

$$\langle xa, y \rangle = \langle x, \sigma(a)y \rangle.$$

(ii) *A bilinear form $\langle -, - \rangle : M \times M \longrightarrow \Bbbk$ is said to be σ-outer-balanced if it satisfies the following condition: for all $a \in A$, $x, y \in M$,*

$$\langle ax, y \rangle = \langle x, y\sigma(a) \rangle.$$

(iii) *A bilinear form $\langle -, - \rangle : M \times M \longrightarrow \Bbbk$ is nondegenerate if $\langle x, y \rangle = 0$ for all $y \in M$ implies that $x = 0$.*

A bimodule with a nondegerate balanced bilinear form has nice dual properties.

Proposition 1. *Let A be an algebra and let ${}_AM_A$ be a finite-dimensional A-bimodule. The following are equivalent:*

(i) *There is a nondegerate σ-inner-balanced bilinear form $\langle -, - \rangle : M \times M \longrightarrow \Bbbk$;*
(ii) *There is an isomorphism of right A-modules $f : M^{\sigma^{-1}} \longrightarrow M^*$;*
(iii) *There is an isomorphism of left A-modules $g : {}^{\sigma}M \longrightarrow M^*$.*

Proof. (i) $\Longrightarrow$ (ii). We define a linear map $f : M \longrightarrow M^*$ by setting $f(x) = \langle x, - \rangle$ for all $x \in M$. For $a \in A$ and $y \in M$, we have

$$f(xa)(y) = \langle xa, y \rangle = \langle x, \sigma(a)y \rangle = f(x)(\sigma(a)y) = (f(x) \cdot \sigma(a))(y),$$

where $f(x) \cdot \sigma(a)$ is the right A-module action on M^*; therefore,

$$f(x\sigma^{-1}(a)) = f(x) \cdot \sigma(\sigma^{-1}(a)) = f(x) \cdot a.$$

Hence $f : M^{\sigma^{-1}} \longrightarrow M^*$ is a right A-module homomorphism. As the bilinear form is nondegenerate, it follows that f is injective. Because M is finite-dimensional, we have $\dim(M) = \dim(M^*)$; hence, f is indeed an isomorphism.

(ii) $\Longrightarrow$ (i). Define a bilinear map $\langle -, - \rangle : M \times M \longrightarrow \Bbbk$ by setting $\langle x, y \rangle = f(x)(y)$. Because $f : M^{\sigma^{-1}} \longrightarrow M^*$ is a right A-module homomorphism, we have

$$\begin{aligned} \langle xa, y \rangle = f(xa)(y) = f(x \diamond \sigma(a))(y) = (f(x) \cdot \sigma(a))(y) \\ = f(x)(\sigma(a)y) = \langle x, \sigma(a)y \rangle. \end{aligned}$$

The injectivity of f implies that $\langle -, - \rangle$ is non-degenerated.

(ii) $\Longrightarrow$ (iii). Taking the vector space dual of the right A-module isomorphism f, we obtain an isomorphism of left A-modules $f^* : (M^*)^* \longrightarrow (M^{\sigma^{-1}})^*$. Note that $(M^{\sigma^{-1}})^* = {}^{\sigma^{-1}}(M^*)$. Let $\tau : M \longrightarrow (M^*)^*$ be the valuation map, that is, $\tau(x)(\alpha) = \alpha(x)$ for all $x \in M$ and $\alpha \in M^*$. Now, for $a \in A$ we have $\tau(ax)(\alpha) = \alpha(ax) = (\alpha \cdot a)(x) = \tau(x)(\alpha \cdot a) = (a \cdot \tau(x))(\alpha)$. Therefore, τ is an isomorphism of left A-module isomorphism. Setting $g = f^* \circ \tau$, g is indeed an isomorphism of left A-modules from ${}^{\sigma}M$ to M^*.

(iii) $\Longrightarrow$ (ii). This case is similar to the previous case. □

The proof of the above proposition shows that the nondegeneracy of the bilinear form defined in Definition 1(iii) is symmetric.

Corollary 1. *If $\langle -, - \rangle$ is a nondegenerate σ-inner balanced bilinear form defined on M, then $\langle x, y \rangle = 0$ for all $x \in M$ implies that $y = 0$.*

Proof. From the proof of Proposition 1, $f = \langle x, - \rangle : M^{\sigma^{-1}} \longrightarrow M^*$ is an isomorphism of right A-modules and $g = f^* \circ \tau : {}^{\sigma}M \longrightarrow M^*$ is an isomorphism of left A-modules. For $y \in M$, we have $g(y) = f^*(\tau(y)) = \langle x, y \rangle$. If $\langle x, y \rangle = 0$ for all $x \in M$, then $g(y) = 0$. Because g is an isomorphism, it follows that $y = 0$. □

Similar to the above proposition, we have the following results for outer-balanced bilinear forms.

Proposition 2. *Let A be an algebra and let ${}_AM_A$ be a finite-dimensional A-bimodule. The following are equivalent:*

(i) *There is a nondegerate σ-outer-balanced bilinear form $\langle -, - \rangle : M \times M \longrightarrow \Bbbk$;*
(ii) *There is an isomorphism of right A-modules $f : M^{\sigma} \longrightarrow M^*$;*
(iii) *There is an isomorphism of left A-modules $g : {}^{\sigma^{-1}}M \longrightarrow M^*$.*

Proof. We only show the following two directions, as the others are similar to the proof of Proposition 1.

(i) $\Longrightarrow$ (iii). Similar to the proof of Proposition 1, set $g(x) = \langle x, - \rangle$ for all $x \in M$. To avoid possible confusion, we use $\diamond$ to denote the left A-action on ${}^{\sigma^{-1}}M$, that is, $a \diamond x = \sigma^{-1}(a)x$ for $a \in A$ and $x \in M$. We have $g(a \diamond x)(y) = \langle \sigma^{-1}(a)x, y \rangle = \langle x, ya \rangle = (a \cdot g(x))(y)$ for $y \in M$; therefore, g is a left A-module homomorphism. The injectivity follows from similar arguments as those in Proposition 1.

(iii) $\Longrightarrow$ (ii). Note that $({}^{\sigma^{-1}}M)^* \cong (M^*)^{\sigma^{-1}}$. Taking the vector dual of the map g, we obtain $g^* : (M^*)^* \longrightarrow ({}^{\sigma^{-1}}M)^* \cong (M^*)^{\sigma^{-1}}$. As in the proof of the Proposition 1, the valuation map $\tau : M \longrightarrow (M^*)^*$ is a right A-module isomorphism. We obtain an isomorphism $f := g^* \circ \tau : M \longrightarrow (M^*)^{\sigma^{-1}}$. Note that f is indeed an isomorphism $M^\sigma \longrightarrow M^*$. $\square$

In view of the propositions above, we make the following definition of Frobenius modules.

Definition 2. *Let A be an algebra and let ${}_AM_A$ be a finite-dimensional A-bimodule. Assume that (ς, σ) is a pair of automorphisms of A.*

(i) *If there is a nondegenerate bilinear form*

$$\langle -, - \rangle : M \times M \longrightarrow \Bbbk$$

which is both ς-inner-balanced and σ-outer-balanced, then we call M a (ς, σ)-Frobenius module, or simply, a Frobenius module.

If the automorphisms $\varsigma = \sigma = id$, then we call M a balanced Frobenius module.

(ii) *If there is an id-inner-balanced nondegenerate bilinear form*

$$\langle -, - \rangle : M \times M \longrightarrow \Bbbk$$

such that $\langle x, y \rangle = \langle y, x \rangle$ for all $x, y \in M$, then we call M a symmetric module.

Remark 1. *In [15], Kadison introduced the notion of a Frobenius bimodule ${}_RM_S$ over rings R and S, where M is assumed to be projective both as a left R-module and as a right S-module (see Definition 2.1 in [15]). We drop these assumptions so that the homological properties of a Frobenius module are not trivial.*

Next, we provide an example of a Frobenius module.

Example 1. *Let $A = \Bbbk\langle x, y \rangle / (xy + yx)$ be a skew polynomial algebra, define an automorphism σ on A by setting $\sigma(x) = -x$ and $\sigma(y) = -y$, and let $M = A/(x^2A + y^2A)$. Then, M is an A-bimodule. Note that M has a basis $\{1, x, y, xy\}$. Defining a bilinear form $\langle -, - \rangle : M \times M \to \Bbbk$ by*

$$\langle k_0 + k_1x + k_2y + k_3xy, l_0 + l_1x + l_2y + l_3xy \rangle = k_0l_3 + k_3l_0 + k_1l_2 + k_2l_1,$$

where $k_0, \dots, k_3, l_0, \dots, l_3 \in \Bbbk$, it is easy to check that M is an (id, σ)-Frobenius A-module.

Condition (ii) in Definition 2 is stronger than the condition of balanced Frobenius algebras. Indeed, we have the following proposition.

Proposition 3. *If ${}_AM_A$ is a symmetric module, then it is a balanced Frobenius module.*

Proof. For any $a \in A$, $x, y \in M$, we have $\langle ax, y \rangle = \langle y, ax \rangle = \langle ya, x \rangle = \langle x, ya \rangle$. Hence, M is a balanced Frobenius module. $\square$

Remark 2. *If A is a Frobenius algebra, then it is a Frobenius module when viewed as an A-bimodule. Indeed, from the definition of a Frobenius algebra, there is a nondegenerate bilinear form $\langle -, - \rangle : A \times A \longrightarrow \Bbbk$ which is id-inner-balanced. Assume σ is the Nakayama automorphism* *of A; then, for all $a, b, c \in A$ we have $\langle ab, c\rangle = \langle a, bc\rangle = \langle bc, \sigma(a)\rangle = \langle b, c\sigma(a)\rangle$. Hence, the bilinear form is σ-outer-balanced and ${}_AA_A$ is a (id, σ)-Frobenius module.*

The next result shows that Frobenius modules exist extensively. Indeed, every finite dimensional bimodule can be viewed as a direct summand of a Frobenius module.

Theorem 3. *Let A be an algebra and let ${}_AM_A$ be a finite dimensional A-bimodule. Let σ be an automorphism of A and set $T_\sigma(M) := {}^\sigma M^{\sigma^{-1}} \oplus M^*$. Then, $T_\sigma(M)$ is a (σ^{-1}, σ)-Frobenius module.*

In particular, $T(M) := M \oplus M^*$ is a symmetric module.

Proof. We define a bilinear map $\langle -, -\rangle : T_\sigma(M) \times T_\sigma(M) \longrightarrow \Bbbk$ by setting

$$\langle (x, \alpha), (y, \beta)\rangle = \alpha(y) + \beta(x)$$

for all $x, y \in M, \alpha, \beta \in M^*$. Now, for $a \in A$ we have

$$\begin{aligned} \langle (x, \alpha) \cdot a, (y, \beta)\rangle &= \langle (x \diamond a, \alpha \cdot a), (y, \beta)\rangle \\ &= (\alpha \cdot a)(y) + \beta(x\sigma^{-1}(a)) \\ &= \alpha(ay) + (\sigma^{-1}(a) \cdot \beta)(x) \\ &= \langle (x, \alpha), (\sigma^{-1}(a) \diamond y, \sigma^{-1}(a) \cdot \beta)\rangle \\ &= \langle (x, \alpha), \sigma^{-1}(a) \cdot (y, \beta)\rangle, \end{aligned}$$

and similarly, we have

$$\langle a \cdot (x, \alpha), (y, \beta)\rangle = \langle (x, \alpha), (y, \beta) \cdot \sigma(a)\rangle.$$

The nondegeneracy of the bilinear form is easy to see. Hence, $T_\sigma(M)$ is a (σ^{-1}, σ)-Frobenius module. □

Propositions 1 and 2 imply the following criteria in order for a bimodule to be Frobenius.

Theorem 4. *Let A be an algebra and let ${}_AM_A$ be a finite-dimensional A-bimodule. Suppose that (ς, σ) is a pair of automorphisms of A. Then, the following are equivalent:*

(i) *M is a (ς, σ)-Frobenius;*

(ii) *There is an A-bimodule isomorphism ${}^{\sigma^{-1}}M^{\varsigma^{-1}} \cong M^*$;*

(iii) *There is an A-bimodule isomorphism ${}^\varsigma M^\sigma \cong M^*$.*

Proof. (i) $\Longrightarrow$ (ii). As was shown in Proposition 1, the map $f : M^{\varsigma^{-1}} \longrightarrow M^*$ is an isomorphism of right A-modules where $f(x) = \langle x, -\rangle$. Proposition 2 shows that f is indeed an isomorphism of left A-modules $f : {}^{\sigma^{-1}}M \longrightarrow M^*$. Hence, f is an A-bimodule isomorphism.

(ii) $\Longrightarrow$ (i). This is similar to the proof of Proposition 1.

(ii) $\Longleftrightarrow$ (iii). This is obtained by taking the vector space dual. □

The following is an immediate consequence of the above theorem.

Corollary 2. *If ${}_AM_A$ is a (ς, σ)-Frobenius module, then M^* is a (σ, ς)-Frobenius module.*

Let ${}_AM_A$ be a (ς, σ)-Frobenius module. Similar to Frobenius algebras, there is a Nakayama automorphism of M. Indeed, from Corollary 1, for an element $x \in M$ there is a unique element $x' \in M$ such that $\langle x, -\rangle = \langle -, x'\rangle$ in M^*, which induces a linear map $\mathfrak{n} : M \to M$ such that

$$\langle x, y\rangle = \langle y, \mathfrak{n}(x)\rangle$$

for all $x, y \in M$. Indeed, from Proposition 1 we have $\mathfrak{n} = g^{-1} \circ f$, where f and g are isomorphisms in Proposition 1; hence, $\mathfrak{n}$ is a linear automorphism. We call $\mathfrak{n}$ the *Nakayama automorphism* of M.

It is clear that ${}_AM_A$ is a symmetric module if and only if the Nakayama automorphism of M is the identity map.

Proposition 4. *Let ${}_AM_A$ be a (ς, σ)-Frobenius module. The Nakayama automorphism $\mathfrak{n}$ is an A-bimodule isomorphism*

$$\mathfrak{n} : {}^{\varsigma\sigma}M^{\sigma\varsigma} \longrightarrow M.$$

Proof. For $x, y \in M$ and $a \in A$, we have

$$\langle y, \mathfrak{n}(xa)\rangle = \langle xa, y\rangle = \langle x, \varsigma(a)y\rangle = \langle \varsigma(a)y, \mathfrak{n}(x)\rangle = \langle y, \mathfrak{n}(x)\sigma(\varsigma(a))\rangle;$$

therefore,

$$\mathfrak{n}(xa) = \mathfrak{n}(x)\sigma\varsigma(a).$$

Similarly,

$$\langle y, \mathfrak{n}(ax)\rangle = \langle ax, y\rangle = \langle x, y\sigma(a)\rangle = \langle y\sigma(a), \mathfrak{n}(x)\rangle = \langle y, \varsigma\sigma(a)\mathfrak{n}(x)\rangle;$$

therefore,

$$\mathfrak{n}(ax) = \varsigma\sigma(a)\mathfrak{n}(x).$$

Hence, the result follows. □

The above proposition implies the following result.

Corollary 3. *Let σ be an automorphism of A. If ${}_AM_A$ is a (σ, σ^{-1})-Frobenius module, then the Nakayama automorphism is an A-bimodule automorphism of ${}_AM_A$.*

3. Homological Properties of Frobenius Modules over Finite-Dimensional Algebras

In this section, we always assume that A is a finite-dimensional algebra. We write A° for the opposite algebra of A. Then, a right A-module can be viewed as a left A°-module. If X and Z are left A-modules, then we write $\mathrm{Hom}_A(X, Z)$ and $\mathrm{Ext}^n_A(X, Z)$ for the Hom-set and extension groups of X and Z, respectively, while if X and Z are right A-modules, then we write $\mathrm{Hom}_{A^\circ}(X, Z)$ and $\mathrm{Ext}^n_{A^\circ}(X, Z)$ for the Hom-set and extension groups of X and Z.

Let (ς, σ) be a pair of automorphisms of A and let ${}_AM_A$ be a (ς, σ)-Frobenius module. Then, take a projective resolution of the left A-module ${}_AM$:

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0, \tag{1}$$

where P_n is a finitely generated projective left A-module for all $n \geq 0$. Twisting the left A-actions on the modules in the above sequence, we obtain a projective resolution of ${}^\sigma M$:

$$\cdots \longrightarrow {}^\sigma P_n \longrightarrow \cdots \longrightarrow {}^\sigma P_1 \longrightarrow {}^\sigma P_0 \longrightarrow {}^\sigma M \longrightarrow 0. \tag{2}$$

Taking the vector space dual of sequence (1), we obtain the following exact sequence:

$$0 \longrightarrow M^* \longrightarrow P_0^* \longrightarrow P_1^* \longrightarrow \cdots \longrightarrow P_n^* \longrightarrow \cdots. \tag{3}$$

Because P_n is a projective left A-module for every $n \geq 0$, it follows that P_n^* is an injective right A-module for every $n \geq 0$. Therefore, the exact sequence (3) is an injective resolution of the right module M^*. From Theorem 4, we have the A-bimodule isomorphism ${}^\varsigma M^\sigma \cong M^*$. Hence, the exact sequence (3) is an injective resolution of the right A-module M^σ. Then, we have the following injective resolution of the right A-module M:

$$0 \longrightarrow M \longrightarrow (P_0^*)^{\sigma^{-1}} \longrightarrow (P_1^*)^{\sigma^{-1}} \longrightarrow \cdots \longrightarrow (P_n^*)^{\sigma^{-1}} \longrightarrow \cdots, \tag{4}$$

or equivalently,

$$0 \longrightarrow M \longrightarrow ({}^{\sigma^{-1}}P_0)^* \longrightarrow ({}^{\sigma^{-1}}P_1)^* \longrightarrow \cdots \longrightarrow ({}^{\sigma^{-1}}P_n)^* \longrightarrow \cdots . \tag{5}$$

Now, let ${}_AY$ be a finite-dimensional left A-module and consider the right A-module $({}^{\sigma^{-1}}Y)^*$. Applying the functor $\mathrm{Hom}_{A^\circ}(({}^{\sigma^{-1}}Y)^*, -)$ to the injective resolution (5), we obtain the following complex:

$$0 \longrightarrow \mathrm{Hom}_{A^\circ}(({}^{\sigma^{-1}}Y)^*, ({}^{\sigma^{-1}}P_0)^*) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_{A^\circ}(({}^{\sigma^{-1}}Y)^*, ({}^{\sigma^{-1}}P_n)^*) \longrightarrow \cdots . \tag{6}$$

Taking the n-th cohomology of complex (6), we obtain the extension group $\mathrm{Ext}^n_{A^\circ}(({}^{\sigma^{-1}}Y)^*, M)$.

Notice that we have

$$\mathrm{Hom}_{A^\circ}(({}^{\theta}X)^*, ({}^{\theta}Z)^*) \cong \mathrm{Hom}_A(Z, X)$$

for any finite-dimensional left A-modules X, Z and any automorphism θ of A. The sequence (6) is equivalent to the following complex:

$$0 \longrightarrow \mathrm{Hom}_A(P_0, Y) \longrightarrow \mathrm{Hom}_A(P_1, Y) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_A(P_n, Y) \longrightarrow \cdots , \tag{7}$$

which is exactly the complex obtained from projective resolution (1) by applying the functor $\mathrm{Hom}_A(-, Y)$. The n-th cohomology of complex (7) is the extension group $\mathrm{Ext}^n_A(M, Y)$. Therefore, we have

$$\mathrm{Ext}^n_A(M, Y) \cong \mathrm{Ext}^n_{A^\circ}(({}^{\sigma^{-1}}Y)^*, M)$$

for all $n \geq 0$.

Notice that from Theorem 4 we also have an A-bimodule isomorphism ${}^{\sigma^{-1}}M^{\varsigma^{-1}}$. Replacing the isomorphism σ^{-1} in sequences (5) and (6) with ς, we finally obtain the isomorphism

$$\mathrm{Ext}^n_A(M, Y) \cong \mathrm{Ext}^n_{A^\circ}(({}^{\varsigma}Y)^*, M)$$

for all $n \geq 0$.

Summarizing the above narratives, we obtain the following result.

Theorem 5. *Let ${}_AM_A$ be a (ς, σ)-Frobenius module and let ${}_AY$ be a finite-dimensional module. For each $n \geq 0$, we have*

$$\mathrm{Ext}^n_A(M, Y) \cong \mathrm{Ext}^n_{A^\circ}(({}^{\sigma^{-1}}Y)^*, M) \cong \mathrm{Ext}^n_{A^\circ}(({}^{\varsigma}Y)^*, M).$$

The above theorem implies that the Ext-groups of a Frobenius module are left–right symmetric.

Corollary 4. *Let ${}_AM_A$ be a (ς, σ)-Frobenius module. For each $n \geq 0$, we have*

$$\mathrm{Ext}^n_A(M, M) \cong \mathrm{Ext}^n_{A^\circ}(M, M).$$

Proof. Note that as a right A-module, from Theorem 4 we have

$$M \cong (M^*)^{\sigma^{-1}} \cong ({}^{\sigma^{-1}}M)^*.$$

The result follows from Theorem 5 by setting $Y = M$. □

Next, let X_A be a right A-module. Applying the functor $(X \otimes_A -)^*$ to the projective resolution (1) of M, we obtain the following complex:

$$0 \longrightarrow (X \otimes_A P_0)^* \longrightarrow (X \otimes_A P_1)^* \longrightarrow \cdots \longrightarrow (X \otimes_A P_n)^* \longrightarrow \cdots , \tag{8}$$

the n-th cohomology of which is equal to $\mathrm{Tor}_n^A(X,M)^*$.

Note that this complex is equivalent to the following complex:

$$0 \longrightarrow \mathrm{Hom}_{A^\circ}(X,(P_0)^*) \longrightarrow \mathrm{Hom}_{A^\circ}(X,(P_1)^*) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_{A^\circ}(X,(P_n)^*) \longrightarrow \cdots. \tag{9}$$

From sequence (4), we can see that complex (9) is indeed obtained by applying the functor $\mathrm{Hom}_{A^\circ}(X,-)$ to the injective resolution of M^σ. Hence, the n-th cohomology of complex (9) is the extension group $\mathrm{Ext}^n_{A^\circ}(X,M^\sigma)$.

Summarizing the above narratives, we obtain the following Tor–Ext translation.

Theorem 6. *Let ${}_AM_A$ be a (ς,σ)-Frobenius module and let X_A be a right A-module. Then, we have the following isomorphisms:*

$$\mathrm{Tor}_n^A(X,M)^* \cong \mathrm{Ext}^n_{A^\circ}(X,M^\sigma)$$

for all $n \geq 0$.

Let ${}_AY$ be a finitely generated left A-module. There is an Auslander–Reiten transpose $Tr(Y)$ of Y (for instance, see [6]) which is defined as follows. Let

$$P_1 \xrightarrow{d} P_0 \longrightarrow Y \longrightarrow 0$$

be a minimal projective resolution of Y; applying the functor $\mathrm{Hom}_A(-,A)$ to the left A-module homomorphism $P_1 \xrightarrow{d} P_0$, we denote the result of the right A-module homomorphism by $\mathrm{Hom}_A(d,A)$, that is, we have

$$\mathrm{Hom}_A(d,A) : \mathrm{Hom}_A(P_0,A) \longrightarrow \mathrm{Hom}_A(P_1,A).$$

The Auslander–Reiten transpose is defined to be the right A-module:

$$Tr(Y) := \mathrm{coker}\,\mathrm{Hom}_A(d,A).$$

Let ${}_AX$ be another finitely generated left A-module. Let $P(Y,X)$ be the subspace of $\mathrm{Hom}_A(Y,X)$ consisting of homomorphisms f which factors through a projective module, that is, there is a projective module P such that $f = gh$ where $h: Y \to P$, $g: P \to X$. Now, we write

$$\underline{\mathrm{Hom}}_A(Y,X) := \mathrm{Hom}_A(Y,X)/P(Y,X).$$

The isomorphism in the next theorem is called the Auslander–Reiten duality.

Theorem 7 ([18])**.** *Let ${}_AX$ and ${}_AY$ be finitely generated left A-modules. There is an isomorphism*

$$\underline{\mathrm{Hom}}_A(Y,X)^* \cong \mathrm{Ext}^1_A(X,Tr(Y)^*).$$

Lemma 1. *Let ${}_AY$ be a finitely generated left A-module. Assume that θ is an automorphism of A. We have $Tr({}^\theta Y) \cong Tr(Y)^\theta$.*

Proof. Let $P_1 \xrightarrow{d} P_0 \longrightarrow Y \longrightarrow 0$ be a minimal projective resolution of Y; then, ${}^\theta P_1 \xrightarrow{{}^\theta d} {}^\theta P_0 \longrightarrow {}^\theta Y \longrightarrow 0$ is a minimal projective resolution of ${}^\theta Y$, where the map ${}^\theta d$ is indeed the same as d. Applying the functor $\mathrm{Hom}_A(-,A)$, we have the following right A-module homomorphism:

$$\mathrm{Hom}_A({}^\theta d,A) : \mathrm{Hom}_A({}^\theta P_0,A) \longrightarrow \mathrm{Hom}_A({}^\theta P_1,A). \tag{10}$$

Because ${}^{\theta^{-1}}A \cong A^\theta$ as A-bimodules, we have the following right A-module isomorphisms:

$$\mathrm{Hom}_A({}^\theta P_0,A) \cong \mathrm{Hom}_A(P_0,{}^{\theta^{-1}}A) \cong \mathrm{Hom}_A(P_0,A^\theta) \cong \mathrm{Hom}_A(P_0,A)^\theta.$$

Then, the right A-module homomorphism in (10) is equivalent to the following map:

$$\mathrm{Hom}_A(d,A)^\theta : \mathrm{Hom}_A(P_0,A)^\theta \longrightarrow \mathrm{Hom}_A(P_1,A)^\theta. \tag{11}$$

Hence,

$$Tr({}^\theta Y) = \operatorname{coker} \mathrm{Hom}_A({}^\theta d, A) \cong \operatorname{coker} \mathrm{Hom}_A(d,A)^\theta = Tr(Y)^\theta.$$

□

Remark. We end this note with the following explanation of Auslander–Reiten duality for Frobenius modules.

Theorem 8. *Let (ς,σ) be a pair of automorphisms of A. Assume that ${}_AM_A$ is a (ς,σ)-Frobenius module and that ${}_AY$ is a finitely generated left A-module. Then, we have the following isomorphism:*

$$\underline{\mathrm{Hom}}_A(Y,M) \cong \mathrm{Tor}_1^A(Tr(Y),M).$$

Proof. From the Auslander–Reiten duality, we have

$$\underline{\mathrm{Hom}}_A(Y,M)^* \cong \mathrm{Ext}_A^1(M, Tr(Y)^*).$$

From Theorem 5,

$$\mathrm{Ext}_A^1(M,Tr(Y)^*) \cong \mathrm{Ext}_{A^\circ}^1(({}^{\sigma^{-1}}(Tr(Y)^*))^*, M).$$

Now, per Lemma 1,

$$({}^{\sigma^{-1}}(Tr(Y)^*))^* \cong (Tr(Y)^{\sigma^{-1}})^*)^* \cong Tr(Y)^{\sigma^{-1}} \cong Tr({}^{\sigma^{-1}}Y);$$

therefore, we have

$$\underline{\mathrm{Hom}}_A(Y,M)^* \cong \mathrm{Ext}_{A^\circ}^1(Tr({}^{\sigma^{-1}}Y),M). \tag{12}$$

Because the functor $(-)^\sigma$ is an auto-equivalence of the Abelian category of right A-modules, we obtain the following isomorphisms:

$$\mathrm{Ext}_{A^\circ}^1(Tr({}^{\sigma^{-1}}Y),M) \cong \mathrm{Ext}_{A^\circ}^1(Tr({}^{\sigma^{-1}}Y)^\sigma, M^\sigma) \cong \mathrm{Ext}_{A^\circ}^1(Tr(Y),M^\sigma). \tag{13}$$

From Theorem 6, we have the isomorphism

$$\mathrm{Ext}_{A^\circ}^1(Tr(Y),M^\sigma) \cong \mathrm{Tor}_1^A(Tr(Y),M)^*. \tag{14}$$

Combining isomorphisms (12)–(14), we obtain

$$\underline{\mathrm{Hom}}_A(Y,M)^* \cong \mathrm{Tor}_1^A(Tr(Y),M)^*.$$

Hence, the result follows. □

Conclusions

In this short note, we have introduced the notion of Frobenius modules over a single algebra, which is a modification of the concept of Frobenius bimodules in the literature. Because a Frobenius module in our sense is not necessary projective as a left module or a right module, it enjoys many nontrivial homological properties. We have proved some symmetric properties of Ext-groups and Tor-groups of Frobenius modules. The following questions deserve further consideration:

(a) Is the differential graded structure on the co-chain complex of a Frobenius module still left–right symmetric?
(b) Does a Frobenius module have certain duality properties between Hochschild cohomology and Hochschild homology, say, Poincare duality?

(c) Does a Frobenius module relate to certain weak versions of Frobenius extensions?

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