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Schatten Index of the Sectorial Operator via the Real Component of Its Inverse

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Abstract: In this paper, we study spectral properties of **non-self-adjoint** operators with the discrete spectrum. The main challenge is to represent a complete description of belonging to the Schatten class through the properties of the Hermitian real component. The method of estimating the singular values is elaborated by virtue of the established asymptotic formulas. The latter fundamental result is advantageous since, of many theoretical statements based upon it, one of them is a concept on the root vectors series expansion, which leads to a wide spectrum of applications in the theory of evolution equations. In this regard, the evolution equations of fractional order with the sectorial operator in the term not containing the time variable are involved. The concrete well-known operators are considered and the advantage of the represented method is convexly shown.

Keywords: strictly accretive operator; Abel–Lidskii basis property; Schatten–von Neumann class; convergence exponent; counting function

MSC: 47B28; 47A10; 47B12; 47B10; 34K30; 58D25

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1. Introduction

Erhard Schmidt, whose advisor had been David Hilbert, studied the integral equations with nonsymmetric kernels and introduced singular values (s-numbers), which afterwards were interpreted by the brilliant Allakhverdiyev theorem as a measure of deviation between a compact operator and finite-dimensional ones. From that time on, singular values have become a most popular tool for studying spectral properties of non-self-adjoint operators. However, although the history could have developed in a different way, the fact is that the eigenvalues of the operator real component are no less suitable for this study. The last idea fully reflects the plot of this paper.

The idea to write this paper originates from the concept of decomposition of an element of the abstract Hilbert space on the root vectors series. The latter concept lies in the framework of abstract functional analysis, and its appearance arises from elaboration of methods of solving evolution equations investigated in the recent century by Lidskii V.B. [1], Markus A.S., Matsaev V.I. [2], Agaranovich M.S. [3], and others. In its simple reduced form, applicably to self-adjoint operators, the concept admits the interpretation through the well-known fact that the eigenvectors of the compact self-adjoint operator form a basis in the closure of its range. The question of what happens in the case when the operator is non-self-adjoint is rather complicated and deserves to be considered as a separate part of the spectral theory.

We should make a brief digression and explain that relevance appears just in the case when a senior term of a considered operator is not self-adjoint, for there is a number of papers [2,4–8] devoted to the perturbed self-adjoint operators. The fact is that most of them deal with a decomposition of the operator on a sum, where the senior term must be either a self-adjoint or normal operator. In other cases, the methods of the papers [9,10] become relevant and allow us to study spectral properties of operators whether we have the abovementioned representation or not; moreover, they have a natural mathematical origin that appears brightly while we are considering abstract constructions expressed in terms of the semigroup theory [10].

Generally, the aims of the mentioned part of the spectral theory are propositions on the convergence of the root vectors series in one or another sense to an element belonging to the closure of the operator range; by this, we mean Bari, Riesz, and Abel–Lidskii senses of the series convergence [11]. The main condition in terms of which the propositions are mostly described is the asyptotics of the operator singular numbers; here, we should note that it is originally formulated in terms of the operator belonging to the Schatten class. However, Agaranovich M.S. made an attempt to express the sufficient conditions of the root vector series basis property, in the abovementioned generalized sense, through the asymptotics of the eigenvalues of the real component [3]. The paper by Markus A.S. and Matsaev V.I. [2] can be also considered within the scope since it establishes the relationship between the asymptotics of the operator eigenvalues absolute value and eigenvalues of the real component.

Thus, the interest in how to express root vectors series decomposition theorems through the asymptotics of the real component eigenvalues arose previously, and the obvious technical advantage in finding the asymptotics creates a prerequisite to investigate the issue properly. We should point out that under the desired relationship between asymptotics, we are able to reformulate theorems on the root vectors series expansion in terms of the assumptions related to the real component of the operator. The latter idea is relevant, since in many cases, the calculation of the real component eigenvalues asymptotics is simpler than direct calculation of the singular numbers' asymptotics.

If we make a comparison analysis between the methods of root vectors decomposition by Lidskii V.B. [1] and Agaranovich M.S. [3], we will see that the first one formulated the conditions in terms of the singular values but the second one did so in terms of the real component eigenvalues. In this regard, we will show that the real component eigenvalue asymptotics are stronger than that of the singular numbers; however, Agaranovich M.S. [3] imposed the additional condition—the spectrum belongs to the domain of the parabolic type. From this point of view, the results by Lidskii V.B. [1] are more advantageous since the convergence in the Abel–Lidskii sense was established for an operator class wider than the class of sectorial operators. Apparently, a reasonable question that may appear is about minimal conditions that guarantee the desired result, which, in particular, is considered in this paper.

Here, we can obviously extend the results devoted to operators with the discrete spectrum to operators with the compact resolvent, for they can be easily reformulated from one realm to another. In this regard, we should give warning that the latter fact does not hold for real components since the real component of the inverse operator does not coincide with the inverse of the operator real component. However, such a complication was diminished due to the results of [9], where the asymptotic equivalence between the eigenvalues of the mentioned operators was established.

The following are a couple of words on the applied relevance of the issue. The abstract approach to the Cauchy problem for the fractional evolution equation is a classic one [12,13]. In its framework, the application of results connected with the basis property covers many problems in the theory of evolution equations [1,10,14–16]. In its general statement, the problem appeals to many applied ones, and we can produce a number of papers dealing with differential equations which can be studied by the abstract methods [17–22]. Apparently, the main advantage of this paper is a method that enables the implementation of the existence and uniqueness theorem abstract condition verification for concrete evolution equations. The latter concept may be interesting for the reader, for it allows broadening of the condition under which the Abel–Lidskii method works, which, in turn, gives a wide spectrum of applications in the theory of differential equations. Thus, we can claim that the offered approach is undoubtedly novel from the abstract theory point of view, and is relevant from the applied one.

2. Preliminaries

Let $C, C_i,\ i \in \mathbb{N}_0$ be real constants. We assume that a value of C is positive and can be different in various formulas, but values of C_i are certain. Denote by $\operatorname{int} M,\ \operatorname{Fr} M$ the interior and the set of boundary points of the set M, respectively. Everywhere further, if the contrary is not stated, we consider linear densely defined operators acting on a separable complex Hilbert space $\mathfrak{H}$. Denote by $\mathcal{B}(\mathfrak{H})$ the set of linear bounded operators on $\mathfrak{H}$. Denote by $\tilde{L}$ the closure of an operator L. We establish the following agreement on using symbols $\tilde{L}^i := (\tilde{L})^i$, where i is an arbitrary symbol. Denote by $\mathrm{D}(L)$, $\mathrm{R}(L)$, $\mathrm{N}(L)$ the *domain of definition* , the *range*, and the *kernel*, or *null space*, of an operator L, respectively. The deficiency (codimension) of $\mathrm{R}(L)$, dimension of $\mathrm{N}(L)$ are denoted by $\operatorname{def} L,\ \operatorname{nul} L$, respectively. In some places, if it is necessary from the stylistic point of view, we use the following notation: $L^{-1} := I/L$. Assume that L is a closed operator acting on $\mathfrak{H}$, $\mathrm{N}(L) = 0$,; let us define a Hilbert space $\mathfrak{H}_L := \big\{f, g \in \mathrm{D}(L),\ (f,g)_{\mathfrak{H}_L} = (Lf, Lg)_{\mathfrak{H}}\big\}$. Considering a pair of complex Hilbert spaces $\mathfrak{H}, \mathfrak{H}_+$, the notation $\mathfrak{H}_+ \subset\subset \mathfrak{H}$ means that $\mathfrak{H}_+$ is dense in $\mathfrak{H}$ as a set of elements and we have a bounded embedding provided by the inequality

$$\|f\|_{\mathfrak{H}} \le C_0 \|f\|_{\mathfrak{H}_+},\ C_0 > 0,\ f \in \mathfrak{H}_+;$$

moreover, any bounded set with respect to the norm $\mathfrak{H}_+$ is compact with respect to the norm $\mathfrak{H}$. Let L be a closed operator for any closable operator S such that $\tilde{S} = L$, its domain $\mathrm{D}(S)$ will be called a core of L. Denote by $\mathrm{D}_0(L)$ a core of a closeable operator L. Let $\mathrm{P}(L)$ be the resolvent set of an operator L and $R_L(\zeta),\ \zeta \in \mathrm{P}(L),\ [R_L := R_L(0)]$ denotes the resolvent of an operator L. Denote by $\lambda_i(L),\ i \in \mathbb{N}$ the eigenvalues of an operator L, we numerate them in order of increasing (decreasing) of their absolute values. Suppose L is a compact operator and $N := (L^*L)^{1/2},\ r(N) := \dim \mathrm{R}(N)$; then the eigenvalues of the operator N are called the *singular values* (*s-numbers*) of the operator L and are denoted by $s_i(L),\ i = 1, 2, \ldots, r(N)$. If $r(N) < \infty$, then we use by definition $s_i = 0,\ i = r(N) + 1, 2, \ldots$. Let $\mathfrak{S}_p(\mathfrak{H}),\ 0 < p < \infty$ be the Schatten–von Neumann class (Schatten class) and $\mathfrak{S}_\infty(\mathfrak{H})$ be the set of compact operators, by definition use

$$\mathfrak{S}_p(\mathfrak{H}) := \left\{L : \mathfrak{H} \to \mathfrak{H},\ \sum_{n=1}^{\infty} s_n^p(L) < \infty,\ 0 < p < \infty\right\}.$$

According to the terminology of the monograph [11], the dimension of the root vectors subspace corresponding to a certain eigenvalue λ_k is called the *algebraic multiplicity* of the eigenvalue λ_k. Let $\nu(L)$ denote the sum of all algebraic multiplicities of an operator L. Denote by $n(r)$ a function equal to a number of the elements of the sequence $\{a_n\}_1^\infty,\ |a_n| \uparrow \infty$ within the circle $|z| < r$. Let A be a compact operator, denoted by $n_A(r)$ *counting function* a function $n(r)$ corresponding to the sequence $\{s_i^{-1}(A)\}_1^\infty$. Let $\mathfrak{S}_p(\mathfrak{H}),\ 0 < p < \infty$ be a Schatten–von Neumann class and $\mathfrak{S}_\infty(\mathfrak{H})$ be the set of compact operators. Suppose L is an operator with a compact resolvent and $s_n(R_L) \le C\, n^{-\mu},\ n \in \mathbb{N},\ 0 \le \mu < \infty$; then we denote by $\mu(L)$ order of the operator L in accordance with the definition given in the paper [8]. Denote by $\mathfrak{Re} L := (L + L^*)/2,\ \mathfrak{Im} L := (L - L^*)/2i$ the real and imaginary components of an operator L, respectively. In accordance with the terminology of the monograph [23], the set $\Theta(L) := \{z \in \mathbb{C} : z = (Lf, f)_{\mathfrak{H}},\ f \in \mathrm{D}(L),\ \|f\|_{\mathfrak{H}} = 1\}$ is called the *numerical range* of an operator L. An operator L is called *sectorial* if its numerical range belongs to a closed sector $\mathfrak{L}_\iota(\theta) := \{\zeta : |\arg(\zeta - \iota)| \le \theta < \pi/2\}$, where ι is the vertex and θ is the semiangle of the sector $\mathfrak{L}_\iota(\theta)$. If we want to stress the correspondence between ι and θ, then we will write θ_ι. An operator L is called *bounded from below* if the following relation holds: $\mathrm{Re}(Lf, f)_{\mathfrak{H}} \ge \gamma_L \|f\|^2_{\mathfrak{H}},\ f \in \mathrm{D}(L),\ \gamma_L \in \mathbb{R}$, where γ_L is called a lower bound of L. An operator L is called *accretive* if $\gamma_L = 0$. An operator L is called *strictly accretive* if $\gamma_L > 0$. An operator L is called *m-accretive* if the following relation holds: $(A + \zeta)^{-1} \in \mathcal{B}(\mathfrak{H}),\ \|(A + \zeta)^{-1}\| \le (\mathrm{Re}\zeta)^{-1},\ \mathrm{Re}\zeta > 0$. An operator L is called *m-sectorial* if L is sectorial and $L + \beta$ is m-accretive for some constant β. An operator L is called *symmetric* if one is densely defined and the following equality holds: $(Lf, g)_{\mathfrak{H}} = (f, Lg)_{\mathfrak{H}}$, $f, g \in \mathrm{D}(L)$. Let B be a bounded operator acting in $\mathfrak{H}$, and assume that $\{\varphi_n\}_1^\infty$, $\{\psi_n\}_1^\infty$ are a pair of orthonormal bases in $\mathfrak{H}$. Define the *absolute operator norm* as follows:

$$\|B\|_2 := \left(\sum_{n,k=1}^{\infty} |(B\varphi_n, \psi_k)_{\mathfrak{H}}|^2 \right)^{1/2} < \infty.$$

Everywhere further, unless otherwise stated, we use notations of the papers [11,23–26].

2.1. Sectorial Sesquilinear Forms and the Hermitian Components

Consider the Hermitian components of an operator (not necessarily bounded):

$$\mathfrak{Re} L := \frac{L + L^*}{2}, \;\; \mathfrak{Im} L := \frac{L - L^*}{2i},$$

where it is clear that in the case when the operator L is unbounded but densely defined we need agreement between the domain of definition of the operator and its adjoint, since in other cases, the real component may be not densely defined. However, the latter claim requires concrete examples; in this regard, we can refer to Remark 4 [10].

Consider a sesquilinear form $t[\cdot,\cdot]$ (see [23]) defined on a linear manifold of the Hilbert space $\mathfrak{H}$. Denote by $t[\cdot]$ the quadratic form corresponding to the sesquilinear form $t[\cdot,\cdot]$. Let

$$\mathfrak{h} = (t + t^*)/2, \, \mathfrak{k} = (t - t^*)/2i$$

be a real and imaginary component of the form t, respectively, where $t^*[u, v] = t\overline{[v, u]}$, $\mathrm{D}(t^*) = \mathrm{D}(t)$. In accordance with the definitions, we have $\mathfrak{h}[\cdot] = \mathrm{Re}\, t[\cdot]$, $\mathfrak{k}[\cdot] = \mathrm{Im}\, t[\cdot]$. Denote by $\tilde{t}$ the closure of a form t. The range of a quadratic form $t[f]$, $f \in \mathrm{D}(t)$, $\|f\|_{\mathfrak{H}} = 1$ is called *range* of the sesquilinear form t and is denoted by $\Theta(t)$. A form t is called *sectorial* if its range belongs to a sector having a vertex ι situated at the real axis and a semiangle $0 \leq \theta_\iota < \pi/2$. Suppose t is a closed sectorial form; then a linear manifold $\mathrm{D}_0(t) \subset \mathrm{D}(t)$ is called the *core* of t, if the restriction of t to $\mathrm{D}_0(t)$ has the closure t (see [23], p. 166).

Suppose L is a sectorial densely defined operator and $t[u, v] := (Lu, v)_{\mathfrak{H}}$, $\mathrm{D}(t) = \mathrm{D}(L)$; then due to Theorem 1.27 ([23], p. 318), the corresponding form t is closable, and due to Theorem 2.7 ([23], p. 323), there exists a unique m-sectorial operator $T_{\tilde{t}}$ associated with the form $\tilde{t}$. In accordance with the definition ([23], p. 325), the operator $T_{\tilde{t}}$ is called a *Friedrichs extension* of the operator L.

Due to Theorem 2.7 ([23], p. 323), there exist unique m-sectorial operators $T_t, T_{\mathfrak{h}}$ associated with the closed sectorial forms $t, \mathfrak{h}$, respectively. The operator $T_{\mathfrak{h}}$ is called a *real part* of the operator T_t and is denoted in accordance with the original definition [23] by $\mathrm{Re}\, T_t$.

Here, we should stress that the construction of the real part in some cases is obviously coincident with that of the real component; however, the latter does not require the agreement between the domain of definitions mentioned above. The condition represented below reflects the nature of uniformly elliptic operators being the direct generalization of the one considered in the context of the theory of Sobolev spaces.

H1: *There exists a Hilbert space $\mathfrak{H}_+ \subset\subset \mathfrak{H}$ and a linear manifold $\mathfrak{M}$ that is dense in $\mathfrak{H}_+$. The closed operator W is defined on $\mathfrak{M}$ and the latter set is its core.*

H2: $|(Wf, g)_{\mathfrak{H}}| \leq C_1 \|f\|_{\mathfrak{H}_+} \|g\|_{\mathfrak{H}_+}, \;\mathrm{Re}(Wf, f)_{\mathfrak{H}} \geq C_2 \|f\|^2_{\mathfrak{H}_+}, \; f, g \in \mathfrak{M}, \; C_1, C_2 > 0.$

Consider a condition $\mathfrak{M} \subset \mathrm{D}(W^*)$; in this case, the real Hermitian component $\mathcal{H} := \mathfrak{Re}\, W$ of the operator is defined on $\mathfrak{M}$, and the fact is that $\tilde{\mathcal{H}}$ is self-adjoint, bounded from below (see Lemma 3 [9]). Hence, a corresponding sesquilinear form (denote this form by h) is symmetric and bounded from below also (see Theorem 2.6 [23], p. 323). The conditions H1, H2 allow us to claim that the form t corresponding to the operator W is a closed sectorial form; consider the corresponding form $\mathfrak{h}$. It can be easily shown that $h \subset \mathfrak{h}$, but, using this fact, we cannot claim in general that $\tilde{\mathcal{H}} \subset H$, where $H := \mathrm{Re}W$ (see [23], p. 330). We just have an inclusion $\tilde{\mathcal{H}}^{1/2} \subset H^{1/2}$ (see [23], p. 332). Note that the fact $\tilde{\mathcal{H}} \subset H$ follows from a condition $\mathrm{D}_0(\mathfrak{h}) \subset \mathrm{D}(h)$ (see Corollary 2.4 [23], p. 323). However, it is proved (see proof of Theorem 4 [9]) that relation H2 guarantees that $\tilde{\mathcal{H}} = H$. Note that the last relation is very useful in applications, since in most concrete cases we can find a concrete form of the operator $\mathcal{H}$.

2.2. Previously Obtained Results

Here, we represent previously obtained results that will undergo thorough study since our principal challenge is to obtain an accurate description of the Schatten–von Neumann class index of a non-self-adjoint operator.

Further, we consider Theorem 1 [10] statements separately under assumptions H1, H2. Note that in terms of Theorem 1 [10] the operator W is a closure of the restriction of the operator L on the set $\mathfrak{M}$. Without loss of generality, we can assume that W is closed since the conditions H1, H2 guarantee that it is closeable. Thus, the given above version of the conditions H1, H2 allows us to avoid redundant notations, more detailed information in this regard is given in the paper [10].

We have the following classification in terms of the operator order μ, where it is defined as follows $\lambda_n(R_H) = O(n^{-\mu})$, $n \to \infty$.

(A) The following Schatten classification holds:

$$R_W \in \mathfrak{S}_p,\ \inf p \le 2/\mu,\ \mu \le 1,\ R_W \in \mathfrak{S}_1,\ \mu > 1.$$

Moreover, under assumptions $\lambda_n(R_H) \ge C\, n^{-\mu}$, $0 \le \mu < \infty$, the following implication holds: $R_W \in \mathfrak{S}_p,\ p \in [1, \infty), \Rightarrow \mu > 1/p$.

Observe that the above-given classification is far from the exact description of the Schatten–von Neumann class index p. However, having analyzed the above implications, we can see that it makes a prerequisite to establish a hypotheses $R_W \in \mathfrak{S}_p$, $\inf p = 1/\mu$. The following narrative is devoted to its verification.

Let us thoroughly analyze the technical tools involved in the proof of the statement in order to absorb and contemplate the scheme of reasonings. Consider the statement, if $\mu \le 1$, then $R_W \in \mathfrak{S}_p$, $\inf p \le 2/\mu$. The main result, on which it is based, is the asymptotic equivalence between the inverse of the real component and the real component of the resolvent. Indeed, due to application of some technicalities, we have a relation

$$(|R_W|^2 f, f)_{\mathfrak{H}} = \|R_W f\|^2_{\mathfrak{H}} \le C \cdot \mathrm{Re}(R_W f, f)_{\mathfrak{H}} = C \cdot (\mathfrak{Re} R_W f, f)_{\mathfrak{H}};$$

using the minimax principle, we obtain the s-numbers asymptotics through the asymptotics of the real component eigenvalues.

Consider the statement that if $\lambda_n(R_H) \ge C\, n^{-\mu}$, $0 \le \mu < \infty$, then the following implication holds: $R_W \in \mathfrak{S}_p,\ p \in [1, \infty), \Rightarrow \mu > 1/p$. The main results that guarantee the fulfilment of the latter relation are inequality (7.9) ([11], p. 123), Theorem 3.5 [10], in accordance with which we obtain

$$\sum_{i=1}^{\infty} |s_i(R_W)|^p \ge \sum_{i=1}^{\infty} |(R_W \varphi_i, \varphi_i)_{\mathfrak{H}}|^p \ge \sum_{i=1}^{\infty} |\mathrm{Re}(R_W \varphi_i, \varphi_i)_{\mathfrak{H}}|^p =$$

$$= \sum_{i=1}^{\infty} |(\mathfrak{Re} R_W \varphi_i, \varphi_i)_{\mathfrak{H}}|^p = \sum_{i=1}^{\infty} |\lambda_i(\mathfrak{Re} R_W)|^p \ge C \sum_{i=1}^{\infty} i^{-\mu p},\ p \ge 1.$$

Thus, we see that estimation of the series is involved; in this regard, we will make a more detailed remark further.

Below, we represent the second statement of Theorem 1 [10], where the peculiar result related to the asymptotics of the eigenvalue absolute value is given.

(B) In the case $\nu(R_W) = \infty$, $\mu \neq 0$, the following relation holds:

$$|\lambda_n(R_W)| = o\big(n^{-\tau}\big),\, n \to \infty,\; 0 < \tau < \mu.$$

It is based on the Theorem 6.1 ([11], p. 81), in accordance with which we have

$$\sum_{m=1}^{k} |\mathrm{Im}\,\lambda_m(B)|^p \leq \sum_{m=1}^{k} |\lambda_m(\mathfrak{Im}B)|^p,\; (k = 1, 2, \ldots, \nu_{\mathfrak{I}}(B)),\, 1 \leq p < \infty, \tag{1}$$

where $\nu_{\mathfrak{I}}(B) \leq \infty$ is the sum of all algebraic multiplicities corresponding to the not-real eigenvalues of the bounded operator B, $\mathfrak{Im}B \in \mathfrak{S}_\infty$ (see [11], p. 79).

Note that the statement (B) allows us to arrange brackets in the series that converges in the Abel–Lidskii sense (see [1,14]), which would be an advantageous achievement in the theory constructed further. However, it has a harmonious correspondence with the case where we do not have the exact index of the Schatten class, for in this case, due to the convergence test, we obtain a relation

$$R_W \in \mathfrak{S}_p, \Rightarrow s_n = o(n^{-1/p}),$$

which gives us a relation $|\lambda_n(R_W)| = o\left(n^{-1/p}\right)$ in accordance with the connection of the asymptotics (see Chapter II, §3 [11]). Note that the latter relation does not contradict (B) if we assume $p > 1/\mu$. Thus, along the abovementioned implication $R_W \in \mathfrak{S}_p$, $p \in [1, \infty), \Rightarrow$ $p > 1/\mu$, it makes the prerequisite to observe the hypotheses $\inf p = 1/\mu$.

Apparently, the used technicalities appeal to the so-called nondirect estimates for singular values realized due to estimates of the series. As we will see further, the main advantage of the series estimation is the absence of the conditions imposed on the type of the asymptotics; it may be not one of the power type. However, we will show that under the restriction imposed on the type of the asymptotics, assuming that one is of the power type, we can obtain direct estimates for singular values. In the reminder, let us note that the classes of differential operators have the asymptotics of the power type, which make the issue rather relevant.

3. Main Results

The Main Refinement of the Result A

The reasonings produced below appeal to a compact operator B, which represents a most general case in the framework of the decomposition on the root vectors theory; however, to obtain more peculiar results, we are compelled to deploy some restricting conditions. In this regard, we involve hypotheses H1, H2 if it is necessary. The result represented below gives us the upper estimate for the singular values; it is based on the result by Ky Fan [27], which can be found as a corollary of the well-known Allakhverdiyev theorem (see Corollary 2.2 [11]).

Lemma 1. *Assume that B is a compact sectorial operator with the vertex situated at the point zero, then*

$$s_{2m-1}(B) \leq \sqrt{2}\sec\theta \cdot \lambda_m(\mathfrak{Re}B),\;\; s_{2m}(B) \leq \sqrt{2}\sec\theta \cdot \lambda_m(\mathfrak{Re}B),\;\; m = 1, 2, \ldots.$$

Proof. Consider the Hermitian components

$$\mathfrak{Re}B := \frac{B + B^*}{2},\;\; \mathfrak{Im}B := \frac{B - B^*}{2i},$$

where it is clear that they are compact self-adjoint operators, since B is compact and due to the technicalities of the given algebraic constructions. Note that the following relation can be established by direct calculation:

$$\mathfrak{Re}^2 B + \mathfrak{Im}^2 B = \frac{B^*B + BB^*}{2},$$

from what follows the inequality

$$\frac{1}{2} \cdot B^*B \leq \mathfrak{Re}^2 B + \mathfrak{Im}^2 B. \tag{2}$$

Having analyzed the latter formula, we see that it is rather reasonable to think over the opportunity of applying the corollary of the minimax principle, pursuing the aim to estimate the singular values of the operator B. For this purpose, consider the following relation: $\mathfrak{Re}^2 B\, f_n = \lambda_n^2 f_n$, where f_n, λ_n are the eigenvectors and the eigenvalues of the operator $\mathfrak{Re}B$, respectively. Since the operator $\mathfrak{Re}B$ is self-adjoint and compact, then its set of eigenvalues form a basis in $\overline{\mathrm{R}(\mathfrak{Re}B)}$. Assume that there exists a nonzero eigenvalue of the operator $\mathfrak{Re}^2 B$ that is different from $\{\lambda_n^2\}_1^\infty$, then, in accordance with the well-known fact of the operator theory, the corresponding eigenvector is orthogonal to the eigenvectors of the operator $\mathfrak{Re}B$. Taking into account the fact that the latter form a basis in $\overline{\mathrm{R}(\mathfrak{Re}B)}$, we come to the conclusion that the eigenvector does not belong to $\overline{\mathrm{R}(\mathfrak{Re}B)}$. Thus, the obtained contradiction proves the fact $\lambda_n(\mathfrak{Re}^2 B) = \lambda_n^2(\mathfrak{Re}B)$. Implementing the same reasonings, we obtain $\lambda_n(\mathfrak{Im}^2 B) = \lambda_n^2(\mathfrak{Im}B)$.

Further, we need a result by Ky Fan [27] (see Corollary 2.2) [11] (Chapter II, § 2.3), in accordance with which we have

$$s_{m+n-1}(\mathfrak{Re}^2 B + \mathfrak{Im}^2 B) \leq \lambda_m(\mathfrak{Re}^2 B) + \lambda_n(\mathfrak{Im}^2 B), \quad m, n = 1, 2, \ldots .$$

Choosing $n = m$ and $n = m + 1$, we obtain, respectively,

$$s_{2m-1}(\mathfrak{Re}^2 B + \mathfrak{Im}^2 B) \leq \lambda_m(\mathfrak{Re}^2 B) + \lambda_m(\mathfrak{Im}^2 B),$$

$$s_{2m}(\mathfrak{Re}^2 B + \mathfrak{Im}^2 B) \leq \lambda_m(\mathfrak{Re}^2 B) + \lambda_{m+1}(\mathfrak{Im}^2 B) \quad m = 1, 2, \ldots .$$

At this stage of reasoning we need involve the sectorial property $\Theta(B) \subset \mathfrak{L}_0(\theta)$, which gives us $|\mathrm{Im}(Bf, f)| \leq \tan\theta \, \mathrm{Re}(Bf, f)$. Applying the corollary of the minimax principle to the latter relation, we obtain $|\lambda_n(\mathfrak{Im}B)| \leq \tan\theta \, \lambda_n(\mathfrak{Re}B)$. Therefore,

$$s_{2m-1}(\mathfrak{Re}^2 B + \mathfrak{Im}^2 B) \leq \lambda_m(\mathfrak{Re}^2 B) + \lambda_m(\mathfrak{Im}^2 B) \leq \sec^2\theta \cdot \lambda_m^2(\mathfrak{Re}B),$$

$$s_{2m}(\mathfrak{Re}^2 B + \mathfrak{Im}^2 B) \leq \sec^2\theta \cdot \lambda_m^2(\mathfrak{Re}B) \quad m = 1, 2, \ldots .$$

Applying the minimax principle to formula (2), we obtain

$$s_{2m-1}(B) \leq \sqrt{2} \sec\theta \cdot \lambda_m(\mathfrak{Re}B), \quad s_{2m}(B) \leq \sqrt{2} \sec\theta \cdot \lambda_m(\mathfrak{Re}B), \quad m = 1, 2, \ldots .$$

This gives us the upper estimate for the singular values of the operator B. □

However, to obtain the lower estimate, we need involve Lemma 3.1 ([23], p. 336), Theorem 3.2 ([23], p. 337). Consider an unbounded operator T, $\Theta(T) \subset \mathfrak{L}_0(\theta)$; in accordance with the first representation theorem ([23], p. 322), we can consider its Friedrichs extension—the m-sectorial operator W, in turn, due to the results ([23], p. 337), it has a real part H which coincides with the Hermitian real component if we deal with a bounded operator. Note that by virtue of the sectorial property, the operator H is non-negative. Further, we consider the case $\mathrm{N}(H) = 0$; it follows that $\mathrm{N}(H^{\frac{1}{2}}) = 0$. To prove this fact we should note that $\mathrm{def} H = 0$; considering inner product with the element belonging to $\mathrm{N}(H^{\frac{1}{2}})$, we easily obtain the fact that it must equal zero. Having analyzed the proof of

Theorem 3.2 ([23], p. 337) we see that its statement remains true in the modified form even in the case where we lift the m-accretive condition; thus, under the sectorial condition imposed upon the closed densely defined operator T, we obtain the following inclusion:

$$T \subset H^{1/2}(I + iG)H^{1/2},$$

where the symbol G denotes a bounded self-adjoint operator in $\mathfrak{H}$. However, to obtain the asymptotic formula established in Theorem 5 [9], we cannot be satisfied by the made assumptions but require the existence of the resolvent at the point zero and its compactness. In spite of the fact that we can proceed our narrative under the weakened conditions regarding the operator W in comparison with H1, H2, we can claim that the statement of Theorem 5 [9] remains true under the assumptions made above, and we prefer to deploy H1, H2, which guarantees the conditions we need and at the same time provides a description of the issue under the natural point of view.

Lemma 2. *Assume that the conditions H1, H2 hold for the operator W, moreover,*

$$\|\mathfrak{Im} W/\mathfrak{Re} W\|_2 < 1,$$

then

$$\lambda^{-1}_{2n}(\mathfrak{Re} W) \leq C s_n(R_W),\ n \in \mathbb{N}.$$

Proof. Firstly, let us show that $\mathrm{D}(W^2)$ is a dense set in $\mathfrak{H}_+$. Since the operator W is closed and strictly accretive, then in accordance with Theorem 3.2 ([23], p. 268), we have $\mathrm{R}(W) = \mathfrak{H}$; hence, there exists the preimage of the set $\mathfrak{M}$—let us denote it by $\mathfrak{M}'$. Consider an arbitrary set of elements $\{x_n\}_0^\infty \subset \mathfrak{H}$ and denote their preimages by x_n'. Using the strictly accretive property of the operator, we have

$$\|x_0 - x_n\|_{\mathfrak{H}} = \|W(x_0' - x_n')\|_{\mathfrak{H}} \geq C\|x_0' - x_n'\|_{\mathfrak{H}_+}.$$

Choosing a sequence

$$\{x_n\}_1^\infty \subset \mathfrak{M},\ x_n \stackrel{\mathfrak{H}}{\to} x_0,$$

we obtain the fact that the set $\mathfrak{M}'$ is dense in $\mathrm{D}(W)$ in the sense of the norm $\mathfrak{H}_+$; hence, it is dense in $\mathfrak{H}_+$ since $\mathfrak{M} \subset \mathrm{D}(W)$ is dense in $\mathfrak{H}_+$ in accordance with condition H1. Therefore, the set $\mathrm{D}(W^2)$ is dense in $\mathfrak{H}_+$ since $\mathfrak{M}' \subset \mathrm{D}(W^2)$. Thus, we have proved the fulfilment of condition H1 for the operator W^2 with respect to the same pair of Hilbert spaces.

Note that under the assumptions H1, H2, using the reasonings of Theorem 3.2 ([23], p. 337), we have the following representation

$$W = H^{1/2}(I + iG)H^{1/2},\ W^* = H^{1/2}(I - iG)H^{1/2}.$$

It follows easily from this formula that the Hermitian components of the operator W are defined, and we have $\mathfrak{Re} W = H,\ \ \mathfrak{Im} W = H^{1/2}GH^{1/2}$. Using the decomposition $W = \mathfrak{Re} W + i\mathfrak{Im} W,\ W^* = \mathfrak{Re} W - i\mathfrak{Im} W$, we easily obtain

$$\left(\frac{W^2 + W^{*2}}{2} f, f\right)_{\mathfrak{H}} = \|\mathfrak{Re} W f\|^2_{\mathfrak{H}} - \|\mathfrak{Im} W f\|^2_{\mathfrak{H}};$$

$$\left(\frac{W^2 - W^{*2}}{2i} f, f\right)_{\mathfrak{H}} = (\mathfrak{Im} W\, \mathfrak{Re} W f, f)_{\mathfrak{H}} + (\mathfrak{Re} W\, \mathfrak{Im} W f, f)_{\mathfrak{H}},\ f \in \mathrm{D}(W^2).$$

Using simple reasonings, we can rewrite the above formulas in terms of Theorem 3.2 ([23], p. 337); we have

$$\mathrm{Re}(W^2 f, f)_{\mathfrak{H}} = \|Hf\|^2_{\mathfrak{H}} - \|H^{1/2}GH^{1/2}f\|^2_{\mathfrak{H}},\ \mathrm{Im}(W^2 f, f)_{\mathfrak{H}} = \mathrm{Re}(H^{1/2}GH^{1/2}f, Hf)_{\mathfrak{H}},$$

$$f \in \mathrm{D}(W^2). \tag{3}$$

Consider a set of eigenvalues $\{\lambda_n\}_1^\infty$ and a complete system of orthonormal vectors $\{e_n\}_1^\infty$ of the operator H, the conditions H1, H2 guarantee the existence of the system $\{e_n\}_1^\infty$ since R_H is compact (see Theorem 3 [10]); using the matrix form of the operator G, we have

$$\|Hf\|_{\mathfrak{H}}^2 = \sum_{n=1}^{\infty} |\lambda_n|^2 |f_n|^2, \; \|H^{1/2}GH^{1/2}f\|_{\mathfrak{H}}^2 = \sum_{n=1}^{\infty} \lambda_n \left| \sum_{k=1}^{\infty} b_{nk}\sqrt{\lambda_k} f_k \right|^2,$$

$$\mathrm{Re}(H^{1/2}GH^{1/2}f, Hf)_{\mathfrak{H}} = \mathrm{Re}\left(\sum_{n=1}^{\infty} \lambda_n^{3/2} f_n \sum_{k=1}^{\infty} b_{nk}\sqrt{\lambda_k} \bar{f}_k \right),$$

where b_{nk} are the matrix coefficients of the operator G. Applying the Cauchy–Schwartz inequality, we obtain

$$\|H^{1/2}GH^{1/2}f\|_{\mathfrak{H}}^2 \leq \sum_{n=1}^{\infty} \lambda_n \left| \sum_{k=1}^{\infty} |\lambda_k f_k|^2 \sum_{k=1}^{\infty} |b_{nk}|^2/\lambda_k \right| \leq \|Hf\|_{\mathfrak{H}}^2 \sum_{n,k=1}^{\infty} |b_{nk}|^2 \lambda_n/\lambda_k;$$

$$|\mathrm{Re}(H^{1/2}GH^{1/2}f, Hf)_{\mathfrak{H}}| \leq \|Hf\|_{\mathfrak{H}} \left(\sum_{n=1}^{\infty} \left| \sum_{k=1}^{\infty} \bar{b}_{nk}\sqrt{\lambda_n\lambda_k} f_k \right|^2 \right)^{1/2} \leq \|Hf\|_{\mathfrak{H}}^2 \left(\sum_{n,k=1}^{\infty} |b_{nk}|^2 \lambda_n/\lambda_k \right)^{1/2}.$$

In accordance with the definition of the sectorial property, we require

$$|\mathrm{Im}(W^2 f, f)_{\mathfrak{H}}| \leq \tan\theta \cdot \mathrm{Re}(W^2 f, f)_{\mathfrak{H}}, \; 0 < \theta < \pi/2.$$

Therefore, the sufficient conditions of the sectorial property can be expressed as follows:

$$\|Hf\|_{\mathfrak{H}}^2 \left(\sum_{n,k=1}^{\infty} |b_{nk}|^2/\lambda_k \right)^{1/2} \leq \|Hf\|_{\mathfrak{H}}^2 \left(1 - \sum_{n,k=1}^{\infty} |b_{nk}|^2 \lambda_n/\lambda_k \right) \tan\theta;$$

$$\sum_{n,k=1}^{\infty} |b_{nk}|^2 \lambda_n/\lambda_k + \cot\theta \left(\sum_{n,k=1}^{\infty} |b_{nk}|^2 \lambda_n/\lambda_k \right)^{1/2} \leq 1,$$

where θ is the semiangle of the supposed sector. Solving the corresponding quadratic equation, we obtain the desired estimate:

$$\left(\sum_{n,k=1}^{\infty} |b_{nk}|^2 \lambda_n/\lambda_k \right)^{1/2} < \frac{1}{2}\left\{ \sqrt{\cot^2\theta + 4} - \cot\theta \right\}. \tag{4}$$

Having noticed the fact that the right-hand side of (4) tends to one from below when θ tends to $\pi/2$, we obtain the condition of the sectorial property expressed in terms of the absolute norm:

$$\|H^{1/2}GH^{-1/2}\|_2 := \left(\sum_{n,k=1}^{\infty} |b_{nk}|^2 \lambda_n/\lambda_k \right)^{1/2} < 1, \tag{5}$$

in this case, we can choose the semiangle of the sector using the following relation:

$$\tan\theta = \frac{N}{1-N^2} + \varepsilon, \; N := \|H^{1/2}GH^{-1/2}\|_2,$$

where ε is an arbitrary small positive number. Thus, we can assume that if the value of the absolute norm is less than one, then the operator W^2 is sectorial and the value of the absolute norm defines the semiangle. Note that coefficients $b_{nk}\sqrt{\lambda_n/\lambda_k},\ \overline{b_{kn}}\sqrt{\lambda_n/\lambda_k}$ correspond to the matrices of the operators, respectively,

$$H^{1/2}GH^{-1/2}f=\sum_{n=1}^{\infty}\lambda_n^{1/2}e_n\sum_{k=1}^{\infty}b_{nk}\lambda_k^{-1/2}f_k,\ H^{-1/2}GH^{1/2}f=\sum_{n=1}^{\infty}\lambda_n^{-1/2}e_n\sum_{k=1}^{\infty}b_{nk}\lambda_k^{1/2}f_k.$$

Thus, if the absolute operator norm exists, i.e.,

$$\|H^{1/2}GH^{-1/2}\|_2<\infty,$$

then both of them belong to the so-called Hilbert–Schmidt class; however, it is clear without involving the absolute norm since the above operators are adjoint. It is remarkable that we can formally write the obtained estimate in terms of the Hermitian components of the operator, i.e.,

$$\|\mathfrak{Im}W/\mathfrak{Re}W\|_2<1.$$

Below, for a convenient form of writing, we will use a short-hand notation $A:=R_W$, where it is necessary. The next step is to establish the asymptotic formula

$$\lambda_n\left(\frac{A^2+A^{2*}}{2}\right)\asymp\lambda_n^{-1}\left(\mathfrak{Re}W^2\right),\ n\to\infty. \tag{6}$$

However, we cannot directly apply Theorem 5 [9] to the operator W^2; thus, we are compelled to modify the proof having taken into account weaker conditions and the additional condition (5).

Let us observe that the compactness of the operator $R_W(\lambda),\ \lambda\in\mathrm{P}(W)$ gives us the compactness of the operator W^{-2}. Since the latter is sectorial, it follows easily that $R_{W^2}(\lambda),\ \lambda\in\mathrm{P}(W^2)$ is compact, since the outside of the sector belongs to the resolvent set and the resolvent compact, at least at one point, is compact everywhere on the resolvent set. Note that due to the reasonings given above, the following relation holds:

$$\mathrm{Re}(W^2f,f)_{\mathfrak{H}}\geq C\|Hf\|^2_{\mathfrak{H}}\geq C\|f\|^2_{\mathfrak{H}_+},\ f\in\mathrm{D}(W^2), \tag{7}$$

where the latter inequality can be obtained easily (see (28) [9]). Thus, we obtain the fact that the operator W^2 is a sectorial, strictly accretive operator; hence, it falls in the scope of the first representation theorem (see Theorem 2.1 [23], p. 322) in accordance with which there exists one-to-one correspondence between the closed densely defined sectorial forms and m-sectorial operators. Using this fact, we can claim that the real part $H_1:=\mathrm{Re}W^2$ is defined and the following relations hold in accordance with the second representation theorem, i.e., Theorem 3.2 ([23], p. 337).

$$W^2=H_1^{1/2}(I+iG_1)H_1^{1/2},\ W^{2*}=H_1^{1/2}(I+iG_2)H_1^{1/2},$$

where G_1,G_2 are self-adjoint bounded operators. Now, by direct calculation, we can verify that $H_1=\mathfrak{Re}W^2$, and we should also note that $\mathrm{D}(W^2)$ is a core of the corresponding closed densely defined sectorial form $\mathfrak{h}$ placed in correspondence to the operator H_1 by virtue of the first representation theorem, i.e., $\mathrm{D}_0(\mathfrak{h})=\mathrm{D}(W^2)$. Let us show that $G_1=-G_2$. We have

$$H_1f=\frac{1}{2}\left[H_1^{\frac{1}{2}}(I+iG_1)+H_1^{\frac{1}{2}}(I+iG_2)\right]H_1^{\frac{1}{2}}=$$

$$=H_1f+\frac{i}{2}H_1^{\frac{1}{2}}(G_1+G_2)H_1^{\frac{1}{2}}f,\ f\in\mathfrak{M}'.$$

By virtue of inequality (7), we see that the operator H_1 is strictly accretive, therefore $\mathrm{N}(H_1) = 0;\ (G_1 + G_2)H_1^{1/2} = 0$. Since

$$\mathfrak{H} = \overline{\mathrm{R}(H_1^{1/2})} \oplus \mathrm{N}(H_1^{1/2}),$$

then $G_1 = G_2 =: G'$. Applying the reasonings represented in Theorem 5 [9], we obtain the fact that $H_1^{-1/2}$ is a bounded operator defined on $\mathfrak{H}$. Using the properties of the operator G', we obtain $\|(I + iG')f\|_{\mathfrak{H}} \cdot \|f\|_{\mathfrak{H}} \geq \mathrm{Re}([I + iG']f, f)_{\mathfrak{H}} = \|f\|^2_{\mathfrak{H}},\ f \in \mathfrak{H}$. Hence, $\|(I + iG')f\|_{\mathfrak{H}} \geq \|f\|_{\mathfrak{H}},\ f \in \mathfrak{H}$. It implies that the operator $I + iG'$ is invertible. The reasonings corresponding to the operator $I - iG'$ are absolutely analogous. Therefore,

$$A^2 = H_1^{-\frac{1}{2}}(I + iG')^{-1}H_1^{-\frac{1}{2}},\ A^{2*} = H_1^{-\frac{1}{2}}(I - iG')^{-1}H_1^{-\frac{1}{2}}. \tag{8}$$

Using simple calculation based upon the operator properties established above, we obtain

$$\mathfrak{Re}A^2 = \frac{1}{2}\,H_1^{-\frac{1}{2}}(I + G'^2)^{-1}H_1^{-\frac{1}{2}}. \tag{9}$$

Therefore,

$$\left(\mathfrak{Re}A^2 f, f\right)_{\mathfrak{H}} = \left(H_1^{-\frac{1}{2}}(I + G'^2)^{-1}H_1^{-\frac{1}{2}}f, f\right)_{\mathfrak{H}} \leq \|(I + G'^2)^{-1}\| \cdot (R_{H_1}f, f)_{\mathfrak{H}},\ f \in \mathfrak{H}.$$

On the other hand, it is easy to see that $((I + G'^2)^{-1}f, f)_{\mathfrak{H}} \geq \|(I + G'^2)^{-1}f\|^2_{\mathfrak{H}}$. At the same time, it is obvious that the operator $I + G'^2$ is bounded and we have $\|(I + G'^2)^{-1}f\|_{\mathfrak{H}} \geq \|I + G'^2\|^{-1}\|f\|_{\mathfrak{H}}$. Applying these estimates, we obtain

$$\left(\mathfrak{Re}A^2 f, f\right)_{\mathfrak{H}} = \left((I + G'^2)^{-1}H_1^{-\frac{1}{2}}f, H_1^{-\frac{1}{2}}f\right)_{\mathfrak{H}} \geq \|(I + G'^2)^{-1}H_1^{-\frac{1}{2}}f\|^2_{\mathfrak{H}} \geq$$

$$\geq \|I + G'^2\|^{-2} \cdot (R_{H_1}f, f)_{\mathfrak{H}},\ f \in \mathfrak{H}.$$

Using relation (7), we obtain the fact that the resolvent R_{H_1} is compact, and the fact that $\mathfrak{Re}A^2$ is compact is obvious. Thus, analogously to the reasonings of Theorem 5 [9], applying the minimax principle, we obtain the desired asymptotic formula (6). Further, we will use the following formula obtained due to the positiveness of the squared Hermitian imaginary component of the operator A, and we have

$$\frac{A^2 + A^{2*}}{2} = \frac{A^2 + A^{*2}}{2} \leq A^*A + AA^*.$$

Applying the corollary of the well-known Allakhverdiyev theorem (Ky Fan [27]), see Corollary 2.2 [11] (Chapter II, § 2.3), we have

$$\lambda_{2n}(A^*A + AA^*) \leq \lambda_n(A^*A) + \lambda_n(AA^*),\ n \in \mathbb{N}.$$

Taking into account the fact $s_n(A) = s_n(A^*)$, using the minimax principle, we obtain the estimate

$$s_n^2(A) \geq C\lambda_{2n}\left(\frac{A + A^{2*}}{2}\right),\ n \in \mathbb{N},$$

and applying (6), we obtain

$$s_n^2(A) \geq C\lambda_{2n}^{-1}\left(\mathfrak{Re}W^2\right),\ n \in \mathbb{N}.$$

Here, it is rather reasonable to apply formula (3), which gives us

$$\|f\|^2_{\mathfrak{H}} \leq \|f\|^2_{\mathfrak{H}_+} \leq \left(\mathfrak{Re}W^2 f, f\right)_{\mathfrak{H}} \leq (Hf, Hf)_{\mathfrak{H}},\ f \in \mathrm{D}(W^2),$$

which, in turn, collaboratively with the minimax principle, leads us to the theorem statement. Here, we should remark that in order to apply the minimax principle, we need a compact embedding of the energetic space, which is provided by the estimate from below. □

Remark 1. *It is remarkable that the central point of the proof is the representation theorems; in accordance with the first one, we have a plain construction of the operator real part equaling the Hermitian real component. These allow us to implement the simplified scheme of reasonings represented in* [9].

Consider a rather wide operator class including the operators having the asymptotics of the resolvent singular values or one of the real component eigenvalues of the power type, i.e.,

$$C_1 n^{\mu} \le \lambda_n \le C_2 n^{\mu},\ \mu < 0.$$

In order to apply the obtained theoretical results to the class, we can reformulate them in the following stylistically convenient form.

Theorem 1. *Assume that the hypotheses H1, H2 hold for the operator W, moreover,*

$$\| \mathfrak{Im} W / \mathfrak{Re} W \|_2 < 1,$$

then

$$s_n(R_W) \asymp \lambda_n^{-1}(\mathfrak{Re} W).$$

Proof. Since conditions H1, H2 hold, then the resolvent R_W is a compact sectorial operator with the vertex situated at the point zero (see Theorem 3 [10]). The estimates from the above and below for the singular values follow from the application of Lemmas 1 and 2, respectively; here, we should take into account the fact that $(Cn)^{\gamma} \asymp n^{\gamma},\ \gamma \in \mathbb{R}$ and the fact that $\lambda_n(\mathfrak{Re} R_W) \asymp \lambda_n^{-1}(\mathfrak{Re} W)$, which is the claim of Theorem 5 [9]. □

4. Mathematical Applications

4.1. The Low Bound for the Schatten Index of the Perturbed Differential Operator

1. Trying to show an application of Lemma 1, we produce an example of a non-self-adjoint operator that is not completely subordinated in the sense of forms (see [8,9]). The pointed-out fact means that we cannot deal with the operator applying methods [8] for they do not work.

Consider a differential operator acting in the complex Sobolev space:

$$\mathcal{L}f := (c_k f^{(k)})^{(k)} + (c_{k-1} f^{(k-1)})^{(k-1)} + \ldots + c_0 f,$$

$$\mathrm{D}(\mathcal{L}) = H^{2k}(I) \cap H_0^k(I),\ k \in \mathbb{N},$$

where $I := (a,b) \subset \mathbb{R}$, and the complex-valued coefficients $c_j(x) \in C^{(j)}(\bar{I})$ satisfy the condition $\operatorname{sign}(\operatorname{Re} c_j) = (-1)^j,\ j = 1, 2, \ldots, k$. Consider a linear combination of the Riemann–Liouville fractional differential operators (see [26], p .44) with the constant real-valued coefficients:

$$\mathcal{D}f := p_n D_{a+}^{\alpha_n} + q_n D_{b-}^{\beta_n} + p_{n-1} D_{a+}^{\alpha_{n-1}} + q_{n-1} D_{b-}^{\beta_{n-1}} + \ldots + p_0 D_{a+}^{\alpha_0} + q_0 D_{b-}^{\beta_0},$$

$$\mathrm{D}(\mathcal{D}) = H^{2k}(I) \cap H_0^k(I),\ n \in \mathbb{N},$$

where $\alpha_j, \beta_j \geq 0,\ 0 \leq [\alpha_j], [\beta_j] < k,\ j = 0, 1, \ldots, n.,$

$$q_j \geq 0,\ \mathrm{sign}\, p_j = \begin{cases} (-1)^{\frac{[\alpha_j]+1}{2}},\ [\alpha_j] = 2m-1,\ m \in \mathbb{N}, \\ (-1)^{\frac{[\alpha_j]}{2}},\ [\alpha_j] = 2m,\ m \in \mathbb{N}_0. \end{cases}$$

The following result is represented in the paper [9]; consider the operator

$$G = \mathcal{L} + \mathcal{D},$$

$$\mathrm{D}(G) = H^{2k}(I) \cap H_0^k(I).$$

It is clear that it is an operator with a compact resolvent; however, for the accuracy we will prove this fact. Moreover, we will produce a pair of Hilbert spaces so that conditions H1, H2 hold. It follows that the resolvent is compact; thus, we are able to observe the problem related to calculating the Schatten index. Apparently, it may happen that the direct calculation of the singular values or their estimation is rather complicated since we have the following relation:

$$GG^* \supset (\mathcal{L} + \mathcal{D})(\mathcal{L}^* + \mathcal{D}^*) \supset \mathcal{L}\mathcal{L}^* + \mathcal{D}\mathcal{L}^* + \mathcal{L}\mathcal{D}^* + \mathcal{D}\mathcal{D}^*,$$

where inclusions must satisfy some conditions connected with the core of the operator form, for in other cases, we have the risk of losing some singular values. In spite of the fact that the shown difficulties, in many cases, can be eliminated, the offered method of singular values estimation becomes apparently relevant.

Let us prove the fulfilment of the conditions H1, H2 under the assumptions $\mathfrak{H} := L_2(I)$, $\mathfrak{H}^+ := H_0^k(I)$, $\mathfrak{M} := C_0^\infty(I)$. The fulfillment of the condition H1 is obvious; let us show the fulfilment of the condition H2. It is easy to see that

$$\mathrm{Re}(\mathcal{L}f, f)_{L_2(I)} \geq \sum_{j=0}^{k} |\mathrm{Re} c_j|\, \|f^{(j)}\|^2_{L_2(I)} \geq C\|f^{(j)}\|^2_{H_0^k(I)},\ f \in \mathrm{D}(\mathcal{L}).$$

On the other hand,

$$|(\mathcal{L}f, f)_{L_2(I)}| = \left| \sum_{j=0}^{k} (-1)^j (c_j f^{(j)}, g^{(j)})_{L_2(I)} \right| \leq \sum_{j=0}^{k} \left| (c_j f^{(j)}, g^{(j)})_{L_2(I)} \right| \leq$$

$$\leq C \sum_{j=0}^{k} \|f^{(j)}\|_{L_2(I)} \|g^{(j)}\|_{L_2(I)} \leq \|f\|_{H_0^k(I)} \|g\|_{H_0^k(I)},\ f \in \mathrm{D}(\mathcal{L}).$$

Consider fractional differential Riemann–Liouville operators of arbitrary non-negative order α (see [26], p. 44) defined by the expressions

$$D^\alpha_{a+} f = \left(\frac{d}{dx}\right)^{[\alpha]+1} I_{a+}^{1-\{\alpha\}} f;\ D^\alpha_{b-} f = \left(-\frac{d}{dx}\right)^{[\alpha]+1} I_{b-}^{1-\{\alpha\}} f,$$

where the fractional integrals of arbitrary positive order α, defined by

$$(I^\alpha_{a+} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt,\ (I^\alpha_{b-} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)}{(t-x)^{1-\alpha}} dt, f \in L_1(I).$$

Suppose $0 < \alpha < 1$, $f \in AC^{l+1}(\bar{I})$, $f^{(j)}(a) = f^{(j)}(b) = 0$, $j = 0, 1, \ldots, l$; then the next formula follows from Theorem 2.2 ([26], p. 46):

$$D^{\alpha+l}_{a+} f = I^{1-\alpha}_{a+} f^{(l+1)},\ D^{\alpha+l}_{b-} f = (-1)^{l+1} I^{1-\alpha}_{b-} f^{(l+1)}. \tag{10}$$

Further, we need the following inequalities (see [28]):

$$Re(D^{\alpha}_{a+}f,f)_{L_2(I)} \geq C\|f\|^2_{L_2(I)},\ f \in I^{\alpha}_{a+}(L_2),$$

$$Re(D^{\alpha}_{b-}f,f)_{L_2(I)} \geq C\|f\|^2_{L_2(I)},\ f \in I^{\alpha}_{b-}(L_2), \tag{11}$$

where $I^{\alpha}_{a+}(L_2), I^{\alpha}_{b-}(L_2)$ are the classes of the functions representable by the fractional integrals (see [26]). Consider the following operator with the constant real-valued coefficients:

$$\mathcal{D}f := p_n D^{\alpha_n}_{a+} + q_n D^{\beta_n}_{b-} + p_{n-1} D^{\alpha_{n-1}}_{a+} + q_{n-1} D^{\beta_{n-1}}_{b-} + \ldots + p_0 D^{\alpha_0}_{a+} + q_0 D^{\beta_0}_{b-},$$

$$\mathrm{D}(\mathcal{D}) = H^{2k}(I) \cap H^k_0(I),\ n \in \mathbb{N},$$

where $\alpha_j, \beta_j \geq 0,\ 0 \leq [\alpha_j], [\beta_j] < k,\ j = 0, 1, \ldots, n.$,

$$q_j \geq 0,\ \operatorname{sign} p_j = \begin{cases} (-1)^{\frac{[\alpha_j]+1}{2}},\ [\alpha_j] = 2m-1,\ m \in \mathbb{N}, \\ \quad (-1)^{\frac{[\alpha_j]}{2}},\ \ [\alpha_j] = 2m,\ \ m \in \mathbb{N}_0. \end{cases}$$

Using (10) and (11), we obtain

$$(p_j D^{\alpha_j}_{a+}f,f)_{L_2(I)} = p_j\left(\left(\frac{d}{dx}\right)^m D^{m-1+\{\alpha_j\}}_{a+}f,f\right)_{L_2(I)} = (-1)^m p_j\left(I^{1-\{\alpha_j\}}_{a+}f^{(m)},f^{(m)}\right)_{L_2(I)} \geq$$

$$\geq C\left\|I^{1-\{\alpha_j\}}_{a+}f^{(m)}\right\|^2_{L_2(I)} = C\left\|D^{\{\alpha_j\}}_{a+}f^{(m-1)}\right\|^2_{L_2(I)} \geq C\left\|f^{(m-1)}\right\|^2_{L_2(I)},$$

where $f \in \mathrm{D}(\mathcal{D})$ is a real-valued function and $[\alpha_j] = 2m-1,\ m \in \mathbb{N}$. Similarly, we obtain for orders $[\alpha_j] = 2m,\ m \in \mathbb{N}_0$

$$(p_j D^{\alpha_j}_{a+}f,f)_{L_2(I)} = p_j\left(D^{2m+\{\alpha_j\}}_{a+}f,f\right)_{L_2(I)} = (-1)^m p_j\left(D^{m+\{\alpha_j\}}_{a+}f,f^{(m)}\right)_{L_2(I)} =$$

$$= (-1)^m p_j\left(D^{\{\alpha_j\}}_{a+}f^{(m)},f^{(m)}\right)_{L_2(I)} \geq C\left\|f^{(m)}\right\|^2_{L_2(I)}.$$

Thus in both cases, we have

$$(p_j D^{\alpha_j}_{a+}f,f)_{L_2(I)} \geq C\left\|f^{(s)}\right\|^2_{L_2(I)},\ s = [[\alpha_j]/2].$$

In the same way, we obtain the inequality

$$(q_j D^{\alpha_j}_{b-}f,f)_{L_2(I)} \geq C\left\|f^{(s)}\right\|^2_{L_2(I)},\ s = [[\alpha_j]/2].$$

Hence, in the complex case, we have

$$\mathrm{Re}(\mathcal{D}f,f)_{L_2(I)} \geq C\|f\|^2_{L_2(I)},\ f \in \mathrm{D}(\mathcal{D}).$$

Combining Theorem 2.6 ([26], p. 53) with (10), we obtain

$$\left\|p_j D^{\alpha_j}_{a+}f\right\|_{L_2(I)} = \left\|I^{1-\{\alpha_j\}}_{a+}f^{([\alpha_j]+1)}\right\|_{L_2(I)} \leq C\left\|f^{([\alpha_j]+1)}\right\|_{L_2(I)} \leq C\|f\|_{H^k_0(I)};$$

$$\left\|q_j D^{\alpha_j}_{b-}f\right\|_{L_2(I)} \leq C\|f\|_{H^k_0(I)},\ f \in \mathrm{D}(\mathcal{D}).$$

Hence, we obtain

$$\|\mathcal{D}f\|_{L_2(I)} \leq C\|f\|_{H_0^k(I)},\ f \in \mathrm{D}(\mathcal{D}).$$

Taking into account the relation

$$\|f\|_{L_2(I)} \leq C\|f\|_{H_0^k(I)},\ f \in H_0^k(I),$$

combining the above estimates, we obtain

$$\mathrm{Re}(Gf,f)_{L_2(I)} \geq C\|f\|^2_{H_0^k(I)},\ |(Gf,g)_{L_2(I)}| \leq \|f\|_{H_0^k(I)}\|g\|_{H_0^k(I)},\ f,g \in C_0^\infty(I).$$

Thus, we have obtained the desired result.

To deploy the minimax principle for eigenvalues estimating, we come to the following relation:

$$C_1\|f\|^2_{H_0^k(I)} \leq (\mathfrak{Re}Gf,f)_{L_2(I)} \leq C_2\|f\|^2_{H_0^k(I)},$$

from which follows easily, due to the asymptotic formula for the eigenvalues of a self-adjoint operator (see [29]), the fact that

$$\lambda_n(\mathfrak{Re}G) \asymp n^{2k},\ n \in \mathbb{N};$$

therefore, applying Lemma 1 collaboratively with the asymptotic equivalence formula (see Theorem 5 [9])

$$\lambda_n^{-1}(\mathfrak{Re}G) \asymp \lambda_n(\mathfrak{Re}R_G),\ n \in \mathbb{N},$$

we obtain the fact that

$$R_G \in \mathfrak{S}_p,\ \inf p \leq 1/2k.$$

Thus, it gives us an opportunity to establish the range of the Schatten index.

2. Let us show the application of Lemma 2; firstly, consider the following reasonings:

$$\|\mathfrak{Im}WH^{-1}\|_2 = \|H^{-1}\mathfrak{Im}W\|_2 = \sum_{n,k=1}^{\infty}\left|(\mathfrak{Im}We_n, H^{-1}e_k)_{\mathfrak{H}}\right|^2 = \sum_{n,k=1}^{\infty}\lambda_n^{-2}(H)|(e_n,\mathfrak{Im}We_k)_{\mathfrak{H}}|^2 =$$

$$= \sum_{n=1}^{\infty}\lambda_n^{-2}(H)||\mathfrak{Im}We_n||^2_{\mathfrak{H}},$$

where $\{e_n\}_1^\infty$ is the orthonormal set of the eigenvectors of the operator H. Thus, we obtain the following condition:

$$\sum_{n=1}^{\infty}\lambda_n^{-2}(H)||\mathfrak{Im}We_n||^2_{\mathfrak{H}} < 1, \tag{12}$$

which guarantees the fulfilment of the conditions expressed in terms of absolute norm in Lemma 2. It is remarkable that this form of the condition is quite convenient if we consider perturbations of differential operators. Below, we observe a simplified case of the operator considered in the previous paragraph. Consider

$$Lf := -f'' + \xi D^{\alpha}_{0+}f,\ \mathrm{D}(L) = H^2(I) \cap H_0^1(I),\ I = (0,\pi),\ \alpha \in (0,1/2),\ \xi \in \mathbb{R},$$

then

$$C_0(L_1f,f)_{L_2(I)} \leq (\mathfrak{Re}Lf,f)_{L_2(I)} \leq C_1(L_1f,f)_{L_2(I)},\ L_1f := -f'',\ \mathrm{D}(L_1) = \mathrm{D}(L).$$

It is a well-known fact that

$$\lambda_n(L_1) = n^2,\ e_n = \sin nx.$$

It is also clear that

$$\mathfrak{Im}L \supset \xi(D^{\alpha}_{0+} - D^{\alpha}_{\pi-})/2i.$$

In accordance with the first representation theorem (see Theorem 2.1 [23], p. 322), we have that $H^2(I) \cap H^1_0(I)$ is a core of the form corresponding to the operator L^*; hence,

$$\mathfrak{Im}L = \xi(D^{\alpha}_{0+} - D^{\alpha}_{\pi-})/2i.$$

Note that

$$(D^{\alpha}_{0+}e_n)(x) = \frac{n}{\Gamma(1-\alpha)}\int\limits_0^x (x-t)^{-\alpha}\cos nt\,dt.$$

Applying the generalized Minkowski inequality, we obtain

$$\left(\int\limits_0^\pi |(D^{\alpha}_{a+}e_n)(x)|^2 dx\right)^{1/2} = \frac{n}{\Gamma(1-\alpha)}\left(\int\limits_0^\pi \left|\int\limits_0^x (x-t)^{-\alpha}\cos nt\,dt\right|^2\right)^{1/2} \leq$$

$$\leq \frac{n}{\Gamma(1-\alpha)}\int\limits_0^\pi \cos nt\,dt\left(\int\limits_t^\pi (x-t)^{-2\alpha}dx\right)^{1/2} = \frac{n}{\sqrt{(1-2\alpha)}\Gamma(1-\alpha)}\int\limits_0^\pi (\pi-t)^{1/2-\alpha}\cos nt\,dt \leq$$

$$\leq \frac{n\pi^{1/2-\alpha}}{\sqrt{(1-2\alpha)}\Gamma(1-\alpha)}.$$

Analogously, we obtain

$$\left(\int\limits_0^\pi |(D^{\alpha}_{\pi-}e_n)(x)|^2 dx\right)^{1/2} \leq \frac{n\pi^{1/2-\alpha}}{\sqrt{(1-2\alpha)}\Gamma(1-\alpha)}.$$

Hence,

$$\|\mathfrak{Im}Le_n\| \leq \frac{n\xi\pi^{1/2-\alpha}}{\sqrt{(1-2\alpha)}\Gamma(1-\alpha)}.$$

Therefore,

$$\sum_{n=1}^{\infty}\lambda_n^{-2}(\mathfrak{Re}L)||\mathfrak{Im}Le_n||^2 < \frac{\xi^2\pi^{1-2\alpha}}{(1-2\alpha)\Gamma^2(1-\alpha)}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\xi^2\pi^{3-2\alpha}}{6(1-2\alpha)\Gamma^2(1-\alpha)}.$$

Using this relation, we can obviously impose a condition on ξ that guarantees the fulfilment of relation (12), i.e.,

$$\xi < \frac{\sqrt{6(1-2\alpha)}\Gamma(1-\alpha)}{\pi^{3/2-\alpha}}.$$

In accordance with Theorem 1, the last condition follows that

$$s_n^{-1}(R_L) \asymp n^2,\ R_L \in \mathfrak{S}_p,\ \inf p = 1/2.$$

4.2. Existence and Uniqueness Theorems for Evolution Equations via Obtained Results

In this paragraph, we consider applications to differential equations in concrete Hilbert spaces and involve such operators as Riemann–Liouville operator, Kipriyanov operator, and Riesz potential, difference operator. Moreover, we produce the artificially constructed normal operator for which the clarification of the Lidskii results relevantly works.

Further, we consider a Hilbert space $\mathfrak{H}$ which consists of element-functions $u : \mathbb{R}_+ \to \mathfrak{H}$, $u := u(t)$, $t \geq 0$ and we assume that if u belongs to $\mathfrak{H}$ then the fact holds for all values of the variable t. Notice that under such an assumption all standard topological properties, such as completeness, compactness, etc., remain correctly defined. We understand such operations as differentiation and integration in the generalized sense that is caused by the topology of the Hilbert space $\mathfrak{H}$; more detailed information can be found in Chapter 4 Krasnoselskii M.A. [30]. Consider an arbitrary compact operator B; we can form the operators corresponding to the groups of its eigenvalues, i.e.,

$$\mathcal{P}_\nu(B,\alpha,t) \Leftrightarrow \lambda_{N_\nu+1}, \lambda_{N_\nu+2}, \ldots, \lambda_{N_{\nu+1}},$$

where $\{N_\nu\}_0^\infty$ is a sequence of natural numbers,

$$\mathcal{P}_\nu(B,\alpha,t) = \frac{1}{2\pi i} \int\limits_{\vartheta_\nu(B)} e^{-\lambda^\alpha t} B(I-\lambda B)^{-1} d\lambda, \; \alpha > 0,$$

$\vartheta_\nu(B)$ is a contour on the complex plain containing the eigenvalues $\lambda_{N_\nu+1}, \lambda_{N_\nu+2}, \ldots, \lambda_{N_{\nu+1}}$ only and no more eigenvalues.

The root vectors of the operator B are called by the Abel–Lidskii basis if

$$\sum_{\nu=0}^{\infty} \mathcal{P}_\nu(B,\alpha,t) \to I, \; t \to 0,$$

where convergence is understood as the operator pointwise convergence in the Hilbert space.

The correspondence between the series and the element, given due to the formula, is known as a convergence in the Abel–Lidskii sense. We can compare this definition with the main principle of the spectral theorem—the unit decomposition. We place the following contour in correspondence to the operator:

$$\vartheta(B) := \{\lambda : \; |\lambda| = r > 0, \; |\arg\lambda| \le \theta + \varepsilon\} \cup \{\lambda : \; |\lambda| > r, \; |\arg\lambda| = \theta + \varepsilon\}.$$

Consider the following hypotheses:

S1: *Under the assumptions* $B \in \mathfrak{S}_p$, $\inf p \le \alpha$, $\Theta(B) \subset \mathfrak{L}_0(\theta)$, *a sequence of natural numbers* $\{N_\nu\}_0^\infty$ *can be chosen so that*

$$\frac{1}{2\pi i} \int\limits_{\vartheta(B)} e^{-\lambda^\alpha t} B(I-\lambda B)^{-1} f d\lambda = \sum_{\nu=0}^{\infty} \mathcal{P}_\nu(B,\alpha,t) f, \; f \in \mathfrak{H},$$

the latter series is absolutely convergent in the sense of the norm.

Combining the generalized integrodifferential operations, we can consider a fractional differential operator in the Riemann–Liouville sense, i.e., in the formal form, we have

$$\mathfrak{D}_-^{1/\alpha} f(t) := -\frac{1}{\Gamma(1-1/\alpha)} \frac{d}{dt} \int\limits_0^\infty f(t+x) x^{-1/\alpha} dx, \; \alpha > 1.$$

Let us study a Cauchy problem:

$$\mathfrak{D}_-^{1/\alpha} u = Wu, \; u(0) = h \in \mathrm{D}(W). \tag{13}$$

Note that it is possible to apply the Abel–Lidskii concept using the methods [1,10,14–16] in the case $R_W \in \mathfrak{S}_p$, $\inf p \le \alpha$. We can assume that the central result of the above-listed papers is to find conditions under which the hypotheses S1 holds. We can generalize the results related to the existence and uniqueness theorem (see Theorem 4 [31], Theorem 1 [16], Theorem 6 [15]), as follows:

Theorem 2. *Assume that S1 holds, then there exists a solution of Cauchy problem (13) in the form*

$$u(t) = \sum_{\nu=0}^{\infty} \mathcal{P}_{\nu}(B, \alpha, t)h.$$

Apparently, under this point of view, the results of the paper become relevant since, applying Theorem 1, we can find the exact value of the Schatten index p. Therefore, we can decrease the value of α, satisfying the condition $\inf p \leq \alpha$ in accordance with S1.

To demonstrate the claimed result, we produce an example dealing with well-known operators. Consider a rectangular domain in the space $\mathbb{R}^n$, defined as follows: $\Omega := \{x_j \in [0, \pi],\ j = 1, 2, \ldots, n\}$; and consider the Kipriyanov fractional differential operator defined in the paper [25] by the formal expression

$$\mathfrak{D}^{\beta} f(Q) = \frac{\beta}{\Gamma(1-\beta)} \int_0^r \frac{[f(Q) - f(T)]}{(r-t)^{\beta+1}} \left(\frac{t}{r}\right)^{n-1} dt + (n-1)! f(Q) r^{-\beta} / \Gamma(n-\beta),$$

$$\beta \in (0,1),\ P \in \partial\Omega,$$

where $Q := P + \mathbf{e}r,\ P := P + \mathbf{e}t,\ \mathbf{e}$ is a unit vector having a direction from the fixed point of the boundary P to an arbitrary point Q belonging to Ω. Consider the perturbation of the Laplace operator by the Kipriyanov operator:

$$L := D^{2k} + \xi \mathfrak{D}^{\beta},\ \mathrm{D}(L) = H_0^k(\Omega) \cap H^{2k}(\Omega),$$

where $\xi > 0$,

$$D^{2k} f = (-1)^k \sum_{j=1}^{n} \mathcal{D}_j^{2k} f.$$

It was proved in the paper [10] that

$$C_0 (D^{2k} f, f)_{L_2(\Omega)} \leq (\mathfrak{Re} L f, f)_{L_2(\Omega)} \leq C_1 (D^{2k} f, f)_{L_2(\Omega)},\ f \in \mathrm{D}(L).$$

Therefore,

$$\lambda_n(\mathfrak{Re} L) \asymp n^{2k/n}.$$

On the other hand, we have the following eigenfunctions of D^{2k} in the rectangular domain:

$$e_{\bar{l}} = \prod_{j=1}^{n} \sin l_j x_j,\ \bar{l} := \{l_1, l_2, \ldots, l_n\},\ l_s \in \mathbb{N},\ s = 1, 2, \ldots, n.$$

It is clear that

$$D^{2k} e_{\bar{l}} = \lambda_{\bar{l}} e_{\bar{l}},\ \lambda_{\bar{l}} = \sum_{j=1}^{n} l_j^{2k}.$$

Since the search for the below-given information in the literature (however, it is a well-known fact) can bring some difficulties, we would like to represent it. Let us prove that the system $\{e_{\bar{l}}\}$ is complete in the Hilbert space $L_2(\Omega)$. We will show it if we prove that the element that is orthogonal to every element of the system is a zero. Assume that

$$\int_0^{\pi} \sin l_1 x_1 dx_1 \int_0^{\pi} \sin l_2 x_2 dx_2 \ldots \int_0^{\pi} \sin l_n x_n f(x_1, x_2, \ldots, x_n) dx_n = (e_{\bar{l}}, f)_{L_2(\Omega)} = 0.$$

In accordance with the fact that the system $\{\sin mx\}_1^\infty$ is a complete system in $L_2(0,\pi)$, we conclude that

$$\int_0^\pi \sin l_2 x_2 dx_2 \ldots \int_0^\pi \sin l_n x_n f(x_1, x_2, \ldots, x_n) dx_n = 0.$$

Having repeated the same reasonings step by step, we obtain the desired result. Taking into account the following inequality (see [10]) and the embedding theorems, we obtain

$$\|\mathfrak{D}^\beta f\|_{L_2(\Omega)} \leq C_\beta \|f\|_{H_0^1(\Omega)} \leq C_{\beta,k,n} \|f\|_{H_0^k(\Omega)}, \tag{14}$$

where the constant C_β is defined through the infinitesimal generator J of the corresponding semigroup of contraction (shift semigroup in the direction) (9) [10]. Now it is clear that the conditions H1, H2 are satisfied, where $\mathfrak{H} := L_2(\Omega)$, $\mathfrak{H}_+ := H_0^k(\Omega)$, $\mathfrak{M} := C_0^\infty(\Omega)$. Using the intermediate inequality (14), by direct calculation, we obtain

$$\sum_{l_1, l_2, \ldots l_n = 1}^\infty \lambda_{\bar{l}}^{-2} (\mathfrak{Re} L)_{L_2(\Omega)} \, \|\mathfrak{Im} L e_{\bar{l}}\|^2_{L_2(\Omega)} \leq (\xi C_\beta)^2 \sum_{l_1, l_2, \ldots l_n = 1}^\infty \frac{\lambda_{\bar{l}}(D^2)}{\lambda_{\bar{l}}^2(D^{2k})}.$$

Therefore, if the following condition holds,

$$\sum_{l_1, l_2, \ldots l_n = 1}^\infty \frac{l_1^2 + l_2^2 + \ldots + l_n^2}{(l_1^{2k} + l_2^{2k} + \ldots + l_n^{2k})^2} < (\xi C_\beta)^{-2}, \tag{15}$$

then the conditions of Lemma 2 are satisfied. Applying Lemma 2, we can consider the values of the parameters k, n such that the last series is convergent, and at the same time, $R_L \in \mathfrak{S}_p$, $\inf p = n/2k > 1$. The latter fact gives us the argument showing the relevance of Lemma 2 since we can find the range of α appropriate for the Abel–Lidskii method applicability. Below, we produce the corresponding reasonings.

Assume that the following condition holds:

$$\frac{n}{2} + 1 < 2k < n.$$

Consider the vector function

$$\psi(\bar{l}) = \frac{(l_1^{2k} + l_2^{2k} + \ldots + l_n^{2k})^2}{l_1^2 + l_2^2 + \ldots + l_n^2},$$

then $\psi(\bar{t}) = nt^{2(2k-1)}$, $\bar{t} = \{t, t, \ldots t\}$. It is clear that the number s of values $\psi(\bar{l})$, $l_i \leq t$ equals t^n, i.e., $s = t^n$. Therefore,

$$\psi(\bar{t}) = ns^{\frac{2(2k-1)}{n}}, \; \psi(\overline{t-1}) = n(s^{1/n} - 1)^{2(2k-1)};$$

$$n(s^{1/n} - 1)^{2(2k-1)} \leq \psi(\bar{l}) \leq ns^{\frac{2(2k-1)}{n}}, \; t - 1 \leq l_i \leq t, \; i = 1, 2, \ldots, n.$$

Having arranged the values in the order corresponding to their absolute value increasing, we obtain

$$n(s^{1/n} - 1)^{2(2k-1)} \leq \psi_j \leq ns^{\frac{2(2k-1)}{n}}, \; (s^{1/n} - 1)^n < j < s.$$

Therefore,

$$\frac{(s^{1/n} - 1)^{2(2k-1)}}{s^{\frac{2(2k-1)}{n}}} < \frac{\psi_j}{nj^{\frac{2(2k-1)}{n}}} < \frac{s^{\frac{2(2k-1)}{n}}}{(s^{1/n} - 1)^{2(2k-1)}},$$

from which follows the convergence of the following series, since if we take into account the condition $n/2+1<2k$, we obtain

$$\sum_{j=1}^{\infty}\psi_j^{-1}<\infty.$$

In other words, we have proved that series (15) is convergent. Thus, we have considered the case showing the relevance of Lemma 2. We can claim that the Abel–Lidskii method in its classical form is not applicable to the fractional evolution equation for the values of α less than $n/2k$. This rather ridiculous result, from one point of view, gives us a better comprehension of methodology and allows us to avoid disturbing calculation and difficulties of any kind connected with the verification of opportunity to apply the method.

5. Conclusions

In this paper, we represent an efficient tool for finding the asymptotics of operator singular values. However, it may be interesting itself since it appeals to the spectral properties of the operator real component, which are undoubtedly relevant in the framework of the abstract spectral theory. Some difficulties in the application of the Abel–Lidskii method were considered under the point of view of the created concept, where the the mathematical applications cover integrodifferential operators of the real order.

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