mathematics

Article

Toeplitz Operators on Harmonic Fock Spaces with Radial Symbols

Zhi-Ling Sun [1], Wei-Shih Du [2,*] and Feng Qi [3,4,5,*]

[1] College of Mathematical Sciences, Inner Mongolia Minzu University, Tongliao 028043, China; zlingsun@126.com
[2] Department of Mathematics, National Kaohsiung Normal University, Kaohsiung 82444, Taiwan
[3] School of Mathematics and Physics, Hulunbuir University, Hulunbuir 021008, China
[4] School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo 454010, China
[5] Independent Researcher, Dallas, TX 75252-8024, USA
* Correspondence: wsdu@mail.nknu.edu.tw (W.-S.D.); honest.john.china@gmail.com (F.Q.)

Abstract: The main aim of this paper is to study new features and specific properties of the Toeplitz operator with radial symbols in harmonic Fock spaces. A new spectral decomposition of a Toeplitz operator with Wick symbols is also established.

Keywords: Toeplitz operators; harmonic Fock space; radial symbol; weighted pluriharmonic Bergman space; Wick symbol

MSC: 47B35; 47B47

1. Introduction and Preliminaries

Let $\mathbb{C}$ denote the complex plane and $\mathrm{d}\mu = \frac{\mathrm{d}v(z)}{\pi\,\mathrm{e}^{z\bar{z}}}$ be the Gaussian measure on $\mathbb{C}$, where $\mathrm{d}v(z) = \mathrm{d}x\,\mathrm{d}y$ is the standard Lebesgue plane measure on $\mathbb{C} \simeq \mathbb{R}^2$. Denote by $L^2(\mathbb{C}, \mathrm{d}\mu)$ the Hilbert space of square Lebesgue integrable functions on $\mathbb{C}$ with $\mathrm{d}\mu$. The Fock space $F^2(\mathbb{C})$ is the Hilbert space of analytic functions which belong to $L^2(\mathbb{C}, \mathrm{d}\mu)$. In fact, the Fock space $F^2(\mathbb{C})$ is a reproducing function space of the reproducing kernel

$$K_z(w) = \mathrm{e}^{\bar{z}w} = \sum_{n=0}^{\infty} \frac{(\bar{z}w)^n}{n!}.$$

The harmonic Fock space $F_h^2(\mathbb{C})$ is the subspace of $L^2(\mathbb{C}, \mathrm{d}\mu)$ consisting of all harmonic functions on $\mathbb{C}$. As in the harmonic Bergman space, it is known (see, e.g., [1]) that

$$F_h^2(\mathbb{C}) = F^2(\mathbb{C}) + \overline{F^2(\mathbb{C})},$$

where $\overline{F^2(\mathbb{C})} = \{\bar{f} \mid f \in F^2(\mathbb{C})\}$. The space $F_h^2(\mathbb{C})$ is also a reproducing function space with the reproducing kernel

$$R_z(w) = K_z(w) + \overline{K_z(w)} - 1, \quad z, w \in \mathbb{C}.$$

The concept and properties of Fock space have been improved and generalized in many various different directions by several authors; for more details, see, e.g., [1–4] and the references therein.

Denote by P the orthogonal Bargmann projection of $L^2(\mathbb{C}, \mathrm{d}\mu)$ into the Fock space $F^2(\mathbb{C})$. Then we have

$$(P\varphi)(z) = \langle \varphi, K_z \rangle$$

for $\varphi \in L^2(\mathbb{C}, \mathrm{d}\mu)$, where the notation $\langle \cdot\,, \cdot \rangle$ denotes the standard inner product in $L^2(\mathbb{C}, \mathrm{d}\mu)$.

check for updates

Citation: Sun, Z.-L.; Du, W.-S.; Qi, F. Toeplitz Operators on Harmonic Fock Spaces with Radial Symbols. *Mathematics* **2024**, *12*, 565. https://doi.org/10.3390/math12040565

Academic Editor: Manuel Gadella

Received: 15 December 2023
Revised: 2 February 2024
Accepted: 11 February 2024
Published: 13 February 2024

Let Q be the harmonic Bargmann projection from $L^2(\mathbb{C}, \mathrm{d}\mu)$ into harmonic Fock space $F_h^2(\mathbb{C})$. For a function $a \in L^2(\mathbb{C}, \mathrm{d}\mu)$, the Toeplitz operator $\mathrm{T}_a : F_h^2(\mathbb{C}) \to F_h^2(\mathbb{C})$ with symbol a is the linear operator defined by

$$\mathrm{T}_a f = Q(af) = \langle af, R_z \rangle = P(af) + \overline{P(\overline{af})} + P(af)(0), \quad f \in F_h^2(\mathbb{C}),$$

where T_a is densely defined and not bounded in general.

The theory of the Toeplitz operator stems from a series of ideas developed in the second half of the 20th century by mathematicians such as Otto Toeplitz, Hermann Hankel, Eberhard Hopf, Norbert Wiener and Gábor Szegö. In recent years, many scholars have obtained many important results for the Toeplitz operator T_a and the Hankel operator in Fock space $F^2(\mathbb{C})$. The boundedness, compactness, algebraic properties and the product properties on Fock-type spaces have been studied extensively. For more details, we refer the readers to, for example, [1,3,5–15]. Because the product of two harmonic functions is often no longer a harmonic function, it is more difficult to study Toeplitz operators in harmonic Fock space than in analytic Fock space. Many of the methods used for operators in the analytic Fock space lose their effectiveness in harmonic Fock space. Therefore, many scholars attempted to provide some new ideas and methods to overcome such situations and generalize Fock spaces. For example, in the paper [2], Chen et alia considered the Toeplitz operator T_a in vector-valued generalized Fock spaces. In the paper [4], He and Wu characterized dual Toeplitz operators in the orthogonal complement of Fock–Sobolev spaces.

There have been some results of the Toeplitz operator T_a in harmonic Bergman space; please refer to [16–18]. In the paper [16], Guo and Zheng characterized compact Toeplitz operators in the unit disk $\mathbb{D}$. In [18], by using the system of integral equations, Lee characterized the commuting Toeplitz operators of holomorphic symbols and pluriharmonic symbols in pluriharmonic Bergman space. In addition, Lee proved in [17] the commutativity of an operator with a radial symbol and an operator with a pluriharmonic symbol in pluriharmonic Bergman space. In the papers [7,19–22], several mathematicians analyzed the influence of the radial component of a symbol of the spectral, compactness and Fredholm properties of Toeplitz operators on Bergman space or Fock space. In the paper [21], Li and Lu characterized radial symbols in Bergman spaces over the polydisk. In [22], the first author of this paper and her coauthor gave some properties about Toeplitz operators in weighted pluriharmonic Bergman spaces with radial symbols.

In this paper, inspired by the above, we are committed to investigating the problem of Toeplitz operators with radial symbols in a harmonic Fock space $F_h^2(\mathbb{C})$. Basing our work on the techniques used in [7,19,20,22], we will construct an operator R whose restriction in harmonic Fock space $F_h^2(\mathbb{C})$ is an isometric isomorphism between $F_h^2(\mathbb{C})$ and ℓ_2, that is,

$$\mathrm{R}\,\mathrm{R}^* = \mathrm{I} : \ell_2 \to \ell_2$$

and

$$\mathrm{R}^*\,\mathrm{R} = Q : L^2(\mathbb{C}, \mathrm{d}\mu) \to F_h^2(\mathbb{C}),$$

where I is the identity operator. Employing the operator R, we will prove that each Toeplitz operator T_a with radial symbols is unitary to the multiplication operator $\gamma_a \mathrm{I}$. Making use of the Berezin concept of Wick and anti-Wick symbols (see [1,5,23]), we will show that, in our particular case (radial symbols), the Wick symbols of a Toeplitz operator give complete information about the operator and provide its spectral decomposition.

2. New Results for Harmonic Fock Spaces and Related Operators

We now recall some notations, definitions and well-known facts about harmonic Fock space. Let $f(z) \in F_h^2(\mathbb{C})$ and write $f = g + \bar{h}$. The harmonic Fock space $F_h^2(\mathbb{C})$ can be described as the closure in $L^2(\mathbb{C}, \mathrm{d}\mu)$ of the set of all smooth functions satisfying the equations

$$\frac{\partial}{\partial \bar{z}} g = \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\,\frac{\partial}{\partial y}\right) g = 0$$

and

$$\frac{\partial}{\partial z} \bar{h} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\,\frac{\partial}{\partial y}\right) \bar{h} = 0,$$

where $z = x + y\,\mathrm{i}$.

As performed in [7], introduce the unitary operator

$$\mathrm{U}_1 : L_2(\mathbb{C}, \mathrm{d}\mu) \to L_2\left(\mathbb{R}^2, \mathrm{d}x\,\mathrm{d}y\right)$$

with the rule

$$(\mathrm{U}_1\,\varphi)(x, y) = \frac{\varphi(x + y\,\mathrm{i})}{\pi^{1/2}\,\mathrm{e}^{(x^2+y^2)/2}} = \frac{\varphi(z)}{\pi^{1/2}\,\mathrm{e}^{z\bar{z}/2}}.$$

Then the image $F^{(1)} = \mathrm{U}_1(F_h^2(\mathbb{C}))$ of the harmonic Fock space $F_h^2(\mathbb{C})$ is the closure of the set of all smooth functions $f = g + \bar{h}$ in $L^2(\mathbb{R}^2, \mathrm{d}x\,\mathrm{d}y)$, which satisfies the equations

$$D^{(1)} g = \mathrm{U}_1\,\frac{\partial}{\partial \bar{z}} \mathrm{U}_1^{-1} g = \left(\frac{\partial}{\partial \bar{z}} + \frac{z}{2}\right) g = \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\,\frac{\partial}{\partial y} + x + y\,\mathrm{i}\right) g = 0$$

and

$$D^{(1)} \bar{h} = \mathrm{U}_1\,\frac{\partial}{\partial z} \mathrm{U}_1^{-1} \bar{h} = \left(\frac{\partial}{\partial z} + \frac{\bar{z}}{2}\right) \bar{h} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\,\frac{\partial}{\partial y} + x - y\,\mathrm{i}\right) \bar{h} = 0.$$

Passing to polar coordinates in $\mathbb{R}^2$, we have

$$L_2(\mathbb{R}^2, \mathrm{d}x\,\mathrm{d}y) = L_2(\mathbb{R}_+, r\,\mathrm{d}r) \otimes L_2([0, 2\pi), \mathrm{d}\alpha) = L_2(\mathbb{R}_+, r\,\mathrm{d}r) \otimes L_2\left(S^1, \frac{\mathrm{d}t}{t\,\mathrm{i}}\right),$$

where S^1 is the unit circle and

$$\frac{\mathrm{d}t}{t\,\mathrm{i}} = \frac{\mathrm{d}\mathrm{e}^{\mathrm{i}\,\alpha}}{\mathrm{i}\,\mathrm{e}^{\mathrm{i}\,\alpha}} = \mathrm{d}\alpha = |\,\mathrm{d}t|$$

is the element of length. In addition,

$$\frac{\partial}{\partial \bar{z}} + \frac{z}{2} = \frac{\mathrm{e}^{\mathrm{i}\,\alpha}}{2}\left(\frac{\partial}{\partial r} + \mathrm{i}\,\frac{1}{r}\frac{\partial}{\partial \alpha} + r\right) = \frac{t}{2}\left(\frac{\partial}{\partial r} - \frac{t}{r}\frac{\partial}{\partial t} + r\right)$$

and

$$\frac{\partial}{\partial z} + \frac{\bar{z}}{2} = \frac{1}{2\,\mathrm{e}^{\mathrm{i}\,\alpha}}\left(\frac{\partial}{\partial r} - \mathrm{i}\,\frac{1}{r}\frac{\partial}{\partial \alpha} + r\right) = \frac{1}{2t}\left(\frac{\partial}{\partial r} + \frac{t}{r}\frac{\partial}{\partial t} + r\right).$$

As performed in [7], introduce the unitary operator

$$\mathrm{U}_2 = \mathrm{I} \otimes \mathcal{F} : L_2(\mathbb{R}_+, r\,\mathrm{d}r) \otimes L_2\left(S^1\right) \to L_2(\mathbb{R}_+, r\,\mathrm{d}r) \otimes \ell_2 = \ell_2(L_2(\mathbb{R}_+, r\,\mathrm{d}r))$$

and the discrete Fourier transform $\mathcal{F} : L_2\left(S^1\right) \to \ell_2$ with

$$\mathcal{F} : f \mapsto c_n = \frac{1}{\sqrt{2\pi}} \int_{S^1} f(t) t^{-n} \frac{\mathrm{d}t}{t\,\mathrm{i}}, \quad n \in \mathbb{Z}.$$

The inverse of $\mathcal{F}$ is given by

$$\mathcal{F}^* = \mathcal{F}^{-1} : \{c_n\} \to f = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} c_n t^n.$$

It is not difficult to see that

$$(\mathrm{I}\otimes\mathcal{F})\frac{t}{2}\left(\frac{\partial}{\partial r}-\frac{t}{r}\frac{\partial}{\partial t}+r\right)(\mathrm{I}\otimes\mathcal{F}^{-1})\{c_n(r)\}_{n\in\mathbb{Z}_+}=\left\{\frac{1}{2}\left(\frac{\partial}{\partial r}-\frac{n-1}{r}+r\right)c_{n-1}(r)\right\}_{n\in\mathbb{Z}_+}$$

and

$$(\mathrm{I}\otimes\mathcal{F})\frac{1}{2t}\left(\frac{\partial}{\partial r}+\frac{t}{r}\frac{\partial}{\partial t}+r\right)(\mathrm{I}\otimes\mathcal{F}^{-1})\{c_n(r)\}_{n\in\mathbb{Z}_-}=\left\{\frac{1}{2}\left(\frac{\partial}{\partial r}+\frac{n+1}{r}+r\right)c_{n+1}(r)\right\}_{n\in\mathbb{Z}_-},$$

where $\mathbb{Z}_+=\{0\}\cup\mathbb{N}$ and $\mathbb{Z}_-=\mathbb{Z}\setminus\mathbb{Z}_+$. Thus, the image $F^{(2)}=\mathrm{U}_2(F^{(1)})$ of the space $F^{(1)}$ can be described as the subspace of $L_2(\mathbb{R}_+,r\,\mathrm{d}r)\otimes l_2$, which is the closure of all sequences $\{c_n(r)\}_{n\in\mathbb{Z}}$ with smooth components satisfying the equations

$$\frac{1}{2}\left(\frac{\partial}{\partial r}-\frac{n}{r}+r\right)c_n(r)=0,\quad n\in\mathbb{Z}_+ \tag{1}$$

and

$$\frac{1}{2}\left(\frac{\partial}{\partial r}-\frac{n}{r}+r\right)c_{-n}(r)=0,\quad n\in\mathbb{Z}_+. \tag{2}$$

Equations (1) and (2) are easy to solve and their general solutions have the form

$$c_n(r)=c'_n\frac{r^n}{\mathrm{e}^{r^2/2}}=c_n\sqrt{\frac{2}{n!}}\,\frac{r^n}{\mathrm{e}^{r^2/2}} \tag{3}$$

and

$$c_{-n}(r)=c'_{-n}\frac{r^n}{\mathrm{e}^{r^2/2}}=c_{-n}\sqrt{\frac{2}{n!}}\,\frac{r^n}{\mathrm{e}^{r^2/2}}. \tag{4}$$

Each function $c_n(r)$ must be in $L_2(\mathbb{R}_+,r\,\mathrm{d}r)$. Therefore, the space $F^{(2)}$ coincides with the space of all two-sided sequences $\{c_n(r)\}_{n\in\mathbb{Z}}$ with

$$c_n(r)=\begin{cases}c_n\sqrt{\frac{2}{n!}}\,\frac{r^n}{\mathrm{e}^{r^2/2}}, & n\in\mathbb{Z}_+\\ c_n\sqrt{\frac{2}{|n|!}}\,\frac{r^{|n|}}{\mathrm{e}^{r^2/2}}, & n\in\mathbb{Z}_-\end{cases}$$

and

$$\|\{c_n(r)\}_{n\in\mathbb{Z}}\|=\left\|\left(\sum_{n\in\mathbb{Z}}|c_n|^2\right)^{1/2}\right\|=\|\{c_n\}_{n\in\mathbb{Z}}\|_{l_2}.$$

For each $n\in\mathbb{Z}$, as performed in [7], introduce the unitary operator

$$\mathrm{u}_n:L_2(\mathbb{R}_+,\mathrm{d}r)\to L_2(\mathbb{R}_+,r\,\mathrm{d}r)$$

with the rule

$$(\mathrm{u}_n f)(r)=\omega_n(r)f(\alpha_n(r)),$$

where

$$\omega_n(r)=r^n\sqrt{\frac{1}{|n|!}\sum_{k=0}^{|n|}\frac{r^{2k}}{k!}}\quad\text{and}\quad\alpha_n(r)=r^2-\ln\sum_{k=0}^{|n|}\frac{r^{2k}}{k!}. \tag{5}$$

As performed in [7], define the unitary operator

$$\mathrm{U}_3:\ell_2(L_2(\mathbb{R}_+,r\,\mathrm{d}r))\to L_2(\mathbb{R}_+)\otimes\ell_2$$

as

$$\mathrm{U}_3:\{c_n(r)\}_{n\in\mathbb{Z}}\mapsto\left\{\left(\mathrm{u}_{|n|}^{-1}c_n\right)(r)\right\}_{n\in\mathbb{Z}}.$$

The space $F^{(3)} = \mathrm{U}_3(F^{(2)})$ coincides with the space of all sequences $\{d_n(r)\}_{n\in\mathbb{Z}}$, where

$$d_n(r) = \mathrm{u}_{|n|}^{-1}\left(c_n\sqrt{\frac{2}{|n|!}}\,\frac{r^{|n|}}{\mathrm{e}^{r^2/2}}\right) = \frac{c_n}{\mathrm{e}^{r/2}}, \quad n \in \mathbb{Z}. \tag{6}$$

Let $l_0(r) = \frac{1}{\mathrm{e}^{r/2}}$. Then $\ell_0(r) \in L_2(\mathbb{R}_+)$ and $\|\ell_0(r)\| = 1$. Denote by L_0 the one-dimensional subspace of $L_2(\mathbb{R}_+)$ generated by $l_0(r)$. Then the one-dimensional projection P_0 of $L_2(\mathbb{R}_+)$ onto L_0 is of the form

$$(P_0 f)(r) = \langle f, l_0\rangle l_0 = \int_{\mathbb{R}_+} \frac{f(\rho)}{\mathrm{e}^{(r+\rho)/2}}\,\mathrm{d}\rho. \tag{7}$$

Now, $F^{(3)} = L_0 \otimes \ell_2$ and the orthogonal projection $P^{(3)}$ of

$$\ell_2(L_2(\mathbb{R}_+)) = L_2(\mathbb{R}_+) \otimes \ell_2$$

onto $F^{(3)}$ is obviously of the form $P^{(3)} = P_0 \otimes \mathrm{I}$.

The above work leads to the following theorem:

Theorem 1. *The unitary operator* $\mathrm{U} = \mathrm{U}_3\,\mathrm{U}_2\,\mathrm{U}_1$ *gives an isometric isomorphism of the space* $L_2(\mathbb{C}, \mathrm{d}\mu)$ *onto* $L_2(\mathbb{R}_+) \otimes \ell_2$ *such that the following statements hold:*

(a) *The harmonic Fock space* $F_h^2(\mathbb{C})$ *is mapped onto* $L_0 \otimes l_2$ *by* $\mathrm{U} : F_h^2(\mathbb{C}) \to L_0 \otimes \ell_2$, *where* L_0 *is the one-dimensional subspace of* $L_2(\mathbb{R}_+)$ *generated by the function* $\ell_0(r) = \frac{1}{\mathrm{e}^{r/2}}$.

(b) *The harmonic Bargmann projection is the unitary equivalent of* $\mathrm{U}\,Q\,\mathrm{U}^{-1} = P_0 \otimes \mathrm{I}$, *where* P_0 *is the one-dimensional projection of* $L_2(\mathbb{R}_+)$ *onto* L_0.

Proof. First, we prove (a). For any $f \in F_h^2(\mathbb{C})$ and $f = g + \overline{h}$, according to the definitions of U_1, U_2 and U_3, as well as Equations (3)–(7), we have

$$\begin{aligned} U_3U_2U_1 : (g+\overline{h}) &\mapsto U_3U_2\frac{1}{\pi^{1/2}\,\mathrm{e}^{z\bar{z}/2}}(g+\overline{h}) \\ &\mapsto U_3\left(\{c_n(r)\}_{n\in\mathbb{Z}_+} + \{c_{-n}(r)\}_{n\in\mathbb{Z}_+}\right) \\ &\mapsto \left\{\frac{c_n}{\mathrm{e}^{r/2}}\right\}_{n\in\mathbb{Z}} \\ &\mapsto L_0 \otimes \ell_2, \end{aligned}$$

which shows (a).

To see (b), according to the definitions of Q and U, we obtain

$$\begin{aligned} UQU^{-1} : L_2(\mathbb{R}_+) \otimes \ell_2 &\mapsto UQ : L_2(\mathbb{C}, \mathrm{d}\mu) \\ &\mapsto U : F_h^2(\mathbb{C}) \\ &\mapsto L_0 \otimes \ell_2. \end{aligned}$$

So, it follows from Equation (7) that

$$P_0 \otimes I : L_2(\mathbb{R}_+) \otimes \ell_2 \mapsto L_0 \otimes \ell_2,$$

which shows (b). The proof of Theorem 1 is completed. □

Introduce the isometric imbedding

$$\mathrm{R}_0 : l_2 \to L_2(\mathbb{R}_+) \otimes \ell_2$$

with the rule

$$\mathrm{R}_0 : \{c_n\} \mapsto \{c_n \ell_0(r)\}_{n\in\mathbb{Z}}.$$

The adjoint operator $\mathrm{R}_0^* : L_2(\mathbb{R}_+) \otimes \ell_2 \to \ell_2$ is given by

$$\mathrm{R}_0^* : \{c_n(r)\}_{n\in\mathbb{Z}} \mapsto \left\{ \int_{\mathbb{R}_+} \frac{c_n(\rho)}{\mathrm{e}^{\rho/2}} \, \mathrm{d}\rho \right\}_{n\in\mathbb{Z}} = \{c_n\}_{n\in\mathbb{Z}},$$

$$\mathrm{R}_0^* \, \mathrm{R}_0 = \mathrm{I} : \ell_2 \to \ell_2,$$

and

$$\mathrm{R}_0 \, \mathrm{R}_0^* = P^{(3)} : L_2(\mathbb{R}_+) \otimes \ell_2 \to F^{(3)} = L_0 \otimes \ell_2.$$

Now, the operator $\mathrm{R} = \mathrm{R}_0^* \, \mathrm{U}$ maps the space $L_2(\mathbb{C}, \mathrm{d}\mu)$ onto ℓ_2 and the restriction

$$\mathrm{R}\,|_{F_h^2(\mathbb{C})} : F_h^2(\mathbb{C}) \to \ell_2$$

is an isometric isomorphism. The adjoint operator

$$\mathrm{R}^* = \mathrm{U}^* \, \mathrm{R}_0 : l_2 \to F_h^2(\mathbb{C}) \subset L_2(\mathbb{C}, \mathrm{d}\mu)$$

is an isometric isomorphism of l_2 in the subspace $F_h^2(\mathbb{C})$ of the space $L_2(\mathbb{C}, \mathrm{d}\mu)$.

Remark 1. *It is not difficult to see that*

$$\mathrm{R}\,\mathrm{R}^* = \mathrm{I} : \ell_2 \to \ell_2$$

and

$$\mathrm{R}^* \, \mathrm{R} = Q : L_2(\mathbb{C}, \mathrm{d}\mu) \to F_h^2(\mathbb{C}).$$

Theorem 2. *The isometric isomorphism* $\mathrm{R}^* = \mathrm{U}^* \, \mathrm{R}_0 : l_2 \to F_h^2(\mathbb{C})$ *is given by*

$$\mathrm{R}^* : \{c_n\} \mapsto \sum_{n\in\mathbb{Z}_+} \frac{c_n}{\sqrt{|n|!}} z^n + \sum_{n\in\mathbb{Z}_-} \frac{c_n}{\sqrt{|n|!}} \bar{z}^{|n|}.$$

Proof. Let $\{c_n\} \in \ell_2$. Then we have

$$\begin{aligned} \mathrm{R}^* = \mathrm{U}_1^* \, \mathrm{U}_2^* \, \mathrm{U}_3^* \, \mathrm{R}_0 : \{c_n\}_{n\in\mathbb{Z}} &\mapsto \mathrm{U}_1^* \, \mathrm{U}_2^* \, \mathrm{U}_3^* \left(\left\{ \frac{c_n}{\mathrm{e}^{r/2}} \right\}_{n\in\mathbb{Z}} \right) \\ &= \mathrm{U}_1^* \, \mathrm{U}_2^* \left(\left\{ c_n \sqrt{\frac{2}{|n|!}} \, \frac{r^{|n|}}{\mathrm{e}^{r^2/2}} \right\}_{n\in\mathbb{Z}} \right) \\ &= \mathrm{U}_1^* \left(\frac{1}{\sqrt{2\pi}} \, \frac{1}{\mathrm{e}^{r^2/2}} \sum_{n\in\mathbb{Z}} c_n \sqrt{\frac{2}{|n|!}} \, r^{|n|} t^n \right) \\ &= \sum_{n\in\mathbb{Z}_+} \frac{c_n}{\sqrt{|n|!}} z^n + \sum_{n\in\mathbb{Z}_-} \frac{c_n}{\sqrt{|n|!}} \bar{z}^{|n|}. \end{aligned}$$

The proof of Theorem 2 is thus complete. □

Corollary 1. *A function*

$$f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$$

belongs to the harmonic Fock space $F_h^2(\mathbb{C})$ *if and only if*

$$\sum_{n\in\mathbb{Z}} |a_n|^2 |n|! < \infty$$

and

$$\|f(z)\| = \left(\sum_{n\in\mathbb{Z}} |a_n|^2 |n|! \right)^{1/2}.$$

Corollary 2. *The inverse isomorphism* $\mathrm{R} : F_h^2(\mathbb{C}) \to \ell_2$ *is given by*

$$R : f(z) \mapsto \left\{ \frac{1}{\sqrt{|n|!}} \int_{\mathbb{C}} f(z) \bar{z}^n \, \mathrm{d}\mu(z) \right\}_{n \in \mathbb{Z}}.$$

3. Toeplitz Operators with Radial Symbols

Denote by $L_1^\infty\left(\mathbb{R}_+, \frac{1}{\mathrm{e}^{r^2}}\right)$ the linear space of all measurable functions $a(r)$ on $\mathbb{R}_+$ for which the integrals

$$\int_{\mathbb{R}_+} \frac{|a(r)| r^{|n|}}{\mathrm{e}^{r^2}} \, \mathrm{d}r < \infty, \quad n \in \mathbb{Z}$$

are finite. In this section, we investigate Toeplitz operators with symbols from $L_1^\infty\left(\mathbb{R}_+, \frac{1}{\mathrm{e}^{r^2}}\right)$ acting in the harmonic Fock space $F_h^2(\mathbb{C})$.

Theorem 3. *Let* $a = a(r)$ *belong to* $L_1^\infty\left(\mathbb{R}_+, \frac{1}{\mathrm{e}^{r^2}}\right)$. *Then the Toeplitz operator* T_a *acting in harmonic Fock space* $F_h^2(\mathbb{C})$ *is the unitary equivalent of the multiplication operator* $\gamma_a I$ *acting on* ℓ_2. *The sequence* $\gamma_a = \{\gamma_a(n)\}_{n \in \mathbb{Z}}$ *is given by*

$$\gamma_a(n) = \frac{1}{|n|!} \int_{\mathbb{R}_+} \frac{a(\sqrt{r}) r^{|n|}}{\mathrm{e}^r} \, \mathrm{d}r, \quad n \in \mathbb{Z}.$$

Proof. The operator T_a is obviously the unitary equivalent of the operator

$$\begin{aligned}
\mathrm{R}\,\mathrm{T}_a\,\mathrm{R}^* &= \mathrm{R}\, QaQ\, \mathrm{R}^* \\
&= \mathrm{R}(\mathrm{R}^*\,\mathrm{R}) a (\mathrm{R}^*\,\mathrm{R})\,\mathrm{R}^* \\
&= (\mathrm{R}\,\mathrm{R}^*)\,\mathrm{R}\, a\, \mathrm{R}^* (\mathrm{R}\,\mathrm{R}^*) \\
&= \mathrm{R}\, a\, \mathrm{R}^* \\
&= \mathrm{R}_0^*\, \mathrm{U}_3\, \mathrm{U}_2\, \mathrm{U}_1\, a(r)\, \mathrm{U}_1^{-1}\, \mathrm{U}_2^{-1}\, \mathrm{U}_3^{-1}\, \mathrm{R}_0 \\
&= \mathrm{R}_0^*\, \mathrm{U}_3 (\mathrm{I} \otimes \mathcal{F}) a(r) (\mathrm{I} \otimes \mathcal{F}^{-1})\, \mathrm{U}_3^{-1}\, \mathrm{R}_0 \\
&= \mathrm{R}_0^*\, \mathrm{U}_3 \{a(r)\}\, \mathrm{U}_3^{-1}\, \mathrm{R}_0 \\
&= \mathrm{R}_0^* \{a(\alpha_n^{-1}(r))\}\, \mathrm{R}_0,
\end{aligned}$$

where the function $\alpha_n(r)$ is given by (5) and $\alpha_n^{-1}(r)$ is the inverse of $\alpha_n(r)$. Therefore, it follows that

$$\begin{aligned}
\mathrm{R}_0^* \{a(\alpha_n^{-1}(r))\}\, \mathrm{R}_0 \{c_n\}_{n \in \mathbb{Z}} &= \left\{ \int_{\mathbb{R}_+} a(\alpha_n^{-1}(r)) \frac{c_n}{\mathrm{e}^r} \, \mathrm{d}r \right\}_{n \in \mathbb{Z}} \\
&= \left\{ c_n \int_{\mathbb{R}_+} a(\alpha_n^{-1}(r)) \frac{1}{\mathrm{e}^r} \, \mathrm{d}r \right\}_{n \in \mathbb{Z}} \\
&= \left\{ c_n \int_{\mathbb{R}_+} \frac{a(r) \alpha_n'(r)}{\mathrm{e}^{\alpha_n(r)}} \, \mathrm{d}r \right\}_{n \in \mathbb{Z}} \\
&= \left\{ \frac{2c_n}{|n|!} \int_{\mathbb{R}_+} \frac{a(r) r^{2|n|+1}}{\mathrm{e}^{r^2}} \, \mathrm{d}r \right\}_{n \in \mathbb{Z}} \\
&= \left\{ \frac{c_n}{|n|!} \int_{\mathbb{R}_+} \frac{a(\sqrt{r}) r^{|n|}}{\mathrm{e}^r} \, \mathrm{d}r \right\}_{n \in \mathbb{Z}} \\
&= \{r_a(n) c_n\}_{n \in \mathbb{Z}}.
\end{aligned}$$

Theorem 3 is proved. □

Theorem 4. *The Toeplitz operator* T_a *with radial symbol* $a = a(r)$ *belonging to* $L_1^\infty\left(\mathbb{R}_+, \frac{1}{\mathrm{e}^{r^2}}\right)$ *is bounded on* $F_h^2(\mathbb{C})$ *if and only if* $\gamma_a = \{\gamma_a(n)\}_{n\in\mathbb{Z}} \in l_\infty$ *and* $\|\mathrm{T}_a\| = \sup_{n\in\mathbb{Z}} |\gamma_a(n)|$.

Moreover, the Toeplitz operator T_a *is compact if and only if* $\lim_{n\to\infty} \gamma_a(n) = 0$.

Proof. This follows directly from Theorem 3. □

Proposition 1 below implies that a Toeplitz operator T_a with symbol

$$a(r) \in L_1^\infty\left(\mathbb{R}_+, \frac{1}{\mathrm{e}^{r^2}}\right)$$

is a well-defined linear operator with a dense domain.

Proposition 1. *Let* $F_0^2(\mathbb{C})$ *and* $\overline{F_0^2(\mathbb{C})}$ *denote the set of all polynomials on* z *and* $\bar{z}$*, respectively. If* $p(z) + \overline{p(z)} \in F_0^2(\mathbb{C}) + \overline{F_0^2(\mathbb{C})}$ *and* $a(r) \in L_1^\infty\left(\mathbb{R}_+, \frac{1}{\mathrm{e}^{r^2}}\right)$*, then*

$$\mathrm{T}_a(p + \overline{p}) \in F_0^2(\mathbb{C}) + \overline{F_0^2(\mathbb{C})} \subset F_h^2(\mathbb{C}).$$

Proof. When

$$p_1(z) = p(z) + \overline{p(z)} = \sum_{n=0}^{m} c_n z^n + \sum_{n=0}^{m} c_n \bar{z}^n \in F_0^2(\mathbb{C}) + \overline{F_0^2(\mathbb{C})}$$

and $a(r) \in L_1^\infty\left(\mathbb{R}_+, \frac{1}{\mathrm{e}^{r^2}}\right)$, we have

$$\begin{aligned}
(\mathrm{T}_a\, p_1)(z) &= \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{\left(a(|\xi|)p(\xi) + \overline{p(\xi)}\right)\left(\mathrm{e}^{\bar{\xi} z} + \mathrm{e}^{\xi \bar{z}} - 1\right)}{\mathrm{e}^{|\xi|^2}}\, \mathrm{d}v(\xi) \\
&= \frac{1}{\pi\,\mathrm{i}} \int_{\mathbb{R}_+} \left[\int_{S^1} \left(\sum_{n=0}^{m} c_n r^n t^n\right) \mathrm{e}^{(rz)/t} \frac{\mathrm{d}t}{t}\right] \frac{a(r) r}{\mathrm{e}^{r^2}}\, \mathrm{d}r \\
&\quad + \frac{1}{\pi\,\mathrm{i}} \int_{\mathbb{R}_+} \left[\int_{S^1} \left(\sum_{n=0}^{m} \frac{c_n r^n}{t^n}\right) \mathrm{e}^{(r\bar{z})t} \frac{\mathrm{d}t}{t}\right] \frac{a(r) r}{\mathrm{e}^{r^2}}\, \mathrm{d}r - c_0 \gamma_a(0) \\
&= \sum_{n=0}^{m} \frac{1}{\pi\,\mathrm{i}} \int_{\mathbb{R}_+} \int_{S^1} t^n \left(\sum_{k=0}^{\infty} \frac{r^k z^k}{k! t^k}\right) \frac{\mathrm{d}t}{t} \frac{a(r) r^{n+1}}{\mathrm{e}^{r^2}}\, \mathrm{d}r \\
&\quad + \sum_{n=0}^{m} \frac{1}{\pi\,\mathrm{i}} \int_{\mathbb{R}_+} \int_{S^1} \frac{1}{t^n} \left(\sum_{k=0}^{\infty} \frac{r^k \bar{z}^k t^k}{k!}\right) \frac{\mathrm{d}t}{t} \frac{a(r) r^{n+1}}{\mathrm{e}^{r^2}}\, \mathrm{d}r - c_0 \gamma_a(0) \\
&= \sum_{n=0}^{m} c_n (z^n + \bar{z}^n) \left(\frac{2}{n!} \int_{\mathbb{R}_+} \frac{a(r) r^{2n+1}}{\mathrm{e}^{r^2}}\right) \mathrm{d}r - c_0 \gamma_a(0) \\
&= \sum_{n=0}^{m} c_n (z^n + \bar{z}^n) \gamma_a(n) - c_0 \gamma_a(0).
\end{aligned}$$

Accordingly, we acquire

$$\mathrm{T}_a\, p_1 \in F_0^2(\mathbb{C}) + \overline{F_0^2(\mathbb{C})} \subset F_h^2(\mathbb{C})$$

and the set $F_0^2(\mathbb{C}) + \overline{F_0^2(\mathbb{C})}$ is the domain for each Toeplitz operator T_a with symbol $a(r)$. The proof of Proposition 1 is thus complete. □

By Proposition 1, the Toeplitz operator T_a with symbol $a(r) \in L_1^\infty\left(\mathbb{R}_+, \frac{1}{\mathrm{e}^{r^2}}\right)$ has a bounded extension to the whole space $F_h^2(\mathbb{C})$ if and only if the sequence $\{\gamma_a(n)\}$ is bounded.

Corollary 3. *The spectrum of a bounded Toeplitz operator* T_a *is given by*

$$\operatorname{sp}\mathrm{T}_a = \overline{\{\gamma_a(n) : n \in \mathbb{Z}\}}$$

and its essential spectrum $\operatorname{ess\,sp}\mathrm{T}_a$ *coincides with the set of all limit points of the sequence* $\{\gamma_a(n)\}_{n\in\mathbb{Z}}$*.*

4. Properties of Toeplitz Operators with Radial Symbols

We start with conditions which guarantee the boundedness or compactness of Toeplitz operators with radial symbols from $L_1^\infty\left(\mathbb{R}_+, \frac{1}{\mathrm{e}^{r^2}}\right)$.

Theorem 5. *Let* $a(r) \in L_1^\infty\left(\mathbb{R}_+, \frac{1}{\mathrm{e}^{r^2}}\right)$*. Then the Toeplitz operator* T_a *is bounded on* $F_h^2(\mathbb{C})$ *if one of the following statements holds:*

(1) *The relation* $a(r) \in L_1^\infty(\mathbb{R}_+)$ *is valid.*

(2) *The sequence* $\gamma_a^{(1)} = \{\gamma_a^{(1)}(n)\}_{n\in\mathbb{Z}}$ *is bounded, where*

$$\gamma_a^{(1)}(n) = \frac{1}{|n|!}\int_{\mathbb{R}_+} \frac{\left|a\left(\sqrt{r}\right)\right| r^{|n|}}{\mathrm{e}^r}\,\mathrm{d}r.$$

(3) *The function*

$$B(r) = \int_r^\infty \left|a\left(\sqrt{r}\right)\right| r^{|n|}\,\mathrm{e}^{r-u}\,\mathrm{d}u$$

is bounded.

Proof. The first statement is well known. Let the second statement hold. Then we have

$$\begin{aligned}
|\gamma_a(n)| &\le \frac{1}{|n|!}\left|\int_{\mathbb{R}_+} \frac{a\left(\sqrt{r}\right) r^{|n|}}{\mathrm{e}^r}\,\mathrm{d}r\right| \\
&\le \frac{1}{|n|!}\int_{\mathbb{R}_+} \frac{\left|a\left(\sqrt{r}\right)\right| r^{|n|}}{\mathrm{e}^r}\,\mathrm{d}r \\
&= \gamma_a^{(1)}(n).
\end{aligned}$$

Accordingly, the second statement is proved.

Finally, integrating by parts yields

$$\begin{aligned}
|\gamma_a(n)| &\le \frac{1}{|n|!}\left|\int_{\mathbb{R}_+} \frac{a\left(\sqrt{r}\right) r^{|n|}}{\mathrm{e}^r}\,\mathrm{d}r\right| \\
&= \frac{1}{|n|!}\left|\int_{\mathbb{R}_+} r^{|n|}\,\mathrm{d}\left[\int_r^\infty \frac{a\left(\sqrt{u}\right)}{\mathrm{e}^u}\,\mathrm{d}u\right]\right| \\
&= \frac{1}{(|n|-1|)!}\left|\int_{\mathbb{R}_+} \frac{r^{|n|-1}}{\mathrm{e}^r}\left[\int_r^\infty a\left(\sqrt{u}\right)\mathrm{e}^{r-u}\,\mathrm{d}u\right]\mathrm{d}r\right| \\
&= \frac{1}{(|n|-1)!}\left|\int_{\mathbb{R}_+} \frac{r^{|n|-1}}{\mathrm{e}^r} B(r)\,\mathrm{d}r\right| \\
&= \left|\gamma_B(|n|-1)\right|.
\end{aligned}$$

Further applying the second statement to the function $B(r)$ leads to the third statement. Theorem 5 is thus proved. □

Theorem 6. *Let $a(r) \in L_1^\infty\left(\mathbb{R}_+, \frac{1}{e^{r^2}}\right)$. Then the Toeplitz operator T_a is compact on $F_h^2(\mathbb{C})$ if one of the following statements holds:*

(1) The limit $\lim_{r\to\infty} a(r) = 0$ is valid.

(2) The limits

$$\lim_{n\to\infty} \gamma_a^{(1)}(n) = \lim_{n\to\infty} \frac{1}{|n|!} \int_{\mathbb{R}_+} \frac{|a(\sqrt{r})| r^{|n|}}{e^r} \, dr = 0$$

are valid.

(3) The limits

$$\lim_{r\to\infty} B(r) = \lim_{r\to\infty} \int_r^\infty |a(\sqrt{r})| r^n e^{r-u} \, du = 0$$

are valid.

Proof. This follows directly from Theorem 5. □

A Toeplitz operator T_a with a symbol $a = a(z)$ acting in the harmonic Fock space $F_h^2(\mathbb{C})$ is an operator with the anti-Wick symbol $a = a(z)$. The function $\tilde{a}(z, \bar{z})$ is called a *Wick symbol of an operator* T if this operator acts on $F_h^2(\mathbb{C})$ as follows:

$$\begin{aligned}(\mathrm{T} f)(z) &= \frac{1}{\pi} \int_{\mathbb{C}} f(\zeta) \left[\frac{\tilde{a}(z, \bar{\zeta})}{e^{\bar{\zeta}(\zeta - z)}} + \frac{\tilde{a}(z, \bar{\zeta})}{e^{\zeta(\bar{\zeta} - \bar{z})}} \right] dv(\zeta) \\ &= \frac{1}{\pi} \int_{\mathbb{C}} f(\zeta) \left[\frac{\tilde{a}(z, \bar{\zeta})}{e^{\bar{\zeta} z}} + \tilde{a}(z, \bar{\zeta})\, e^{\zeta \bar{z}} \right] d\mu(\zeta).\end{aligned}$$

The Wick and anti-Wick symbols of the same operator are connected by the formula

$$\tilde{a}(z, \bar{z}) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{a(\zeta)}{e^{(z-\zeta)(\bar{z}-\bar{\zeta})}} \, dv(\zeta).$$

Denote by $\mathcal{M}$ the linear subspace of $L_1^\infty\left(\mathbb{R}_+, \frac{1}{e^{r^2}}\right)$ such that for each $a(r) \in \mathcal{M}$ the Toeplitz operator $\mathrm{T}_{a(r)}$ is bounded on $F_h^2(\mathbb{C})$. Denote by $\mathcal{T}(\mathcal{M})$ the C^*-algebra generated by all Toeplitz operators T_a with symbols $a \in \mathcal{M}$.

The system of functions

$$L_n(z) = \begin{cases} \dfrac{z^n}{\sqrt{n!}}, & n \in \mathbb{Z}_+ \\ \dfrac{\bar{z}^{|n|}}{\sqrt{|n|!}}, & n \in \mathbb{Z}_- \end{cases}$$

is an orthonormal basis for the harmonic Fock space $F_h^2(\mathbb{C})$. Denote by $L_{(n)}$ the one-dimensional space generated by the function $l_{(n)}(z)$. The orthogonal projection $P_{(n)} : F_h^2(\mathbb{C}) \to L_{(n)}$ is obviously of the forms

$$(P_{(n)} f)(z) = \langle f(\zeta), l_{(n)}(z) \rangle l_{(n)}(z) = \frac{z^n}{n!} \frac{1}{\pi} \int_{\mathbb{C}} \frac{f(\zeta) \bar{\zeta}^n}{e^{|\zeta|^2}} \, dv(\zeta), \quad n \in \mathbb{Z}_+$$

and

$$(P_{(n)} f)(z) = \langle f(\zeta), l_{(n)}(z) \rangle l_{(n)}(z) = \frac{\bar{z}^{|n|}}{|n|!} \frac{1}{\pi} \int_{\mathbb{C}} \frac{f(\zeta) \zeta^{|n|}}{e^{|\zeta|^2}} \, dv(\zeta), \quad n \in \mathbb{Z}_-.$$

They are Toeplitz operators with the symbol from $\mathcal{M}$.

For any $n \in \mathbb{Z}$, the one-dimensional space $L_{(n)}$ is an eigenspace for the Toeplitz operator T_a with $a(r) \in \mathcal{M}$, and the corresponding eigenvalue is equal to $\gamma_a(n)$.

Theorem 7. *Let $a(r) \in \mathcal{M}$. Writing the Toeplitz operator T_a in the form of an operator with a Wick symbol gives the spectral decomposition of the operator T_a:*

$$\mathrm{T}_a = \sum_{n=-\infty}^{\infty} \gamma_a(n) P_{(n)}.$$

Proof. For $n \in \mathbb{Z}$ and $f \in F_h^2(\mathbb{C})$, consider the operator with the Wick symbol of the form $p_{(n)}(z,\bar{z}) = \mathrm{e}^{z\bar{z}} \frac{z^n \bar{z}^n}{n!}$. Then

$$\begin{aligned}(\mathrm{T}_a f)(z) &= \frac{1}{\pi} \int_{\mathbb{C}} f(\zeta) \left[\frac{\tilde{a}(z,\bar{\zeta})}{\mathrm{e}^{\bar{\zeta}(\zeta - z)}} + \frac{\tilde{a}(z,\bar{\zeta})}{\mathrm{e}^{\zeta(\bar{\zeta}-\bar{z})}} \right] \mathrm{d}v(\zeta) \\ &= \frac{1}{\pi} \sum_{n=0}^{\infty} \gamma_a(n) \frac{z^n}{n!} \int_{\mathbb{C}} \frac{\bar{\zeta}^n f(\zeta)}{\mathrm{e}^{|\bar{\zeta}|^2}} \mathrm{d}v(\zeta) + \frac{1}{\pi} \sum_{n=-\infty}^{0} \gamma_a(n) \frac{\bar{z}^{|n|}}{|n|!} \int_{\mathbb{C}} \frac{\zeta^{|n|} f(\zeta)}{\mathrm{e}^{|\bar{\zeta}|^2}} \mathrm{d}v(\zeta) \\ &= \sum_{n=-\infty}^{\infty} \gamma_a(n) (P_{(n)} f)(z).\end{aligned}$$

The required proof is complete. □

5. Conclusions

This paper is devoted to studying specific properties (such as the boundedness, compactness, algebraic properties, spectral decomposition and others) of the Toeplitz operator with radial symbols in harmonic Fock spaces. On the basis of analytic functions theory, we present several problems of harmonic functions and expand the scope of the past study. In summary, new important results and features for Toeplitz operators with radial symbols in harmonic Fock spaces are established (see Theorems 5–7). We believe that these newly discovered results will help us study the problems in pluriharmonic Fock spaces or polydisk Fock spaces in future studies.

Author Contributions: Writing—original draft, Z.-L.S., W.-S.D. and F.Q.; writing—review and editing, Z.-L.S., W.-S.D. and F.Q. All authors contributed equally to the manuscript. All authors have read and agreed to the published version of the manuscript.

Funding: Zhi-Ling Sun is partially supported by the Foundation Inner Mongolia Minzu University (Grant No. NMDYB19058). Wei-Shih Du is partially supported by Grant No. NSTC 112-2115-M-017-002 of the 368 National Science and Technology Council of the Republic of China.

Data Availability Statement: Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Acknowledgments: The authors wish to express their hearty thanks to the anonymous referees for their valuable suggestions and comments.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Zhu, K. *Analysis on Fock Spaces*; Graduate Texts in Mathematics, Volume 263; Springer: New York, NY, USA, 2012. [CrossRef]
2. Chen, J.; Wang, X.; Xia, J. Toeplitz operators with positive operator-valued symbols on vector-valued generalized Fock spaces. *Acta Math. Sci.* **2020**, *40*, 625–640. [CrossRef]
3. Cho, H.R. Toeplitz operators on generalized Fock spaces. *Bull. Korean Math. Soc.* **2016**, *53*, 711–722. [CrossRef]
4. He, L.; Wu, B. Dual Toeplitz operators on the orthogonal complement of the Fock–Sobolev space. *J. Math.* **2023**, *2023*, 1679173. [CrossRef]
5. Chalendar, I.; Fricain, E.; Gürdal, M.; Karaev, M. Compactness and Berezin symbols. *Acta Sci. Math.* **2012**, *78*, 315–329. [CrossRef]
6. Cho, H.R.; Park, J.-D.; Zhu, K. Products of Toeplitz operators on the Fock space. *Proc. Am. Math. Soc.* **2014**, *142*, 2483–2489. [CrossRef]
7. Grudsky, S.M.; Vasilevski, N.L. Toeplitz operators on the Fock space: Radial component effects. *Integral Equ. Oper. Theory* **2002**, *44*, 10–37. [CrossRef]
8. Isralowitz, J.; Zhu, K. Toeplitz operators on the Fock space. *Integral Equ. Oper. Theory* **2010**, *66*, 593–611. [CrossRef]

9. Karaev, M.T.; Iskenderov, N.S.H. Berezin number of operators and related questions. *Meth. Funct. Anal. Topo.* **2013**, *19*, 68–72.
10. Le, T.; Li, B. Compact Toeplitz operators on Segal–Bargmann type spaces. *N. Y. J. Math.* **2011**, *17A*, 213–224.
11. Mustafayev, H. Some convergence theorems for operator sequences. *Integral Equ. Oper. Theory* **2020**, *92*, 36. [CrossRef]
12. Stroethoff, K. Hankel and Toeplitz operators on the Fock space. *Mich. Math. J.* **1992**, *39*, 3–16. [CrossRef]
13. Tapdigoglu, R. New Berezin symbol inequalities for operators on the reproducing kernel Hilbert space. *Oper. Matrices* **2021**, *15*, 1031–1043. [CrossRef]
14. Wang, X.; Cao, G.; Zhu, K. Boundedness and compactness of operators on the Fock space. *Intergral Equ. Oper. Theory* **2013**, *77*, 355–370. [CrossRef]
15. Yan, F.; Zheng, D. Products of Toeplitz and Hankel operators on Fock spaces. *Intergral Equ. Oper. Theory* **2020**, *92*, 22. [CrossRef]
16. Guo, K.; Zheng, D. Toeplitz algebra and Hankel algebra on the harmonic Bergman space. *J. Math. Anal. Appl.* **2002**, *276*, 213–230. [CrossRef]
17. Lee, Y.J. Commuting Toeplitz operators on the pluriharmonic Bergman space. *Czechoslov. Math. J.* **2004**, *54*, 535–544. [CrossRef]
18. Lee, Y.J.; Zhu, K. Some differential and integral equations with application to Toeplitz operators. *Integral Equ. Oper. Theory* **2002**, *44*, 466–479. [CrossRef]
19. Grudsky, S.; Karapetyants, A.; Vasilevski, N. Toeplitz operators on the unit ball in $\mathbb{C}^n$ with radial symbols. *J. Oper. Theory* **2003**, *49*, 325–346.
20. Grudsky, S.; Vasilevski, N. Bergman–Toeplitz operators: Radial component influence. *Integral Equ. Oper. Theory* **2001**, *40*, 16–33. [CrossRef]
21. Li, R.; Lu, Y.F. Radial operators on the weighted Bergman spaces over the polydisk. *Acta Math. Sin. (Engl. Ser.)* **2019**, *35*, 227–238. [CrossRef]
22. Sun, Z.L.; Lu, Y.F. Toeplitz operators on the weighted pluriharmonic Bergman space with radial symbols. *Abstr. Appl. Anal.* **2011**, *2011*, 210596. [CrossRef]
23. Berezin, F.A. Wick and Anti-Wick Symbols. Mathematics of the USSR-Sbornik, Volume 15, Number 4, 577. Available online: https://iopscience.iop.org/article/10.1070/SM1971v015n04ABEH001564/pdf (accessed on 12 December 2023).