Evaluating Infinite Series Involving Harmonic Numbers by Integration

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Abstract: Eight infinite series involving harmonic-like numbers are coherently and systematically reviewed. They are evaluated in closed form exclusively by integration together with calculus and complex analysis. In particular, a mysterious series \( W \) is introduced and shown to be expressible in terms of the trilogarithm function. Several remarkable integral values and difficult infinite series identities are shown as consequences.

Keywords: Euler sum; harmonic number; trilogarithm; Catalan’s constant; Riemann zeta function

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1. Introduction and Outline

Let \( \mathbb{N} \) be the set of natural numbers with \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). For \( n \in \mathbb{N}_0 \), define four classes of the harmonic-like numbers (cf. [1–5]) as follows:

- **Harmonic number**
  \[ H_n = \sum_{k=1}^{n} \frac{1}{k}. \]

- **Skew harmonic number**
  \[ O_n = \sum_{k=1}^{n} \frac{1}{2k-1}. \]

- **Alternating harmonic number**
  \[ \bar{H}_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}. \]

- **Alternating skew harmonic number**
  \[ \bar{O}_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{2k-1}. \]

In 1775, Euler discovered the following well-known identity

\[ \sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\zeta(3), \]

where \( \zeta \) stands for the Riemann zeta function

\[ \zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \quad \text{for} \quad \Re(z) > 1. \]
Since then, numerous series involving harmonic-like numbers (nowadays called Euler sums, cf. [5–7]) have been evaluated, for example, by generating functions [2,8,9], partial fraction decompositions [10] and the hypergeometric series method [4]. Some of them can also be done by computing integrals (cf. [11–13]). Even though there is already a full-scale exploration for the series about harmonic numbers \( \{H_n, O_n\} \), examinations for the series involving alternating harmonic numbers \( \{\bar{H}_n, \bar{O}_n\} \) are much fewer. This motivates the authors to investigate, in this paper, four classes of Euler sums containing not only \( \{H_n, O_n\} \), but also \( \{\bar{H}_n, \bar{O}_n\} \) (especially) in their summands. Our approach will exclusively depend upon integral representations, which consists of three main steps:

- For a given Euler sum “\( S = \sum T(n) \)”, figure out an integral representation for a “key” factor of the summand \( T(n) \).
- Exchange the order between summation and integration, and then work out a definite integral expression for the Euler sum \( S \).
- Evaluate the integral in closed form as long as possible in order to find an exact value for the Euler sum \( S \).

The rest of the paper will be organized as follows. In the next section, we shall review two identities about harmonic numbers, including the aforementioned Euler’s identity and its alternating companion

\[
\sum_{n=1}^{\infty} (-1)^{n-1} \frac{H_n}{n^2} = \frac{5}{8} \zeta(3).
\]

Then, for skew harmonic numbers, the integration method will be employed in Section 2 to rederive two known formulae below:

\[
\sum_{n=1}^{\infty} \frac{O_n}{n^2} = \frac{7}{4} \zeta(3) \quad \text{and} \quad \sum_{n=1}^{\infty} (-1)^{n-1} \frac{O_n}{n^2} = \pi G - \frac{7}{4} \zeta(3),
\]

where \( G \) denotes Catalan’s constant (cf. [14])

\[
G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \approx 0.915965594.
\]

To the authors’ knowledge, most results (including Theorems 5–8 and integral expressions for \( \mathfrak{W} \)) in the remaining parts of the paper seem new. In Section 3, we shall prove the two identities about alternating harmonic numbers:

\[
\sum_{n=1}^{\infty} \frac{\bar{H}_n}{n^2} = \frac{\pi^2}{4} \ln 2 - \frac{1}{4} \zeta(3) \quad \text{and} \quad \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\bar{H}_n}{n^2} = \frac{\pi^2}{4} \ln 2 - \frac{5}{8} \zeta(3).
\]

As a preliminary work for Section 5, the following mysterious series is introduced and examined in Section 4 (where the hypergeometric \( _4F_3 \)-series will be defined)

\[
\mathfrak{W} = \sum_{n=0}^{\infty} \frac{(2n)^2}{16^n(2n+1)^2} = _4F_3 \left[ \begin{array}{c} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{1}{2}, \frac{1}{2}, 1 \end{array} \right].
\]

We find three integral expressions and two remarkably explicit formulæe

\[
\frac{\pi \mathfrak{W}}{4} = 43 \left\{ \text{Li}_3(1+i) \right\} - \frac{\pi^3}{8} - \frac{\pi}{4} \ln^2 2,
\]

\[
\frac{\pi}{4} \mathfrak{W} = \frac{3\pi^3}{32} + \frac{\pi \ln^2 2}{8} - 43 \left\{ \text{Li}_3 \left( \frac{1+i}{2} \right) \right\},
\]
where the polylogarithm function \( \text{Li}_n(z) \) is defined by (cf. Lewin [15])

\[
\text{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n} \quad \text{with} \quad \text{Li}_3\left(\frac{1+i}{2}\right) \approx 0.486159537 + 0.570077407i.
\]

Finally, in Section 5, the two difficult series concerning alternating skew harmonic numbers will explicitly be evaluated in closed form:

\[
\sum_{n=1}^{\infty} \frac{O_n}{n^2} = \frac{\pi^2}{4} + \frac{\pi^3}{16} - 2G\ln 2 \quad \text{and} \quad \sum_{n=1}^{\infty} (-1)^{n-1} \frac{O_n}{n^2} = 4G\ln 2 - \frac{\pi^2}{2}.
\]

In the course of working on these aforementioned series, a number of bisection series identities are shown and several significant integrals emerge as byproducts whose evaluations are recorded as applications.

2. Infinite Series Containing \( H_n \)

This section is devoted to reviewing a few series involving harmonic numbers \( H_n \) by definite integrals. First, we shall examine two well-known expressions of \( \zeta(3) \) as positive and alternating series. Then we shall evaluate two further bisection series by their combinations.

2.1. Positive Series

As a warm up, we first review Euler’s well-known identity. For different proofs, the reader can refer to [4–6,11] and the references therein.

**Theorem 1** (Euler, 1775).

\[
\sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\zeta(3).
\]

**Proof.** According to the integral representation

\[
\int_0^1 x^{n-1} \ln(1-x)dx = - \frac{H_n}{n},
\]

we can express the series as an integral and then evaluate it as follows:

\[
\sum_{n=1}^{\infty} \frac{H_n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \int_0^1 x^{n-1} \ln(1-x)dx
\]

\[
= \int_0^1 \ln(1-x) \sum_{n=1}^{\infty} \frac{(-x^n)}{n} dx
\]

\[
= \int_0^1 \ln(1-x) \frac{x^n}{x^n} dx = \int_0^1 \ln(1-y) dy
\]

\[
= \sum_{n=1}^{\infty} \frac{1}{n^{1}} \ln^2 y dy = 2\zeta(3).
\]

Hereafter, exchanging the order of summation and integration is justified by Lebesgue’s dominated convergence theorem [16] (§11.32). \( \square \)

Instead, by making use of the generating function

\[
\sum_{n=1}^{\infty} H_n x^n = \ln(1-x) \quad \frac{1}{x-1},
\]
we can alternatively deduce the same formula as before:

\[
\sum_{n=1}^{\infty} \frac{H_n}{n^2} = \int_0^1 \frac{dy}{y} \int_0^y \frac{\ln(1-x)}{x(x-1)} \, dx = \int_0^1 \frac{\ln x \ln(1-x)}{x(1-x)} \, dx \\
= \int_0^1 \frac{\ln x \ln(1-x)}{x} \, dx + \int_0^1 \frac{\ln x \ln(1-x)}{1-x} \, dx \\
= 2 \int_0^1 \frac{\ln x \ln(1-x)}{x} \, dx \\
= -2 \sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 x^{n-1} \ln x \, dx = 2\zeta(3).
\]

2.2. Alternating Series

Next, we examine the alternating series counterpart of Theorem 1 as below.

**Theorem 2** (cf. [7,11]).

\[
\sum_{n=1}^{\infty} (-1)^n \frac{H_n}{n^2} = \frac{5}{8}\zeta(3).
\]

**Proof.** By making use of (1), we can express the series as an integral

\[
\sum_{n=1}^{\infty} (-1)^n \frac{H_n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \int_0^1 x^{n-1} \ln(1-x) \, dx \\
= \int_0^1 \frac{\ln(1-x)}{x} \sum_{n=1}^{\infty} \frac{(-x)^n}{n} \, dx \\
= -\int_0^1 \frac{\ln(1-x) \ln(1+x)}{x} \, dx.
\]

Recalling the generating function

\[
\frac{\ln^2(1+x)}{2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} H_{n-1} x^n
\]

and then applying the integral formula

\[
2\zeta(3) = \int_0^1 \frac{\ln^2(1-x)}{x} \, dx = \int_0^1 \frac{\ln^2 y}{1-y} \, dy,
\]

we can also express the series as

\[
\sum_{n=1}^{\infty} (-1)^n \frac{H_n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{H_{n-1}}{n^2} \\
= \frac{3}{4}\zeta(3) - \int_0^1 \frac{\ln^2(1+x)}{2x} \, dx \\
= \frac{7}{4}\zeta(3) - \frac{1}{2} \int_0^1 \frac{\ln^2(1+x) + \ln^2(1-x)}{x} \, dx.
\]
By putting the two expressions together, we have
\[ 2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} H_n = \frac{7}{4} \zeta(3) - \frac{1}{2} \int_{0}^{1} \frac{\ln^2(1-x)}{x} \, dx \]
\[ = \frac{7}{4} \zeta(3) - \frac{1}{4} \int_{0}^{1} \frac{\ln^2(1-y)}{y} \, dy \]
\[ = \frac{7}{4} \zeta(3) - \frac{1}{2} \zeta(3) = \frac{5}{4} \zeta(3), \]

which proves the desired formula stated in Theorem 2. A different proof by the hypergeometric series method can be found in [4].

As byproducts, we have three similar integral formulae
\[ -\frac{5}{8} \zeta(3) = \int_{0}^{1} \frac{\ln(1+x) \ln(1-x)}{x} \, dx, \quad (3) \]
\[ \frac{1}{4} \zeta(3) = \int_{0}^{1} \frac{\ln^2(1+x)}{x} \, dx, \quad (4) \]
\[ \frac{7}{2} \zeta(3) = \int_{0}^{1} \frac{\ln^2 \frac{1+x}{x}}{x} \, dx. \quad (5) \]

Furthermore, by making use of the generating function
\[ \ln \frac{1}{1-x} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} H_n x^n, \]
we can deduce the third integral representation
\[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} H_n x^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} H_n x^n, \]
\[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} H_n x^n = \int_{0}^{1} \frac{\ln(1+x) \ln(1-x)}{x(1+x)} \, dx = \int_{0}^{1} \frac{\ln(1+x)}{x(1+x)} \, dx \int_{0}^{1} \frac{\ln y}{y} \, dy \]
\[ = -\int_{0}^{1} \frac{\ln x \ln(1+x)}{x(1+x)} \, dx = \frac{3}{4} \zeta(3) + \int_{0}^{1} \frac{\ln x \ln(1+x)}{1+x} \, dx, \]

which yields yet another integral identity
\[ \frac{\zeta(3)}{8} = \int_{0}^{1} \frac{\ln x \ln(1+x)}{1+x} \, dx. \quad (6) \]

2.3. Bisection Series

By combining Theorems 1 and 2, we deduce two further identities
\[ \sum_{n=1}^{\infty} \frac{H_{2n}}{n^2} = 2 \sum_{n=1}^{\infty} \frac{H_n}{n^2} - 2 \sum_{n=1}^{\infty} (-1)^{n-1} \frac{H_n}{n^2} = \frac{11}{4} \zeta(3), \quad (7) \]
\[ \sum_{n=1}^{\infty} \frac{H_{2n-1}}{(2n-1)^2} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{H_n}{n^2} + \frac{1}{2} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{H_n}{n^2} = \frac{21}{16} \zeta(3). \quad (8) \]

3. Infinite Series Containing \( O_n \)

Further series about skew harmonic numbers \( O_n \) will be examined in this section.

3.1. Positive Series

For series containing skew harmonic numbers \( O_n \), Doedler [7] derived, by manipulating the digamma function, the following identity (see [4] for a proof by the hypergeometric series approach).
Theorem 3 (cf. [6,9]).
\[ \sum_{n=1}^{\infty} \frac{O_n}{n^2} = \frac{7}{4} \zeta(3). \]

Proof. Applying the integral representation
\[ \int_0^1 x^{2n-1} \ln \frac{1+x}{1-x} \, dx = \frac{O_n}{n}, \quad (9) \]
we can evaluate the series as follows:
\[ \sum_{n=1}^{\infty} \frac{O_n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 x^{2n-1} \ln \frac{1+x}{1-x} \, dx = \int_0^1 \frac{1}{x} \ln \frac{1+x}{1-x} \sum_{n=1}^{\infty} \frac{x^{2n}}{n} \, dx \]
\[ = \int_0^1 \frac{\ln(1-x^2)}{x} \ln \frac{1-x}{1+x} \, dx = \int_0^1 \frac{\ln^2(1-x) - \ln^2(1+x)}{x} \, dx \]
\[ = 2\zeta(3) - \frac{1}{4} \zeta(3) = \frac{7}{4} \zeta(3), \]
where we have invoked (2) and (4). This proves Theorem 3. \(\square\)

3.2. Alternating Series

Now, we turn to examining the alternating series.

Theorem 4 (cf. [7,12]).
\[ \sum_{n=1}^{\infty} (-1)^{n-1} \frac{O_n}{n^2} = \pi G - \frac{7}{4} \zeta(3). \]

Proof. According to the Maclaurin series
\[ \arctan x = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k-1} x^{2k-1}, \quad (10) \]
it is routine to expand the product
\[ \arctan^2 x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} O_n}{n} x^{2n}. \quad (11) \]
Multiplying both sides by \( \frac{2}{x} \), and then integrating over \([0, 1]\), we confirm the summation formula in Theorem 4 as follows:
\[ \sum_{n=1}^{\infty} (-1)^{n-1} \frac{O_n}{n^2} = 2 \int_0^1 \frac{\arctan^2 x}{x} \, dx = \pi G - \frac{7}{4} \zeta(3), \]
where the above integral has been evaluated in [17,18]. \(\square\)

There exists another generating function
\[ \sum_{n=1}^{\infty} (-1)^{n-1} O_n x^{2n-1} = \frac{\arctan x}{1+x^2}. \]
By integrating it twice, we can deduce
\[ \sum_{n=1}^{\infty} (-1)^{n-1} \frac{O_n}{n^2} = 4 \int_0^1 \frac{dy}{y} \int_0^y \frac{\arctan x}{1+x^2} \, dx \]
\[ = -4 \int_0^1 \ln x \arctan x \, dx, \]
which implies the following integral identity:

\[
\int_0^1 \ln x \arctan x \frac{dx}{1 + x^2} = \frac{7}{16} \zeta(3) - \frac{\pi}{4} G. \tag{12}
\]

Alternatively, the same series can be reformulated, by means of (9), as

\[
\sum_{n=1}^{\infty} (-1)^{n-1} \frac{O_n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \int_0^1 x^{2n-1} \ln \frac{1 + x}{1 - x} dx
\]

\[
= \int_0^1 \frac{1}{x} \ln \frac{1 + x}{1 - x} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n}}{n} dx
\]

\[
= \int_0^1 \ln \frac{1 + x^2}{x} \ln \frac{1 + x}{1 - x} dx.
\]

This leads us to yet another integral formula

\[
\int_0^1 \ln \left(\frac{1 + x^2}{x} \right) \ln \frac{1 + x}{1 - x} dx = \pi G - \frac{7}{4} \zeta(3). \tag{13}
\]

In the course of proving Theorem 4, the following integral formula was crucial:

\[
\int_0^1 \frac{\arctan^2 x}{x} dx = \frac{\pi G}{2} - \frac{7}{8} \zeta(3). \tag{14}
\]

Here we provide an easier proof. Under the change in variables

\[
\frac{1 + ix}{1 - ix} \to y \quad \iff \quad x \to i \left(\frac{1}{1 + y} - 1\right)
\]

the integral can be reformulated and then evaluated as follows:

\[
\int_0^1 \frac{\arctan^2 x}{x} dx = -\frac{1}{4} \int_0^1 \frac{1}{x} \ln^2 \frac{1 + iy}{1 - iy} dx = \frac{1}{2} \int_1^i \frac{\ln^2 y}{1 - y^2} dy
\]

\[
= \frac{1}{2y} \sum_{n=1}^{\infty} \left( y^{2n} \ln^2 y - \frac{2y^{2n} \ln y}{2n - 1} - \frac{2y^{2n}}{(2n - 1)^2} \right) \bigg|_1^i
\]

which simplifies into the closed expression \(\frac{\pi G}{2} - \frac{7}{8} \zeta(3)\) as in (14).

### 3.3. Bisection Series

Combining Theorem 3 with Theorem 4 yields two further identities:

\[
\sum_{n=1}^{\infty} \frac{O_{2n}}{n^2} = 2 \sum_{n=1}^{\infty} \frac{O_n}{n^2} - 2 \sum_{n=1}^{\infty} (-1)^{n-1} \frac{O_n}{n^2} = 7\zeta(3) - 2\pi G, \tag{15}
\]

\[
\sum_{n=1}^{\infty} \frac{O_{2n-1}}{(2n-1)^2} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{O_n}{n^2} + \frac{1}{2} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{O_n}{n^2} = \frac{\pi G}{2}. \tag{16}
\]

By further employing the two relations below

\[H_{2n} = O_n + \frac{1}{2} H_n \quad \text{and} \quad \bar{H}_{2n} = O_n - \frac{1}{2} H_n,\]
we deduce from Theorems 2 and 4 the following two identities:

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1} H_{2n}}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} O_n}{n^2} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} H_n}{n^2} = \pi G - \frac{23}{16} \zeta(3),
\]

(17)

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1} H_{2n}}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} O_n}{n^2} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} H_n}{n^2} = \pi G - \frac{33}{16} \zeta(3).
\]

(18)

In addition, for the quadratic skew harmonic numbers defined by

\[ O_n^{(2)} = \sum_{k=1}^{n} \frac{1}{(2k-1)^2}, \]

we can prove the following identity, which should be a known one, even though we failed to locate it in the literature:

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} O_n^{(2)} = \frac{7}{4} \zeta(3) - \frac{\pi G}{2}.
\]

(19)

In fact, recalling (10) and then applying integration by parts, we can express

\[
\int_{0}^{1} \frac{\arctan^2 x}{x} \, dx = \int_{0}^{1} \frac{\arctan x \sum_{k=1}^{\infty} (-1)^{k+1} x^{2k-2}}{2k-1} \, dx
\]

\[
= \frac{\pi G}{4} + \sum_{k=1}^{\infty} \frac{(-1)^{k}}{(2k-1)^2} \int_{0}^{1} \frac{x^{2k-1}}{1+x^2} \, dx
\]

\[
= \frac{\pi G}{4} + \sum_{k=1}^{\infty} \frac{(-1)^{k+1-1}}{(2k-1)^2} \int_{0}^{1} x^{2k+2j-3} \, dx
\]

\[
= \frac{\pi G}{4} + \sum_{k=1}^{\infty} \frac{(-1)^{k+1-1}}{2(2k-1)^2(2k+2j-1)} \]

\[
\left[ k+j-1 \to n \right]
\]

\[
= \frac{\pi G}{4} + \frac{1}{2n} \sum_{k=1}^{n} \frac{(-1)^{n}}{n} O_n^{(2)}.
\]

In view of (14), we can confirm identity (19) as follows:

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} O_n^{(2)} = \frac{\pi G}{2} - 2 \int_{0}^{1} \frac{\arctan^2 x}{x} \, dx = \frac{7}{4} \zeta(3) - \frac{\pi G}{2}.
\]

4. Infinite Series Containing $H_n$

Analogously by the integration approach, we shall evaluate in this section similar infinite series involving alternating harmonic numbers $H_n$ in closed form. For further series about $H_n$, the reader can refer to [2].

4.1. Positive Series

First, we consider the series as in the following theorem.

Theorem 5.

\[
\sum_{n=1}^{\infty} \frac{H_n}{n^2} = \frac{\pi^2}{4} - \ln 2 - \frac{1}{4} \zeta(3).
\]
Proof. For the sake of brevity, let $\chi$ stand for the logical function defined by $\chi(\text{true}) = 1$ and $\chi(\text{false}) = 0$. Denote by $i \equiv_m j$ the congruence relation between $i$ and $j$ modulo $m$. Recall the integral representation

$$\int_0^1 x^{n-1} \ln(1 + x)dx = (-1)^n \frac{\mathcal{H}_n}{n} + \frac{2\ln 2}{n} \chi(n \equiv 2 1). \tag{20}$$

We can express the series as a single integral

$$\sum_{n=1}^{\infty} \frac{\mathcal{H}_n}{n^2} = \sum_{n=1}^{\infty} \frac{2\ln 2}{n^2} \chi(n \equiv 2 1) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \int_0^1 x^{n-1} \ln(1 + x)dx$$

$$= \sum_{n=1}^{\infty} \frac{2\ln 2}{(2n-1)^2} - \int_0^1 \frac{\ln^2(1 + x)}{x} dx$$

$$= \frac{\pi^2}{4} \ln 2 - \frac{1}{4} \zeta(3),$$

where the rightmost integral is carried out by (4). This proves Theorem 5. \qed

Alternatively, by invoking the generating function

$$\ln(1 + x) = \sum_{n=1}^{\infty} \mathcal{H}_n x^n,$$

we can also express the series as

$$\sum_{n=1}^{\infty} \frac{\mathcal{H}_n}{n^2} = \int_0^1 \frac{dy}{y} \int_0^y \frac{\ln(1 + x)}{x(1-x)} dx = \int_0^1 \frac{\ln(1 + x)}{x(1-x)} dx \int_0^1 \frac{dy}{y}$$

$$= \int_0^1 \ln x \ln(1 + x) \ln(1 - x) dx = \frac{3}{4} \zeta(3) - \int_0^1 \frac{\ln x \ln(1 + x)}{1 - x} dx.$$

By appealing to Theorem 5, we derive the following integral formula:

$$\int_0^1 \frac{\ln x \ln(1 + x)}{1 - x} dx = \zeta(3) - \frac{\pi^2}{4} \ln 2. \tag{21}$$

4.2. Alternating Series

Next, we examine the alternating series.

Theorem 6.

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\mathcal{H}_n}{n^2} = \frac{\pi^2}{4} \ln 2 - \frac{5}{8} \zeta(3).$$

Proof. Reformulate the series by (20) analogously, we can evaluate it as follows:

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\mathcal{H}_n}{n^2} = \sum_{n=1}^{\infty} \frac{2\ln 2}{n^2} \chi(n \equiv 2 1) - \sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 x^{n-1} \ln(1 + x)dx$$

$$= \sum_{n=1}^{\infty} \frac{2\ln 2}{(2n-1)^2} + \int_0^1 \frac{\ln(1 + x) \ln(1 - x)}{x} dx$$

$$= \frac{\pi^2}{4} \ln 2 - \frac{5}{8} \zeta(3),$$

where we have employed (3). Therefore, the identity in Theorem 6 is confirmed. \qed
Instead, by making use of the generating function
\[-\ln(1-x) \over 1+x = \sum_{n=1}^{\infty} (-1)^{n-1} H_n x^n,\]
we can also express the series as
\[\sum_{n=1}^{\infty} (-1)^{n-1} {H_n \over n^2} = \int_0^1 {dy \ln(1-x) \over x(1+x)} + \int_0^1 {\ln(1-x) \over x(1+x)} dx = \int_0^1 {\ln(1-x) \over x(1+x)} dx = \zeta(3) - \int_0^1 {\ln(1-x) \over 1+x} dx.\]
By applying Theorem 6, we derive the following integral formula
\[\int_0^1 {\ln(1-x) \over 1+x} dx = {13 \over 8} \zeta(3) - {\pi^2 \over 4} \ln 2.\] (22)

4.3. Bisection Series

By combining Theorem 5 with Theorem 6, we can derive, without difficulty, the following pair of identities:
\[\sum_{n=1}^{\infty} {H_{2n} \over n^2} = 2 \sum_{n=1}^{\infty} {H_n \over n^2} - 2 \sum_{n=1}^{\infty} (-1)^{n-1} {H_n \over n^2} = {3 \over 4} \zeta(3),\] (23)
\[\sum_{n=1}^{\infty} {H_{2n-1} \over (2n-1)^2} = {1 \over 2} \sum_{n=1}^{\infty} {H_n \over n^2} + {1 \over 2} \sum_{n=1}^{\infty} (-1)^{n-1} {H_n \over n^2} = {\pi^2 \over 4} \ln 2 - {7 \over 16} \zeta(3).\] (24)

Putting (7) and (15) together gives rise to two further identities:
\[\sum_{n=1}^{\infty} {H_{4n} \over n^2} = \sum_{n=1}^{\infty} {O_{4n} \over n^2} + {1 \over 2} \sum_{n=1}^{\infty} {H_{2n} \over n^2} = {67 \over 8} \zeta(3) - 2\pi G,\] (25)
\[\sum_{n=1}^{\infty} {H_{4n} \over n^2} = \sum_{n=1}^{\infty} {O_{4n} \over n^2} - {1 \over 2} \sum_{n=1}^{\infty} {H_{2n} \over n^2} = {45 \over 8} \zeta(3) - 2\pi G.\] (26)

Furthermore, by relating (7) to (17) and (18) to (23), respectively, we also find the following two identities:
\[\sum_{n=1}^{\infty} {H_{4n-2} \over (2n-1)^2} = {1 \over 2} \sum_{n=1}^{\infty} {H_{2n} \over n^2} + {1 \over 2} \sum_{n=1}^{\infty} (-1)^{n-1} {H_{2n} \over n^2} = {\pi G \over 2} + {21 \over 32} \zeta(3),\] (27)
\[\sum_{n=1}^{\infty} {H_{4n-2} \over (2n-1)^2} = {1 \over 2} \sum_{n=1}^{\infty} {H_{2n} \over n^2} - {1 \over 2} \sum_{n=1}^{\infty} (-1)^{n-1} {H_{2n} \over n^2} = {\pi G \over 2} - {21 \over 32} \zeta(3).\] (28)

5. Mysterious Series \(\mathfrak{M}\)

When making attempts to evaluate series involving the alternating skew harmonic numbers \(\mathcal{O}_n\) (see the next section), we came across the following infinite series:
\[\mathfrak{M} := \sum_{k=0}^{\infty} \left(2k-1\right)^2 {2 \choose k}^2 {1 \over 16^k (2k+1)^2} = 4 F_3 \left[ {1 \over 2, 1 \over 2, 1 \over 2 \over 1, 2, 2 \over 2} ; {1 \over 1, 2, 3} \right] \approx 1.037947764,
\]
where the classical hypergeometric series (cf. Bailey [19] (§2.1)) is defined by
\[p F_q \left[ {a_1, a_2, \ldots, a_p \over b_1, b_2, \ldots, b_q} ; z \right] = \sum_{n=0}^{\infty} \left( (a_1)_n (a_2)_n \cdots (a_p)_n \right) z^n \left( b_1 \right)_n \left( b_2 \right)_n \cdots (b_q)_n \left( n! \right)^{-1}.\]
with the shifted factorial being given by

\[(x)_0 = 1 \quad \text{and} \quad (x)_n = x(x+1)\cdots(x+n-1) \quad \text{for} \quad n \in \mathbb{N}.\]

The series \( \mathcal{W} \) is mysterious because it does not seem to have a closed-form formula, but admits several remarkable expressions in terms of definite integrals and other infinite series. As a preparation for the computations in the next section, we record below what we have accomplished about this series \( \mathcal{W} \).

5.1. Integral Representations

Recall the binomial series

\[
\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^n}{n!} x^n = \frac{1}{\sqrt{1-x}}.
\]

Writing in terms of the \( \Gamma \)-function (cf. Rainville [20] (§8))

\[
\frac{\left(\frac{1}{2}\right)_n}{n!} = \frac{\Gamma\left(n + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{n!\Gamma(n+1)} = \frac{1}{\pi} \int_0^1 \frac{y^{n-\frac{1}{2}}}{\sqrt{1-y}} dy,
\]

we can express the series as the following triple integral:

\[
4\pi \mathcal{W} = \sum_{n=0}^{\infty} \frac{4\pi \left(\frac{1}{2}\right)^2}{16^n(2n+1)^2} = \sum_{n=0}^{\infty} \frac{4\pi \left(\frac{1}{2}\right)^2}{(n!)^2(2n+1)^2} = \int_0^1 \frac{dy}{y\sqrt{1-y}} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n y^{n+\frac{1}{2}}}{(n!)^2(n+\frac{1}{2})^2} = \int_0^1 \frac{dy}{y\sqrt{1-y}} \int_0^y \frac{dT}{T} \int_0^T \frac{dx}{\sqrt{x(1-x)}}.
\]

- First, integrating with respect to \( y \) over \([T, 1]\) gives that

\[
2\pi \mathcal{W} = \int_0^1 \int_0^T \frac{\text{arctanh} \sqrt{1-T} dxdT}{T\sqrt{x(1-x)}}.
\]

- Second, integrating with respect to \( x \) over \([0, T]\) yields that

\[
\pi \mathcal{W} = \int_0^1 \frac{\text{arctanh} \sqrt{1-T} \arccos \sqrt{1-T} dT}{T}.
\]

- Third, making the change in variables \( \sqrt{1-T} \to x \) results in

\[
\frac{\pi}{2} \mathcal{W} = \int_0^1 \frac{x \text{arctanh} x \arccos x}{1-x^2} dx. \quad (29)
\]

According to the power series expansion

\[
\frac{x \text{arctanh} x}{1-x^2} = \sum_{n=1}^{\infty} \mathcal{O}_n x^{2n},
\]

we can derive an alternative expression

\[
\frac{\pi}{2} \mathcal{W} = \sum_{n=1}^{\infty} \mathcal{O}_n \int_0^1 x^{2n} \arccos x dx = \sum_{n=1}^{\infty} \frac{4^n O_n}{(2n+1)^2} \quad (30)
\]
where we have employed the integral identity
\[
\int_0^1 x^{2n} \arccos x \, dx = \frac{4^n}{(2n)(2n+1)^2}.
\]

- Fourth, making the change in variables \( x \to \cos 2\theta \), we find a familiar expression:
\[
\frac{\pi}{8} \mathfrak{M} = \int_0^{\frac{\pi}{4}} \theta \cot(2\theta) \ln \cot \theta \, d\theta.
\] (31)

5.2. First Expression of \( \mathfrak{M} \) in Polylogarithm

By Mathematica, the following integral value is detected:
\[
\int_0^{\frac{\pi}{4}} \theta \cot(2\theta) \ln \cot \theta \, d\theta = 2i \Li_3 \left( \frac{1+i}{2} \right) + \frac{3\pi^3}{64} + \frac{\pi \ln^2 2}{16} - \frac{35i}{32} \zeta(3) - \frac{i \ln^3 2}{24} + \frac{5i\pi^2}{96} \ln 2,
\]

Consequently, we arrive at the expression in terms of the trilogarithm function:
\[
\frac{\pi}{4} \mathfrak{M} = \frac{3\pi^3}{32} + \frac{\pi \ln^2 2}{8} - 43 \left\{ \Li_3 \left( \frac{1+i}{2} \right) \right\}.
\] (32)

This trilogarithm value \( \Li_3 \left( \frac{1+i}{2} \right) \) was extensively studied in [21], where it is called a natural companion of Catalan’s constant. The reader can refer to “Closed Form for the Imaginary Part of \( \Li_3 \left( \frac{1+i}{2} \right) \)” (https://math.stackexchange.com/questions/918680, accessed on 11 November 2023) for a discussion about its real and imaginary parts.

However, a rigorous proof for (32) is indispensable.

By applying integration by parts, we can rewrite (29) as
\[
\frac{\pi}{2} \mathfrak{M} = \int_0^1 \ln(1 - x^2) \arccos x \, dx - \int_0^1 \ln(1 - x^2) \arctanh x \, dx
\]
\[
= \int_0^{\frac{\pi}{2}} y \ln \sin y \, dy + \int_0^{\frac{\pi}{2}} \ln \sin y \ln \tan \left( \frac{y}{2} \right) \, dy.
\] (x → \cos y)

The above two integrals can further be reformulated as
\[
I = \int_0^{\frac{\pi}{2}} y \ln \sin y \, \frac{dy}{\sin y} = \int_0^{\frac{\pi}{2}} 2i y \ln \frac{e^{iy} - e^{-iy}}{2iy} \, dy
\]
\[
= 2i \int_1^\infty \frac{\ln x}{1 - x^2} \ln \left( \frac{-2ix}{1 - x^2} \right) \, dx,
\]
\[
J = \int_0^{\frac{\pi}{2}} \ln \sin y \ln \tan \left( \frac{y}{2} \right) \, dy = \int_0^{\frac{\pi}{2}} \frac{\ln \left( 1 - e^{iy} \right)}{2i} \ln \left( \frac{1 - e^{iy}}{1 + e^{iy}} \right) \, dy
\]
\[
= -2i \int_0^1 \ln \left( -\frac{ix}{1 - x^2} \right) \, dx.
\]
Taking into account that
\[ \begin{align*}
I - J &= 2i \int_1^1 \frac{\ln x}{1 - x^2} \ln \left( -\frac{2ix}{1 - x^2} \right) dx + 2i \int_0^1 \frac{\ln(-ix)}{1 - x^2} \ln \left( -\frac{2ix}{1 - x^2} \right) dx \\
&= 2i \int_0^1 \frac{\ln x}{1 - x^2} \ln \left( -\frac{2ix}{1 - x^2} \right) dx + 2i \int_0^1 \frac{\pi}{1 - x^2} \ln \left( -\frac{2ix}{1 - x^2} \right) dx \\
&= \left( \frac{i\pi^2}{4} \ln 2 - \frac{\pi^3}{8} \right) - \frac{i\pi^2}{4} \ln 2 = -\frac{\pi^3}{8},
\end{align*} \]
we deduce the following expression:
\[ \frac{\pi}{2} \mathcal{M} = I + J = 2I + \frac{\pi^3}{8}. \]

Hence, we need only to compute \( I \). In order to do that, we rewrite it as
\[ I = 2i \int_1^1 \frac{\ln x \ln(-2ix)}{1 - x^2} dx - 2i \int_1^1 \ln x \ln(1 - x^2) dx. \]

Evaluating separately the two integrals
\[ \begin{align*}
\int_1^1 \frac{\ln x \ln(-2ix)}{1 - x^2} dx &= \sum_{n=1}^{\infty} \int_1^1 x^{2n-2} \ln x \ln(-2ix) dx \\
&= \frac{7\zeta(3)}{4} - \frac{\pi G}{2} + iG \ln 2, \\
\int_1^1 \frac{\ln x \ln(1 - x^2)}{1 - x^2} dx &= \int_0^1 \frac{\ln(1-y) \ln(1+y)}{1+y} dy \\
&= \int_0^{-1} \ln^2(1+y) dy - \int_0^{-1} \ln(1+y) \ln(1-y) dy \\
&\quad + \int_0^{-1} 2 \ln(1+y) dy - \int_0^{-1} \ln(1+y) dy \\
&\quad + \int_0^{-1} \ln x \ln(1+y) dy - \int_0^{-1} \ln x \ln(1+y) dy \\
&= \frac{3\pi^3}{64} + \frac{i\pi}{16} \ln^2 2 - iG \ln 2 - 2 \text{Li}_3 \left( \frac{1+i}{2} \right) \\
&\quad + \frac{\ln^3 2}{24} - \frac{5\pi^2}{96} \ln 2 - \frac{\pi G}{2} + 2\zeta(3) + \left\{ \frac{27}{32} \zeta(3) - \frac{\pi G}{2} \right\} \\
&\quad + \left\{ iG \ln 2 + \frac{\pi^2}{48} \ln 2 \right\} + \left\{ iG \ln 2 - \frac{\pi^2}{48} \ln 2 \right\} \\
&\quad + \left\{ \pi G - \frac{i\pi^3}{48} - \frac{3\zeta(3)}{64} \right\} + \left\{ \pi G - \frac{i\pi^3}{96} + \frac{3\zeta(3)}{64} \right\} \\
&= \frac{91}{32} \zeta(3) + \frac{\ln^3 2}{24} - \frac{5\pi^2}{96} \ln 2 - \frac{\pi G}{2} \\
&\quad - 2 \text{Li}_3 \left( \frac{1+i}{2} \right) + \frac{i\pi^3}{64} + \frac{i\pi}{16} \ln^2 2 + iG \ln 2.
\end{align*} \]
After some routine simplification, we find the closed formula for \( I \):
\[ I = 4i \text{Li}_3 \left( \frac{1+i}{2} \right) + \frac{\pi^3}{32} + \frac{\pi}{8} \ln^2 2 - \frac{35i}{16} \zeta(3) - \frac{i\ln^3 2}{12} + \frac{5i\pi^2}{48} \ln 2. \]
As a bonus, we also have the following closed formula for J:

$$J = 4i \text{Li}_3\left(\frac{1+i}{2}\right) + \frac{5\pi^3}{32} + \frac{\pi}{8} \ln^2 2 - \frac{35i}{16} \zeta(3) - \frac{i \ln^3 2}{12} + \frac{5i\pi^2}{48} \ln 2.$$ 

In conclusion, we have established the formula

$$\frac{\pi}{2} W = 2i + \frac{\pi^3}{8} = 8i \text{Li}_3\left(\frac{1+i}{2}\right) + \frac{3\pi^3}{16} + \frac{\pi}{4} \ln^2 2 - \frac{35i}{8} \zeta(3) - \frac{i \ln^3 2}{6} + \frac{5i\pi^2}{24} \ln 2,$$

which is equivalent to the formula enunciated in (32). □

5.3. Another Expression of $W$ in Polylogarithm

As shown in (32), the series $W$ is expressed in $\text{Li}_3\left(\frac{1+i}{2}\right)$. It can also be expressed in $\text{Li}_3\left(\frac{1+i}{2}\right)$.

Recall the formula below about the trilogarithm function (cf. Lewin [15] (A2.6))

$$\text{Li}_3(z) - \text{Li}_3\left(\frac{1}{2}\right) = -\frac{\pi^2}{6} \ln(-z) - \frac{\ln^3(-z)}{6}.$$

By letting $z = 1 + i$ and then taking into account the particular value

$$\ln(-1-i) = \frac{\ln 2}{2} - \frac{3\pi}{4}i,$$

we deduce the following relation

$$\text{Li}_3(1+i) - \text{Li}_3\left(\frac{1-i}{2}\right) = \frac{11\pi^2}{192} \ln 2 - \frac{\ln^3 2}{48} + \frac{7i\pi^3}{128} + \frac{3i\pi}{32} \ln^2 2.$$

Keeping in mind that

$$\Re\left\{\text{Li}_3(1+i)\right\} = \Re\left\{\text{Li}_3(1-i)\right\},$$

$$\Im\left\{\text{Li}_3(1+i)\right\} = -\Im\left\{\text{Li}_3(1-i)\right\};$$

$$\Re\left\{\text{Li}_3\left(\frac{1+i}{2}\right)\right\} = \Re\left\{\text{Li}_3\left(\frac{1-i}{2}\right)\right\},$$

$$\Im\left\{\text{Li}_3\left(\frac{1+i}{2}\right)\right\} = -\Im\left\{\text{Li}_3\left(\frac{1-i}{2}\right)\right\};$$

we obtain two relations:

$$\Re\left\{\text{Li}_3(1+i)\right\} - \Re\left\{\text{Li}_3\left(\frac{1+i}{2}\right)\right\} = \frac{11\pi^2}{192} \ln 2 - \frac{\ln^3 2}{48},$$

$$\Im\left\{\text{Li}_3(1+i)\right\} + \Im\left\{\text{Li}_3\left(\frac{1+i}{2}\right)\right\} = \frac{7\pi^3}{128} + \frac{3\pi}{32} \ln^2 2.$$ (33)

Vălean [22] (2019) succeeded in extracting the real part

$$\Re\left\{\text{Li}_3\left(\frac{1+i}{2}\right)\right\} = \frac{35}{64} \zeta(3) - \frac{5\pi^2}{192} \ln 2 + \frac{\ln^3 2}{48}.$$

Consequently, we deduce the following closed formula:

$$\Re\left\{\text{Li}_3(1+i)\right\} = \frac{\pi^2}{32} \ln 2 + \frac{35}{64} \zeta(3).$$
Finally, by relating (32) to (33), we find another expression:

\[
\frac{\pi \mathbb{W}}{4} = 4 \Im \left\{ \text{Li}_3(1 + i) \right\} - \frac{\pi^3}{8} - \frac{\pi}{4} \ln^2 2. \tag{34}
\]

To facilitate the subsequent use in the next section, we invert (32) and (34) as follows:

\[
4 \Im \left\{ \text{Li}_3 \left( \frac{1 + i}{2} \right) \right\} = \frac{3 \pi^3}{32} + \frac{\pi \ln^2 2}{8} - \frac{\pi \mathbb{W}}{4}, \tag{35}
\]

\[
4 \Im \left\{ \text{Li}_3(1 + i) \right\} = \frac{\pi^3}{8} + \frac{\pi}{4} \ln^2 2 + \frac{\pi \mathbb{W}}{4}. \tag{36}
\]

6. Infinite Series Containing $\bar{O}_n$

Finally, in this section, more difficult series containing alternating skew harmonic numbers $\bar{O}_n$ will be investigated, whose values are expressed in the series $\mathbb{W}$ examined in the last section.

6.1. Positive Series

Now we are going to show the following elegant formula.

**Theorem 7.**

\[
\sum_{n=1}^{\infty} \frac{\bar{O}_n}{n^2} = \frac{\pi \mathbb{W}}{4} + \frac{\pi^3}{16} - 2G \ln 2.
\]

**Proof.** First, it is not difficult to verify the integral representation

\[
\int_{0}^{1} x^{2m-1} \arctan x dx = \frac{\pi}{4m} \chi(m \equiv 2 \ 1) + (-1)^m \frac{\bar{O}_m}{2m}. \tag{37}
\]

Then we can manipulate the series

\[
\sum_{n=1}^{\infty} \frac{\bar{O}_n}{n^2} = 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left\{ \int_{0}^{1} x^{2n-1} \arctan x dx - \frac{\pi}{4n} \chi(n \equiv 2 \ 1) \right\}
\]

\[
= \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} + 2 \int_{0}^{1} \frac{\arctan x}{x} dx \sum_{n=1}^{\infty} \frac{(-x^2)^n}{n}
\]

\[
= \frac{\pi^3}{16} - 2 \int_{0}^{1} \frac{\ln(1 + x^2) \arctan x}{x} dx.
\]

To evaluate the above integral, rewrite the function

\[
\ln(1 + x^2) \arctan x = \frac{\ln(1 + xi)(1 - xi)}{2i} \ln \frac{1 + xi}{1 - xi}
\]

\[
= \frac{\ln^2(1 + xi) - \ln^2(1 - xi)}{2i}
\]

\[
= 3 \left\{ \ln^2(1 + xi) - \ln^2(1 - xi) \right\}
\]

\[
= 3 \left\{ \ln^2(1 + xi) \right\}.
\]
Then we can evaluate the integral
\[ \int_0^1 \ln^2(1 + xi) \frac{dx}{x} = -\int_0^{1+i} \ln^2 \frac{y}{1-y} \, dy \quad y \to 1 + xi \]
\[ = -\sum_{n=1}^{\infty} \int_0^{1+i} y^{n-1} \ln^2 y \, dy \]
\[ = -\sum_{n=1}^{\infty} \left\{ \frac{y^n}{n} \ln^2 y - 2 \frac{y^n}{n^2} \ln y + 2 \frac{y^n}{n^3} \right\}_{1+i}^{1+i} \]
\[ = \left\{ \ln(1 - y) \ln^2 y + 2 \text{Li}_2(y) \ln y - 2 \text{Li}_3(y) \right\}_{1+i}^{1+i} \]
\[ = \ln(-i) \ln^2(1 + i) + 2 \text{Li}_2(1 + i) \ln(1 + i) - 2 \text{Li}_3(1 + i) + 2\zeta(3). \]

By making use of (36) and the following three known values
\[ \ln(-i) = -\frac{\pi i}{2}, \]
\[ \ln(1 + i) = \frac{\ln 2}{2} + \frac{\pi i}{4}, \]
\[ \text{Li}_2(1 + i) = \frac{\pi^2}{16} + G + \frac{\pi i}{4} \ln 2, \]
we find the integral value below:
\[ \int_0^1 \frac{\ln(1 + x^2) \arctan x}{x} \, dx = G \ln 2 - \frac{\pi W}{8}, \quad (38) \]
and consequently, the infinite series identity in Theorem 7. \qed

Alternatively, consider the generating function
\[ \sum_{n=1}^{\infty} \bar{O}_n x^{2n-1} = \frac{\arctan x}{1 - x^2}. \]

We can express the series as
\[ \sum_{n=1}^{\infty} \frac{\bar{O}_n}{n^2} = 4 \int_0^1 \frac{dy}{y} \int_0^y \frac{\arctan x}{1 - x^2} \, dx \]
\[ = 4 \int_0^1 \frac{\arctan x}{1 - x^2} \, dx \int_x^1 \frac{dy}{y} \]
\[ = -4 \int_0^1 \ln x \arctan x \, dx. \]

By appealing to Theorem 7, we derive an analogous integral formula to (12):
\[ \int_0^1 \frac{\ln x \arctan x}{1 - x^2} \, dx = \frac{G}{2} \ln 2 - \frac{\pi^3}{64} - \frac{\pi W}{16}. \quad (39) \]

6.2. Alternating Series

We have also the alternating series identity.

**Theorem 8.**
\[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \bar{O}_n}{n^2} = 4G \ln 2 - \frac{\pi W}{2}. \]
Proof. According to (37), we can reformulate the series
\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \bar{O}_n}{n^2} = \sum_{n=1}^{\infty} \frac{2}{n} \int_0^1 x^{2n-1} \arctan x \, dx.
\]
This integral can alternatively be rewritten as
\[
\int_0^1 \frac{\ln(1-x^2) \arctan x}{x} \, dx = \int_0^1 \frac{1}{2ix} \left[ \ln(1+x) \ln(1+ix) - \ln(1+x) \ln(1-ix) \right] \, dx.
\]
By making use of the algebraic identity \(2AB = A^2 + B^2 - (A - B)^2\), we can further reformulate the two integrals:
\[
2 \int_0^1 \frac{\ln(1+x) \ln(1+ix)}{x} \, dx = \int_0^1 \frac{\ln^2(1+x)}{x} \, dx
\]
\[
+ \int_0^1 \frac{\ln^2(1+ix)}{x} \, dx - \int_0^1 \frac{\ln^2(1-x)}{x} \, dx
\]
\[
2 \int_0^1 \frac{\ln(1-x) \ln(1+ix)}{x} \, dx = \int_0^1 \frac{\ln^2(1-x)}{x} \, dx
\]
\[
+ \int_0^1 \frac{\ln^2(1+ix)}{x} \, dx - \int_0^1 \frac{\ln^2(1-x)}{x} \, dx.
\]
Now we are going to compute these six integrals on the right-hand sides displayed in the above two equations. First, recalling (2) and (4), we have immediately
\[
\int_0^1 \frac{\ln^2(1-x)}{x} \, dx = 2\zeta(3) \quad \text{and} \quad \int_0^1 \frac{\ln^2(1+x)}{x} \, dx = \frac{\zeta(3)}{4}.
\]
Then, from the proof of Theorem 7, we have already obtained
\[
\int_0^1 \frac{\ln^2(1+ix)}{x} \, dx = 2\zeta(3) - 2\text{Li}_3(1+i) + 2\text{Li}_2(1+i) \ln(1+i) - \frac{i\pi}{2} \ln^2(1+i).
\]
The remaining two integrals are evaluated as follows:

\[
\int_0^1 \frac{1}{x} \ln^2 \frac{1 + x}{1 + ix} \, dx = \int_1^{1-i} \frac{(1 + i) \ln^2 y}{(y - 1)(y + i)} \, dy \\
= \int_1^{1-i} \frac{\ln^2 y}{y - 1} \, dy - \int_1^{1-i} \frac{\ln^2 y}{y + i} \, dy \\
= \left\{ \ln(1 - y) \ln^2 y + 2 \ln y \text{Li}_2(y) - 2 \text{Li}_3(y) \right\}_1^{1-i} \\
- \left\{ \ln(1 - iy) \ln^2 y + 2 \ln y \text{Li}_2(iy) - 2 \text{Li}_3(iy) \right\}_1^{1-i} \\
= \frac{35}{16} \zeta(3) + 2 \text{Li}_3(1 + i) - 2 \text{Li}_3(1 - i) - \pi G - 2iG \ln 2 - \frac{i\pi^3}{8} - \frac{i\pi}{4} \ln^2 2, \\
\]

\[
\int_0^1 \frac{1}{x} \ln^2 \frac{1 - x}{1 + ix} \, dx = \int_0^1 \frac{(1 - i) \ln^2 y}{(1 - y)(y - i)} \, dy \\
= \int_0^1 \frac{\ln^2 y}{1 - y} \, dy + \int_0^1 \frac{\ln^2 y}{y - i} \, dy \\
= \left\{ 2 \text{Li}_3(y) - 2 \ln y \text{Li}_2(y) - \ln(1 - y) \ln^2 y \right\}_0^1 \\
- \left\{ 2 \text{Li}_3(-iy) - 2 \ln y \text{Li}_2(-iy) - \ln(1 + iy) \ln^2 y \right\}_0^1 \\
= 2\zeta(3) - 2 \text{Li}_3(-i). \\
\]

By substitution, we find the following two integral identities:

\[
\int_0^1 \frac{\ln(1 + x) \ln(1 + ix)}{x} \, dx = \text{Li}_3(1 - i) - 2 \text{Li}_3(1 + i) + \frac{\zeta(3)}{32} + \frac{\pi G}{4} \\
+ \frac{\pi^2}{32} \ln 2 + \frac{3i\pi^3}{32} + \frac{3iG}{2} \ln 2 + \frac{3i\pi}{16} \ln^2 2, \\
\int_0^1 \frac{\ln(1 - x) \ln(1 + ix)}{x} \, dx = \frac{i\pi}{16} \ln^2 2 + \frac{iG}{2} \ln 2 - \text{Li}_3(1 + i) \\
+ \frac{29\zeta(3)}{32} + \frac{\pi^2}{32} \ln 2 - \frac{\pi G}{4}. \\
\]

Putting the last two expressions together and then making some simplifications, we find that

\[
\int_0^1 \frac{\ln(1 - x^2) \arctan x}{x} \, dx = \Im \left\{ \frac{\pi G}{4} + \frac{33\zeta(3)}{32} + \frac{\pi^2}{32} \ln 2 \\
+ \text{Li}_3(1 - i) - 3 \text{Li}_3(1 + i) + \text{Li}_3(-i) + \text{Li}_2(1 + i) \ln(1 + i) \\
+ \frac{3i\pi^3}{32} + \frac{3iG}{2} \ln 2 + \frac{3i\pi}{16} \ln^2 2 - \frac{i\pi}{4} \ln^2 (1 + i) \right\} \\
= 3 \left\{ \text{Li}_3(1 - i) - 3 \text{Li}_3(1 + i) + 2iG \ln 2 \\
+ \frac{3i\pi^3}{32} + \frac{i\pi}{4} \ln^2 2 + \frac{15\zeta(3)}{16} + \frac{\pi^2}{16} \ln 2 \right\} \\
= 2G \ln 2 + \frac{3\pi^3}{32} + \frac{\pi}{4} \ln^2 2 - 4\Im \left\{ \text{Li}_3(1 + i) \right\}. \\
\]

By invoking (36), we finally confirm the claimed integral formula (40). □
According to the above proof, further two refined formulae below can be established:

\[
\int_0^1 \frac{\ln(1 + x) \arctan x}{x} \, dx = \frac{3 \log 2}{2} - \frac{3\pi\gamma}{16}, \tag{41}
\]

\[
\int_0^1 \frac{\ln(1 - x) \arctan x}{x} \, dx = \frac{\log 2}{2} - \frac{\pi\gamma}{16} - \frac{\pi^3}{32}. \tag{42}
\]

The proofs for them are not difficult and are illustrated by

\[
\int_0^1 \frac{\ln(1 + x) \arctan x}{x} \, dx = \int_0^1 \frac{\ln(1 + x) \ln(1 + ix)}{1 - ix} \, dx
\]

\[
= 3 \left\{ \int_0^1 \frac{\ln(1 + x) \ln(1 + ix)}{x} \, dx \right\}
\]

\[
= 3 \left\{ \text{Li}_3(1 - i) - 2 \text{Li}_3(1 + i) + \frac{\zeta(3)}{32} + \frac{\pi G}{4}
\]

\[
+ \frac{\pi^2}{32} \log 2 + \frac{3\pi^3}{32} + \frac{3\log 2}{2} + \frac{3\pi}{16} \log^2 2 \right\}
\]

\[
= \frac{3\pi^3}{32} + \frac{3\log 2}{2} + \frac{3\pi}{16} \log^2 2 - 3 \left\{ \text{Li}_3(1 + i) \right\},
\]

\[
= 3 \log 2 - \frac{3\pi\gamma}{16},
\]

and

\[
\int_0^1 \frac{\ln(1 - x) \arctan x}{x} \, dx = \int_0^1 \frac{\ln(1 - x) \ln(1 + ix)}{1 - ix} \, dx
\]

\[
= 3 \left\{ \int_0^1 \frac{\ln(1 - x) \ln(1 + ix)}{x} \, dx \right\}
\]

\[
= 3 \left\{ \frac{29\zeta(3)}{32} + \frac{\pi^2}{32} \log 2 - \frac{\pi G}{4}
\]

\[
- \text{Li}_3(1 + i) + \frac{4G}{2} \log 2 + \frac{i\pi}{16} \log^2 2 \right\}
\]

\[
= \frac{G}{2} \log 2 + \frac{\pi\gamma}{16} - 3 \left\{ \text{Li}_3(1 + i) \right\}
\]

\[
= \frac{G}{2} \log 2 - \frac{\pi\gamma}{16} - \frac{\pi^3}{32}.
\]

Alternatively, by making use of the power series expansion

\[
\sum_{n=1}^{\infty} (-1)^{n-1} O_n x^{2n-1} = \frac{\ln \frac{1+x}{1-x}}{2(1+x^2)},
\]

we can reformulate the series

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1} O_n}{n^2} = 2 \int_0^1 \frac{dy}{y} \int_0^y \frac{\ln \frac{1+x}{1-x}}{1+x^2} \, dx
\]

\[
= 2 \int_0^1 \ln \frac{1+x}{1-x} \int_0^1 \frac{dy}{y}
\]

\[
= 2 \int_0^1 \frac{\ln x \ln \frac{1-x}{1+x}}{1+x^2} \, dx.
\]
In view of Theorem 8, we find the following integral identity:

$$\int_0^1 \ln x \ln \frac{1-x}{1+x} dx = 2G \ln 2 - \frac{\pi \zeta(2)}{4}. \quad (43)$$

6.3. Bisection Series

Combining Theorem 7 with Theorem 8 yields two further bisection series identities:

$$\sum_{n=1}^{\infty} \frac{O_{2n}^{(2)}}{n^2} = 2 \sum_{n=1}^{\infty} \frac{O_{n}^{(2)}}{n^2} - 2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}O_n}{n^2} = \frac{\pi^3}{8} + \frac{3\pi \zeta(2)}{2} - 12G \ln 2, \quad (44)$$

$$\sum_{n=1}^{\infty} \frac{O_{2n-1}^{(2)}}{(2n-1)^2} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{O_{n}^{(2)}}{n^2} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}O_n}{n^2} = \frac{\pi^3}{32} - \frac{\pi \zeta(2)}{8} + G \ln 2. \quad (45)$$

7. Concluding Comments

By making use of the integration approach, we have reviewed (Theorems 1–4) and found (Theorems 5–8) eight remarkable infinite series identities involving harmonic numbers. The same approach can possibly be utilized to treat the series containing harmonic numbers of higher order. The related integrals would concern the polylogarithm function, which will increase the difficulty level for evaluation. For instance, Vălean [23] discovered (among others), by evaluating integrals involving the dilogarithm function, the following four values for the series containing quadratic harmonic numbers:

$$\sum_{n=1}^{\infty} \frac{H_{2n}^{(2)}}{n^3} = 3\zeta(2)\zeta(3) - \frac{9}{2} \zeta(5),$$

$$\sum_{n=1}^{\infty} \frac{H_{2n}^{(2)}}{n^3} = \frac{7}{2} \zeta(5) - \zeta(2)\zeta(3);$$

$$\sum_{n=1}^{\infty} (-1)^n \frac{H_{2n}^{(2)}}{n^3} = \frac{11}{32} \zeta(5) - \frac{5}{8} \zeta(2)\zeta(3),$$

$$\sum_{n=1}^{\infty} (-1)^n \frac{H_{2n}^{(2)}}{n^3} = \frac{11}{8} \zeta(2)\zeta(3) + \frac{19}{32} \zeta(5) - 4 \ln 2 \text{Li}_4\left(\frac{1}{2}\right) - 4 \text{Li}_3\left(\frac{1}{2}\right)$$
$$- \frac{7 \ln^2 \zeta(3)}{4} + \frac{2 \ln^3 \zeta(2)}{3} - \frac{2 \ln^2 \zeta(2)}{15}.$$  

A natural question is what would happen if the ordinary harmonic numbers \(\{H_{2n}^{(2)}, H_{2n}^{(2)}\}\) were replaced by the skew harmonic numbers \(\{O_{2n}^{(2)}, O_{2n}^{(2)}\}\) or their alternating counterparts \(\{H_{2n}^{(2)}, H_{2n}^{(2)}; O_{2n}^{(2)}, O_{2n}^{(2)}\}\)? In order to facilitate the reader to tackle some of these problems, the following account of plausible approaches may be helpful. Up to now, there are several ways to deal with infinite series containing harmonic numbers. The following three methods are typical and have been shown to be efficient:

- Abel’s lemma on summation by parts for Euler sums (cf. [24,25]).
- Partial fraction decompositions for binomial series (cf. [10]).
- The “coefficient extraction” technique from hypergeometric series (cf. [4]) for more involved series.

In addition, there are other methods available in the literature, for example, the Cauchy residue method (cf. [5]) and Fourier series expansions (cf. [6]), as well as manipulations of the polylogarithm function (cf. [7,9,17]). The interested reader is encouraged to make further exploration.

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