mathematics

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Further Results on the Input-to-State Stability of a Linear Disturbed System with Control Delay

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Abstract: In this paper, a theorem is obtained that gives sufficient input-to-state stability conditions for linear systems with control delay and additive disturbances. Stabilizing feedback is considered available in the absence of delay and disturbances. The mathematical tools are the Lyapunov–Krasovskii functional, the Jensen inequality and the double Hadamard inequality. The critical delay is highlighted.

Keywords: control delay; input-to-state-stability; Lyapunov–Krasovskii functionals; linear disturbed systems; critical delay

MSC: 39A30; 93C05; 93C43; 93D05; 93D15; 93D25

1. Introduction

The present paper continues recent works of the authors [1–4] on the topic of the equilibrium stability of switched nonlinear or linear [5] systems with input delay. A brief introduction to the various types of delay (on state, on input or on output) is given (since we are not interested here, we do not elaborate on the concept of switched systems). In the bibliography of the field, the delayed systems are sometimes called hereditary systems, or systems with dead time [6]. Classical authors talk about differential equations: delayed [7], with deviating argument [8], functional delayed [9], with partial derivatives, also called equations with distributed parameters [10]. Both the latter and the equations with delay or with lumped parameters have in common the property that they are infinite dimensional systems. The study of delayed equations marks an important point within the work [11], in which Pontryagin gives a fundamental theorem regarding the stability of equations with delay. References [12–15] are also important, and it is worth mentioning that interest in the theory of automatic systems with control delay has appeared relatively recently [16–20].

We briefly present some works with different approaches compared to those in [21–26], which are closer to the present work. The paper [27] discusses linear systems with distributed delay, taking into account the kernel, but also systems with a discrete delay. The paper uses a polytopic representation to deal with nonlinear input saturation and utilizes a certain integral inequality to obtain the reduction of conservatism. The paper [28] addresses the finite-time synchronization problem of complex networks under a new communication constraint. In the work [29], it can be noted that the Bessel–Legendre inequality is efficient in reducing conservatism and in investigating scalable methods. This scalable methods were also used in [5]. In the work [30], it is demonstrated that the Wirtinger inequality encompasses Jensen's inequality and facilitates, if it is used, the reduction of conservatism. The approach from [31] allows an extension to various practical systems, such as jet engines, flight systems and robots, if the mathematical models are presented in a lower triangular form. Finally, in a recent paper [32], the probabilistic control of discrete-time stochastic systems with multiple control inputs and state delays was considered.

Citation: Enciu, D.; Toader, A.; Ursu, I. Further Results on the Input-to-State Stability of a Linear Disturbed System with Control Delay. *Mathematics* **2024**, *12*, 634. https://doi.org/10.3390/math12050634

Academic Editors: Hebatallah J. Alsakaji and Snezhana Hristova

Received: 12 January 2024
Revised: 7 February 2024
Accepted: 18 February 2024
Published: 21 February 2024

In this paper, in a different mathematical framework compared to the works [1–5], we consider a class of linear systems with additive disturbances on the state, without switching and with input (control) delay, for which stabilizing feedback is considered available in the absence of delay and disturbances. In this context, the problem arises of finding some conditions that the system parameters must meet to ensure stability in the presence of control delays and additive disturbances on the state model. The input-to-state stability (ISS) solution is obtained with the following mathematical tools: the Lyapunov–Krasovskii functional, the Jensen inequality and the double Hadamard inequality. The critical delay is also given.

The mathematical apparatus necessary for formulating the problem given in Section 3, which will be solved in this work, assumes some specific notions of functional analysis (such as "ess sup" or "essentially bounded measurable function") or some special classes of functions. These elements appear disparately in works such as [21–25], as if these were of general use and easily understood by themselves, which is not necessarily the case for the readers. In Section 2, these concepts are introduced as definitions, in a quasi-exhaustive approach, in the interest of the reader, mainly to clarify the physical meaning of the mathematical model that appears in the problem presented in Section 3 and to clarify the definition of ISS stability given there. Readers interested in other details will be directed to monographs such as [33,34]. The main contribution of the paper is the ISS stability theorem in Section 4. The novelty of the paper consists in emphasizing the linear models [5,35], important in the usual application of active control, as well as using the double Hadamard inequality. In this context, a particular set of (sufficient) stability conditions is obtained, which is not found in the literature.

A summary exercise for evaluating the conditions of ISS stability is presented in Section 5. Some final remarks in Section 6 close the paper.

2. Notation and Preliminaries

A function $\rho : [0, a) \to [0, \infty)$ is of class $\mathcal{K}$ if $\rho(0) = 0$ is continuous and strictly increasing; of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$, $a = \infty$ and $\rho(r) \to \infty$ as $r \to \infty$; of class $\mathcal{L}$ if it is continuous and strictly decreasing to zero, as its argument tends to $+\infty$. A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is of class $\mathcal{KL}$ if (1) for each fixed s, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r and (2) for each fixed r, the mapping $\beta(r, s)$ is decreasing with respect to s; in other words, $\beta(r, s) \to 0$ as $s \to \infty$. The definitions are classical and can be found in [21–25].

In mathematics, a measure is a generalization of concepts such as length, area and volume. In measure theory and functional analysis, there appear properties that are not valid for all elements of a set, but only almost everywhere, which means that they are valid, except for a set of zero Lebesgue measure. (Finite collections of points and countable sets are examples of sets of zero Lebesgue measure.) The ess sup (essential supremum) of a function as defined in [33]: $\operatorname{ess\,sup}_t = \sup\{f(t) | t \in (a, b),$ isolated points of $f(t)$ being disregarded$\}$. In other words, ess sup of a function is the smallest value that is greater than or equal to the values of the function everywhere, except for a set of zero Lebesgue measure. The supremum (abbreviated sup) of a subset S of a partially ordered set is the smallest element that is greater than or equal to every element of P, if such an element exists. The concepts of infimum and supremum refer to those of minimum and maximum, but characterize sets that may not have a minimum or maximum better [34].

Consider a set X and a σ-algebra $\mathcal{F}$ [36] of X space subsets X_i. Then the tuple $(X, \mathcal{F})$ is called a measurable space; then any set X_i is a measurable set. A (positive) measure is a function μ defined on $\sigma-$ algebra $\mathcal{F} = \{X_i\}, i = 1, 2, \ldots$ of disjoint sets X_i, with the image of μ in $[0, \infty]$, and μ is additive countably $\mu\left(\bigcup_{i=1}^{\infty} X_i\right) = \sum_{i=1}^{\infty} \mu(X_i)$. Let $f : X \to \mathbb{R}$ be an $\mathcal{F}$-measurable function. We say that f is essentially bounded if and only if there exists a real number c, such that $\mu(\{x \in X : |f(x) > c|\}) = 0$. It is easy to see that in the definition, the possible isolated points x where $f(x) > c$ are ignored, since they have zero measure.

A function $f : X \to \mathbb{R}$ is measurable if, for every real number a, the set $\{x \in X : f(x) > 0\}$ is measurable. If f is a measurable function on X, it is defined $\|f\|_\infty$ as ess sup of f, whereas $L^\infty(\mu)$ is defined as the set of all functions f for which $\|f\|_\infty < \infty$. The elements of $L^\infty(\mu)$ are sometimes called essential bounded measurable functions on X. An $L^2(\mu)$ space may be defined as a space of measurable functions for which the 2-nd power of the absolute value is Lebesgue integrable. In other words, when $p \neq \infty$, consider the set $L^p(S,\mu)$ of all measurable functions f from S to $\mathbb{R}$, such that $\|f\|_p := \left(\int_S |f|^p d\mu\right)^{1/p} < \infty$. For $t \in \mathbb{R}^+$, the notation $\boldsymbol{x}_t(s) \in C\left(\left[-\overline{h}, 0\right]; \mathbb{R}^n\right)$ means $\boldsymbol{x}_t(s) := \boldsymbol{x}(t+s), s \in [-\overline{h}, 0]$. The notation is specific for differential equations with delay. The initial function $\boldsymbol{\varphi}$ (in the sense of definition (1), and by analogy with the notion of initial condition in the framework of the Cauchy problem) is differentiable, belonging to the Banach space $\Phi[-\overline{h}, 0]$ of absolutely continuous functions $\boldsymbol{\varphi}$:$[-\overline{h}, 0] \to \mathbb{R}^n$, equipped with the norm $\|\boldsymbol{\varphi}\|_{\overline{h}} = \max_{m \in [-\overline{h},0]} |\varphi(m)| + \left(\int_{-\overline{h}}^0 |\dot{\boldsymbol{\varphi}}(s)|^2 ds\right)^{1/2}$ [23,37] and with $\dot{\boldsymbol{\varphi}} \in L^2\left(\left[-\overline{h}, 0\right] \to \mathbb{R}^n\right)$. A function $f : I \to R$ is absolutely continuous on I if, for every $\varepsilon > 0$, there is $\delta > 0$, such that whenever a finite sequence of pairwise disjoint subintervals (x_k, y_k) on I with $x_k < y_k \in I$ satisfies $\sum_k (y_k - x_k) < \delta$, then $\sum_k |f(y_k) - f(x_k)| < \varepsilon$.

It is useful to note that the symbol $|\,|$ (modulus) is used when algebraic-geometric operations are suggested. The symbol $\|\,\|$ (norm) rather suggests operator algebras and functional analysis operations. Another interesting aspect is that there is an analogy between the concepts of topological space, open set and continuous function on the one hand, and measurable space, measurable set and measurable function, respectively, on the other hand [33,34]. In Section 4, the notation $|\,|$, with the meaning of norm $\|\,\|$, is used for convenience.

A function $V : \mathbb{R}^n \to [0, \infty)$ is called proper and positive definite provided that there are $\underline{\alpha}$, $\overline{\alpha} \in \mathcal{K}_\infty$, such that $\underline{\alpha}^2(|\boldsymbol{x}(t)|) \leq V(\boldsymbol{x}(t)) \leq \overline{\alpha}^2(|\boldsymbol{x}(t)|)$ for all $\boldsymbol{x} \in \mathbb{R}^n$. Choosing $V(\boldsymbol{x}) = \boldsymbol{x}^{\mathrm{T}} \boldsymbol{P} \boldsymbol{x}$, with symmetric $\boldsymbol{P} > 0$, the following functions can be chosen $\underline{\alpha}(|\boldsymbol{x}(t)|) = \sqrt{\lambda_{\min}(\boldsymbol{P})} |\boldsymbol{x}(t)| \in \mathcal{K}_\infty$,$\overline{\alpha}(|\boldsymbol{x}(t)|) = \sqrt{\lambda_{\max}(\boldsymbol{P})} |\boldsymbol{x}(t)| \in \mathcal{K}_\infty$. Other notations that appear in Section 3 are usual.

3. Problem Formulation

Consider the perturbed linear system with delayed state feedback control

$$\begin{gathered} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}(\boldsymbol{u}(\boldsymbol{x}(t-h)) + \boldsymbol{w}(t)), \boldsymbol{A} \in \mathbb{R}^{n \times n}, \boldsymbol{B} \in \mathbb{R}^{n \times p} \\ \boldsymbol{x}(\theta; 0, \varphi) = \varphi(\theta), \text{ for } -\overline{h} \leq \theta \leq 0, \overline{h} > 0,\ h \in [0, \overline{h}] \end{gathered} \tag{1}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$ is the state vector, $\boldsymbol{u}(\boldsymbol{x}) \in \mathbb{R}^p$ of class C^1 is the predesigned state feedback stabilizing control, with $\boldsymbol{u}(0) = 0$, set for $h = \boldsymbol{w} = 0$; h denotes the control time delay satisfying $0 \leq h \leq \overline{h}, \overline{h}$ finite. The disturbance function $\boldsymbol{w} : R^+ \to R^p$ is measurable and essentially bounded, with the norm $\|\boldsymbol{w}\|_\infty :=$ ess $\sup_{t \geq 0} |\boldsymbol{w}(t)|$. If $\|\boldsymbol{w}\|_\infty < \infty$, we write $\boldsymbol{w}(t) \in L^\infty(\mu)$. In what follows, the matrix $\boldsymbol{A}$ is considered to be stable.

Remark 1. *In control theory, the definition of matched uncertainty assumes that uncertainty enters the system through the same channel as the control [38]. Therefore, from the way system (1) was written, it follows that $\boldsymbol{w}(t)$ was considered a matched disturbance. There are various techniques for the treatment of matched disturbances, one of which is the sliding mode control paradigm [39].*

Definition 1. ([21]). *System (1) is said to be input-to-state stable (ISS) if there exist functions $\beta \in \mathcal{KL}$ and $\nu, \gamma \in \mathcal{K}_\infty$, such that for any initial condition $\varphi \in \Phi[-\overline{h}, 0]$ and for any $\boldsymbol{w} \in L^\infty(\mu)$, the solution of (1) exists over $[0, +\infty)$ and satisfies*

$$\nu(|\boldsymbol{x}(t)|) \leq \beta\left(\|\boldsymbol{\varphi}\|_{\overline{h}}\right) + \gamma(\|\boldsymbol{w}[0,t]\|), t \geq 0 \tag{2}$$

The definition extends identically from an associated nonlinear system to a linear one (1)

$$\begin{aligned} &\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t)) + \boldsymbol{g}(\boldsymbol{x}(t))(\boldsymbol{u}(\boldsymbol{x}(t-h)) + \boldsymbol{w}(t)) \\ &\boldsymbol{x}(\theta; 0, \varphi) = \varphi(\theta),\ \text{for } -\overline{h} \leq \theta \leq 0, \overline{h} > 0,\ h \in \left[0, \overline{h}\right] \end{aligned} \tag{3}$$

Remark 2. *The specific notions of functional and the special classes of functions presented in Section 2 are now justified, as they appear in the system data (1) and in the ISS definition. The membership of disturbance* $\boldsymbol{w}$ *to the class of functions* $L^{\infty}(\mu)$ *and the membership of the initial condition function* $\boldsymbol{\varphi}$ *to Banach space* $\Phi[-\overline{h}, 0]$ *of absolutely continuous functions* $\boldsymbol{\varphi}\colon \left[-\overline{h}, 0\right] \to \mathbb{R}^n$ *, with* $\|\boldsymbol{\varphi}\|_{\overline{h}} = \max_{m \in [-\overline{h}, 0]} |\varphi(m)| + \left(\int_{-\overline{h}}^{0} |\dot{\boldsymbol{\varphi}}(s)|^2 ds\right)^{1/2}$ *, ensure for the system (1) the existence of a unique solution [14,34], belonging to the class* $C\left(\left[-\overline{h}, 0\right], \mathbb{R}^n\right)$ *of continuous functions defined on* $\left[-\overline{h}, 0\right]$ *, with values in* $\mathbb{R}^n$ *equipped with norm* $\|\boldsymbol{x}(t)\|_{\overline{h}} = \sup_{-\overline{h} \leq s \leq 0} \{|\boldsymbol{x}(t+s), \dot{\boldsymbol{x}}(t+s)|\}$ *(for instance, [21]). It is worth emphasizing this because in the whole series of cited works [21–25], the reason for choosing these notions and classes of functions is not clearly argued.*

4. Main Result: ISS Stability with Respect to Disturbances; Critical Time Delay

The following assumptions are introduced below.

1. There is known a C^1 proper and positive definite function $V : \mathbb{R}^n \to \mathbb{R}^+$, such that

Assumption 1: there is a constant $\eta > 0$ such that, for all $\boldsymbol{x}(t) \in \mathbb{R}^n$ and $t \geq 0$ and for a stabilizing state feedback $\boldsymbol{u}(\boldsymbol{x}(t)) = \boldsymbol{K}\boldsymbol{x}(t)$ of the system, in the absence of delay and disturbance $\boldsymbol{w}(t)$, the inequality occurs

$$\frac{\partial V(\boldsymbol{x}(t))}{\partial \boldsymbol{x}(t)} (\boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(\boldsymbol{x}(t))) \leq -\eta V(\boldsymbol{x}(t)) \tag{4}$$

2. There exist constants $c > 0,\ k_j \geq 0, j = 1, \ldots, 4$ and a function $\underline{\alpha}(|\boldsymbol{x}(t)|) \in \mathcal{K}_{\infty}$, such that, for all $\boldsymbol{x}(t) \in \mathbb{R}^n, t \geq 0$, the following inequalities hold:

Assumption 2: $\left|\frac{\partial V(\boldsymbol{x}(t))}{\partial \boldsymbol{x}(t)} \boldsymbol{B}\right| \leq k_1 \underline{\alpha}(|\boldsymbol{x}(t)|),\ \left|\frac{\partial \boldsymbol{u}(\boldsymbol{x}(t))}{\partial \boldsymbol{x}(t)}\right| \leq c$

Assumption 3: $|\boldsymbol{A}\boldsymbol{x}(t)|^2 \leq k_2 \underline{\alpha}^2(|\boldsymbol{x}(t)|), |\boldsymbol{B}|^2 \leq k_3$

Assumption 4: $(|\boldsymbol{B}||\boldsymbol{u}(x(t-h))|)^2 \leq k_4 \underline{\alpha}^2(|\boldsymbol{x}(t-h)|)$

Consider the following Lyapunov–Krasovskii functionals $V_i, i = 1, 2$, defined along the trajectories of (1), with $\boldsymbol{P}$ a symmetric positive definite matrix and d_1, d, a, b positive constants

$$\begin{gathered} V(\boldsymbol{x}) = \boldsymbol{x}^{\mathrm{T}} \boldsymbol{P} \boldsymbol{x},\ \ V_1(\boldsymbol{x}_t) = d_1 \int_{t-\overline{h}}^{t} e^{-a(t-s)} \underline{\alpha}^2(|\boldsymbol{x}(s)|) ds \\ V_2(\boldsymbol{x}_t) = d \int_{-\overline{h}}^{0} \int_{t+\theta}^{t} e^{-b(t-s)} \dot{\boldsymbol{x}}^{\mathrm{T}}(s) \dot{\boldsymbol{x}}(s) ds d\theta \\ U(\boldsymbol{x}_t) = V(\boldsymbol{x}) + V_1(\boldsymbol{x}_t) + V_2(\boldsymbol{x}_t) \end{gathered} \tag{5}$$

Remark 3. *In order for the Lyapunov–Krasovskii functional* $U(\boldsymbol{x}_t)$ *to have a negative defined derivative, so as to ensure the ISS stability of the system, the following conditions must be fulfilled: the two assumptions, the properties of the proper and positive defined function* $V(\boldsymbol{x})$*, as well as the sufficient conditions from the theorem. In addition, the accuracy of the sufficient conditions and their degree of conservatism depends on the choice of the expression of the functionals* $V_1(\boldsymbol{x}_t),\ V_2(\boldsymbol{x}_t)$ *(the number and expression of these functionals varies from approach to approach). We emphasize that these assumptions and Lyapunov–Krasovskii functionals expressions have no physical meaning, but only mathematical meaning. Only the functions* $\boldsymbol{\varphi},\ \boldsymbol{w}$ *can have physical meaning.*

Now, we will evaluate the actual conditions that must be fulfilled by the system (1) in order to satisfy the assumptions 1-4.

Assumption 1. *With $V(\boldsymbol{x}) = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}$, the condition to be fulfilled is*

$$\begin{aligned}\tfrac{dV(x)}{dt} &= \dot{\boldsymbol{x}}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{x}(t) + \boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{P}\dot{\boldsymbol{x}}(t) = \\ &= \boldsymbol{x}^{\mathrm{T}}(t)(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K})^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}(t) + \boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{P}(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K})\boldsymbol{x}(t) \leq \\ &\leq -\eta\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{x}(t)\end{aligned}$$

thus,

$$(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K})^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K}) \leq -\eta\boldsymbol{P}. \tag{6}$$

Remark 4. *By virtue of the synthesis of the stabilizing control law $\boldsymbol{u}(\boldsymbol{x}(t)) = \boldsymbol{K}\boldsymbol{x}(t)$, the matrix $\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K}$ is Hurwitz; therefore, for the Lyapunov matrix equation $(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K})^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K}) = -\boldsymbol{Q}$, there are symmetric positive definite matrices $\boldsymbol{P}$ satisfying this equation for some symmetric positive definite matrices $\boldsymbol{Q}$. It remains to identify that positive η, if it exists, so that the inequality (6) is fulfilled.*

Assumption 2. *Consider $\boldsymbol{K}$, the gain of a state feedback $\boldsymbol{u}(\boldsymbol{x}(t)) = \boldsymbol{K}\boldsymbol{x}(t)$, stabilizing the linear system without delay and disturbances $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(\boldsymbol{x}(t))$.*

The two conditions to be fulfilled are the following:

$$\left|\frac{\partial V(\boldsymbol{x}(t))}{\partial \boldsymbol{x}(t)}\boldsymbol{B}\right| = \left|2\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{B}\right| \leq 2|\boldsymbol{P}\boldsymbol{B}||\boldsymbol{x}| \leq k_1\underline{\alpha}(|\boldsymbol{x}|), \left|\frac{\partial \boldsymbol{u}(\boldsymbol{x})}{\partial \boldsymbol{x}}\right| \leq c$$

thus,

$$2|\boldsymbol{P}\boldsymbol{B}| \leq k_1\sqrt{\lambda_{\min}(\boldsymbol{P})}, \quad |\boldsymbol{K}| \leq c. \tag{7}$$

Assumption 3. *The following conditions must be met here:*

$$\begin{aligned}&|\boldsymbol{A}\boldsymbol{x}(t)|^2 = \left|\boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x}(t)\right| \leq \sigma_{\max}(\boldsymbol{A})|\boldsymbol{x}(t)|^2 = \tfrac{\sigma_{\max}(\boldsymbol{A})}{\lambda_{\min}(\boldsymbol{P})}\lambda_{\min}(\boldsymbol{P})|\boldsymbol{x}(t)|^2 = \\ &= \tfrac{\sigma_{\max}(\boldsymbol{A})}{\lambda_{\min}(\boldsymbol{P})}\underline{\alpha}^2(|\boldsymbol{x}(t)|) := k_2\underline{\alpha}^2(|\boldsymbol{x}(t)|)\end{aligned}$$

thus,

$$|\boldsymbol{A}\boldsymbol{x}|^2 \leq k_2\underline{\alpha}^2(|\boldsymbol{x}|), k_2 := \frac{\sigma_{\max}(\boldsymbol{A})}{\lambda_{\min}(\boldsymbol{P})}, |\boldsymbol{B}|^2 \leq k_3, k_3 := |\boldsymbol{B}|^2. \tag{8}$$

Assumption 4. *Finally, the following conditions must be met:*

$$\begin{gathered}(|\boldsymbol{B}||\boldsymbol{u}(x(t-h))|)^2 = (|\boldsymbol{B}||\boldsymbol{K}\boldsymbol{x}(t-h)|)^2 \leq |\boldsymbol{B}\boldsymbol{K}|^2|\boldsymbol{x}(t-h)|^2 \leq \\ \leq k_4\underline{\alpha}^2(|\boldsymbol{x}(t-h)|), \left(\underline{\alpha}^2(|\boldsymbol{x}(t-h)|) = \lambda_{\min}(\boldsymbol{P})|\boldsymbol{x}(t-h)|^2\right) \\ (|\boldsymbol{B}||\boldsymbol{u}(x(t-h))|)^2 \leq k_4\underline{\alpha}^2(|\boldsymbol{x}(t-h)|)\end{gathered}$$

thus,

$$|\boldsymbol{A}\boldsymbol{x}|^2 \leq k_2\underline{\alpha}^2(|\boldsymbol{x}|), k_2 := \frac{\sigma_{\max}(\boldsymbol{A})}{\lambda_{\min}(\boldsymbol{P})}, |\boldsymbol{B}|^2 \leq k_3, k_3 := |\boldsymbol{B}|^2. \tag{9}$$

Let's summarize these conditions (6)–(9), to which those from the stability theorem, stated below, will be added.

$$\begin{gathered}(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K})^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K}) \leq -\eta\boldsymbol{P} \\ 2|\boldsymbol{P}\boldsymbol{B}| \leq k_1\sqrt{\lambda_{\min}(\boldsymbol{P})},\ |\boldsymbol{K}| \leq c\ , k_2 = \tfrac{\sigma_{\max}(\boldsymbol{A})}{\lambda_{\min}(\boldsymbol{P})}, k_3 = |\mathbf{B}|^2,\ |\boldsymbol{B}\boldsymbol{K}|^2 \leq k_4\lambda_{\min}(\boldsymbol{P}).\end{gathered} \tag{10}$$

Of course, η, k_1, k_4 will be determined in such a way as to satisfy the respective inequalities (10). Those values that lead to less conservative conditions will be chosen; as can be seen from the examples provided in the literature, the higher the value of $\overline{h}$ is, the less conservative the sufficient stability conditions are.

We are now ready to give an ISS theorem, derived to obtain delay-dependent stability conditions, as little conservative as possible. The following propositions are involved in the proof of the theorem:

Proposition 1. (Jensen's inequality, [26]). *For any $n \times n$ matrix $\boldsymbol{R} > 0$, a scalar $h > 0$ and a vector function $\boldsymbol{\phi} : [-h, 0] \to \mathbb{R}^n$, such that the integrations concerned are well defined, the following holds:*

$$\int_{-h}^{0} \boldsymbol{\phi}^{\mathrm{T}}(s)\boldsymbol{R}\boldsymbol{\phi}(s)ds \geq \frac{1}{h}\int_{-h}^{0} \boldsymbol{\phi}^{\mathrm{T}}(s)ds\boldsymbol{R}\int_{-h}^{0} \boldsymbol{\phi}(s)ds \tag{11}$$

Proposition 2. *The Leibniz integral rule gives a formula for the differentiation of a definite integral whose limits are functions of the differential variable:*

$$\frac{d}{d\xi}\int_{a(\xi)}^{b(\xi)} f(\xi, x)dx = \int_{a(\xi)}^{b(\xi)} \frac{\partial f(\xi, x)}{\partial \xi}dx + f(\xi, b(\xi))\frac{db(\xi)}{d\xi} - f(\xi, a(\xi))\frac{da(\xi)}{d\xi} \tag{12}$$

Theorem 1. *Let the conditions in (10), which describe the value of the constant k_2, be fulfilled; the matrices $\boldsymbol{P}$ and $\boldsymbol{K}$, in which $\boldsymbol{P}$ is a positive definite symmetric matrix, and $\boldsymbol{K}$ is the gain of a state feedback obtained for the linear system* (3)*, without delay and disturbance, and let values $k_1 = 2|\boldsymbol{PB}|/\sqrt{\lambda_{\min}(\boldsymbol{P})}$, $k_2 = \frac{\sigma_{\max}(\boldsymbol{A})}{\lambda_{\min}(\boldsymbol{P})}$, $c = |\boldsymbol{K}|$, $k_4 = |\boldsymbol{BK}|^2/\lambda_{\min}(\boldsymbol{P})$.If there are positive constants d_1, d, a, b, η, such that*

$$\begin{gathered} 4k_4 - d_1 e^{-a\overline{h}} = 0,\ \eta > \frac{k_1^2}{4} + k_1^2 \frac{\overline{h}e^{b\overline{h}}}{4d}c^2 + d_1 + 2k_2 d\overline{h} \\ (\boldsymbol{A} + \boldsymbol{BK})^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}(\boldsymbol{A} + \boldsymbol{BK}) \leq -\eta\boldsymbol{P} \end{gathered} \tag{13}$$

then system (1) admits functional (5) and is ISS with critical delay value $\overline{h}$.

Proof of Theorem 1. Step 1. The calculation of the $\dot{V}$, $\dot{V}_1$, $\dot{V}_2$ derivatives represents an intermediate step for obtaining the differential equation to which the parameters of the problem are subjected to ensure the ISS:

$$\begin{gathered} \dot{V}(\boldsymbol{x}(t)) = \frac{\partial V(\boldsymbol{x}(t))}{\partial \boldsymbol{x}(t)}(\boldsymbol{Ax}(t) + \boldsymbol{Bu}(\boldsymbol{x}(t)) + \boldsymbol{Bu}(\boldsymbol{x}(t-h)) - \boldsymbol{Bu}(\boldsymbol{x}(t)) + \boldsymbol{Bw}(t)) \leq \\ \leq -\eta V(\boldsymbol{x}(t)) + \frac{\partial V(\boldsymbol{x}(t))}{\partial \boldsymbol{x}(t)}(\boldsymbol{Bu}(\boldsymbol{x}(t-h)) - \boldsymbol{Bu}(\boldsymbol{x}(t)) + \boldsymbol{Bw}(t)) \leq \\ -\eta V(\boldsymbol{x}(t)) + \left|\frac{\partial V(\boldsymbol{x}(t))}{\partial \boldsymbol{x}(t)}\boldsymbol{B}\right||\boldsymbol{w}(t)| + \left|\frac{\partial V(\boldsymbol{x}(t))}{\partial \boldsymbol{x}(t)}\boldsymbol{B}\right|c\left|\int_{t-h}^{t} \dot{\boldsymbol{x}}(s)ds\right| \\ \left|\frac{\partial V(\boldsymbol{x}(t))}{\partial \boldsymbol{x}(t)}\boldsymbol{B}\right|c\left|\int_{t-h}^{t} \dot{\boldsymbol{x}}(s)ds\right| \leq \frac{\overline{h}e^{b\overline{h}}}{4d}\left|\frac{\partial V(\boldsymbol{x}(t))}{\partial \boldsymbol{x}(t)}\boldsymbol{B}\right|^2 c^2 + \frac{d}{\overline{h}e^{b\overline{h}}}\left|\int_{t-h}^{t} \dot{\boldsymbol{x}}(s)ds\right|^2 \\ \dot{V}_1(\boldsymbol{x}(t)) = d_1\underline{\alpha}^2(|\boldsymbol{x}(t)|) - d_1 e^{-a\overline{h}}\underline{\alpha}^2(|\boldsymbol{x}(t-h)|) - aV_1(\boldsymbol{x}(t)) \\ \dot{V}_2(\boldsymbol{x}_t) \leq -bV_2(\boldsymbol{x}) + d\overline{h}|\dot{\boldsymbol{x}}(t)|^2 - d\int_{-\overline{h}}^{0} e^{b\theta}\dot{\boldsymbol{x}}^{\mathrm{T}}(t+\theta)\dot{\boldsymbol{x}}(t+\theta)d\theta \\ \int_{-\overline{h}}^{0} e^{b\theta}\dot{\boldsymbol{x}}^{\mathrm{T}}(t+\theta)\dot{\boldsymbol{x}}(t+\theta)d\theta = \int_{t-\overline{h}}^{t} e^{b(s-t)}\dot{\boldsymbol{x}}^{\mathrm{T}}(s)\dot{\boldsymbol{x}}(s)ds \\ -d\int_{t-\overline{h}}^{t} e^{b(s-t)}\dot{\boldsymbol{x}}^{\mathrm{T}}(s)\dot{\boldsymbol{x}}(s)ds \leq -d\frac{e^{-b\overline{h}}}{\overline{h}}\left|\int_{t-\overline{h}}^{t} \dot{\boldsymbol{x}}(s)ds\right|^2. \end{gathered} \tag{14}$$

Summarizing:

$$\begin{aligned}&\dot{U}(\boldsymbol{x}_t) = \dot{V}(\boldsymbol{x}) + \dot{V}_1(\boldsymbol{x}_t) + \dot{V}_2(\boldsymbol{x}_t) \leq \\ &\leq -\eta V(\boldsymbol{x}(t)) + \left|\frac{\partial V(\boldsymbol{x}(t))}{\partial \boldsymbol{x}(t)}\boldsymbol{B}\right||\boldsymbol{w}(t)| + \frac{\overline{h}e^{b\overline{h}}}{4d}\left|\frac{\partial V(\boldsymbol{x}(t))}{\partial \boldsymbol{x}(t)}\boldsymbol{B}\right|^2 c^2 + \\ &+ \frac{d}{\overline{h}e^{b\overline{h}}}\left|\int_{t-h}^{t}\dot{\boldsymbol{x}}(s)ds\right|^2 + +d_1\underline{\alpha}^2(|\boldsymbol{x}(t)|) - d_1 e^{-ah}\underline{\alpha}^2(|\boldsymbol{x}(t-h)|) - \\ &-aV_1(\boldsymbol{x}(t)) - bV_2(\boldsymbol{x}) + d\overline{h}|\dot{\boldsymbol{x}}(t)|^2 - d\frac{e^{-b\overline{h}}}{\overline{h}}\left|\int_{t-\overline{h}}^{t}\dot{\boldsymbol{x}}(s)ds\right|^2.\end{aligned} \tag{15}$$

Above, the Leibniz integral rule (12) for differentiation under the integral sign was applied for the three derivatives, with the obvious extension to the double integral. An inequality of the form $xy \leq qx^2/2 + y^2/(2q),\ 2q := \overline{h}e^{b\overline{h}}/d$ and the integral Jensen inequality were used, for $\dot{V}(\boldsymbol{x}(t))$ and for $\dot{V}_2(\boldsymbol{x}(t))$, respectively. In addition, for $\dot{V}_2(\boldsymbol{x}(t))$, the inequality $e^{b(s-t)} \geq e^{-b\overline{h}}$, valid for $s \in \left[t-\overline{h}, t\right]$, is easy to get.

Step 2. First, by using (iii)–(iv) in the expression of $\left|\dot{\boldsymbol{x}}(t)\right|^2$, after a tedious calculation we have the following:

$$\begin{aligned}&\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}(\boldsymbol{u}(\boldsymbol{x}(t-h)) + \boldsymbol{w}(t)) \\ &\left|\dot{\boldsymbol{x}}(t)\right|^2 = |\boldsymbol{A}\boldsymbol{x}(t)|^2 + (\boldsymbol{A}\boldsymbol{x}(t))^{\mathrm{T}}\boldsymbol{B}(\boldsymbol{u}(\boldsymbol{x}(t-h)) + \boldsymbol{w}(t)) + \\ &+(\boldsymbol{u}(\boldsymbol{x}(t-h)))^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x}(t) + (\boldsymbol{u}(\boldsymbol{x}(t-h)))^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{B}(\boldsymbol{u}(\boldsymbol{x}(t-h)) + \boldsymbol{w}(t)) + \\ &+\boldsymbol{w}(t)^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{w}(t)^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{B}(\boldsymbol{u}(\boldsymbol{x}(t-h)) + \boldsymbol{w}(t)) = \\ &= |\boldsymbol{A}\boldsymbol{x}(t)|^2 + 2(\boldsymbol{A}\boldsymbol{x}(t))^{\mathrm{T}}\boldsymbol{B}(\boldsymbol{u}(\boldsymbol{x}(t-h)) + \boldsymbol{w}(t)) + (\boldsymbol{u}(\boldsymbol{x}(t-h)))^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{B}\boldsymbol{u}(\boldsymbol{x}(t-h)) + \\ &+\boldsymbol{w}(t)^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{B}\boldsymbol{w}(t) + 2(\boldsymbol{u}(\boldsymbol{x}(t-h)))^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{B}\boldsymbol{w}(t) \leq \\ &\leq 2|\boldsymbol{A}\boldsymbol{x}(t)|^2 + |\boldsymbol{B}(\boldsymbol{u}(\boldsymbol{x}(t-h)) + \boldsymbol{w}(t))|^2 + 2|\boldsymbol{B}\boldsymbol{u}(\boldsymbol{x}(t-h))|^2 + 2|\boldsymbol{B}\boldsymbol{w}(t)|^2 \leq \\ &\leq 2|\boldsymbol{A}\boldsymbol{x}(t)|^2 + 4\left(|\boldsymbol{B}\boldsymbol{u}(\boldsymbol{x}(t-h))|^2 + |\boldsymbol{B}\boldsymbol{w}(t)|^2\right)\end{aligned} \tag{16}$$

$$\left|\dot{\boldsymbol{x}}(t)\right|^2 \leq 2k_2\underline{\alpha}^2(|\boldsymbol{x}|) + 4k_4\underline{\alpha}^2(|\boldsymbol{x}(t-h)|) + 4k_3|\boldsymbol{w}|^2 \tag{17}$$

Then, (15) is combined with Assumption 2 and (13) to give a majorant function for $\dot{U}(\boldsymbol{x}_t)$

$$\begin{aligned}&\dot{U}(\boldsymbol{x}_t) = \dot{V}(\boldsymbol{x}(t)) + \dot{V}_1(\boldsymbol{x}_t) + \dot{V}_2(\boldsymbol{x}_t) \leq \\ &\leq -\eta V(\boldsymbol{x}(t)) + k_1\underline{\alpha}(|\boldsymbol{x}(t)|)|\boldsymbol{w}(t)| + \frac{\overline{h}e^{b\overline{h}}}{4d}k_1^2 c^2\underline{\alpha}^2(|\boldsymbol{x}(t)|) + \\ &+d_1\underline{\alpha}^2(|\boldsymbol{x}(t)|) - d_1 e^{-a\overline{h}}\underline{\alpha}^2(|\boldsymbol{x}(t-h)|) - aV_1(\boldsymbol{x}(t)) - bV_2(\boldsymbol{x}(t)) + \\ &+d\overline{h}\left(2k_2\underline{\alpha}^2(|\boldsymbol{x}(t)|) + 4k_4\underline{\alpha}^2(|\boldsymbol{x}(t-h)|) + 4k_3|\boldsymbol{w}(t)|^2\right) \leq \\ &\leq -\eta V(\boldsymbol{x}(t)) + \frac{k_1^2}{4}\underline{\alpha}^2(|\boldsymbol{x}(t)|) + |\boldsymbol{w}(t)|^2 + \frac{\overline{h}e^{b\overline{h}}}{4d}k_1^2 c^2\underline{\alpha}^2(|\boldsymbol{x}(t)|) + \\ &+d_1\underline{\alpha}^2(|\boldsymbol{x}(t)|) - aV_1(\boldsymbol{x}(t)) - bV_2(\boldsymbol{x}(t)) + d\overline{h}\left(2k_2\underline{\alpha}^2(|\boldsymbol{x}(t)|) + 4k_3|\boldsymbol{w}(t)|^2\right) = \\ &= -\eta V(\boldsymbol{x}(t)) - aV_1(\boldsymbol{x}(t)) - bV_2(\boldsymbol{x}(t)) + \\ &+\left(\frac{k_1^2}{4} + \frac{\overline{h}e^{b\overline{h}}}{4d}k_1^2 c^2 + d_1 + 2k_2 d\overline{h}\right)\underline{\alpha}^2(|\boldsymbol{x}(t)|) + (1+4k_3)|\boldsymbol{w}(t)|^2\end{aligned} \tag{18}$$

Step 3. The inequality in (18) is manipulated by a (legitimate) majorant of the term in the second member to highlight the functional $U(\boldsymbol{x}_t)$ here; the reasoning scheme is the following: $\dot{U}(\boldsymbol{x}_t) \le A,\ A \le B \Rightarrow \dot{U}(\boldsymbol{x}_t) \le B$ (A, B negative quantities, A is substituted by B) and the result is

$$\begin{gathered}\dot{U}(\boldsymbol{x}_t) \le -\mu U(\boldsymbol{x}_t) + \delta(|\boldsymbol{w}(t)|), \\ \mu := \min\{\eta - \left[\tfrac{k_1^2}{4} + k_1^2 \tfrac{\overline{h} e^{b\overline{h}}}{4d} c^2 + d_1 + 2k_2 d\overline{h}\right], a, b\} \\ \delta(|\boldsymbol{w}(t)|) = (1 + 4k_3)|\boldsymbol{w}(t)|^2.\end{gathered} \tag{19}$$

The relation $\underline{\alpha}^2(\|\boldsymbol{x}(t)\|) \le V(\boldsymbol{x}(t))$ and the first inequality from (13) were used. Next, by integrating the differential inequality, the following inequality holds:

$$U(\boldsymbol{x}_t) \le e^{-\mu t} U(\boldsymbol{x}_0) + \delta(\|\boldsymbol{w}[0,t]\|)/\mu. \tag{20}$$

We know that $\|\boldsymbol{w}[0,t]\|$ is finite from the hypothesis; thus, $\|\boldsymbol{w}[0,t]\| := \|\boldsymbol{w}[0,t]\|_\infty$ and $\boldsymbol{w}(t) \in L^\infty$. The calculations proceed as follows:

$$\begin{gathered}U(\boldsymbol{x}_t) \le e^{-\mu t} U(\boldsymbol{x}_0) + \int_0^t e^{-\mu(t-s)} \delta(|\boldsymbol{w}(s)|) ds \le e^{-\mu t} U(\boldsymbol{x}_0) + e^{-\mu t} \int_0^t e^{\mu s} ds\, \delta(\|\boldsymbol{w}(0,t)\|) \le \\ \le e^{-\mu t} U(\boldsymbol{x}_0) + \delta(\|\boldsymbol{w}(0,t)\|)/\mu.\end{gathered}$$

A second relationship suitable for the evaluation of $U(\boldsymbol{x}_0)$ can be established starting from (5):

$$\begin{gathered}U(\boldsymbol{x}_t) \le \hat{\alpha}(\|\boldsymbol{x}\|_{\overline{h}}) = \overline{\alpha}^2(\|\boldsymbol{x}\|_{\overline{h}}) + d_1 \overline{h} \underline{\alpha}^2(\|\boldsymbol{x}\|_{\overline{h}}) + d \tfrac{\overline{h}^2}{2} \|\boldsymbol{x}\|_{\overline{h}}^2 \\ \overline{\alpha}(|\boldsymbol{x}|) = \sqrt{\lambda_{\max}(\boldsymbol{P})} |\boldsymbol{x}|.\end{gathered} \tag{21}$$

Indeed,

$$\begin{gathered}V(\boldsymbol{x}) \le \overline{\alpha}^2(|\boldsymbol{x}|) \\ V_1(\boldsymbol{x}_t) = d_1 \int_{t-\overline{h}}^t e^{-a(t-s)} \underline{\alpha}^2(|\boldsymbol{x}(s)|) ds \le d_1 \overline{h} \underline{\alpha}^2(\|\boldsymbol{x}\|_{\overline{h}}) \\ \|\boldsymbol{x}\|_{\overline{h}} := \sup_{s \in [t-\overline{h}, t]} |\boldsymbol{x}(s)|.\end{gathered} \tag{22}$$

If $f : [a,b] \to R$ is a convex function, then the double inequality of Hadamard holds [40]:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_a^b f(x) dx \le \frac{f(a) + f(b)}{2}. \tag{23}$$

We use the second Hadamard inequality in the case of the functional $V_2(\boldsymbol{x}_t)$, which we rewrite as follows:

$$V_2(\boldsymbol{x}_t) = d \int_{-\overline{h}}^0 f(\theta; t) d\theta, f(\theta; t) := e^{-bt} \int_{t+\theta}^t e^{bs} \dot{\boldsymbol{x}}^{\mathrm{T}}(s) \dot{\boldsymbol{x}}(s) ds.$$

For fixed t, the continuous function $f(\theta; t)$ is a convex function, since it represents a real line segment; its derivatives are given below:

$$\begin{gathered}\dot{f}_\theta(\theta; t) = -d e^{b\theta} \left|\boldsymbol{x}^{\mathrm{T}}(t+\theta)\right|^2 \\ \ddot{f}_\theta(\theta; t) = -d e^{b\theta} \left[b \left|\dot{\boldsymbol{x}}(t+\theta)\right|^2 + \ddot{\boldsymbol{x}}^{\mathrm{T}}(t+\theta) \dot{\boldsymbol{x}}(t+\theta) + \dot{\boldsymbol{x}}^{\mathrm{T}}(t+\theta) \ddot{\boldsymbol{x}}(t+\theta)\right].\end{gathered} \tag{24}$$

They are nonzero, as functions of θ. It follows that

$$\begin{gathered}d \int_{-\overline{h}}^0 f(\theta; t) d\theta \le \tfrac{d\overline{h}}{2} \left(f(0; t) + f\left(-\overline{h}; t\right)\right); f(0; t) = 0 \Rightarrow \\ f\left(-\overline{h}; t\right) = \int_{t-\overline{h}}^t e^{-b(t-s)} \dot{\boldsymbol{x}}^{\mathrm{T}}(s) \dot{\boldsymbol{x}}(s) ds \le \overline{h} \|\dot{\boldsymbol{x}}\|_{\overline{h}}^2 \Rightarrow \\ V_2(\boldsymbol{x}_t) \le d \tfrac{\overline{h}^2}{2} \|\dot{\boldsymbol{x}}\|_{\overline{h}}^2.\end{gathered}$$

The two relations, (20) and (21), will lead to (2) and with this, the theorem is proved. Indeed, based on (1), $\boldsymbol{x}_0 := \boldsymbol{x}(\theta; 0, \varphi) = \varphi(\theta)$ (in the article we preferred to choose $t_0 = 0$) and finally,

$$\begin{aligned} \underline{\alpha}^2(|\boldsymbol{x}(t)|) \le U(\boldsymbol{x}_t) \le e^{-\mu t} U(\boldsymbol{x}_0) + \delta(\|\boldsymbol{w}[0,t]\|)/\mu \le \\ \le e^{-\mu t} \hat{\alpha}\left(\|\varphi\|_{\bar{h}}\right) + \delta(\|\boldsymbol{w}[0,t]\|)/\mu \\ \underline{\alpha}^2(|\boldsymbol{x}(t)|) \le e^{-\mu t} \hat{\alpha}\left(\|\varphi\|_{\bar{h}}\right) + \delta(\|\boldsymbol{w}[0,t]\|)/\mu \end{aligned} \tag{25}$$

In other words, the definition relation (2) of the ISS for the system (1) is obtained, where

$$v(|\boldsymbol{x}(t)|) := \underline{\alpha}^2(|\boldsymbol{x}(t)|),\ \beta\left(\|\boldsymbol{\varphi}\|_{\bar{h}}, t\right) := e^{-\mu t} \hat{\alpha}\left(\|\varphi\|_{\bar{h}}\right),\ \gamma(\|\boldsymbol{w}[0,t]\|) := \delta(\|\boldsymbol{w}[0,t]\|)/\mu.$$

The functions $v, \gamma \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$. The theorem is proved. □

Remark 5. *The membership of functions $v(|\boldsymbol{x}(t)|)$, $\gamma(\|\boldsymbol{w}[0,t]\|)$ to class $\mathcal{K}_\infty$ and that of the function $\beta\left(\|\boldsymbol{\varphi}\|_{\bar{h}}, t\right)$ to class $\mathcal{KL}$ derive, on the one hand, from structural considerations, since these functions are generated by solving a differential equation of the form (1), and on the other hand, this fact is beneficial, since it generates a bounded, ISS-stable solution, as a response to bounded initial conditions $\boldsymbol{\varphi}$ and essentially bounded disturbances $\boldsymbol{w}$.*

5. Numerical Example with Simulation

Application of the result expressed by the theorem from Section 4 to a real-world model, as in the works [5,35], will be the subject of a subsequent work, being in itself a rather complex operation. Here, we will present only a simple exercise to analyze the respective ISS-stability conditions and to perform a minimal numerical simulation. It is obvious that the fulfilment of the inequality (6), also present in (10) and (13), represents a key point of the analysis, which will finally provide starting values for the parameters $\boldsymbol{K}, \boldsymbol{P}$ and η. Next, the analysis focuses on the first two relationships from (13). The parameter d will be removed from the inequality $\eta > \frac{k_1^2}{4} + k_1^2 \frac{\bar{h} e^{b\bar{h}}}{4d} c^2 + d_1 + 2k_2 d\bar{h}$, and the calculations proceed as follows:

$$\begin{aligned} \eta &> \tfrac{4|\boldsymbol{PB}|^2}{\lambda_{\min}(\boldsymbol{P})}\left(\tfrac{1}{4} + \tfrac{\bar{h}e^{b\bar{h}}}{4d}|\boldsymbol{K}|^2\right) + \tfrac{4|\boldsymbol{BK}|^2}{\lambda_{\min}(\boldsymbol{P})} e^{a\bar{h}} + 2\tfrac{\sigma_{\max}(\boldsymbol{A})}{\lambda_{\min}(\boldsymbol{P})} d\bar{h} = \\ &= \tfrac{1}{\lambda_{\min}(\boldsymbol{P})}\left[4|\boldsymbol{PB}|^2\left(\tfrac{1}{4} + \tfrac{\bar{h}e^{b\bar{h}}}{4d}|\boldsymbol{K}|^2\right) + 4|\boldsymbol{BK}|^2 e^{a\bar{h}} + 2\sigma_{\max}(\boldsymbol{A}) d\bar{h}\right]. \end{aligned}$$

The next strategy is to minimize the sum of the two terms on the right that contain the parameter d by introducing $d_{\min} = |\boldsymbol{PB}||\boldsymbol{K}|\sqrt{\frac{e^{b\bar{h}}}{2\sigma_{\max}(\boldsymbol{A})}}$. In the end, the condition below is obtained:

$$\eta > \frac{1}{\lambda_{\min}(\boldsymbol{P})}\left(|\boldsymbol{PB}|^2 + 2\bar{h}|\boldsymbol{PB}||\boldsymbol{K}|\sqrt{2e^{b\bar{h}}\sigma_{\max}(\boldsymbol{A})} + 4|\boldsymbol{BK}|^2 e^{a\bar{h}}\right) \tag{26}$$

to be added to the condition (6). Next, the strategy for evaluating a maximal admissible critical delay $\bar{h}_{\max}$ for the preservation of ISS stability involves reducing the positive parameters a, b to the maximum, so that the inequality is valid at the limit. We use a scalar approach to system (1) to illustrate the operability of the theorem, and we choose $\boldsymbol{A} := -4$, $\boldsymbol{B} =: 1$, $\boldsymbol{K} := 1$. The inequality $(\boldsymbol{A} + \boldsymbol{BK})^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}(\boldsymbol{A} + \boldsymbol{BK}) \le -\eta\boldsymbol{P}$ returns to $2(\boldsymbol{A} + \boldsymbol{BK})\boldsymbol{P} \le -\eta\boldsymbol{P}$, and we choose $\boldsymbol{P} = 1$. It results that $2(\boldsymbol{A} + \boldsymbol{BK}) \le -\eta$. Next, we choose the equality, and it results that $\eta = 6$. From (26), it results that $\lambda_{\min}(\boldsymbol{P}) = 1$, $\sigma_{\max}(\boldsymbol{A}) = 4$, and inequality (26) is written as $6 > 1 + 2\bar{h}\sqrt{2e^{b\bar{h}}\sigma_{\max}(\boldsymbol{A})} + 4e^{a\bar{h}}$. At the limit, we have the value of the critical delay h_{cr}, by considering a, b convergent to 0: $6 > 5 + 4\bar{h}\sqrt{2}$, $\bar{h} := h_{cr} = 0.17677$ s.

With the choice made for system (1), we have to numerically simulate the scalar system:

$$\dot{x}(t) = -4x(t) + x(t-h) + w(t)$$
$$x(\theta; 0, \varphi) = \varphi(\theta), \text{ for } -\overline{h} \le \theta \le 0, \overline{h} > 0,\ h \in \left[0, \overline{h}\right] \tag{27}$$

For the disturbance, we choose a sinusoidal signal: $w(t) = x_0 \sin(2\pi f t)$. Figure 1 shows the time histories of state x. To illustrate the results, only three numerical simulations were retained for the three delays: $h = h_{cr}, h = 0.3$ s, $h = 0.5$ s. The stability seems to be influenced only by the size of h, not by the parameters of $w(t)$, an essentially bounded function. As expected, the theorem provides significant conservative stability conditions, the ISS instability being written down at $h = 0.5$ s.

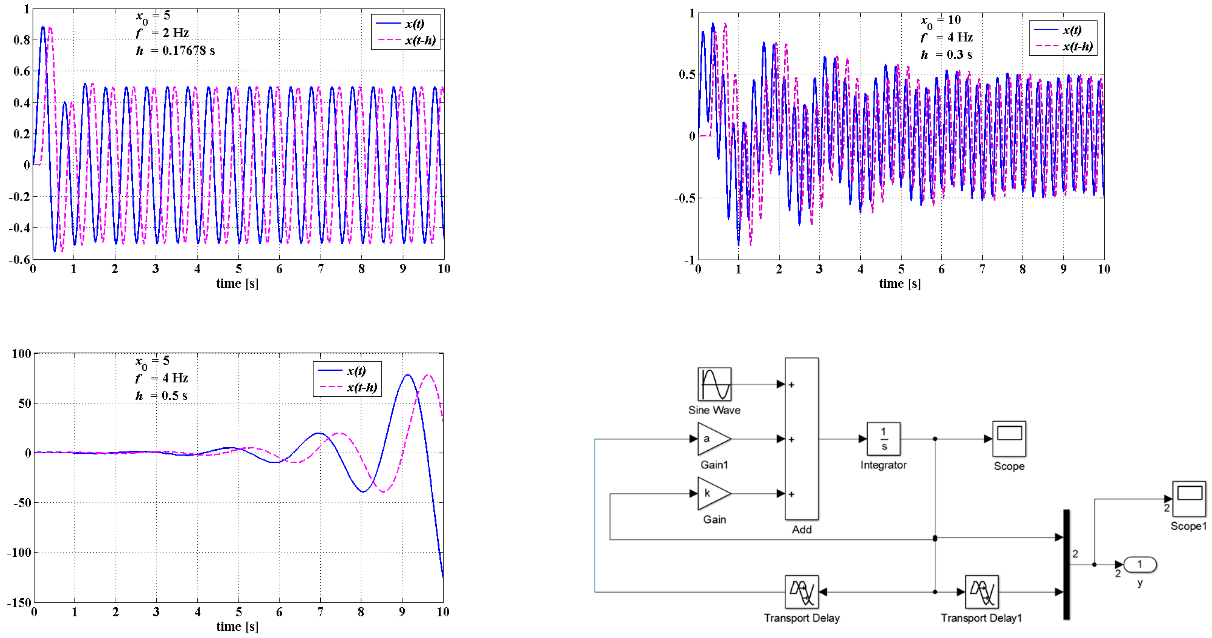

Figure 1. Time histories of the system (3), for three values of h: $h = h_{cr}$, h = 0.3 s, h = 0.5 s (ISS unstable system); bottom right—simulation block diagram.

6. Concluding Remarks

This work is in line with either older or newer concerns of the authors, in which theoretical studies find their applications mainly in the fields of electrohydraulic servomechanisms actuating flight controls and active control systems within aerospace structures. In relation to the models discussed in the works [1–5], there was, on the one hand, a simplification to a mathematical model without switching, and on the other hand, the mathematical model of the system was extended to one with additive-matched disturbances on the state $w(t)$. This led to a radical change in the mathematical tools, by calling for appropriate Lyapunov–Krasovskii functionals, inspired by the works [21,23], but which were applied to a linear model. In aerospace structures, linear models are more common than nonlinear ones, as they appear naturally by applying finite element methods, or by analytical-experimental identification, as in [5]. The basic mathematical model (1) involves the call for a definition of input-to-state stability (ISS) and the introduction of some notions of functional analysis (such as "essential supremum" or "essentially bounded measurable function"), as well as of some special classes or sets of functions $\mathcal{K}$, $\mathcal{K}_\infty$, $\mathcal{L}$, $\mathcal{KL}$, $L^\infty(\mu)$, $L^\infty(\mu)$. It is important to emphasize that all these were claimed by the definition of a mathematical model of disturbances $w(t)$, admissible in the sense of the validity of a theorem on the existence and uniqueness of the solution to the system (1), as given in the work [14]. A novelty compared to work [23] is the use of Jensen's inequality, of Leibniz's rule and of a Hadamard-type double inequality in the mechanism of the proof of the theorem in Section 4.

The work does not bring a new paradigm in the matter of ISS stability of delay systems. What we believe has been achieved in the paper concerns the thorough approach to the problem, so that a reader who is not familiar with the topic can understand the definitions, hypotheses and proof of the theorem, without the need to refer to other sources, as in the case with other works. Moreover, the approached linear model is not systematically present in the literature, as can be seen from the works [21–25], related to the present work, being unjustifiably ignored as being trivial. In reality, we believe that things are different. The reality/practice is not of nonlinear models, but, in the computer era, paradoxically perhaps, primarily of linear models, as it results from our work [5,35]. Control science achieved a qualitative leap with the launch of MATLAB Toolbox Control and linear state models. The linear model in this work essentially involves solving the matrix inequality (6), which is more or less simple to solve. We believe that the article [41] drew attention to an important question: "Control Theory: Is it a Theory of Model or Control?" (see also [42]).

There are several problems that are worth further study. Indeed, the objective of the current work was mainly to propose an ISS-stability theorem for the linear model (1) and less to carry out related numerical simulations on real-world systems because we know from experience that in this case, things become complicated. In a future paper, the aim will be to obtain less conservative stability conditions. We also intend to apply the ISS stability theorem from Section 4 to an active control system, as it appears in the paper [5]. Then, it would be interesting to develop an ISS-stability theorem for the nonlinear system described in works [1–4]. We mention, however, that in the case of a system from the real world, as it was considered in [2], major difficulties appear when compared to the case of didactic examples, illustrated with simple matrices, of the order of 2×2, as they frequently appear in academic works.

Author Contributions: D.E.: Conceptualization, writing—review and editing, funding acquisition. A.T.: Validation, writing—review and editing, funding acquisition. I.U.: Methodology, validation, writing—original draft, writing—review and editing, funding acquisition. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by grants of the Romanian Ministry of Research, Innovation and Digitization, CNCS–UEFISCDI, project number PN-III-P1-1.1-PD-2021-0610, project number PN-III-P2-2.1-PED-2021-2265, project number PN-III-P2-2.1-PED-2021-4147, and by NUCLEU Program project codes PN 23-17-02-03 and PN 23-17-07-01, Ctr. 36 N/12.01.2023.

Data Availability Statement: Data available on request.

Conflicts of Interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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