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A Blow-Up Criterion for 3D Compressible Isentropic Magnetohydrodynamic Equations with Vacuum

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Abstract: In this paper, we investigate a blow-up criterion for compressible magnetohydrodynamic equations. It is shown that if density and velocity satisfy $(\|\rho\|_{L^\infty(0,T;L^\infty)} + \|\mathbf{u}\|_{C([0,T];L^3)} < \infty)$, then the strong solutions to isentropic magnetohydrodynamic equations can exist globally over $[0, T]$. Notably, our analysis accommodates the presence of an initial vacuum.

Keywords: compressible magnetohydrodynamic equations; strong solution; blow up; vacuum

MSC: 76W05; 76N10; 35B40; 35B45



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1. Introduction

Magnetohydrodynamic (MHD) equations concern the dynamics of electrically conducting fluid (plasma) in the magnetic fields. These equations provide a mathematical description of the complex interactions between fluid flow and electromagnetic fields. The mathematical formulation of MHD equations combines Navier–Stokes equations for fluid dynamics with Maxwell’s equations for electromagnetism, creating a set of coupled, nonlinear partial differential equations. Analyzing these equations provides insights into the fundamental principles governing the behavior of conducting fluids in the presence of magnetic fields. The study of existence and blow-up criteria for solutions to MHD equations is a part of the broader effort to understand the mathematical properties and physical implications of these complex systems.

In this paper, we consider the following 3D compressible magnetohydrodynamic (MHD) equations:

$$\begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{u}) = 0, \\ (\rho \mathbf{u})_t + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla P(\rho) = (\nabla \times \mathbf{H}) \times \mathbf{H} + \mu \Delta \mathbf{u} + (\lambda + \mu) \nabla (\nabla \cdot \mathbf{u}), \\ \mathbf{H}_t - \nabla \times (\mathbf{u} \times \mathbf{H}) = -\nabla \times (\nu \nabla \times \mathbf{H}), \\ \nabla \cdot \mathbf{H} = 0. \end{cases} \quad (1)$$

The variables are defined as follows: ρ represents density, $\mathbf{u} = (u^1, u^2, u^3) \in \mathbb{R}^3$ denotes velocity, $P = P(\rho)$ signifies pressure, and $\mathbf{H} = (H^1, H^2, H^3) \in \mathbb{R}^3$ stands for the magnetic field. Additionally, $\nu > 0$ denotes the magnetic diffusion coefficient, while constants μ and λ correspond to the shear and bulk viscosity coefficients of the flow, respectively. These coefficients must adhere to certain physical constraints:

$$\mu > 0, \quad \text{and} \quad 2\mu + 3\lambda \geq 0. \quad (2)$$

Specifically, we focus exclusively on the isentropic magnetohydrodynamic flows characterized by an equation of state of the form

$$P(\rho) = A\rho^\gamma \quad \text{with} \quad A > 0, \gamma > 1. \quad (3)$$

Because of the significance and unique nature of the MHD system (1)–(3), physicists and mathematicians have extensively investigated the compressible MHD problems in [1–8] and the references therein. Here, we briefly review some findings of the multidimensional cases. Fan et al. [9] proved the local existence of classical solutions to (1). In Lions' framework [10], Hu et al. [11] considered the global existence of the weak solutions and the large-time behavior of the three-dimensional MHD system with large data. Additionally, B. Ducomet et al. [3] examined a multi-dimensional non-isentropic MHD system for gaseous stars coupled with the Poisson equation. This system incorporates viscosity coefficients dependent on temperature, and pressure depends on density asymptotically. Zhang et al. [12] obtained the optimal decay estimates of classical solutions to (1) in the L^p frame where the initial data closely approximate a non-vacuum equilibrium state.

When density ρ remains constant, System (1)–(3) simplifies to homogeneous incompressible MHD equations. In this context, Duvaut et al. [4] introduced global weak solutions akin to Leray–Hopf weak solutions for three-dimensional Navier–Stokes equations. Sermange et al. [13] established the local existence of the strong solution with $(\mathbf{u}_0, \mathbf{H}_0) \in H^m(\mathbb{R}^3)$ with $m \geq 2$. For the two-dimensional case, when the partial derivatives of the viscosity and resistivity are equal to zero, Cao et al. [14] established a global classical solution. For the nonhomogeneous case (1)–(4), there has also been a lot of literature which includes the existence, uniqueness, and regularity of solutions [15–17]. Gerbeau et al. [5] and Desjardins et al. [2] investigated the global weak solutions of finite energy in the whole space or the torus. Abidi et al. [18] proposed the existence of strong solutions when initiated with small data within some Besov spaces. In a recent paper, Huang et al. [19] established the global strong solutions' uniqueness for the general initial conditions to (1)–(3) in two dimensions.

The primary objective of this paper is to present a criterion for the blow-up of strong solutions for the compressible MHD system (1) with the initial and boundary conditions:

$$(\rho, \mathbf{u}, \mathbf{H})(x, 0) = (\rho_0, \mathbf{u}_0, \mathbf{H}_0)(x), \quad \text{for all } x \in \mathbb{R}^3, \quad (4)$$

and far-field conditions:

$$(\rho, \mathbf{u}, \mathbf{H}) \rightarrow (0, 0, 0), \quad \text{as } |x| \rightarrow \infty. \quad (5)$$

Before presenting the key outcome of this paper, for $k \in \mathbb{Z}^+$ and $r > 1$, we first denote the standard homogeneous and inhomogeneous Sobolev spaces for scalar/vector functions as follows:

$$\begin{cases} L^r = L^r(\mathbb{R}^3), \quad D^{k,r} = \{u \in L^1_{\text{loc}}(\mathbb{R}^3) \mid \|\nabla^k u\|_{L^r} < \infty\}, \quad W^{k,r} = L^r \cap D^{k,r}, \\ H^k = W^{k,2}, \quad D^k = D^{k,2}, \quad D^1 = \{u \in L^6 \mid \|\nabla u\|_{L^2} < \infty\}, \quad \|u\|_{D^{k,r}} = \|\nabla^k u\|_{L^r}. \end{cases}$$

Second, we state a local existence of strong solutions to (1)–(4), which was established in [9].

Proposition 1. Assume that for some $q \in (3, 6]$, and the initial data $(\rho_0, \mathbf{u}_0, \mathbf{H}_0)$ satisfy

$$\rho_0 \geq 0, \rho_0 \in L^1 \cap W^{1,q}, \mathbf{u}_0 \in D^1 \cap D^2, \mathbf{H}_0 \in H^2, \operatorname{div} \mathbf{H}_0 = 0, \quad (6)$$

and

$$-\Delta u_0 + \nabla P_0 - (\nabla \times H_0) \times H_0 = \rho_0^{1/2} g \quad (7)$$

for some $g \in L^2$. Then, there exists time $T_* > 0$ and a unique strong solution $(\rho, \mathbf{u}, \mathbf{H})$ to (1)–(4) such that

$$\begin{cases} \rho \geq 0, \rho \in C([0, T_*]; W^{1,q}), \rho_t \in C([0, T_*]; L^q), \\ \mathbf{u} \in C([0, T_*]; D^1 \cap D^2) \cap L^2(0, T_*; D^{2,q}), \sqrt{\rho} \mathbf{u}_t \in L^\infty(0, T_*; L^2) \\ (\mathbf{u}_t, \mathbf{H}_t) \in L^2(0, T_*; D^1), \mathbf{H} \in C([0, T_*]; H^2) \cap L^2(0, T_*; W^{2,q}). \end{cases} \quad (8)$$

While substantial advancements have been achieved in investigating three-dimensional compressible MHD systems, numerous physically significant and mathematically fundamental issues persist due to the absence of a smoothing mechanism and the presence of strong nonlinearity. Analogous to challenges encountered in three-dimensional compressible Navier–Stokes equations, the question of whether the unique local strong solution, as outlined in Proposition 1, can persist globally remains a formidable open problem, particularly when considering general initial data in three dimensions. Consequently, a plethora of studies have been dedicated to exploring the blow-up criterion for weak/classical solutions to compressible Navier–Stokes equations (see [20–22]) and to incompressible/compressible MHD systems (see [23–28]).

We note that He et al. [24] established a blow-up criteria for nonhomogeneous incompressible MHD equations: the local solution extends globally if the locally smooth solution $(\mathbf{u}, \mathbf{H})$ satisfies one of the following two conditions:

$$(A) \quad \mathbf{u} \in L^s(0, T; L^r), \quad \frac{2}{s} + \frac{3}{r} \leq 1, \quad 3 < r \leq \infty,$$

$$(B) \quad u \in C([0, T]; L^3).$$

We wonder whether these results can hold for compressible magnetohydrodynamic equations or not. Indeed, conditions (A) are proved by Xu and Zhang in [28]: if $T^* \in (0, \infty)$ is the maximal time of existence for strong solutions $(\rho, \mathbf{u}, \mathbf{H})$ to the compressible MHD Equation (4), then

$$\lim_{T \rightarrow T^*} \left(\|\rho\|_{L^\infty(0, T; L^\infty)} + \|\mathbf{u}\|_{L^s(0, T; L_w^r)} \right) = \infty \quad (9)$$

for any r and s satisfying

$$\frac{2}{s} + \frac{3}{r} \leq 1, \quad 3 < r \leq \infty, \quad (10)$$

where L_w^r denotes the weak L^r -space.

In this paper, we do our best to generalize condition (B) to compressible MHD Equation (1). Then, the principal outcome of this paper is articulated as follows.

Theorem 1. Suppose that the assumptions in Proposition 1 are satisfied. Let $(\rho, \mathbf{u}, \mathbf{H})$ be a strong solution to (1)–(4) with Regularity (9). If $T^* < \infty$ is the maximal time of existence, then

$$\lim_{T \rightarrow T^*} \left(\|\rho\|_{L^\infty(0, T; L^\infty)} + \|\mathbf{u}\|_{C([0, T]; L^3)} \right) = \infty. \quad (11)$$

Remark 1. The proof of Theorem 1.1 is mainly taken from the ideas for Navier–Stokes equations (cf. [6,20]) and MHD equations (cf. [24,28]).

Remark 2. When the initial density is permitted to be zero, the system’s nonlinearity can lead to singularities and instability in the solutions. This may bring some mathematical difficulties such as division by zero, which needs more mathematical techniques. It is more challenging than $\rho > 0$.

To prove Theorem 1, the main step is to estimate the L^2 -norm of $\nabla \mathbf{u}$ and $\nabla \mathbf{H}$. First, we provide initial estimations of the effective viscous flux, $F = (2\mu + \lambda) \operatorname{div} \mathbf{u} - P(\rho) - \frac{1}{2} |\mathbf{H}|^2$ (see Lemma 1), which is used to control the L^p -norm of $\nabla \mathbf{u}$ in the proof of Lemma 4. And then, under the assumption that the right-hand side of (11) is bounded from above

and by a key observation that for $U \in C([0, T]; L^3)$, there exists $U^1 \in C([0, T]; L^3)$ and $U^2 \in L^\infty(0, T; L^\infty)$ such that for any $\delta > 0$,

$$U = U^1 + U^2, \quad \|U^1\|_{C([0, T]; L^3)} \leq \delta, \quad \|U^2\|_{L^\infty(0, T; L^\infty)} \leq C(\delta, M), \quad (12)$$

where $M \triangleq \|u\|_{C([0, T]; L^3)}$ and $C(\delta, M)$ is a positive constant depending only on δ, M , we improve the integrability of $\mathbf{H}$ by choosing $\delta > 0$ suitably small. Finally, we succeed in obtaining the estimates on $\|\nabla u\|_{L^2}$ and $\|\nabla \mathbf{H}\|_{L^2}$ by utilizing the preliminary estimates of the effective viscous flux, $F = (2\mu + \lambda)\operatorname{div} u - P(\rho) - \frac{1}{2}|\mathbf{H}|^2$ and (12) to control the L^p -norm of ∇u in Lemma 4. Having obtained the estimates of $\|\nabla u\|_{L^2}$ and $\|\nabla \mathbf{H}\|_{L^2}$, we can then provide a higher-order estimate for $(\rho, u, \mathbf{H})$, thereby concluding the proof of Theorem 1.

2. Proof of Theorem 1

We let $(\rho, u, \mathbf{H})$ be strong solutions to (1)–(4) as stated in Theorem 1. If we assume for any $0 < T < T^* < \infty$,

$$\lim_{T \rightarrow T^*} \left(\|\rho\|_{L^\infty(0, T; L^\infty)} + \|u\|_{C([0, T]; L^3)} \right) \leq M < \infty. \quad (13)$$

Thus, we deduce a contradiction to the maximality of T^* .

In the rest of this paper, C represents different positive constants in this context, which rely on the initial data, M and T . Any specific dependency is explicitly highlighted if required.

As pointed out in Section 1, the key step is to obtain a delicate L^p -estimate for the gradient of velocity u and magnetic field $\mathbf{H}$. To achieve this, we first introduce two important canonical variables, that is, effective viscous flux F and vorticity ω , which are defined as follows:

$$F = (2\mu + \lambda)\operatorname{div} u - P(\rho) - \frac{1}{2}|\mathbf{H}|^2, \quad \text{and} \quad \omega = \nabla \times u. \quad (14)$$

Due to identity

$$(\nabla \times \mathbf{H}) \times \mathbf{H} = \mathbf{H} \cdot \nabla \mathbf{H} - |\mathbf{H}|^2/2,$$

it follows from (1)₂, (1)₃ that

$$\Delta F = \operatorname{div}(\rho u_t + \rho u \cdot \nabla u - \mathbf{H} \cdot \nabla \mathbf{H}), \quad \mu \Delta \omega = \nabla \times (\rho u_t + \rho u \cdot \nabla u - \mathbf{H} \cdot \nabla \mathbf{H}), \quad (15)$$

and

$$-\nu \Delta \mathbf{H} = -\mathbf{H}_t + u \cdot \nabla \mathbf{H} + \mathbf{H} \cdot \nabla u - \operatorname{div} u \mathbf{H}. \quad (16)$$

We now present the subsequent elementary estimates, derived from the standard L^p -estimate for Systems (15) and (16), which are instrumental in acquiring the estimates of $\|\nabla u\|_{L^2}$ and $\|\nabla \mathbf{H}\|_{L^2}$ in Lemma 4.

Lemma 1. Under Assumption (13), we have

$$\|\nabla F\|_{L^2} + \|\nabla \omega\|_{L^2} \leq C \left(\|\rho^{1/2} u_t\|_{L^2} + \|u \cdot \nabla u\|_{L^2} + \|\mathbf{H} \cdot \nabla \mathbf{H}\|_{L^2} \right), \quad (17)$$

and

$$\|\nabla \mathbf{H}\|_{L^6} \leq C(\|\mathbf{H}_t\|_{L^2} + \|u \cdot \nabla \mathbf{H}\|_{L^2} + \|\mathbf{H} \cdot \nabla u\|_{L^2}), \quad (18)$$

where the letter C denotes a positive constant depending only on $\mu, \lambda, \nu, \gamma, M, A$ and the initial data.

Proof. Applying the standard L^p -estimate of elliptic System (15) leads to

$$\|\nabla F\|_{L^2} + \|\nabla \omega\|_{L^2} \leq C(\|\rho u_t\|_{L^2} + \|\rho u \cdot \nabla u\|_{L^2} + \|\mathbf{H} \cdot \nabla \mathbf{H}\|_{L^2})$$

$$\leq C \left(\|\rho^{1/2} \mathbf{u}_t\|_{L^2} + \|\mathbf{u} \nabla \mathbf{u}\|_{L^2} + \|\mathbf{H} \nabla \mathbf{H}\|_{L^2} \right).$$

Similarly, we also deduce from the H^2 -estimate for elliptic System (16) and imbedding inequality that

$$\|\nabla \mathbf{H}\|_{L^6} \leq C \|\nabla^2 \mathbf{H}\|_{L^2} \leq C (\|\mathbf{H}_t\|_{L^2} + \|\mathbf{u} \nabla \mathbf{H}\|_{L^2} + \|\mathbf{H} \nabla \mathbf{u}\|_{L^2}).$$

Hence, Lemma 1 is completed. $\square$

The next lemma is the standard energy estimate.

Lemma 2. Let $(\rho, \mathbf{u}, \mathbf{H})$ be a smooth solution of (1)–(4) on $\mathbb{R}^3 \times (0, T]$. Then, there exists a constant C such that

$$\sup_{0 \leq t \leq T} \int \left(\rho^\gamma + \frac{1}{2} \rho |\mathbf{u}|^2 + \frac{1}{2} |\mathbf{H}|^2 \right) dx + \int_0^T \left(\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{H}\|_{L^2}^2 \right) dt \leq C. \quad (19)$$

Proof. The proof of the lemma is standard, so detail is omitted. $\square$

To derive the L^2 -estimate of $\nabla \mathbf{u}$ and $\nabla \mathbf{H}$ under Assumption (13), we also need to improve the integrability of magnetic field $\mathbf{H}$.

Lemma 3. Under Assumption (13), for any $T < T^*$, it holds that

$$\|\mathbf{H}\|_{L^\infty(0, T; L^q)} \leq C, \quad \forall q \in [2, \infty). \quad (20)$$

Proof. As in [28], multiplication of (1)₃ by $q|\mathbf{H}|^{q-2}\mathbf{H}$ ($q \geq 2$) and integration of the resulting equation over $\mathbb{R}^3$ lead to

$$\begin{aligned} & \frac{d}{dt} \int |\mathbf{H}|^q dx + \nu \int \left(q |\mathbf{H}|^{q-2} |\nabla \mathbf{H}|^2 + \frac{q(q-2)}{2} |\mathbf{H}|^{q-4} |\nabla |\mathbf{H}|^2|^2 \right) dx \\ &= - \int \left(q |\mathbf{H}|^{q-2} \mathbf{H} \cdot \nabla \mathbf{H} \cdot \mathbf{u} + \frac{q(q-2)}{2} |\mathbf{H}|^{q-4} (\mathbf{H} \cdot \nabla |\mathbf{H}|^2) (\mathbf{u} \cdot \mathbf{H}) \right) dx \\ &\leq \frac{\nu}{2} \left(\int \left(q |\mathbf{H}|^{q-2} |\nabla \mathbf{H}|^2 + \frac{q(q-2)}{2} |\mathbf{H}|^{q-4} |\nabla |\mathbf{H}|^2|^2 \right) dx \right) + C \int |\mathbf{u}|^2 |\mathbf{H}|^q dx, \end{aligned}$$

which immediately implies that

$$\frac{d}{dt} \int |\mathbf{H}|^q dx + \int |\nabla |\mathbf{H}|^{q/2}|^2 dx \leq C \int |\mathbf{u}|^2 |\mathbf{H}|^{q/2} dx. \quad (21)$$

Due to the fact that $\mathbf{u} \in C([0, T]; L^3)$, we can decompose u into the following two parts:

$$\mathbf{u} \triangleq U^1 + U^2$$

with

$$\|U^1\|_{C([0, T]; L^3)} \leq \delta, \quad \|U^2\|_{L^\infty(0, T; L^\infty)} \leq C(\delta, M_0) \quad (22)$$

for $M_0 \triangleq \|\mathbf{u}\|_{C([0, T]; L^3)}$ and any $\delta \in (0, 1)$.

From (21) and (22) and using Hölder inequality and imbedding inequality, we have

$$\begin{aligned} & \frac{d}{dt} \int |\mathbf{H}|^q dx + \int |\nabla |\mathbf{H}|^{q/2}|^2 dx \\ &\leq C \|U^1\|_{L^3}^2 \left\| |\mathbf{H}|^{q/2} \right\|_{L^6}^2 + \|U^2\|_{L^\infty((0, T) \times (\mathbb{R}^3))}^2 \int |\mathbf{H}|^q dx \\ &\leq C \delta \left\| |\nabla |\mathbf{H}|^{q/2}|^2 \right\|_{L^2}^2 + C \int |\mathbf{H}|^q dx. \end{aligned}$$

Selecting δ suitably small and employing Gronwall's inequality, we promptly attain the desired Estimate (20). $\square$

Under Assumption (13) and with the help of Lemmas 1–3, we establish the subsequent key lemma concerning the estimates of $\nabla \mathbf{u}$ and $\nabla \mathbf{H}$.

Lemma 4. Under Assumption (13), for any $T < T^*$, it holds that

$$\sup_{0 \leq t \leq T} \left(\|\nabla \mathbf{u}\|_{L^2}^2 + \|\mathbf{H}\|_{H^1}^2 \right) + \int_0^T \left(\|\rho^{1/2} \mathbf{u}_t\|_{L^2}^2 + \|\mathbf{H}_t\|_{L^2}^2 + \|\mathbf{H}\|_{H^2}^2 \right) dt \leq C. \quad (23)$$

Proof. Multiplying (1)₂ by $\mathbf{u}_t$ and integrating the resulting equation over $\mathbb{R}^3$ produce

$$\begin{aligned} & \frac{d}{dt} \int \left(\frac{\mu}{2} |\nabla \mathbf{u}|^2 + \frac{\lambda + \mu}{2} (\operatorname{div} \mathbf{u})^2 \right) dx + \int \rho |\mathbf{u}_t|^2 dx \\ &= \int P \operatorname{div} \mathbf{u}_t dx + \int \left(\mathbf{H} \cdot \nabla \mathbf{H} - \frac{1}{2} \nabla |\mathbf{H}|^2 \right) \cdot \mathbf{u}_t dx - \int \rho \mathbf{u} \cdot \nabla \mathbf{u} \cdot \mathbf{u}_t dx. \end{aligned} \quad (24)$$

On the other hand, it follows from (1)₃ that

$$\frac{d}{dt} \nu \|\nabla \mathbf{H}\|_{L^2}^2 + \left(\|\mathbf{H}_t\|_{L^2}^2 + \nu \|\Delta \mathbf{H}\|_{L^2}^2 \right) = \int |\mathbf{H} \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{H} - \operatorname{div} \mathbf{u} \mathbf{H}|^2 dx. \quad (25)$$

Putting (24) and (25) together leads to

$$\begin{aligned} & \frac{d}{dt} \left(\frac{\mu}{2} |\nabla \mathbf{u}|^2 + \frac{\lambda + \mu}{2} (\operatorname{div} \mathbf{u})^2 + \nu \|\nabla \mathbf{H}\|_{L^2}^2 \right) + \left(\|\rho^{1/2} \mathbf{u}_t\|_{L^2}^2 + \|\mathbf{H}_t\|_{L^2}^2 + \nu \|\Delta \mathbf{H}\|_{L^2}^2 \right) \\ &= \int P \operatorname{div} \mathbf{u}_t dx + \int \left(\mathbf{H} \cdot \nabla \mathbf{H} - \frac{1}{2} \nabla |\mathbf{H}|^2 \right) \cdot \mathbf{u}_t dx - \int \rho \mathbf{u} \cdot \nabla \mathbf{u} \cdot \mathbf{u}_t dx \\ &+ \int |\mathbf{H} \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{H} - \operatorname{div} \mathbf{u} \mathbf{H}|^2 dx = \sum_{i=1}^4 J_i. \end{aligned} \quad (26)$$

It is now the position to estimate the terms on the right-hand of (26) term by term. First, taking (1)₁ and $P(\rho) = A\rho^\gamma$ into consideration, we determine that

$$P_t + \operatorname{div}(P\mathbf{u}) + (\gamma - 1)P \operatorname{div} \mathbf{u} = 0.$$

Then, by the definition of F , we deduce from the integration of parts that

$$\begin{aligned} J_1 &= \frac{d}{dt} \int P \operatorname{div} \mathbf{u} dx + \int \operatorname{div}(P\mathbf{u}) \operatorname{div} \mathbf{u} dx + (\gamma - 1) \int P (\operatorname{div} \mathbf{u})^2 dx \\ &= \frac{d}{dt} \int P \operatorname{div} \mathbf{u} dx - \frac{1}{2\mu + \lambda} \int P \mathbf{u} \cdot \nabla F dx + \frac{1}{2(2\mu + \lambda)} \int P^2 \operatorname{div} \mathbf{u} dx \\ &\quad - \frac{1}{2(2\mu + \lambda)} \int P \mathbf{u} \cdot \nabla |\mathbf{H}|^2 dx + (\gamma - 1) \int P (\operatorname{div} \mathbf{u})^2 dx \\ &\leq \frac{d}{dt} \int P \operatorname{div} \mathbf{u} dx + C \|P\|_{L^3} \|\mathbf{u}\|_{L^6} \|\nabla F\|_{L^2} + C \|P\|_{L^\infty} \|P\|_{L^2} \|\nabla \mathbf{u}\|_{L^2} \\ &\quad + C \|P\|_{L^\infty} \|\mathbf{u}\|_{L^6} \|\mathbf{H}\|_{L^3} \|\nabla \mathbf{H}\|_{L^2} + C \|P\|_{L^\infty} \|\nabla \mathbf{u}\|_{L^2}^2 \\ &\leq \frac{d}{dt} \int P \operatorname{div} \mathbf{u} dx + \varepsilon \left(\|\rho^{1/2} \mathbf{u}_t\|_{L^2}^2 + \|\mathbf{u} \nabla \mathbf{u}\|_{L^2}^2 + \|\mathbf{H} \nabla \mathbf{H}\|_{L^2}^2 \right) \\ &\quad + C(\varepsilon) \left(\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{H}\|_{L^2}^2 + 1 \right), \end{aligned}$$

where we also use (13), (20) and (17). Similarly, using integration by parts and the fact that $\operatorname{div} \mathbf{H} = 0$, we have

$$\begin{aligned} J_2 &= - \int \left(\mathbf{H} \cdot \nabla \mathbf{u}_t \cdot \mathbf{H} - \frac{1}{2} |\mathbf{H}|^2 \operatorname{div} \mathbf{u}_t \right) dx \\ &= - \frac{d}{dt} \int \left(\mathbf{H} \cdot \nabla \mathbf{u} \cdot \mathbf{H} - \frac{1}{2} |\mathbf{H}|^2 \operatorname{div} \mathbf{u} \right) dx \\ &\quad + \int (\mathbf{H}_t \cdot \nabla \mathbf{u} \cdot \mathbf{H} + \mathbf{H} \cdot \nabla \mathbf{u} \cdot \mathbf{H}_t - \mathbf{H} \cdot \mathbf{H}_t \operatorname{div} \mathbf{u}) dx \\ &\leq - \frac{d}{dt} \int \left(\mathbf{H} \cdot \nabla \mathbf{u} \cdot \mathbf{H} - \frac{1}{2} |\mathbf{H}|^2 \operatorname{div} \mathbf{u} \right) dx + \frac{1}{2} \|\mathbf{H}_t\|_{L^2}^2 + C \|\mathbf{H}\| \|\nabla \mathbf{u}\|_{L^2}^2. \end{aligned}$$

Under Assumption (13), we have, following from Young inequality,

$$J_3 \leq \frac{1}{4} \|\rho^{1/2} \mathbf{u}_t\|_{L^2}^2 + C \|\mathbf{u}\| \|\nabla \mathbf{u}\|_{L^2}^2,$$

and

$$J_4 = \int |\mathbf{H} \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{H} - \operatorname{div} \mathbf{u} \mathbf{H}|^2 dx \leq C \left(\|\mathbf{u}\| \|\nabla \mathbf{H}\|_{L^2}^2 + \|\mathbf{H}\| \|\nabla \mathbf{u}\|_{L^2}^2 \right).$$

Substituting J_1 – J_4 into (25) and combining this resulting with J_1 by choosing $\varepsilon > 0$ small enough, we find

$$\begin{aligned} &\frac{d}{dt} \left(\frac{\mu}{2} |\nabla \mathbf{u}|^2 + \frac{\lambda + \mu}{2} (\operatorname{div} \mathbf{u})^2 + \nu \|\nabla \mathbf{H}\|_{L^2}^2 \right) + \left(\frac{1}{2} \|\rho^{1/2} \mathbf{u}_t\|_{L^2}^2 + \frac{1}{2} \|\mathbf{H}_t\|_{L^2}^2 + \nu \|\Delta \mathbf{H}\|_{L^2}^2 \right) \\ &\leq \frac{d}{dt} \int \left((P + \frac{1}{2} |\mathbf{H}|^2) \operatorname{div} \mathbf{u} - \mathbf{H} \cdot \nabla \mathbf{u} \cdot \mathbf{H} \right) dx + C (\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{H}\|_{L^2}^2 + 1) \\ &\quad + C_1 \left(\|\mathbf{u}\| \|\nabla \mathbf{u}\|_{L^2}^2 + \|\mathbf{H}\| \|\nabla \mathbf{H}\|_{L^2}^2 + \|\mathbf{u}\| \|\nabla \mathbf{H}\|_{L^2}^2 + \|\mathbf{H} \nabla \mathbf{u}\|_{L^2}^2 \right), \end{aligned} \quad (27)$$

where C_1 is a positive constant depending only on the initial data.

By (20) and (22), we can estimate the last term on the right-hand side of (27) as follows:

$$\begin{aligned} \|\mathbf{u}\| \|\nabla \mathbf{H}\|_{L^2}^2 &\leq \int |U^1|^2 |\nabla \mathbf{H}|^2 dx + \int |U^2|^2 |\nabla \mathbf{H}|^2 dx \\ &\leq \|U^1\|_{L^3}^2 \|\nabla \mathbf{H}\|_{L^6}^2 + \|U^2\|_{L^\infty}^2 \|\nabla \mathbf{H}\|_{L^2}^2 \\ &\leq \delta^2 \|\nabla \mathbf{H}\|_{L^6}^2 + C(\delta, M) \|\nabla \mathbf{H}\|_{L^2}^2, \end{aligned}$$

$$\begin{aligned} \|\mathbf{u}\| \|\nabla \mathbf{u}\|_{L^2}^2 &\leq \int |U^1|^2 |\nabla \mathbf{u}|^2 dx + \int |U^2|^2 |\nabla \mathbf{u}|^2 dx \\ &\leq \|U^1\|_{L^3}^2 \|\nabla \mathbf{u}\|_{L^6}^2 + \|U^2\|_{L^\infty}^2 \|\nabla \mathbf{u}\|_{L^2}^2 \\ &\leq \delta^2 \|\nabla \mathbf{u}\|_{L^6}^2 + C(\delta, M) \|\nabla \mathbf{u}\|_{L^2}^2, \end{aligned}$$

$$\begin{aligned} \|\mathbf{H}\| \|\nabla \mathbf{u}\|_{L^2}^2 &\leq C \|\mathbf{H}\|_{L^6}^2 \|\nabla \mathbf{u}\|_{L^3}^2 \leq C \|\nabla \mathbf{u}\|_{L^2} \|\nabla \mathbf{u}\|_{L^6} \\ &\leq \delta^2 \|\nabla \mathbf{u}\|_{L^6}^2 + C(\delta) \|\nabla \mathbf{u}\|_{L^2}^2, \end{aligned}$$

$$\begin{aligned} \|\mathbf{H}\| \|\nabla \mathbf{H}\|_{L^2}^2 &\leq C \|\mathbf{H}\|_{L^6}^2 \|\nabla \mathbf{H}\|_{L^3}^2 \leq C \|\nabla \mathbf{H}\|_{L^2} \|\nabla \mathbf{H}\|_{L^6} \\ &\leq \delta^2 \|\nabla \mathbf{H}\|_{L^6}^2 + C(\delta) \|\nabla \mathbf{H}\|_{L^2}^2. \end{aligned}$$

Hence,

$$\begin{aligned} & \| |\mathbf{u}| |\nabla \mathbf{u}| \|_{L^2}^2 + \| |\mathbf{H}| |\nabla \mathbf{H}| \|_{L^2}^2 + \| |\mathbf{u}| |\nabla \mathbf{H}| \|_{L^2}^2 + \| |\mathbf{H}| |\nabla \mathbf{u}| \|_{L^2}^2 \\ & \leq \delta^2 \left(\| \nabla \mathbf{u} \|_{L^6}^2 + \| \nabla \mathbf{H} \|_{L^6}^2 \right) + C(\delta) \left(\| \nabla \mathbf{u} \|_{L^2}^2 + \| \nabla \mathbf{H} \|_{L^2}^2 \right). \end{aligned} \quad (28)$$

Next, we turn to estimate $\| \nabla \mathbf{u} \|_{L^6}$ and $\| \nabla \mathbf{H} \|_{L^6}$. Using (17), we determine that

$$\begin{aligned} \| \nabla \mathbf{u} \|_{L^6} & \leq C(\| \operatorname{div} \mathbf{u} \|_{L^6} + \| \nabla \times \mathbf{u} \|_{L^6}) \\ & \leq C \left(\| F \|_{L^6} + \| \omega \|_{L^6} + \| P \|_{L^6} + \| |\mathbf{H}|^2 \|_{L^6} \right) \\ & \leq C \left(\| \nabla F \|_{L^2} + \| \nabla \omega \|_{L^2} + \| \rho \|_{L^1} + \| \mathbf{H} \|_{L^{12}}^2 \right) \\ & \leq C \left(\| \rho^{1/2} \mathbf{u}_t \|_{L^2} + \| \mathbf{u} \nabla \mathbf{u} \|_{L^2} + \| \mathbf{H} \nabla \mathbf{H} \|_{L^2} + 1 \right), \end{aligned}$$

which, together with (18), implies that

$$\| \nabla \mathbf{u} \|_{L^6} + \| \nabla \mathbf{H} \|_{L^6} \leq C \left(\| \rho^{1/2} \mathbf{u}_t \|_{L^2} + \| \mathbf{H}_t \|_{L^2} + \| |\mathbf{u}| |\nabla \mathbf{u}| \|_{L^2} + \| |\mathbf{u}| |\nabla \mathbf{H}| \|_{L^2} + \| |\mathbf{H}| |\nabla \mathbf{u}| \|_{L^2} \right). \quad (29)$$

Placing (29) into (28), we obtain the following inequality by choosing $\delta > 0$ to be sufficiently small:

$$\begin{aligned} & \| |\mathbf{u}| |\nabla \mathbf{u}| \|_{L^2}^2 + \| |\mathbf{H}| |\nabla \mathbf{H}| \|_{L^2}^2 + \| |\mathbf{u}| |\nabla \mathbf{H}| \|_{L^2}^2 + \| |\mathbf{H}| |\nabla \mathbf{u}| \|_{L^2}^2 \\ & \leq \frac{1}{4C_1} \left(\| \rho^{1/2} \mathbf{u}_t \|_{L^2}^2 + \| \mathbf{H}_t \|_{L^2}^2 \right) + C(\delta) \left(\| \nabla \mathbf{u} \|_{L^2}^2 + \| \nabla \mathbf{H} \|_{L^2}^2 \right), \end{aligned} \quad (30)$$

where C_1 is given in (27).

Substituting (30) into (27) leads to

$$\begin{aligned} & \frac{d}{dt} \left(\frac{\mu}{2} \| \nabla \mathbf{u} \|^2 + \frac{\mu + \lambda}{2} \| \nabla \mathbf{H} \|_{L^2}^2 \right) + \frac{1}{4} \left(\| \rho^{1/2} \mathbf{u}_t \|_{L^2}^2 + \| \mathbf{H}_t \|_{L^2}^2 + \nu \| \Delta \mathbf{H} \|_{L^2}^2 \right) \\ & \leq - \frac{d}{dt} \int \left(\left(P + \frac{1}{2} |\mathbf{H}|^2 \right) \operatorname{div} \mathbf{u} - \mathbf{H} \cdot \nabla \mathbf{u} \cdot \mathbf{H} \right) dx + C \left(\| \nabla \mathbf{u} \|_{L^2}^2 + \| \nabla \mathbf{H} \|_{L^2}^2 \right). \end{aligned} \quad (31)$$

By (19) and (20) and using Young inequality, we easily see that

$$\left| \int \left(\left(P + \frac{1}{2} |\mathbf{H}|^2 \right) \operatorname{div} \mathbf{u} - \mathbf{H} \cdot \nabla \mathbf{u} \cdot \mathbf{H} \right) dx \right| \leq \frac{\mu}{4} \| \nabla \mathbf{u} \|_{L^2}^2 + C.$$

Considering this, we can conclude from (31) and the Gronwall inequality, for any $0 \leq T < T^*$, that the following holds:

$$\sup_{0 \leq t \leq T} \left(\| \nabla \mathbf{u} \|_{L^2}^2 + \| \mathbf{H} \|_{H^1}^2 \right) + \int_0^T \left(\| \rho^{1/2} \mathbf{u}_t \|_{L^2}^2 + \| \mathbf{H}_t \|_{L^2}^2 + \| \Delta \mathbf{H} \|_{L^2}^2 \right) dt \leq C. \quad (32)$$

This lemma is completed. $\square$

With the help of Lemmas 2–4, we can improve regularity estimates on ρ , $\mathbf{u}$ and $\mathbf{H}$ using the same method as in [28] (Lemmas 3.6–3.8). For completeness and simplicity, we only state these estimates without proof.

Lemma 5. Under Assumption (13), it holds for any $0 \leq T < T^*$ that

$$\sup_{0 \leq t \leq T} \left(\| \rho^{1/2} \mathbf{u}_t \|_{L^2}^2 + \| \mathbf{H}_t \|_{L^2}^2 + \| \nabla \mathbf{H} \|_{H^1}^2 \right) + \int_0^T \left(\| \nabla \mathbf{u}_t \|_{L^2}^2 + \| \nabla \mathbf{H}_t \|_{L^2}^2 \right) dt \leq C,$$

$$\sup_{0 \leq t \leq T} (\|\rho\|_{H^1 \cap W^{1,q}} + \|\nabla \mathbf{u}\|_{H^1}) + \int_0^T (\|\nabla \mathbf{u}\|_{L^\infty} + \|\nabla \mathbf{u}\|_{W^{1,q}}^2) dt \leq C,$$

and

$$\sup_{0 \leq t \leq T} \|\rho^{1/2} \mathbf{u}_t\|_{L^2}^2 + \int_0^T \|\nabla \mathbf{u}_t\|_{L^2}^2 dt \leq C.$$

With all the a priori estimates in Section 2, Lemmas 2–5 and the local existence theorem (cf. Proposition 1), it becomes straightforward to extend the strong solutions of $(\rho, \mathbf{u}, \mathbf{H})$ beyond $t > T^*$ by the standard method. Consequently, this contradicts the assumption made on T^* .

Proof of Theorem 1. From Lemma 5, we can find that functions $(\rho, \mathbf{u}, \mathbf{H}, \theta)(\mathbf{x}, t = T^*) = \lim_{t \rightarrow T^*} (\rho, \mathbf{u}, \mathbf{H}, \theta)$ have the same regularities imposed on initial data (6) at time $t = T^*$. Furthermore,

$$\begin{aligned} -\Delta u_0 + \nabla P_0 - (\nabla \times H_0) \times H_0 &= \rho_0^{1/2} g|_{t=T^*} \\ &= \lim_{t \rightarrow T^*} \rho^{1/2} (\rho^{1/2} \mathbf{u}_t + \rho^{1/2} \mathbf{u} \cdot \nabla \mathbf{u}) \triangleq \rho^{1/2} g, \end{aligned}$$

with $g \triangleq (\rho^{1/2} \mathbf{u}_t + \rho^{1/2} \mathbf{u} \cdot \nabla \mathbf{u})|_{t=T^*} \in L^2$. That means functions $(\rho, \mathbf{u}, \mathbf{H}, \theta)|_{t=T^*}$ satisfy compatibility Condition (7) at time T^* . Subsequently, by considering $(\rho, \mathbf{u}, \mathbf{H}, \theta)|_{t=T^*}$ as the initial data, we can employ the local existence theorem (refer to Proposition 1) to prolong the local strong solutions beyond T^* . This contradicts the assumption on T^* as the maximum time of existence. Consequently, the proof of Theorem 1 is concluded. $\square$

The theorem relies on proof by contradiction; if various estimates of involved quantities remain finite, strong solutions can be extended beyond T^* .

The proof of the existence of solutions for MHD equations plays a vital role in affirming the physical relevance of these equations. The existence of solutions under specified conditions ensures that the mathematical model accurately represents essential physical phenomena in magnetohydrodynamics. Additionally, the blow-up criterion contributes to understand the stability and potential singularities in the solutions. This understanding is crucial for evaluating the long-term behavior of MHD systems and predicting conditions under which solutions may become unbounded.

In summary, establishing the existence of solutions or blow-up criteria for MHD equations enhances our comprehension of magnetohydrodynamic processes. It not only validates mathematical models but also guides numerical simulations and informs practical applications in engineering and science.

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