Anisotropic Moser–Trudinger-Type Inequality with Logarithmic Weight

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Abstract: Our main purpose in this paper is to study the anisotropic Moser–Trudinger-type inequalities with logarithmic weight \( \omega_{\beta}(x) = [-\ln F_0(x)]^{(n-1)\beta} \). This can be seen as a generation result of the isotropic Moser–Trudinger inequality with logarithmic weight. Furthermore, we obtain the existence of extremal function when \( \beta \) is small. Finally, we give Lions’ concentration-compactness principle, which is the improvement of the anisotropic Moser–Trudinger-type inequality.

Keywords: anisotropic Moser–Trudinger-type inequality; logarithmic weight; existence of extremal function

MSC: 35A23; 35A01; 35B38

1. Introduction

It is well-known that important geometric inequalities, for example, the Sobolev inequality, Moser–Trudinger inequality, etc., and the existence of extreme functions play a key role to study partial differential equations. For a bounded domain \( \Omega \subset \mathbb{R}^n \) with \( n \geq 2 \), we have \( W^{1,p}_0(\Omega) \subset L^q(\Omega) \), \( 1 \leq q \leq \frac{np}{n-p} \), for \( 1 \leq p < n \) by the classical Sobolev embedding theorem. Particularly, for \( p = n \), \( W^{1,n}_0(\Omega) \subset L^q(\Omega) \), \( \forall q \geq 1 \). But \( W^{1,n}_0(\Omega) \not\subset L^\infty(\Omega) \). For the borderline case \( p = n \), the Moser–Trudinger inequality is the perfect replacement. In 1971, Moser [1] proved the sharpening of Trudinger’s inequality as follows:

\[
\sup_{u \in W^{1,n}_0(\Omega), \|\nabla u\|_{n-1} \leq 1} \int_{\Omega} e^{a\|u\|_{n-1}^n} \, dx \leq C,
\]

for \( \forall a \leq a_n = n\omega_{n-1}^{1/n} \) and \( \omega_{n-1} \) stands for area of the \( (n-1) \)-sphere. Moreover, \( a_n \) is sharp, which means that if \( a > a_n \), then the inequality (1) can no longer hold. Inequality (1) is the so-called Moser–Trudinger inequality, the extremal of which is related to the existence of solutions of some semi-linear Liouville-type equations.

As far as we know, there have been many important studies related to the Moser–Trudinger inequality, for example, [2–8], etc. In the references listed above, readers can see the Moser–Trudinger inequality in \( \mathbb{R}^n \) and in hyperbolic spaces, the existence of an extremal function for the Moser–Trudinger inequality, etc. These important geometric inequalities play a key role in geometry analysis, calculus of variations, and PDEs; we refer to [9–14] and references therein. And recently, the authors of [15] studied a system of Kirchhoff type driven by the \( Q \)-Laplacian in the Heisenberg group \( \mathbb{H}^n \). They obtained the existence of solutions via variational methods based on a new Moser–Trudinger-type inequality for the Heisenberg group \( \mathbb{H}^n \). Moreover, in [16], the authors also focus on a Kirchhoff-type problem and establish the existence of a radial solution in the subcritical growth case by the Moser–Trudinger inequality and minimax method.
Let \( \varrho_\beta(x) = (-\ln |x|)\beta(n-1), 0 \leq \beta < 1 \). Clainchi and Ruf \([17,18]\) proved the weighed Moser–Trudinger-type inequality involving the radial functions in unit ball \( B \):

\[
\sup_{u \in W^{1,n}_{0,rad}(B, \varrho_\beta), \|u\|_{\varrho_\beta} \leq 1} \int_B e^{\lambda |u|^{\frac{n}{n-1}(1-\beta)}} \, dx < \infty
\]

for any \( \alpha \leq \alpha_{\beta,n} = n(1 - \beta)\omega_{n-1}^{1/n} \), \( \|u\|_{\varrho_\beta} = (\int_B |\nabla u|^n \varrho_\beta(x) \, dx)^{\frac{1}{n}} \). Moreover, the constant \( \alpha_{\beta,n} \) is sharp, i.e., if \( \alpha > \alpha_{\beta,n} \), the supremum in (2) will be infinite. We note that the authors applied Leckband’s inequality \([19]\) to prove the weighed Moser–Trudinger-type inequality (2). Note that when \( \beta = 0 \), by the Pólya–Szegö principle, (2) recovers the classical Moser–Trudinger inequality (1). Furthermore, Roy \([20]\) proved the existence of an extremal function for inequality (2).

Recently, many researchers have intended to establish anisotropic Moser–Trudinger-type inequalities. Let \( F \in C^2(\mathbb{R}^n \setminus \{0\}) \) be a nonnegative and convex function, the polar \( F^\circ(x) \) of which represents a Finsler metric on \( \mathbb{R}^n \). By \( F(x) \), a Finsler–Laplacian operator \( \Delta_F \) is defined by

\[
\Delta_F u := \sum_{i=1}^n \frac{\partial}{\partial x_i} (F(\nabla u) F_{i\beta}(\nabla u)),
\]

where \( F_{i\beta} = \frac{\partial F}{\partial x_i} \). In Euclidean modulus, \( \Delta_F \) is nothing but the common Laplacian. The Finsler–Laplacian operator is closely related to the Wulff shape, which was initiated in Wulff’s work \([21]\). More details about the properties of \( F(x) \) and \( F^\circ(x) \) can be seen in Section 2.

For a bounded smooth domain \( \Omega \subset \mathbb{R}^n \), Wang and Xia \([22]\) proved that, for \( \forall \lambda \leq \lambda_n = n \pi^{\frac{1}{n}} \kappa_n^{\frac{1}{n}} \),

\[
\sup_{u \in W^{1,n}_{0,rad}(\Omega), \int_{\Omega} F^\circ(\nabla u) dx \leq 1} \int_\Omega e^{\lambda |u|^{\frac{n}{n-1}}} \, dx \leq C,
\]

(3)

where

\[
\kappa_n = |\{ x \in \mathbb{R}^n : F^\circ(x) \leq 1 \}|
\]

(4)

denotes the volume of a unit Wulff ball in \( \mathbb{R}^n \).

In this paper, we intend to establish the anisotropic Moser–Trudinger-type inequality with logarithmic weight. We believe that these sharp inequalities will be the key tools to study the existence of solutions for some quasi-linear elliptic equations, such as the Finsler–Laplacian equation. For \( \beta \in [0,1) \), we let

\[
\omega_\beta(x) = [-\ln F^\circ(x)]^{\beta(n-1)},
\]

which is the weight of logarithmic type defined on a unit Wulff ball \( \mathcal{W}_1 = \{ x \in \mathbb{R}^n : F^\circ(x) < 1 \} \). And \( W^{1,n}_{0}(\mathcal{W}_1, \omega_\beta) \) represents the functions of completion of \( C^1_0(\mathcal{W}_1) \) with respect to the norm

\[
\|u\|_{\omega_\beta} = \left( \int_{\mathcal{W}_1} F^\circ(\nabla u) \omega_\beta(x) \, dx \right)^{\frac{1}{\beta}}, \quad u \in C^1_0(\mathcal{W}_1).
\]

Let \( W^{1,n}_{0,rad}(\mathcal{W}_1, \omega_\beta) \) be the subspace of \( W^{1,n}_{0}(\mathcal{W}_1, \omega_\beta) \) of all radial functions with respect to \( F \). In this paper, radial functions with respect to \( F \) means that \( u(x) = \bar{u}(r) \), where \( r = F^\circ(x) \).

In the following, for convenience, we denote

\[
AMT(n, \lambda, \beta) = \sup_{u \in W^{1,n}_{0,rad}(\mathcal{W}_1, \omega_\beta), \|u\|_{\omega_\beta} \leq 1} \int_{\mathcal{W}_1} e^{\lambda \|u\|_{\omega_\beta}^{\frac{n}{n-1}(1-\beta)}} \, dx.
\]

(5)
We now state our main results.

**Theorem 1.** For any
\[
\lambda \leq \lambda_{\beta,n} = n^{1+\frac{1}{p-1} - \frac{1}{|\beta|}} \left( \frac{1}{n^{|\beta|-1}} \right)^{\frac{1}{|\beta|}} ,
\]
we have AMT(n, \lambda, \beta) < \infty. Moreover, this constant \( \lambda_{\beta,n} \) is sharp, i.e., if \( \lambda > \lambda_{\beta,n} \) AMT(n, \lambda, \beta) is infinite.

Next, we prove the existence of an extremal function for the anisotropic Moser–Trudinger-type inequality with logarithmic weight.

**Theorem 2.** There exists \( \beta_0 \in [0,1) \) such that, for \( \forall \beta \in [0, \beta_0) \), AMT(n, \lambda, \beta) is attained.

Finally, we establish the Lions-type concentration-compactness property, which can be seen as an improvement of the anisotropic Moser–Trudinger-type inequality in Theorem 1 for some situations.

**Theorem 3.** Let \( \{u_k\} \) be a sequence in \( W^{1,n}_0(\mathbb{W}, \omega_\beta) \) such that \( \|u_k\|_{\omega_\beta} = 1 \) and \( u_k \rightharpoonup u_0 \) in \( W^{1,n}_0(\mathbb{W}, \omega_\beta) \). Then we have
\[
\limsup_{k \to \infty} \int_{\mathbb{W}} e^{p|\lambda_n|u_k^{p-1-n-\beta}} \, dx < \infty,
\]
for any \( p < p(u_0) := \left(1 - \|u_0\|_{\omega_\beta}^n \right)^{-\frac{1}{p-1} - \frac{1}{|\beta|}} \).

2. Preliminaries

In this section, we give preliminaries involving the Finsler-Laplacian, co-area formula with respect to \( F \) and convex symmetrization \( u^* \) of \( u \) with respect to \( F \).

Let \( F : \mathbb{R}^n \to \mathbb{R} \) be a function that is \( C^2(\mathbb{R}^n \setminus \{0\}) \), convex, and even. And \( F(x) \) is a homogenous function, that is, for any \( t \in \mathbb{R}, \xi \in \mathbb{R}^n \),
\[
F(t\xi) = |t|F(\xi).
\]

Furthermore, we assume for any \( \xi \neq 0, F(\xi) > 0 \).

By the homogeneity property of \( F \), we can find two positive constants \( 0 < c_1 \leq c_2 < \infty \) such that \( c_1|\xi| \leq F(\xi) \leq c_2|\xi|, \forall \xi \in \mathbb{R}^n \).

The operator
\[
\Delta_F u := \sum_{i=1}^n \frac{\partial}{\partial x_i} (F(\nabla u)F_{i}(\nabla u))
\]
is called Finsler–Laplacian, which was studied by many mathematicians. For some important works involving the Finsler-Laplacian, we refer to [22–26] and the references therein.

\( F^0(x) \) is the support function of \( F(x) \), which is defined as \( F^0(x) := \sup_{\xi \in K} (x, \xi), \) where \( K = \{x \in \mathbb{R}^n : F(x) \leq 1\} \). Then we can check that \( F^0(x) \) is also a function that is \( C^2(\mathbb{R}^n \setminus \{0\}) \). And \( F^0(x) \) is also a convex and homogeneous function. What is more, \( F^0(x) \) is dual to \( F(x) \) in the sense that
\[
F^0(x) = \sup_{\xi \neq 0} \frac{(x, \xi)}{F(\xi)}, \quad F(x) = \sup_{\xi \neq 0} \frac{(x, \xi)}{F^0(\xi)}
\]

Denote the unit Wulff ball of center at origin as
\[
\mathbb{W}_1 := \{x \in \mathbb{R}^n | F^0(x) \leq 1\}
and

\[ \kappa_n := |W_1|, \]

which is the volume of a unit Wulff ball \( W_1 \). Also, we denote \( W_r \) as the Wulff ball of center at origin with radius \( r \), i.e.,

\[ W_r := \{ x \in \mathbb{R}^n | F^0(x) \leq r \}. \]

For later use, by the assumptions of \( F(x) \), we can obtain some properties of the function \( F(x) \); see also [25,27,28].

**Lemma 1.** We have

(i) \( |F(m) - F(n)| \leq F(m + n) \leq F(m) + F(n) \);

(ii) \( \frac{1}{C} \leq |\nabla F(m)| \leq C \), and \( \frac{1}{C} \leq |\nabla F^0(m)| \leq C \) for some \( C > 0 \) and \( m \neq 0 \);

(iii) \( \langle m, \nabla F(m) \rangle = F(m), \langle m, \nabla F^0(m) \rangle = F^0(m) \) for \( m \neq 0 \);

(iv) \( F(\nabla F^0(m)) = 1, F^0(\nabla F(m)) = 1 \) for \( m \neq 0 \);

(v) \( F^0(m) F_\xi(\nabla F^0(m)) = m \) for \( m \neq 0 \).

Now, we give the co-area formula and isoperimetric inequality with respect to \( F \), respectively. For a domain \( \Omega \subset \mathbb{R}^n, G \subset \Omega \), let \( u \in BV(\Omega) \), which we denote as a function of bounded variation. The anisotropic bounded variation of \( u \) with respect to \( F \) is defined by

\[ \int_\Omega |\nabla u|_F = \sup \left\{ \int_\Omega u \, \text{div} \tau \, dx, \tau \in C^1_0(\Omega; \mathbb{R}^n), F^0(\tau) \leq 1 \right\}, \]

and the anisotropic perimeter of \( G \) with respect to \( F \) is defined by

\[ H_F(G) := \int_\Omega |\nabla X_G|_F dx, \]

where \( X_G \) is the characteristic function defined on the subset \( G \). Then we have the co-area formula (see [26])

\[ \int_\Omega |\nabla u|_F = \int_0^\infty H_F(|u| > t) \, dt \tag{8} \]

and the isoperimetric inequality

\[ H_F(G) \geq n\kappa_n^{\frac{1}{n}} |G|^{1-\frac{1}{n}}. \tag{9} \]

Furthermore, (9) becomes an equality if and only if \( G \) is a Wulff ball.

### 3. Anisotropic Moser–Trudinger-Type Inequality with Logarithmic Weight

In this section, we prove Theorem 1. Firstly, we give a useful formula involving the change in functions in a unit Wulff ball \( W_1 \). For \( u \in W_{0,rad}^{1,n}(W_1, \omega_\beta) \) and any \( 0 \leq \tilde{\beta} < \beta \), we let

\[ v(x) = \left( \frac{\lambda_{\tilde{\beta},n}}{\lambda_{\beta,n}} \right)^{\frac{(n-1)(1-\tilde{\beta})}{n}} u(x) |u(x)|^{\frac{\beta-\tilde{\beta}}{n}}. \tag{10} \]

Then we have the following lemma.

**Lemma 2.** Let \( u \in W_{0,rad}^{1,n}(W_1, \omega_\beta) \) with \( \|u\|_{\omega_\beta} \leq 1 \). Define \( v \) by (10); then we have \( \|v\|_{\omega_\tilde{\beta}} \leq 1 \).
Proof. By the property of $F(x)$ in Lemma 1, we have
\[
F^n(\nabla u) = \left(\frac{\lambda_{\beta,n}(n-1)}{\lambda_{\beta,n}}\right)^{n-1} F^n(\nabla u) \frac{u(x)}{u(x)} \left(\frac{n(1-\beta)}{n(1-\beta)}\right)
\leq \left(\frac{1-\beta}{(1-\beta)}\right)^{n-1} \int_{W_1} F^n(\nabla u) \frac{u(x)}{u(x)} \left(\frac{n(1-\beta)}{n(1-\beta)}\right) \frac{u(x)}{u(x)} dy.
\]

Hence, by the co-area formula (8), we have
\[
\left\|v\right\|^2 = \int_{W_1} F^n(\nabla u) \omega_\beta(x) dx 
\leq \left(\frac{1-\beta}{(1-\beta)}\right)^{n-1} \int_{W_1} F^n(\nabla u) \omega_\beta(x) \left(\int_{t} n \kappa_n |u'(s)|^n \omega_\beta(s) d\nu dx \right) \frac{n(1-\beta)}{n(1-\beta)}
\leq \left(\frac{1-\beta}{(1-\beta)}\right)^{n-1} \int_{W_1} F^n(\nabla u) \omega_\beta dx \leq 1.
\]

Next, in this paper, we frequently need to change the variable in the following way. For $u \in W_{0,rad}^{1,n}(W_1, \omega_\beta)$, we change the variable as follows:
\[
F^0(x) = e^{-\frac{t}{4}}
\]
and set
\[
\psi(t) = \kappa_{\beta,n}^{\frac{1}{2}} n^{\frac{1-(n-1)(1-\beta)}{2}} (1-\beta)^{\frac{n-1}{2}} u(x).
\]

Then we have $\psi'(t) = -n^{-2} \kappa_{\beta,n}^{\frac{1}{2}} (1-\beta)^{-1} u'(e^{-\frac{t}{4}}) e^{-\frac{t}{4}}$. By Lemma 1 and co-area formula (8), we can transform the norm as follows:
\[
\int_{W_1} F^n(\nabla u) \log F^n(\nabla u) F^n(\nabla u) \log F^n(\nabla u) |u'(F^n(\nabla u))|^{1/2} F^n(\nabla u) dx 
\leq \int_{W_1} F^n(\nabla u) \log F^n(\nabla u) \frac{n \kappa_n |u'(s)|^n \omega_\beta(s) d\nu dx \right) \frac{n(1-\beta)}{n(1-\beta)}
\leq \int_{W_1} F^n(\nabla u) \omega_\beta dx \leq 1.
\]

The functional changes as follows:
\[
\frac{1}{\kappa_n} \int_{W_1} e^{\lambda_{\beta,n} |u|^{\frac{n}{n-1}} - \frac{n}{n-1} \frac{\beta}{(1-\beta)} \int_{0}^{+\infty} e^{\psi(t)} dt
\]
\[
= \int_{0}^{+\infty} e^{\psi(t)} dt
\]

(13)
we obtain
\[
\lambda \frac{1}{\kappa_n} \int_{W_1} e^{\lambda |u|^n (n-1)/(1-\beta)} \, dx = \int_0^{+\infty} e^{\lambda |\varphi|^{(n-1)/(1-\beta)} - t} \, dt, \tag{14}
\]
where $\bar{\lambda} = \frac{\lambda}{\kappa_n}$. 

Now it is easy to prove Theorem 1 by Lemma 2.

**Proof of Theorem 1.** Let $u \in W_{0,rad}^{1,n}(W_1, \omega_\beta)$ with $\|u\|_{\omega_\beta} \leq 1$. Define $v$ by (10). By Lemma 2, we have $\|v\|_{\omega_\beta} \leq 1$ for $\forall \bar{\beta}$. By the definition of $AMT(n, \lambda_{\bar{\beta},n, \bar{\beta}})$, we obtain

\[
\int_{W_1} e^{\lambda_{\bar{\beta},n} |u|^{(n-1)/(1-\beta)}} \, dx = \int_{W_1} e^{\lambda_{\bar{\beta},n} |\varphi|^{(n-1)/(1-\beta)}} \, dx \leq AMT(n, \lambda_{\bar{\beta},n, \bar{\beta}}). \tag{15}
\]

Since (15) holds for $\forall u \in W_{0,rad}^{1,n}(W_1, \omega_\beta)$ with $\|u\|_{\omega_\beta} \leq 1$, then we have

\[
AMT(n, \lambda_{\bar{\beta},n, \bar{\beta}}) \leq AMT(n, \lambda_{\bar{\beta},n, \bar{\beta}}),
\]

for any $0 \leq \bar{\beta} \leq \beta < 1$. Hence, we obtain that the function $\beta \mapsto AMT(n, \lambda_{\bar{\beta},n, \bar{\beta}})$ is decreasing on $[0, 1)$. Thus, by the anisotropic Moser–Trudinger-type inequality (3), we obtain $AMT(n, \lambda_{\bar{\beta},n, \bar{\beta}}) < \infty$.

Now we prove the constant $\lambda_{\bar{\beta},n}$ is sharp. We need to show that, if $\lambda > \lambda_{n, \beta}$, $AMT(n, \lambda, \beta)$ is infinite. By (14), we only need to test

\[
\int_0^{+\infty} e^{\lambda |\varphi|^{(n-1)/(1-\beta)} - t} \, dt,
\]

where $\bar{\lambda} > 1$.

Consider the family of functions of Moser’s type

\[
\eta_m(t) = \begin{cases} \frac{t^{1-\beta}}{1-\beta}, & t \leq m, \\ \frac{m^{1-\beta}t^{(n-1)/n}}{1-\beta}, & t \geq m. \end{cases}
\]

By direct computation, we have $\int_0^{+\infty} \frac{\eta_m'(t)}{(1-\beta)m^{n-1}} \, dt = 1$. However, as $m \to +\infty$,

\[
\int_0^{+\infty} e^{\lambda |\eta_m|^{(n-1)/(1-\beta)} - t} \, dt \geq \int_m^{+\infty} e^{\lambda m^{-\beta} - t} \, dt \to +\infty, \quad \text{if } \bar{\lambda} > 1.
\]

The proof of Theorem 1 is completed. $\square$

4. Existence of the Extremal Function

In this section, we complete the proof of Theorem 2. Firstly, we give a uniform bound for $u \in W_{0,rad}^{1,n}(W_1, \omega_\beta)$. For $u \in W_{0,rad}^{1,n}(W_1, \omega_\beta)$, we denote by $u(r)$ the value of $u(x)$ with $r = F^\beta(x)$. By the Hölder inequality and co-area Formula (8), for any $0 < r < s \leq 1$ and $u \in W_{0,rad}^{1,n}(W_1, \omega_\beta)$, we have

\[
\begin{align*}
|u(r) - u(s)| & = |\int_s^r u'(t) \, dt| \\
& \leq \int_s^r |u'(t)| t^{1-\beta} \, dt \leq \left( \int_0^1 |u'(t)| t^{1-\beta} \, dt \right) \left( \int_0^1 t^{-\frac{n-1}{\beta}} \, dt \right)^{-1} \\
& \leq \left( \kappa_n \right)^{-\frac{1}{\beta}} \left( \frac{1}{1-\beta} \right)^{\frac{n-1}{\beta}} \left( \int_{W_1 \setminus W_1} F^n(\nabla u) \omega_\beta \, dx \right)^{\frac{1}{\beta}} \left( \ln \frac{r}{s} \right)^{\frac{(n-1)(1-\beta)}{\beta}} \\
& = \left( \frac{\kappa_n}{\lambda_{n, \beta}} \right)^{\frac{1}{\beta}} \left( \frac{n}{1-\beta} \right)^{\frac{n-1}{\beta}} \left( \int_{W_1 \setminus W_1} F^n(\nabla u) \omega_\beta \, dx \right)^{\frac{1}{\beta}} \left( \ln \frac{r}{s} \right)^{\frac{(n-1)(1-\beta)}{\beta}} \frac{1}{\beta}.
\end{align*}
\]
In particular, when $s = 1$, for any $0 < r \leq 1$ and $u \in W^{1,n}_{0,\text{rad}}(W_1, \omega_\beta)$, we have

$$|u(r)| \leq \left(\frac{r}{\Lambda_{\beta,n}}\right)^{(n-1)(1-\beta)/n} \left(\int_{W_1 \setminus W_1 r} F^n(\nabla u) \omega_\beta dx\right)^{1/n} \left(-\ln r\right)^{(n-1)(1-\beta)/n}. \quad (17)$$

The definition of $\psi(t)$ in (11) and (12) shows that the anisotropic norm changes as

$$\Gamma(\psi) := \int_0^{+\infty} |\psi'(n)\beta(n-1)/(1-\beta)| \frac{dt}{n} = \int_{W_1} F^n(\nabla u) \log F^n(x) \beta(n-1) dx.$$  

and (13) shows that the functional $I_\beta(\psi)$ and $I_\beta(u)$ changes as

$$I_\beta(\psi) := \int_0^{+\infty} e^{\left|\psi\right|^n \beta(n-1)/(1-\beta)} t dt = \frac{1}{k_n} \int_{W_1} e^{\lambda_{\beta,\beta} |u|^n \beta(n-1)} dx := I_\beta(u). \quad (18)$$

For $\delta \in [0, 1)$, we define

$$\tilde{\Lambda}_\beta = \{ \psi \in C^1[0,\infty) \mid \psi(0) = 0, \Gamma(\psi) \leq \delta \}.$$  

Then the existence of an extremal function in Theorem 1 reduces to find $\psi_0 \in \tilde{\Lambda}_1$ such that

$$Q_\beta := I_\beta(\psi_0) = \sup_{\psi \in \Lambda_1} I_\beta(\psi). \quad (19)$$

Let $\tilde{g}_k(x)$ be a maximizing sequence of (19), that is, $I_\beta(\tilde{g}_k) \rightarrow Q_\beta$. Since

$$\int_{W_1} F^n(\nabla \tilde{g}_k) \log F^n(x) \beta(n-1) dx \leq 1,$$

then there exist a subsequence (still denoted by $\tilde{g}_k$) and a function $\tilde{g}_0 \in W^{1,n}_{0,\text{rad}}(W_1, \omega_\beta)$ such that

$$\tilde{g}_k \rightharpoonup \tilde{g}_0, \quad \tilde{g}_k \rightarrow \tilde{g}_0 \text{ pointwise.} \quad (20)$$

Next, we give an inequality and we will use it several times. For any $h \in C^1[0,\infty)$ and $t \geq A \geq 0$, by the Hölder inequality, we have

$$h(t) = h(A) + \int_A^h t'(s) ds$$

$$= h(A) + \int_A^h t'(s) s^{(n-1)/(1-\beta)} ds$$

$$\leq h(A) + \left(\int_A^h \left|h'(s)\right|^n s^{\beta(n-1)} ds\right)^{1/n} \left(\int_A^h s^{-\beta} ds\right)^{n-1}$$

$$= h(A) + \left(\int_A^h \left|h'(s)\right|^n s^{\beta(n-1)} ds\right)^{1/n} \left(1 - A^{-1-\beta}\right)^{n-1}. \quad (21)$$

Now we give a lemma involving concentration-compactness alternative, by which we only need to prove that the maximizing sequence $\tilde{g}_k(x)$ in (20) does not concentrate at 0, and then we can pass to the limit in the functional. Firstly, we give a definition.

We say a sequence of functions $u_k \in W^{1,n}_{0,\text{rad}}(W_1, \omega_\beta)$ concentrates at $x = 0$, denoted by

$$F^n(\nabla u_k) \omega_\beta dx \rightarrow \delta_0,$$

if $\|u_k\|_{\omega_\beta} \leq 1$ and any $1 > r > 0$, $\int_{W_1 \setminus W_1 r} F^n(\nabla u_k) \omega_\beta dx \rightarrow 0$.

**Lemma 3.** [Concentration-compactness alternative] For any sequence $\tilde{v}_k$, $\tilde{v} \in W^{1,n}_{0,\text{rad}}(W_1, \omega_\beta)$, such that $\tilde{v}_k \rightharpoonup \tilde{v}$ in $W^{1,n}_{0,\text{rad}}(W_1, \omega_\beta)$, then up to a subsequence (still denoted by $\tilde{v}_k$), either (i) $I_\beta(\tilde{v}_k) \rightarrow I_\beta(\tilde{v})$, or (ii) $\tilde{v}_k$ concentrates at $x = 0$. 

Proof. We assume that (ii) does not hold; then we only need to show that (i) holds. Since (ii) does not hold, then there exist $A > 0$ and $\delta \in (0, 1)$ such that for sufficiently large $k$,

$$
\int_{\mathcal{W}_k} F^n (\nabla \tilde{v}_k) |\log F^o (x)|^{\beta(n-1)} dx = \int_0^A \frac{|v'_k|^n t^{\beta(n-1)}}{(1 - \beta)^{n-1}} dt \geq \delta,
$$

where we use the variable of change

$$
F^o (x) = e^{-\frac{x}{n}} \text{ and } \lambda_{\eta, \beta}^{(n-1)(1-\beta)} \delta_k (x) = v_k (t).
$$

By (21), we have

$$
|v_k (t) - v_k (A)| \leq (1 - \delta)^{\frac{1}{n}} (t^{1-\beta} - A^{1-\beta})^{\frac{n-1}{n}} \leq (1 - \delta)^{\frac{1}{n}} t^{(1-\beta) \frac{n-1}{n}}.
$$

Since for any $k$,

$$
|v_k (A)| \leq A^{\frac{(n-1)(1-\beta)}{n}},
$$

we have for $t \geq T$, $T$ sufficiently large,

$$
\begin{align*}
|v_k (t)|^{\frac{n}{(1-\beta)}} &\leq \left[ A^{\frac{(n-1)(1-\beta)}{n}} + (1 - \delta)^{\frac{1}{n}} t^{(1-\beta) \frac{n-1}{n}} \right]^{\frac{\beta}{n-1}} \\
&\leq A + (1 - \delta)^{\frac{1}{n}} \left( \frac{1}{(1-\beta)} \right)^{\frac{1}{n-1}} t. 
\end{align*}
$$

We note that in (23), we applied the inequality if $a > b > 0$, $p > 1$. Then, for $x \in \mathbb{R}$ large enough, $(1 + ax)^p \leq 1 + b^p x^p$.

We split the integral $I_\beta (v_k) = I_1 (v_k) + I_2 (v_k)$, where

$$
I_1 (v_k) = \int_0^T e^{t v_k (t)} \left( \frac{(n-1)(1-\beta)}{n} \right) dt,
$$

and

$$
I_2 (v_k) = \int_T^{\infty} e^{t v_k (t)} \left( \frac{(n-1)(1-\beta)}{n} \right) dt.
$$

Since $\delta_k$ converges pointwise to $\delta$, then $v_k$ also converges pointwise to $v$. Then, by $|v_k (t)| \leq t^{\frac{(n-1)(1-\beta)}{n}}$ and the dominated convergence theorem, we have that $I_1 (v_k) \to I_1 (v)$.

By (23), we have for any small $\epsilon > 0$ and $T$ large enough,

$$
\begin{align*}
I_2 (v_k) &= \int_T^{\infty} e^{t v_k (t)} \left( \frac{(n-1)(1-\beta)}{n} \right) dt \\
&\leq e^A \int_T^{\infty} e^{(1 - \frac{\delta}{n})(n-1-t)} dt,
\end{align*}
$$

which is smaller than $\epsilon$. Then $I_\beta (v_k) \to I_\beta (v)$, that is, $I_\beta (\tilde{v}_k) \to I_\beta (\tilde{\delta})$.

The following lemma is proved in [29]. For $\delta, a > 0$, let

$$
\Lambda^\delta_a = \{ \phi \in C^1 [0, \infty) | \phi (0) = 0, \int_a^{\infty} |\phi|^{1+\delta} dt \leq \delta \}.
$$

Lemma 4 ([29]). For each $a > 0$ and $\phi (t) \in \Lambda^\delta_a$, we have

$$
\int_a^{\infty} e^{\phi^{\frac{1}{1+\delta}} (t) - t} dt \leq \frac{e^{\phi^{\frac{1}{1+\delta}} (a) - a}}{1 - \delta^{\frac{1}{1+\delta}}} e^{a^{\frac{1}{1+\delta}}} e^{a^{\frac{1}{1+\delta}} + \frac{1}{2} \frac{1}{1+\delta} + \cdots + \frac{1}{\delta}},
$$

where $\beta_n = (1 - \delta^{\frac{1}{1+\delta}})^{-n+1}$ and $c = \frac{1}{n+1} \phi^{\frac{1}{1+\delta}} (a)$. The inequality tends to an equality if $c^n \beta_n \to \infty, a \to \infty$ and $\delta \to 0$. 

Let \( \tilde{f}_k(x) \in W^{1,p}_{\text{rad}}(W_1, \omega_\beta) \) such that \( \tilde{f}_k(x) \) concentrates at 0, that is, \( \|\tilde{f}_k\|_{\omega_\beta} \leq 1 \), \( F^n(\nabla \tilde{f}_k)|_{\omega_\beta} \rightarrow \delta_0 \). Define \( f_k(t) \) from \( \tilde{f}_k(x) \) by the same transformation as in (22). Then, since \( \tilde{f}_k(x) \) concentrates at 0, we have that \( \tilde{f}_k(x) \rightarrow 0 \) in \( W^{1,p}_{\text{rad}}(W_1, \omega_\beta) \) and converges pointwise to 0.

**Lemma 5.** Let \( f_k(t) \) be as above. Then one of the following alternatives holds:

(i) We can find points \( a_k \in [1, \infty) \) such that

\[
|f_k(a_k)|^{\frac{n}{(n-1)(1-\beta)}} - a_k = -2 \log a_k;
\]

(ii) If such \( a_k \) does not exist, then

\[
\limsup_{k \rightarrow \infty} \int_0^\infty e^{\frac{|f_k(t)|^{\frac{n}{(n-1)(1-\beta)}} - t}{t}} dt = 1.
\]

What is more, if the first alternative (i) holds, we can find \( a_k \) to be the first point in \([1, \infty)\) satisfying (26) and satisfying \( a_k \rightarrow \infty \) as \( k \rightarrow \infty \).

**Proof.** Since \( |f_k(t)| \leq t^{\frac{(n-1)(1-\beta)}{n}} \), then if \( t \in [0, 1) \), \( |f_k(t)|^{\frac{n}{(n-1)(1-\beta)}} - t \leq 0 \). However, if \( t \in [0, 1), -2 \log t > 0 \), then \( |f_k(t)|^{\frac{n}{(n-1)(1-\beta)}} - t < -2 \log t \), which implies that we cannot find \( a_k \) satisfying (26) in \([0, 1)\).

Now we assume (i) does not hold. Then we have \( |f_k(t)|^{\frac{n}{(n-1)(1-\beta)}} - t < -2 \log t \), \( t \in [1, \infty) \). Furthermore, we have

\[
e^{\frac{|f_k(t)|^{\frac{n}{(n-1)(1-\beta)}} - t}{t}} \leq t^{-2}, \text{ if } t \in [1, \infty).
\]

Define the dominating function as follows:

\[
h(t) = \begin{cases} 
1, & t \in (0, 1), \\
\frac{1}{t^2}, & t \in [1, \infty).
\end{cases}
\]

Then, by the dominated convergence theorem, we obtain that \( I_\beta(f_k) \rightarrow 1 \).

Let (i) hold. We choose the first \( a_k \geq 1 \) satisfying (26). We now prove that \( a_k \rightarrow \infty \) as \( k \rightarrow \infty \). For any large number \( M \), we need to prove that there exist \( k_0 \in \mathbb{N} \), such that for any \( k \geq k_0 \), \( a_k \geq M \). Firstly, we choose \( \mu \) small, such that

\[
\mu t < -2 \log t + t, \text{ for } t \in [0, M).
\]

Now, since \( \tilde{f}_k \) concentrates, we have for \( t \in [0, M) \) and any \( k \geq k_0 \),

\[
|f_k(t)|^{\frac{n}{(n-1)(1-\beta)}} \leq \left( \int_0^M \frac{|f_k^n|^n \beta^{n-1}}{(1-\beta)^{n-1}} dt \right)^{\frac{1}{n-1}} < \mu t \leq t - 2 \log t.
\]

Then we obtain for any \( k \geq k_0 \), \( a_k \geq M \). \( \square \)

Now we define the concentration level at 0,

\[
f_\beta(0) = \sup_{\tilde{f}_k \in W^{1,p}_{\text{rad}}(W_1, \omega_\beta)} \{ \limsup_{k \rightarrow \infty} I_\beta(f_k) | F^n(\nabla f_k)|_{\omega_\beta} \rightarrow \delta_0 \}.
\]

We can give the estimate for the concentration level.

**Lemma 6.** For \( \beta \in [0, 1) \), we have that

\[
f_\beta(0) \leq 1 + e^{1 + \frac{1}{2} + \cdots + \frac{1}{n-1}}.
\]

(27)
Proof. To prove the lemma, it is sufficient to assume the sequences \( \tilde{f}_k \) satisfy the first alternative in Lemma 5, because if \( \tilde{f}_k \) satisfy the second alternative, we can obtain the inequality (27) by Lemma 5.

Firstly, we show that

\[
\lim_{k \to \infty} \int_{0}^{a_k} e^{|f_k(t)| \frac{\beta}{(n-1)(1-\beta)}} - t dt = 1,
\]

where \( f_k \) and \( a_k \) are as in Lemma 5. Since \( F^n(\nabla \tilde{f}_k)\omega_k \to 0 \) and (21), we have that \( f_k \to 0 \) uniformly on compact subsets of \( \mathbb{R}^+ \). Then for any \( \epsilon, A > 0 \), we obtain \( |f_k(t)| \frac{\beta}{(n-1)(1-\beta)} \leq \epsilon \) for \( t \leq A \) and \( k \) large enough. By the property of \( a_k \), that is, for \( t \leq a_k \), \( |f_k(t)| \frac{\beta}{(n-1)(1-\beta)} \leq t - 2 \log t \), we obtain

\[
\int_{0}^{a_k} e^{|f_k(t)| \frac{\beta}{(n-1)(1-\beta)}} - t dt = \int_{0}^{A} e^{|f_k(t)| \frac{\beta}{(n-1)(1-\beta)}} - t dt + \int_{a_k}^{A} e^{|f_k(t)| \frac{\beta}{(n-1)(1-\beta)}} - t dt \\
\leq e^\epsilon \int_{0}^{A} e^{-t} dt + \int_{a_k}^{A} e^{-2 \log t} dt \\
= e^\epsilon (1 - e^{-A}) + (\frac{1}{A} - \frac{1}{a_k}).
\]

Therefore,

\[
\limsup_{k \to \infty} \int_{0}^{a_k} e^{|f_k(t)| \frac{\beta}{(n-1)(1-\beta)}} - t dt \leq e^\epsilon (1 - e^{-A}) + \frac{1}{A}.
\]

Now, as \( \epsilon \to 0 \) and \( A \to \infty \), we have

\[
\limsup_{k \to \infty} \int_{0}^{a_k} e^{|f_k(t)| \frac{\beta}{(n-1)(1-\beta)}} - t dt \leq 1.
\]

On the other hand,

\[
\limsup_{k \to \infty} \int_{0}^{a_k} e^{|f_k(t)| \frac{\beta}{(n-1)(1-\beta)}} - t dt \geq \int_{0}^{a_k} e^{-t} dt = 1 - e^{-a_k} \to 1.
\]

Next, we prove that

\[
\lim_{k \to \infty} \int_{a_k}^{\infty} e^{|f_k(t)| \frac{\beta}{(n-1)(1-\beta)}} - t dt \leq e^{1 + \frac{1}{2} + \cdots + \frac{1}{n-1}}.
\]

Set \( \delta_k = \int_{a_k}^{\infty} \frac{|f_k(t)| \frac{\beta}{(n-1)(1-\beta)}}{(1-\beta)^{n-1}} dt \). Then, by (21) with \( A = 0 \) and \( t = a_k \), we have

\[
\delta_k = 1 - \int_{0}^{a_k} \frac{|f_k(t)| \frac{\beta}{(n-1)(1-\beta)}}{(1-\beta)^{n-1}} dt \\
= 1 - \frac{(|f_k(a_k)| \frac{\beta}{(n-1)(1-\beta)})(1-\beta)(n-1)}{a_k} \\
= 1 - \frac{2 \log a_k}{a_k} (1-\beta)(n-1).
\]

(28)

Define the function \( \tilde{g}_k(t) = |f_k(t)|^{\frac{\beta}{n}} \). Then

\[
\int_{a_k}^{\infty} e^{|f_k(t)| \frac{\beta}{(n-1)(1-\beta)}} - t dt = \int_{a_k}^{\infty} e^{\tilde{g}_k(t)^n} - t dt.
\]

(29)
By \(|f_k^{(n-1)-\beta}(t)| \leq t| \), we have

\[
\int_{a_k}^{\infty} |g_k'|^{\nu} dt = \frac{1}{(1 - \beta)^{\nu}} \int_{a_k}^{\infty} f_k^{(n-\beta)} |f_k'|^{\nu} dt \\
\leq \frac{1}{(1 - \beta)^{\nu}} \int_{a_k}^{\infty} (n-\beta)|f_k'|^{\nu} dt \\
\leq \frac{\delta_k}{1 - \beta} := \delta_k^+ \to 0.
\]

(30)

Now, applying Lemma 4 with \(\delta = \delta_k^+\) and \(a = a_k\), we obtain

\[
\int_{a_k}^{\infty} e^{\|f_k(t)\|^{\nu}(1 - \beta)^{\nu}} dt \\
\leq \frac{\delta_k(a_k)^{\nu}}{1 - |\delta_k^+|^{\nu}} - a_k \frac{\delta_k(a_k)^{\nu}}{1 - |\delta_k^+|^{\nu}} (n-1)\beta_a + \frac{1}{\beta_a} + \cdots + \frac{1}{\beta_a^{n-1}},
\]

where \(\beta_a = \delta_k^+(1 - |\delta_k^+|^{\nu})^{-n+1}\) and \(c = \frac{n}{\beta_a} \delta_k(a_k)^{\nu}\). Therefore, it is left to show that

\[
\limsup_{k \to \infty} G_k := \limsup_{k \to \infty} \frac{|g_k(a_k)|^{\nu} - a_k + \frac{g_k(a_k)^{\nu}}{n-1}(1 - |\delta_k^+|)^{n-1}}{(n-1)(1 - |\delta_k^+|^{\nu})^{n-1}} \leq 0.
\]

We split \(G_k\) as follows:

\[
G_k = -2 \log a_k + \frac{(a_k - 2 \log a_k) \delta_k}{(n-1)(1 - \beta)^{1 - |\delta_k^+|^{\nu}}^{n-1}} \\
= -2 \log a_k + \frac{a_k \delta_k}{(n-1)(1 - \beta)(1 - |\delta_k^+|^{\nu})^{n-1}} \\
- \frac{2 \log a_k \delta_k}{(n-1)(1 - \beta)(1 - |\delta_k^+|^{\nu})^{n-1}} \\
= \{ -2 \log a_k + \frac{a_k \delta_k}{(n-1)(1 - \beta)} \} + \frac{a_k \delta_k (1 - |\delta_k^+|^{\nu})^{n-1}}{(n-1)(1 - \beta)(1 - |\delta_k^+|^{\nu})^{n-1}} \\
= t_k^1 + t_k^2 - t_k^3.
\]

We make use of the Maclaurin series expansion. Firstly,

\[
\delta_k = 1 - \left( - \frac{2 \log a_k}{a_k} + 1 \right)(1-\beta)(n-1) \\
= (1 - \beta)(n-1) \frac{2 \log a_k}{a_k} + C \left( \frac{2 \log a_k}{a_k} \right)^2 + o \left( \left( \frac{2 \log a_k}{a_k} \right)^2 \right),
\]

for some positive constant \(C\), which depends only on \(\beta, n\). Thus, we have

\[
|t_k^1| = 4C \left( \frac{\log a_k}{a_k} \right)^2 + a_k o \left( \left( \frac{\log a_k}{a_k} \right)^2 \right) \to 0, \text{ as } k \to \infty.
\]

Also,

\[
|t_k^3| \leq C_2 \frac{(\log a_k)^2}{a_k^2} + C_2 \frac{(\log a_k)^3}{a_k^3} + o \left( \frac{(\log a_k)^2}{a_k} \right) \to 0, \text{ as } k \to \infty.
\]

To estimate \(t_k^2\), we first use the binomial expansion of \((1 - |\delta_k^+|^{\nu})^{n-1}\) to obtain \(|t_k^2| \leq C_3 \delta_k |\delta_k^+|^{\nu}. \text{ Now, using } (30) \text{ and } (33), \text{ we obtain}

\[
|t_k^2| \leq C \frac{(\log a_k)^{\nu}}{a_k^{\nu}} \to 0, \text{ as } k \to \infty.
\]

Then we have completed the proof of the Lemma.
Proof of Theorem 2. We assume \( I_\beta(\hat{g}_k) \) does not converge to \( I_\beta(\hat{g}_0) \), where \( \hat{g}_k, \hat{g} \) is as in (20). Thus, by Lemma 6, we obtain

\[
M_\beta = \lim_{k \to \infty} I_\beta(\hat{g}_k) \leq 1 + e^{1 + \frac{1}{\alpha} + \cdots + \frac{1}{\beta}}.
\]

If we can find some \( \phi \in \tilde{A}_1 \) such that

\[
I_\beta(\phi) > 1 + e^{1 + \frac{1}{\alpha} + \cdots + \frac{1}{\beta}},
\]

then clearly \( Q_\beta > 1 + e^{1 + \frac{1}{\alpha} + \cdots + \frac{1}{\beta}} \) and thus, we obtain a contradiction. Consider the function \( h_n(t) \) as follows:

\[
h_n(t) = \begin{cases} 
(1 - \frac{1}{n})(n - 1)^{-\frac{1}{n}} t, & 0 \leq t \leq n, \\
(t - 1)^{1 - \frac{1}{n}} t, & n \leq t \leq T_n, \\
(T_n - 1)^{1 - \frac{1}{n}}, & t \geq T_n,
\end{cases}
\]

where \( T_n = (n - 1)e^{1 + \frac{1}{\alpha} + \cdots + \frac{1}{\beta}} + 1 \). It has been proved in [29] that \( \int_0^\infty |h_n'|^\alpha dt \leq 1 \) and

\[
\int_0^\infty e^{a h_n(t) \frac{t}{1 - \alpha}} dt = 1 + e^{1 + \frac{1}{\alpha} + \cdots + \frac{1}{\beta} + \gamma(n)} \geq 1 + e^{1 + \frac{1}{\alpha} + \cdots + \frac{1}{\beta}}.
\]

Set \( \phi_n^\alpha(\alpha) = [ah_n(t)]^{1-\alpha} \) for \( \alpha \in (0, 1) \). Then

\[
I_\beta(\phi_n^\alpha) = \int_0^\infty e^{[ah_n(t)]^{\frac{\alpha}{1-\alpha}} - t} dt = \int_0^\infty e^{(\frac{\alpha}{1-\alpha}) h_n(t) \frac{t}{1-\alpha} - t} dt \geq e^{(\frac{\alpha}{1-\alpha}) h_n ||h_n|| \alpha (1 + e^{1 + \frac{1}{\alpha} + \cdots + \frac{1}{\beta} + \gamma(n))}. \]

Now we can choose \( \alpha = \alpha_s \) sufficiently close to 1 such that \( I_\beta(\phi_n^\alpha) > 1 + e^{1 + \frac{1}{\alpha} + \cdots + \frac{1}{\beta}} \). Let us estimate the term \( \Gamma(\phi_n^\alpha) \). Since \( (\phi_n^\alpha)' = 0 \) for \( t \geq T_n \), we have

\[
\Gamma(\phi_n^\alpha) = \int_0^{T_n} \frac{|(\phi_n^\alpha)'|^n|\phi_n^\alpha(\alpha)|^{n\alpha}}{(1-\beta)^{n\alpha}} dt = \int_0^{T_n} \frac{|(\phi_n^\alpha)'|^n|\phi_n^\alpha(\alpha)|^{n\alpha}}{(1-\beta)^{n\alpha}} dt = \int_0^{T_n} \frac{|(\phi_n^\alpha)'|^n|\phi_n^\alpha(\alpha)|^{n\alpha}}{(1-\beta)^{n\alpha}} dt (35)
\]

Now, by direct calculation, we obtain

\[
I_1(\beta) = (1 - \beta) \alpha \int_0^{T_n} h_n^{1-\beta}|h_n(\alpha)|^{n\alpha} dt = \alpha \int_0^{T_n} h_n^{1-\beta}|h_n(\alpha)|^{n\alpha} dt = \alpha \cdot \frac{1}{n-1}(1 - \beta)^{n\alpha} \alpha \cdot \frac{1}{n-1}(n-1)\beta^{n\alpha-1} - \frac{1}{n} \beta^{n\alpha-1} = \alpha \cdot \frac{1}{n-1}(1 - \beta)^{n\alpha} \alpha \cdot \frac{1}{n-1}(n-1)\beta^{n\alpha-1} - \frac{1}{n} \beta^{n\alpha-1}.
\]
and

\[ I_2(\beta) = a_s^{(1-\beta)n}(1-\beta) \int_n^{T_n} h_n^{-n\beta} |u_n|^n \beta^{(n-1)} dt \]

\[ = a_s^{(1-\beta)n}(1-\beta)(1 - \frac{1}{n})^n \int_{n-1}^{T_n-1} \frac{1}{s} (1 + \frac{1}{n-1}) \beta^{(n-1)} ds \]

\[ \leq a_s^{(1-\beta)n}(1-\beta)(1 - \frac{1}{n})^n (1 + \frac{1}{n-1}) \beta^{(n-1)} \int_{n-1}^{T_n-1} \frac{1}{s} ds \]

\[ \leq a_s^{(1-\beta)n}(1-\beta)(1 - \frac{1}{n})^n - \frac{n}{n-1} \]

\[ = a_s^{(1-\beta)n}(1-\beta)(1 - \frac{1}{n})^n - \frac{n}{n-1} B_n, \]

where \( B_n = (\frac{n}{n-1})^{n-1} - 1. \) Note that by the above estimates, we have

\[ I_1(0) + I_2(0) \leq a_0^k < 1. \]

Thus, we can choose \( \beta = \beta_s \), depending only on \( n \), such that \( I_1(\beta_s) + I_2(\beta_s) \leq 1. \) Thus, we have finished the proof of the Theorem. \( \square \)

5. Improvement of the Anisotropic Moser–Trudinger Inequality

In this section, we complete the proof of Theorem 3, which can be seen as an improvement of the anisotropic Moser–Trudinger inequality when \( u_k \rightarrow u_0 \).

Proof of Theorem 3. If \( u_0 \equiv 0 \), then we can directly obtain (7) by Theorem 1. Thus, it is left to consider the case \( u_0 \neq 0 \). By (16), we have that \( u_k \rightarrow u_0 \) uniformly on \( \mathcal{W}_1 \setminus \mathcal{W}_r, \forall r \in (0,1) \). Then, by (17) and dominated convergence theorem, we have \( u_k \rightarrow u_0 \) in \( L^q(\mathcal{W}_1) \) for any \( q < \infty \).

For any \( R > 0, k \in \mathbb{N} \), we define the functions

\[ v_{R,k} = \min\{ |u_k|, L \} \text{sign}(u_k) \quad \text{and} \quad w_{R,k} = u_k - v_{R,k}. \]

Since \( \lim_{k \to \infty} \|v_{R,0}\|^n_{\omega_{\beta}} = \|u_0\|^n_{\omega_{\beta}} \) for \( \forall p < p(u_0) \), then there exist \( R \) large enough such that

\[ p_0 := p(1 - \|v_{R,0}\|^n_{\omega_{\beta}})^{(n-1)/(1-\beta)} < 1. \]

Since \( v_{R,0} \rightarrow v_{R,0} \) a.e. in \( \mathcal{W}_1 \) as \( k \to \infty \) and \( v_{R,k} \) is bounded in \( W_{0,rad}^{1,n}(\mathcal{W}_1, \omega_{\beta}) \), up to a subsequence, we can assume that \( v_{R,k} \rightarrow v_{R,0} \) weakly in \( W_{0,rad}^{1,n}(\mathcal{W}_1, \omega_{\beta}) \). Then we have

\[ \liminf_{k \to \infty} \|v_{R,k}\|^n_{\omega_{\beta}} \geq \|v_{R,0}\|^n_{\omega_{\beta}} \]

and

\[ \limsup_{k \to \infty} \|w_{R,k}\|^n_{\omega_{\beta}} = 1 - \liminf_{k \to \infty} \|v_{R,k}\|^n_{\omega_{\beta}} \leq 1 - \|v_{R,0}\|^n_{\omega_{\beta}}. \]

Then we can find \( k_0 \in \mathbb{N} \) such that for \( \forall k \geq k_0 \), we have

\[ p \|w_{R,k}\|^n_{\omega_{\beta}} \leq \frac{p_0 + 1}{2} < 1. \]  \( (36) \)

Using \( u_k = w_{R,k} + v_{R,k} \) and \( |v_{R,k}| \leq R \), we obtain

\[ |u_k|^{(n-1)/(1-\beta)} \leq (1 + \epsilon) |w_{R,k}|^{(n-1)/(1-\beta)} + C(n, \beta, \epsilon) R^{(n-1)/(1-\beta)}, \]

where

\[ C(n, \beta, \epsilon) = (1 - (1 + \epsilon)^{-\frac{(n-1)/(1-\beta)}{n\beta+1-\beta}}) - \frac{n\beta+1-\beta}{n\beta+1-\beta}. \]  \( (37) \)
Now we choose $\epsilon > 0$ such that $\frac{(1+\epsilon)(1+p\beta)}{2} < 1$. By (36), we have
\begin{align}
&\int_{W_1} e^{\rho_{\beta,\epsilon}|u_1|^{\frac{\beta}{(n-1)(1-\beta)}}} dx \\
&\leq \int_{W_1} e^{\rho_{\beta,\epsilon}|u_1|^{\frac{\beta}{(n-1)(1-\beta)}} + p\rho_{\beta,\epsilon} C(n,\beta,\epsilon) R^{(1-\beta)(1-\beta)}} dx \\
&\leq C \int_{W_1} e^{\rho_{\beta,\epsilon}(1+\epsilon)|u_1|^{\frac{\beta}{(n-1)(1-\beta)}}} dx \\
&\leq C \int_{W_1} e^{\rho_{\beta,\epsilon}(1+\epsilon)|u_1|^{\frac{\beta}{(n-1)(1-\beta)}}} dx
\end{align}
for any $k \geq k_0$, where $C$ depends only on $n, \beta, \epsilon, p$ and $R$. Combining (38) with Theorem 1 and the choice of $\epsilon$, we obtain (7). The proof is completed. \qed

6. Conclusions

In this paper, we mainly study the anisotropic Moser–Trudinger-type inequality for radial Sobolev space with logarithmic weight $\omega_{\beta}(x) = (-\ln F^\beta(x))^\beta(n-1), \beta \in [0,1)$. Moreover, we obtain the existence of an extremal function when $\beta$ is small. The extremal function is densely related to the existence of solutions of Finsler–Liouville-type equations. Finally, we obtain the Lions-type concentration-compactness principle, which is the improvement of an anisotropic Moser–Trudinger-type inequality. However, we note that the singular anisotropic Moser–Trudinger-type inequality with logarithmic weight in a unit Wulff ball $W_1$ and the anisotropic Moser–Trudinger-type inequality with logarithmic weight in $\mathbb{R}^n$ are still open questions.

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