

Article

Maps on the Mirror Heisenberg–Virasoro Algebra

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Abstract: Using the first cohomology from the mirror Heisenberg–Virasoro algebra to the twisted Heisenberg algebra (as the mirror Heisenberg–Virasoro algebra module), in this paper, we determined the derivations on the mirror Heisenberg–Virasoro algebra. Based on this result, we proved that any two-local derivation on the mirror Heisenberg–Virasoro algebra is a derivation. All half-derivations are described, and as corollaries, we have descriptions of transposed Poisson structures and local (two-local) half-derivations on the mirror Heisenberg–Virasoro algebra.

Keywords: derivation; two-local derivation; half-derivation; Lie algebra

MSC: 17B40; 17B65; 17B68



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1. Introduction

The mirror Heisenberg–Virasoro algebra, whose structure is similar to that of the twisted Heisenberg–Virasoro algebra, is the even part of mirror $N = 2$ superconformal algebra (see [1]). As is well known, representation theory is essential in the research of Lie algebras. The Whittaker modules, $\mathcal{U}(\mathbb{C}d_0)$ -free modules (d_0 is one of the standard basis vectors for the mirror Heisenberg–Virasoro algebra), and the tensor products of Whittaker modules and $\mathcal{U}(\mathbb{C}d_0)$ -free modules on the mirror Heisenberg–Virasoro algebra, which are non-weight modules, are determined. Sufficient and necessary conditions are given for these non-weight modules on the mirror Heisenberg–Virasoro algebra to be irreducible (see [2]). Also, the tensor product weight modules on the mirror Heisenberg–Virasoro algebra are studied, and some examples of irreducible weight modules on the mirror Heisenberg–Virasoro algebra are given (see [3]). All Harish–Chandra modules on the mirror Heisenberg–Virasoro algebra are classified (see [4]). Likewise, structure theory is also very important in the research of Lie algebras. For example, the problems of derivations and the relative topics for some algebras are discussed in the papers [5–11]. The derivation algebra and automorphism group on the twisted Heisenberg–Virasoro algebra have been determined (see [12]). Two-local derivations on the twisted Heisenberg–Virasoro algebra are also given in [13]. The mirror Heisenberg–Virasoro algebra is defined as $\frac{1}{2}\mathbb{Z}$ -graded algebra, and its $\frac{1}{2}\mathbb{Z}$ -graded derivation algebras are given in [14]. Different from the graded structure proposed by reference [14], in this paper, using $\mathbb{Z}$ -graded structure on the mirror Heisenberg–Virasoro algebra introduced by [15], the first cohomology from the mirror Heisenberg–Virasoro algebra to the twisted Heisenberg algebra (as the mirror Heisenberg–Virasoro algebra-module) is determined. Then, using the exact sequence of low-degree terms associated with the Hochschild–Serre spectral sequence

$$E_2^{pq} = H^p(\mathcal{D}/\mathcal{H}, H^q(\mathcal{H}, \mathcal{D}/\mathcal{H})) \Rightarrow H^{p+q}(\mathcal{D}, \mathcal{D}/\mathcal{H}),$$

the exact sequence related to $\mathcal{H}$

$$0 \rightarrow \mathcal{H} \rightarrow \mathcal{D} \rightarrow \mathcal{D}/\mathcal{H} \rightarrow 0,$$

and the exact sequence

$$H^1(\mathfrak{D}, \mathcal{H}) \rightarrow H^1(\mathfrak{D}, \mathfrak{D}) \rightarrow H^1(\mathfrak{D}, \mathfrak{D}/\mathcal{H}),$$

the first cohomology from the mirror Heisenberg–Virasoro algebra to itself is equal to the first cohomology from the mirror Heisenberg–Virasoro algebra to its ideal the twisted Heisenberg algebra. Hence, the $\mathbb{Z}$ -graded derivations on the mirror Heisenberg–Virasoro algebra are determined. Furthermore, based on the result, we prove that every two-local derivation on the mirror Heisenberg–Virasoro algebra is a derivation. In the end, we give the description of all $\frac{1}{2}$ -derivations, and as corollaries, we have descriptions of transposed Poisson structures and local (two-local) half-derivations of the mirror Heisenberg–Virasoro algebra.

2. Derivations on the Mirror Heisenberg–Virasoro Algebra

Hereafter, we denote by $\mathbb{C}$, $\mathbb{Z}$ the set of all complex numbers and integers, respectively. All vector spaces, algebras, and their tensor products are assumed to be over the field $\mathbb{C}$. Kronecker delta $\delta_{i,j}$ equals 1 if $i = j$ and 0 otherwise. We consider the following Lie algebras, which are referred to as the *mirror Heisenberg–Virasoro algebra* $\mathfrak{D}$ studied by [1–4,15]. The Lie algebra has a basis $\{d_m, h_r, \mathbf{c}, \mathbf{1} \mid m \in \mathbb{Z}, r \in \frac{1}{2} + \mathbb{Z}\}$ with the following nontrivial multiplication table:

$$[d_m, d_n] = (m - n)d_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}\mathbf{c}, [d_m, h_r] = -rh_{m+r}, [h_r, h_s] = r\delta_{r+s,0}\mathbf{1},$$

where $m, n \in \mathbb{Z}, r, s \in \frac{1}{2} + \mathbb{Z}$. From [15], suppose $\deg(d_n) = n$, $\deg(h_r) = r - \frac{1}{2}$, $\deg(\mathbf{c}) = 0$, $\deg(\mathbf{1}) = -1$. Let

$$\mathfrak{D}_0 = \langle d_0, h_{\frac{1}{2}}, \mathbf{c} \rangle, \mathfrak{D}_{-1} = \langle d_{-1}, h_{-\frac{1}{2}}, \mathbf{1} \rangle, \mathfrak{D}_n = \langle d_n, h_{n+\frac{1}{2}} \rangle_{n \in \mathbb{Z} \setminus \{0, -1\}}.$$

Then, $\mathfrak{D} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{D}_n$ is a $\mathbb{Z}$ -graded algebra. Note that $\mathfrak{D}$ is a generalization of the Virasoro algebra $\mathcal{V}$ and the twisted Heisenberg algebra $\mathcal{H}$. Furthermore, the Lie algebra spanned by $\{d_m, \mathbf{c} \mid m \in \mathbb{Z}\}$ is isomorphic to the Virasoro algebra $\mathcal{V}$ and the Lie algebra spanned by $\{h_r, \mathbf{1} \mid r \in \frac{1}{2} + \mathbb{Z}\}$ is isomorphic to the twisted Heisenberg algebra $\mathcal{H}$. It is necessary to note that $\mathcal{H}$ is the ideal of $\mathfrak{D}$ and $\mathfrak{D}$ is the semi-direct product of $\mathcal{V}$ and $\mathcal{H}$, that is, $\mathfrak{D} = \mathcal{V} \ltimes \mathcal{H}$. The center of $\mathfrak{D}$ is spanned by $\{\mathbf{c}, \mathbf{1}\}$, denoted by $C(\mathfrak{D})$. Let $\mathcal{U}(\mathcal{V})$ denote the universal enveloping algebra of $\mathcal{V}$.

Proposition 1. Let $\mathfrak{D} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{D}_n$ be the mirror Heisenberg–Virasoro algebra and $\mathcal{H}$ be the twisted Heisenberg algebra. Suppose $\mathcal{H}_{-1} = \mathbb{C}h_{-\frac{1}{2}} \oplus \mathbb{C}\mathbf{1}$ and $\mathcal{H}_n = \mathbb{C}h_{n+\frac{1}{2}}, n \in \mathbb{Z} \setminus \{-1\}$; then,

- (i) $\mathcal{H} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}h_{n+\frac{1}{2}} \oplus \mathbb{C}\mathbf{1}$ is a $\mathbb{Z}$ -graded Lie algebra.
- (ii) $\mathcal{H}$ is a $\mathbb{Z}$ -graded $\mathfrak{D}$ -module.

Let L be a Lie algebra and V be an L -module. A linear map φ from L to V is called a *derivation* if it satisfies $\varphi([x, y]) = x \cdot \varphi(y) - y \cdot \varphi(x)$ for any $x, y \in L$. A linear map $\varphi : x \mapsto x \cdot v$ is called an *inner derivation* for any $x \in L$ and $v \in V$. If L and V are $\mathbb{Z}$ -graded and $\varphi(L_n) \subset V_{m+n}$ for all $m, n \in \mathbb{Z}$, then the *order* of φ is m , denoted by $\deg(\varphi) = m$. Denoted by $\text{Der}_{\mathbb{C}}(L, V)$, the vector space consists of all derivations from L to V . Denoted by $\text{Inn}_{\mathbb{C}}(L, V)$, the vector space consists of all inner derivations from L to V .

Let

$$H^1(\mathfrak{D}, \mathcal{H}) = \text{Der}_{\mathbb{C}}(\mathfrak{D}, \mathcal{H}) / \text{Inn}_{\mathbb{C}}(\mathfrak{D}, \mathcal{H}).$$

The exact sequence given by $\mathcal{H}$

$$0 \rightarrow \mathcal{H} \rightarrow \mathfrak{D} \rightarrow \mathfrak{D}/\mathcal{H} \rightarrow 0$$

induces an exact sequence

$$H^1(\mathfrak{D}, \mathcal{H}) \rightarrow H^1(\mathfrak{D}, \mathfrak{D}) \rightarrow H^1(\mathfrak{D}, \mathfrak{D}/\mathcal{H}) \quad (1)$$

of $\mathbb{Z}$ -graded vector spaces. The right side can be calculated from the exact sequence of low-degree terms

$$0 \rightarrow H^1(\mathfrak{D}/\mathcal{H}, \mathfrak{D}/\mathcal{H}) \rightarrow H^1(\mathfrak{D}, \mathfrak{D}/\mathcal{H}) \rightarrow H^1(\mathcal{H}, \mathfrak{D}/\mathcal{H})^{\mathfrak{D}/\mathcal{H}} \quad (2)$$

associated with Hochschild–Serre spectral sequence

$$E_2^{pq} = H^p(\mathfrak{D}/\mathcal{H}, H^q(\mathcal{H}, \mathfrak{D}/\mathcal{H})) \Rightarrow H^{p+q}(\mathfrak{D}, \mathfrak{D}/\mathcal{H}).$$

Here, according to $H^1(\mathfrak{D}/\mathcal{H}, \mathfrak{D}/\mathcal{H}) = 0$ (see [16]) and

$$H^1(\mathcal{H}, \mathfrak{D}/\mathcal{H})^{\mathfrak{D}/\mathcal{H}} \cong \text{Hom}_{\mathbb{C}}(\mathcal{H}/[\mathcal{H}, \mathcal{H}], \mathfrak{D}/\mathcal{H})^{\mathfrak{D}/\mathcal{H}} = \text{Hom}_{\mathfrak{D}/\mathcal{H}}(\mathcal{H}/[\mathcal{H}, \mathcal{H}], \mathfrak{D}/\mathcal{H}),$$

$H^1(\mathcal{H}, \mathfrak{D}/\mathcal{H})^{\mathfrak{D}/\mathcal{H}}$ can be embedded in $\text{Hom}_{\mathcal{U}(\mathcal{V})}(\mathcal{H}/[\mathcal{H}, \mathcal{H}], \mathcal{V})$.

Proposition 2. Let $\mathfrak{D} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{D}_n$ be the mirror Heisenberg–Virasoro algebra and $\mathcal{H}$ the twisted Heisenberg algebra. If $\mathcal{H} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}h_{n+\frac{1}{2}} \oplus \mathbb{C}\mathbf{1}$ is a $\mathbb{Z}$ -graded $\mathfrak{D}$ -module, then

$$\begin{aligned} \text{Der}_{\mathbb{C}}(\mathfrak{D}, \mathcal{H}) &= \bigoplus_{n \in \mathbb{Z}} \text{Der}_{\mathbb{C}}(\mathfrak{D}, \mathcal{H})_n, \text{ where} \\ \text{Der}_{\mathbb{C}}(\mathfrak{D}, \mathcal{H})_n &:= \{\varphi \in \text{Der}_{\mathbb{C}}(\mathfrak{D}, \mathcal{H}) \mid \deg(\varphi) = n\}. \end{aligned}$$

Lemma 1. Let $\mathfrak{D} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{D}_n$ be the mirror Heisenberg–Virasoro algebra and $\mathcal{H} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}h_{n+\frac{1}{2}} \oplus \mathbb{C}\mathbf{1}$ the twisted Heisenberg algebra. Then,

$$H^1(\mathfrak{D}_0, \mathcal{H}_n) = 0, \quad n \in \mathbb{Z} \setminus \{0, -1\}.$$

Proof. Since $H^1(\mathfrak{D}_0, \mathcal{H}_n) = \text{Der}_{\mathbb{C}}(\mathfrak{D}_0, \mathcal{H}_n) / \text{Inn}_{\mathbb{C}}(\mathfrak{D}_0, \mathcal{H}_n)$ for $\varphi \in \text{Der}_{\mathbb{C}}(\mathfrak{D}_0, \mathcal{H}_n)$, then it is enough to verify that $\varphi \in \text{Inn}_{\mathbb{C}}(\mathfrak{D}_0, \mathcal{H}_n)$.

For $\varphi \in \text{Der}_{\mathbb{C}}(\mathfrak{D}_0, \mathcal{H}_n)$, given by

$$\varphi(d_0) = a_n h_{n+\frac{1}{2}}, \varphi(h_{\frac{1}{2}}) = b_n h_{n+\frac{1}{2}}, \quad \varphi(\mathbf{c}) = c_n h_{n+\frac{1}{2}},$$

then

$$\begin{aligned} \varphi([d_0, h_{\frac{1}{2}}]) &= [\varphi(d_0), h_{\frac{1}{2}}] + [d_0, \varphi(h_{\frac{1}{2}})] = [a_n h_{n+\frac{1}{2}}, h_{\frac{1}{2}}] + [d_0, b_n h_{n+\frac{1}{2}}] \\ &= -(n + \frac{1}{2})b_n h_{n+\frac{1}{2}} = -\frac{1}{2}\varphi(h_{\frac{1}{2}}). \end{aligned}$$

Hence,

$$\varphi(h_{\frac{1}{2}}) = 2(n + \frac{1}{2})b_n h_{n+\frac{1}{2}} = b_n h_{n+\frac{1}{2}}.$$

This concludes that $b_n = 0$, that is, $\varphi(h_{\frac{1}{2}}) = 0$. In addition,

$$\varphi([d_0, \mathbf{c}]) = [\varphi(d_0), \mathbf{c}] + [d_0, \varphi(\mathbf{c})] = [d_0, c_n h_{n+\frac{1}{2}}] = -c_n(n + \frac{1}{2})h_{n+\frac{1}{2}} = 0.$$

It follows that $c_n = 0$, that is, $\varphi(\mathbf{c}) = 0$.

Let $E_n = -\frac{a_n}{n+\frac{1}{2}}h_{n+\frac{1}{2}}$. Then,

$$\varphi(d_0) = [d_0, E_n] = [d_0, -\frac{a_n}{n+\frac{1}{2}}h_{n+\frac{1}{2}}] = a_n h_{n+\frac{1}{2}}.$$

Thus $\varphi: \mathfrak{D}_0 \rightarrow \mathcal{H}_n$ can be given by

$$d_0 \mapsto [d_0, E_n]h_{\frac{1}{2}} \mapsto [h_{\frac{1}{2}}, E_n], \quad \mathbf{c} \mapsto [\mathbf{c}, E_n].$$

Hence, $H^1(\mathfrak{D}_0, \mathcal{H}_n) = 0, n \in \mathbb{Z} \setminus \{0, -1\}$. $\square$

Lemma 2. Let $\mathfrak{D} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{D}_n$ be the mirror Heisenberg–Virasoro algebra and $\mathcal{H} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}h_{n+\frac{1}{2}} \oplus \mathbb{C}\mathbf{1}$ the twisted Heisenberg algebra. Then,

$$\text{Hom}_{\mathfrak{D}_0}(\mathfrak{D}_m, \mathcal{H}_n) = 0 \text{ for all } m \neq n, m \neq 0.$$

Proof. Recall that

$$\text{Hom}_{\mathfrak{D}_0}(\mathfrak{D}_m, \mathcal{H}_n) = \{f: \mathfrak{D}_m \rightarrow \mathcal{H}_n \mid x \cdot f(y) = f(x \cdot y), x \in \mathfrak{D}_0, y \in \mathfrak{D}_m\}.$$

Case 1: If $m \in \mathbb{Z} \setminus \{0, -1\}$, then $\mathfrak{D}_m = \langle d_m, h_{m+\frac{1}{2}} \rangle$. If $n \neq -1$, suppose $f(d_m) = kh_{n+\frac{1}{2}}$; then,

$$\begin{aligned} [d_0, f(d_m)] &= f([d_0, d_m]) = -mf(d_m) = -mkh_{n+\frac{1}{2}} \\ &= [d_0, kh_{n+\frac{1}{2}}] = -k(n + \frac{1}{2})h_{n+\frac{1}{2}}. \end{aligned}$$

Hence, $k = 0$.

Suppose $f(h_{m+\frac{1}{2}}) = kh_{n+\frac{1}{2}}$; then,

$$\begin{aligned} [d_0, f(h_{m+\frac{1}{2}})] &= f([d_0, h_{m+\frac{1}{2}}]) = -(m + \frac{1}{2})f(h_{m+\frac{1}{2}}) \\ &= -(m + \frac{1}{2})kh_{n+\frac{1}{2}} = -(n + \frac{1}{2})kh_{n+\frac{1}{2}}. \end{aligned}$$

Hence, $k = 0$.

When $n = -1$, suppose $f(d_m) = k_1h_{-\frac{1}{2}} + k_2\mathbf{1}$ and $f(h_{m+\frac{1}{2}}) = k_3h_{-\frac{1}{2}} + k_4\mathbf{1}$; then,

$$[d_0, f(d_m)] = f([d_0, d_m]) = [d_0, k_1h_{-\frac{1}{2}} + k_2\mathbf{1}] = \frac{1}{2}k_1h_{-\frac{1}{2}} = -mk_1h_{-\frac{1}{2}} - mk_2\mathbf{1}.$$

Hence, $k_1 = k_2 = 0$. Furthermore, we have

$$[d_0, f(h_{m+\frac{1}{2}})] = f([d_0, h_{m+\frac{1}{2}}]) = [d_0, k_3h_{-\frac{1}{2}} + k_4\mathbf{1}] = \frac{1}{2}k_3h_{-\frac{1}{2}} = -(m + \frac{1}{2})k_3h_{-\frac{1}{2}} - (m + \frac{1}{2})k_4\mathbf{1}.$$

Hence, $k_3 = k_4 = 0$.

Case 2: When $m = -1$, since $m \neq n$, then $n \neq -1$. Here, $\mathfrak{D}_{-1} = \langle d_{-1}, h_{-\frac{1}{2}}, \mathbf{1} \rangle$. Let

$$f(d_{-1}) = k_1h_{n+\frac{1}{2}}, \quad f(h_{-\frac{1}{2}}) = k_2h_{n+\frac{1}{2}}, \quad f(\mathbf{1}) = k_3h_{n+\frac{1}{2}}.$$

Then, we have

$$[d_0, f(d_{-1})] = f([d_0, d_{-1}]) = f(d_{-1}) = k_1h_{n+\frac{1}{2}} = k_1[d_0, h_{n+\frac{1}{2}}] = -k_1(n + \frac{1}{2})h_{n+\frac{1}{2}}.$$

So, $k_1 = 0$. Furthermore, we have

$$[d_0, f(h_{-\frac{1}{2}})] = f([d_0, h_{-\frac{1}{2}}]) = \frac{1}{2}k_2h_{n+\frac{1}{2}} = k_2[d_0, h_{n+\frac{1}{2}}] = -(n + \frac{1}{2})k_2h_{n+\frac{1}{2}}.$$

Hence $k_2 = 0$. In addition, we have

$$[d_0, f(\mathbf{1})] = f([d_0, \mathbf{1}]) = k_3[d_0, h_{n+\frac{1}{2}}] = -k_3(n + \frac{1}{2})h_{n+\frac{1}{2}} = 0.$$

Hence $k_3 = 0$. Thus, $f = 0$, that is, $\text{Hom}_{\mathfrak{D}_0}(\mathfrak{D}_m, \mathcal{H}_n) = 0$ for $m \neq n, m \neq 0$. $\square$

Lemma 3. Let $\mathfrak{D} = \oplus_{n \in \mathbb{Z}} \mathfrak{D}_n$ be the mirror Heisenberg–Virasoro algebra and $\mathcal{H} = \oplus_{n \in \mathbb{Z}} \mathbb{C}h_{n+\frac{1}{2}} \oplus \mathbb{C}\mathbf{1}$ the twisted Heisenberg algebra. If $m = 0$ and $n \neq 0, -1$, then $\text{Hom}_{\mathfrak{D}_0}(\mathfrak{D}_0, \mathcal{H}_n) = 0$.

Proof. Suppose $f(d_0) = k_1h_{n+\frac{1}{2}}, f(h_{\frac{1}{2}}) = k_2h_{n+\frac{1}{2}}, f(\mathbf{c}) = k_3h_{n+\frac{1}{2}}$; then,

$$f([d_0, d_0]) = [d_0, f(d_0)] = k_1[d_0, h_{n+\frac{1}{2}}] = -k_1(n + \frac{1}{2})h_{n+\frac{1}{2}} = 0.$$

Hence, $k_1 = 0$. Furthermore, we have

$$f([d_0, h_{\frac{1}{2}}]) = [d_0, f(h_{\frac{1}{2}})] = k_2[d_0, h_{n+\frac{1}{2}}] = -k_2(n + \frac{1}{2})h_{n+\frac{1}{2}} = -\frac{1}{2}f(h_{\frac{1}{2}}) = -\frac{1}{2}k_2h_{n+\frac{1}{2}}.$$

Hence $k_2 = 0$. In addition, we have

$$f([d_0, \mathbf{c}]) = [d_0, f(\mathbf{c})] = k_3[d_0, h_{n+\frac{1}{2}}] = -k_3(n + \frac{1}{2})h_{n+\frac{1}{2}}.$$

Hence, $k_3 = 0$. So, $f = 0$, that is, $\text{Hom}_{\mathfrak{D}_0}(\mathfrak{D}_0, \mathcal{H}_n) = 0$ for $n \neq 0, -1$. $\square$

Lemma 4. Let $\mathfrak{D}$ be the mirror Heisenberg–Virasoro algebra and $\mathcal{H}$ the twisted Heisenberg algebra. Then, $D \in \text{Der}_{\mathbb{C}}(\mathfrak{D}, \mathcal{H})_{-1}$ if and only if D is a linear map from $\mathfrak{D}$ to $\mathcal{H}$ satisfying the following conditions:

- (i) $D(d_n) = ch_{n-\frac{1}{2}}$;
- (ii) $D(\mathbf{c}) = 0$;
- (iii) $D(h_{n+\frac{1}{2}}) = \delta_{n,0}\mathbf{c}\mathbf{1}$;
- (iv) $D(\mathbf{1}) = 0$;

where $n \in \mathbb{Z}$, $c \in \mathbb{C}$. Furthermore, D is an inner derivation.

Proof. Sufficiency: Let D be a linear map that satisfies (i)–(iv). Then, it is easy to prove that D is a derivation from $\mathfrak{D}$ to $\mathcal{H}$.

Necessity: (i) For $n \neq 0$, set $D(d_n) = f(n)h_{n-\frac{1}{2}}$. If $n \neq \pm m$, on the one hand,

$$D([d_m, d_n]) = (m - n)D(d_{m+n}) = (m - n)f(m + n)h_{m+n-\frac{1}{2}},$$

and on the other hand,

$$\begin{aligned} D([d_m, d_n]) &= [D(d_m), d_n] + [d_m, D(d_n)] = f(m)[h_{m-\frac{1}{2}}, d_n] + f(n)[d_m, h_{n-\frac{1}{2}}] \\ &= (m - \frac{1}{2})f(m)h_{m+n-\frac{1}{2}} - (n - \frac{1}{2})f(n)h_{m+n-\frac{1}{2}}, \end{aligned}$$

then

$$(m - n)f(m + n) = (m - \frac{1}{2})f(m) - (n - \frac{1}{2})f(n).$$

Let $f(1) = f(-1) = c, c \in \mathbb{C}$. Using induction on n , assuming $n - 1$, the conclusion holds, that is, $f(n - 1) = c$. Then,

$$(n - 2)f(n) = (n - \frac{3}{2})f(n - 1) - \frac{1}{2}f(1) = (n - 2)c.$$

Therefore, $f(n) = c$ since $D([d_1, d_{-1}]) = 2D(d_0)$ and

$$D([d_1, d_{-1}]) = [D(d_1), d_{-1}] + [d_1, D(d_{-1})] = c[h_{\frac{1}{2}}, d_{-1}] + c[d_1, h_{-\frac{3}{2}}] = 2ch_{-\frac{1}{2}}.$$

Hence, $D(d_0) = ch_{-\frac{1}{2}}$. So, we have $D(d_n) = ch_{n-\frac{1}{2}}$ for $n \in \mathbb{Z}$.

(ii) For $n \neq 0$, let $D(h_{n+\frac{1}{2}}) = g(n)h_{n-\frac{1}{2}}$. If $m \neq -n, n \neq 0$, on the one hand,

$$D([d_m, h_{n+\frac{1}{2}}]) = -(n + \frac{1}{2})D(h_{n+m+\frac{1}{2}}) = -(n + \frac{1}{2})g(n+m)h_{n+m-\frac{1}{2}},$$

and on the other hand,

$$\begin{aligned} D([d_m, h_{n+\frac{1}{2}}]) &= [D(d_m), h_{n+\frac{1}{2}}] + [d_m, D(h_{n+\frac{1}{2}})] \\ &= c[h_{m-\frac{1}{2}}, h_{n+\frac{1}{2}}] + g(n)[d_m, h_{n-\frac{1}{2}}] = -(n - \frac{1}{2})g(n)h_{n+m-\frac{1}{2}}. \end{aligned}$$

Therefore, $(n + \frac{1}{2})g(n+m) = (n - \frac{1}{2})g(n)$.

Let $g(1) = d, d \in \mathbb{C}$. Then $g(n) = \frac{d}{2^{n-1}}$. Using induction on n , if $n-1$, the conclusion holds, that is, $g(n-1) = \frac{d}{2^{(n-1)-1}}$. Then,

$$(n - \frac{1}{2})g(n) = (n - \frac{3}{2})g(n-1) = (n - \frac{3}{2})\frac{d}{2^{n-3}}.$$

Hence, $g(n) = \frac{d}{2^{n-1}}$. Since $D([d_1, h_{-\frac{1}{2}}]) = \frac{1}{2}D(h_{\frac{1}{2}})$ and

$$D([d_1, h_{-\frac{1}{2}}]) = [D(d_1), h_{-\frac{1}{2}}] + [d_1, D(h_{-\frac{1}{2}})] = c[h_{\frac{1}{2}}, h_{-\frac{1}{2}}] + g(-1)[d_1, h_{-\frac{3}{2}}] = \frac{1}{2}c\mathbf{1} - \frac{1}{2}dh_{-\frac{1}{2}},$$

we have $D(h_{\frac{1}{2}}) = c\mathbf{1} - dh_{-\frac{1}{2}}$. So, it follows that

$$D(h_{n+\frac{1}{2}}) = \frac{d}{2n-1}h_{n-\frac{1}{2}} + \delta_{n,0}c\mathbf{1}.$$

Again, on the one hand,

$$D([d_0, h_{\frac{1}{2}}]) = [D(d_0), h_{\frac{1}{2}}] + [d_0, D(h_{\frac{1}{2}})] = [ch_{-\frac{1}{2}}, h_{\frac{1}{2}}] + [d_0, c\mathbf{1} - dh_{-\frac{1}{2}}] = -\frac{1}{2}c\mathbf{1} - \frac{1}{2}dh_{-\frac{1}{2}},$$

and on the other hand,

$$D([d_0, h_{\frac{1}{2}}]) = -\frac{1}{2}D(h_{\frac{1}{2}}) = -\frac{1}{2}c\mathbf{1} + \frac{1}{2}dh_{-\frac{1}{2}}.$$

Comparing the two formulas above, we have $d = 0$. Therefore, $D(h_{n+\frac{1}{2}}) = \delta_{n,0}c\mathbf{1}$.

(iii) Since $D([d_2, d_{-2}]) = 4D(d_0) + \frac{1}{2}D(c) = 4ch_{-\frac{1}{2}} + \frac{1}{2}D(c)$ and

$$D([d_2, d_{-2}]) = [D(d_2), d_{-2}] + [d_2, D(d_{-2})] = c[h_{\frac{3}{2}}, d_{-2}] + c[d_2, h_{-\frac{5}{2}}] = 4ch_{-\frac{1}{2}},$$

we have $D(c) = 0$.

(iv) On the one hand, $D([h_{\frac{1}{2}}, h_{-\frac{1}{2}}]) = \frac{1}{2}D(\mathbf{1})$, and on the other hand,

$$D([h_{\frac{1}{2}}, h_{-\frac{1}{2}}]) = [D(h_{\frac{1}{2}}), h_{-\frac{1}{2}}] + [h_{\frac{1}{2}}, D(h_{-\frac{1}{2}})] = [c\mathbf{1}, h_{-\frac{1}{2}}] = 0.$$

Then, we have $D(\mathbf{1}) = 0$.

Let D be given by

$$\begin{aligned} D(d_n) &= [d_n, 2ch_{-\frac{1}{2}}] = ch_{n-\frac{1}{2}}, D(\mathbf{c}) = [c, 2ch_{-\frac{1}{2}}] = 0, \\ D(h_{n+\frac{1}{2}}) &= [h_{n+\frac{1}{2}}, 2ch_{-\frac{1}{2}}] = c\delta_{n,0}\mathbf{1}, D(\mathbf{1}) = [1, 2ch_{-\frac{1}{2}}] = 0. \end{aligned}$$

Then, it is easy to see that D is an inner derivation. $\square$

Theorem 1. Let $\mathfrak{D} = \oplus_{n \in \mathbb{Z}} \mathfrak{D}_n$ be the mirror Heisenberg–Virasoro algebra and $\mathcal{H}$ the twisted Heisenberg algebra. Then,

$$\text{Der}_{\mathbb{C}}(\mathfrak{D}, \mathcal{H}) = \text{Der}_{\mathbb{C}}(\mathfrak{D}, \mathcal{H})_0 \oplus \text{Inn}_{\mathbb{C}}(\mathfrak{D}, \mathcal{H}).$$

Proof. Suppose $\varphi: \mathfrak{D} \rightarrow \mathcal{H}$ is a derivation, according to Proposition 2; then, φ can be decomposed into homogeneous elements: $\varphi = \sum_{n \in \mathbb{Z}} \varphi_n$ where $\varphi_n \in \text{Der}_{\mathbb{C}}(\mathfrak{D}, \mathcal{H})_n$.

Let $n \in \mathbb{Z} \setminus \{0, -1\}$. Then, $\varphi_n|_{\mathfrak{D}_0}$ is a derivation of $\mathcal{H}_n$ from $\mathfrak{D}_0$ to $\mathfrak{D}_0$ -module. By virtue of Lemma 1, $\varphi_n|_{\mathfrak{D}_0}$ is an inner derivation, that is, there exists $E_n \in \mathcal{H}_n$ such that $\varphi_n(x) = [x, E_n]$, $x \in \mathfrak{D}_0$. Considering $\psi_n(x) = \varphi_n(x) - [x, E_n]$, $x \in \mathfrak{D}$, then ψ_n is an inner derivation, $\deg \psi_n = n$, and $\psi_n(\mathfrak{D}_0) = 0$. Therefore, ψ_n is $\mathfrak{D}_0$ -module homomorphism. Since $x \in \mathfrak{D}_0, y \in \mathfrak{D}_n$,

$$\begin{aligned} \psi_n([x, y]) &= \varphi_n([x, y]) - [[x, y], E_n] = [\varphi_n(x), y] + [x, \varphi_n(y)] - [x, [y, E_n]] - [y, [E_n, x]] \\ &= [x, \varphi_n(y)] - [x, [y, E_n]] = [x, \varphi_n(y) - [y, E_n]] = [x, \psi_n(y)]. \end{aligned}$$

By Lemma 2, we have $\text{Hom}_{\mathfrak{D}_0}(\mathfrak{D}_m, \mathcal{H}_n) = 0$ for $m \neq n, m \neq 0$ and $\psi_n|_{\mathfrak{D}_m} = 0$. At last, by virtue of Lemma 3, we obtain $\psi_n|_{\mathfrak{D}_0} = 0$. Furthermore, for $m \in \mathbb{Z}$, $\psi_n|_{\mathfrak{D}_m} = 0$. Therefore, $\varphi_n \in \text{Inn}_{\mathbb{C}}(\mathfrak{D}, \mathcal{H})$. Also, by Lemma 4, the conclusion is proved. $\square$

Lemma 5. Let $\mathfrak{D}$ be the mirror Heisenberg–Virasoro algebra and $\mathcal{H}$ be the twisted Heisenberg algebra. Then, $D \in \text{Der}_{\mathbb{C}}(\mathfrak{D}, \mathcal{H})_0$ if and only if D is a linear map from $\mathfrak{D}$ to $\mathcal{H}$ satisfying

- (i) $D(d_n) = ah_{n+\frac{1}{2}}$;
- (ii) $D(\mathbf{c}) = 0$;
- (iii) $D(h_{n+\frac{1}{2}}) = bh_{n+\frac{1}{2}} + \delta_{n,-1}a\mathbf{1}$;
- (iv) $D(\mathbf{1}) = 2b\mathbf{1}$;

where $n \in \mathbb{Z}$, $a, b \in \mathbb{C}$.

Proof. Sufficiency: Suppose D is a linear map that satisfies (i)–(iv); then, it is easy to verify that D is a derivation from $\mathfrak{D}$ to $\mathcal{H}$.

Necessity: (i) Set $D \in \text{Der}_{\mathbb{C}}(\mathfrak{D}, \mathcal{H})$. When $n \neq -1$, suppose $D(d_n) = f(n)h_{n+\frac{1}{2}}$ for $m \neq \pm n$. On the one hand,

$$D([d_m, d_n]) = (m - n)D(d_{m+n}) = (m - n)f(m + n)h_{m+n+\frac{1}{2}}.$$

On the other hand,

$$\begin{aligned} D([d_m, d_n]) &= [D(d_m), d_n] + [d_m, D(d_n)] = f(m)[h_{m+\frac{1}{2}}, d_n] + f(n)[d_m, h_{n+\frac{1}{2}}] \\ &= (m + \frac{1}{2})f(m)h_{m+n+\frac{1}{2}} - (n + \frac{1}{2})f(n)h_{m+n+\frac{1}{2}}. \end{aligned}$$

Then,

$$(m - n)f(m + n) = (m + \frac{1}{2})f(m) - (n + \frac{1}{2})f(n).$$

Let $m = 1$, $n = 0$. Then, $f(0) = f(1) = a$ ($a \in \mathbb{C}$). Next, using induction on n , assuming $n - 1$, the conclusion holds. Then,

$$(n-2)f(n) = (n-2)f((n-1)+1) = (n-\frac{1}{2})f(n-1) - \frac{3}{2}f(1) = (n-2)a.$$

Hence, $f(n) = a$ for $n \in \mathbb{Z}$. If $n = -1$, then on the one hand, $D([d_1, d_{-2}]) = 3D(d_{-1})$, and on the other hand,

$$D([d_1, d_{-2}]) = [D(d_1), d_{-2}] + [d_1, D(d_{-2})] = a[h_{\frac{3}{2}}, d_{-2}] + a[d_1, h_{-\frac{3}{2}}] = 3ah_{-\frac{1}{2}}.$$

Therefore, $D(d_{-1}) = ah_{-\frac{1}{2}}$. Thus, $D(d_n) = ah_{n+\frac{1}{2}}$, $a \in \mathbb{C}$.

(ii) On the one hand, we have

$$D([d_2, d_{-2}]) = 4D(d_0) + \frac{1}{2}D(\mathbf{c}) = 4ah_{\frac{1}{2}} + \frac{1}{2}D(\mathbf{c}),$$

and on the other hand,

$$D([d_2, d_{-2}]) = [D(d_2), d_{-2}] + [d_2, D(d_{-2})] = a[h_{\frac{5}{2}}, d_{-2}] + a[d_2, h_{-\frac{3}{2}}] = 4ah_{\frac{1}{2}}.$$

Therefore, $D(\mathbf{c}) = 0$.

(iii) For $n \neq -1$, let $D(h_{n+\frac{1}{2}}) = g(n)h_{n+\frac{1}{2}}$. Since

$$D([d_n, h_{m+\frac{1}{2}}]) = -(m+\frac{1}{2})D(h_{m+n+\frac{1}{2}}) = -(m+\frac{1}{2})g(n+m)h_{m+n+\frac{1}{2}}.$$

and

$$\begin{aligned} D([d_n, h_{m+\frac{1}{2}}]) &= [D(d_n), h_{m+\frac{1}{2}}] + [d_n, D(h_{m+\frac{1}{2}})] = a[h_{n+\frac{1}{2}}, h_{m+\frac{1}{2}}] + g(m)[d_n, h_{m+\frac{1}{2}}] \\ &= -(m+\frac{1}{2})g(m)h_{m+n+\frac{1}{2}}, \end{aligned}$$

then $g(n+m) = g(m)$, $n \neq -1$, $m \neq -n-1, -1$.

Set $g(1) = b$, $b \in \mathbb{C}$. Then, $g(n) = b$ for $n \in \mathbb{Z} \setminus \{-1\}$. Since $D([d_{-1}, h_{\frac{1}{2}}]) = -\frac{1}{2}D(h_{-\frac{1}{2}})$ and

$$D([d_{-1}, h_{\frac{1}{2}}]) = [D(d_{-1}), h_{\frac{1}{2}}] + [d_{-1}, D(h_{\frac{1}{2}})] = a[h_{-\frac{1}{2}}, h_{\frac{1}{2}}] + b[d_{-1}, h_{\frac{1}{2}}] = -\frac{1}{2}a\mathbf{1} - \frac{1}{2}bh_{-\frac{1}{2}},$$

then $D(h_{-\frac{1}{2}}) = a\mathbf{1} + bh_{-\frac{1}{2}}$. Thus, $D(h_{n+\frac{1}{2}}) = bh_{n+\frac{1}{2}} + \delta_{n,-1}a\mathbf{1}$.

(iv) Since $D([h_{-\frac{1}{2}}, h_{\frac{1}{2}}]) = -\frac{1}{2}D(\mathbf{1})$ and

$$D([h_{-\frac{1}{2}}, h_{\frac{1}{2}}]) = [D(h_{-\frac{1}{2}}), h_{\frac{1}{2}}] + [h_{-\frac{1}{2}}, D(h_{\frac{1}{2}})] = b[h_{-\frac{1}{2}}, h_{\frac{1}{2}}] + b[h_{-\frac{1}{2}}, h_{\frac{1}{2}}] = -b\mathbf{1},$$

we obtain $D(\mathbf{1}) = 2b\mathbf{1}$. $\square$

According to Lemma 5, D_i ($i = 1, 2$) is given by

$$\begin{aligned} D_1: D_1|_{\mathcal{V}} &= 0, D_1(h_{n+\frac{1}{2}}) = h_{n+\frac{1}{2}}, & D_1(\mathbf{1}) &= 2\mathbf{1} \ (n \in \mathbb{Z}), \\ D_2: D_2(d_n) &= h_{n+\frac{1}{2}}, D_2(h_{n+\frac{1}{2}}) = \delta_{n,-1}l, & D_2(\mathbf{c}) &= D_2(\mathbf{1}) = 0. \end{aligned}$$

Lemma 6. Let $\mathfrak{D}$ be the mirror Heisenberg–Virasoro algebra and $\mathcal{H}$ the twisted Heisenberg algebra. Then,

$$H^1(\mathfrak{D}, \mathcal{H}) = \mathbb{C}D_1 \oplus \mathbb{C}D_2.$$

Proof. Let us suppose that there are k_1 and k_2 such that $k_1 D_1 + k_2 D_2 = 0$. Then, for $h_{-\frac{1}{2}}$, we have

$$0 = (k_1 D_1 + k_2 D_2)(h_{-\frac{1}{2}}) = k_1 h_{-\frac{1}{2}} + k_2 l.$$

Therefore, $k_1 = k_2 = 0$, that is, $\{D_i \mid i = 1, 2\}$ are linearly independent. And, for any $D \in \text{Der}_{\mathbb{C}}(\mathfrak{D}, \mathcal{H})_0$, we have

$$\begin{aligned} D(d_n) &= ah_{n+\frac{1}{2}} = aD_2(d_n), D(\mathbf{c}) = D_1(\mathbf{c}) = 0, \\ D(h_{n+\frac{1}{2}}) &= bh_{n+\frac{1}{2}} + \delta_{n,-1} a\mathbf{l} = bD_1(h_{n+\frac{1}{2}}) + aD_2(h_{n+\frac{1}{2}}), D(\mathbf{l}) = 2b\mathbf{l} = bD_1(\mathbf{l}). \end{aligned}$$

By virtue of Lemma 1, we obtain

$$\text{Der}_{\mathbb{C}}(\mathfrak{D}, \mathcal{H}) = \mathbb{C}D_1 \oplus \mathbb{C}D_2 \oplus \text{Inn}_{\mathbb{C}}(\mathfrak{D}, \mathcal{H}) \text{ and } H^1(\mathfrak{D}, \mathcal{H}) = \text{Der}_{\mathbb{C}}(\mathfrak{D}, \mathcal{H}) / \text{Inn}_{\mathbb{C}}(\mathfrak{D}, \mathcal{H}).$$

□

Lemma 7. Let $\mathcal{V}$ be the Virasoro algebra and $\mathcal{H}$ the twisted Heisenberg algebra. Then,

$$\text{Hom}_{\mathcal{U}(\mathcal{V})}(\mathcal{H}/[\mathcal{H}, \mathcal{H}], \mathcal{V}) = 0.$$

Proof. By the definition of $\mathcal{H}$, we have $[\mathcal{H}, \mathcal{H}] = \mathbb{C}\mathbf{l}$. Set $f \in \text{Hom}_{\mathcal{U}(\mathcal{V})}(\mathcal{H}/[\mathcal{H}, \mathcal{H}], \mathcal{V})$ for $m \in \mathbb{Z}$. Suppose

$$f(h_{m+\frac{1}{2}}) = \sum a_i(m) d_{m_i} + k\mathbf{c}.$$

On the one hand,

$$[d_0, f(h_{m+\frac{1}{2}})] = [d_0, \sum a_i(m) d_{m_i} + k\mathbf{c}] = \sum a_i(m) [d_0, d_{m_i}] = -\sum a_i(m) m_i d_{m_i},$$

and on the other hand,

$$\begin{aligned} [d_0, f(h_{m+\frac{1}{2}})] &= f([d_0, h_{m+\frac{1}{2}}]) = -(m + \frac{1}{2})f(h_{m+\frac{1}{2}}) \\ &= -(m + \frac{1}{2})(\sum a_i(m) d_{m_i} + k\mathbf{c}) = -(m + \frac{1}{2}) \sum a_i(m) d_{m_i} - (m + \frac{1}{2})k\mathbf{c}. \end{aligned}$$

Then,

$$a_i(m) m_i = (m + \frac{1}{2}) a_i(m), \quad k = 0.$$

Since $m_i \in \mathbb{Z}$, $m \in \mathbb{Z}$, $m_i \neq m + \frac{1}{2}$, then $a_i(m) = 0$. Furthermore, we obtain $\text{Hom}_{\mathcal{U}(\mathcal{V})}(\mathcal{H}/[\mathcal{H}, \mathcal{H}], \mathcal{V}) = 0$. □

Theorem 2. Let $\mathfrak{D}$ be the mirror Heisenberg–Virasoro algebra and $\mathcal{H}$ the twisted Heisenberg algebra. Then,

$$\text{Der}_{\mathbb{C}} \mathfrak{D} = \mathbb{C}D_1 \oplus \mathbb{C}D_2 \oplus \text{Inn} \mathfrak{D}.$$

Proof. Since

$$H^1(\mathcal{H}, \mathfrak{D}/\mathcal{H})^{\mathfrak{D}/\mathcal{H}} \cong \text{Hom}_{\mathbb{C}}(\mathcal{H}/[\mathcal{H}, \mathcal{H}], \mathfrak{D}/\mathcal{H})^{\mathfrak{D}/\mathcal{H}} = \text{Hom}_{\mathfrak{D}/\mathcal{H}}(\mathcal{H}/[\mathcal{H}, \mathcal{H}], \mathfrak{D}/\mathcal{H}),$$

by virtue of Lemma 7, then $H^1(\mathcal{H}, \mathfrak{D}/\mathcal{H})^{\mathfrak{D}/\mathcal{H}} = 0$. According to the exact sequence (2)

$$0 \rightarrow H^1(\mathfrak{D}/\mathcal{H}, \mathfrak{D}/\mathcal{H}) \rightarrow H^1(\mathfrak{D}, \mathfrak{D}/\mathcal{H}) \rightarrow H^1(\mathcal{H}, \mathfrak{D}/\mathcal{H})^{\mathfrak{D}/\mathcal{H}}$$

and $H^1(\mathfrak{D}/\mathcal{H}, \mathfrak{D}/\mathcal{H}) = 0$, we obtain $H^1(\mathfrak{D}, \mathfrak{D}/\mathcal{H}) = 0$. Then, by virtue of Lemma 6 and the exact sequence (1)

$$H^1(\mathfrak{D}, \mathcal{H}) \rightarrow H^1(\mathfrak{D}, \mathfrak{D}) \rightarrow H^1(\mathfrak{D}, \mathfrak{D}/\mathcal{H}),$$

we have

$$H^1(\mathfrak{D}, \mathcal{H}) = H^1(\mathfrak{D}, \mathfrak{D}).$$

Therefore,

$$H^1(\mathfrak{D}, \mathfrak{D}) = \mathbb{C}D_1 \oplus \mathbb{C}D_2.$$

Since $H^1(\mathfrak{D}, \mathfrak{D}) = \text{Der}\mathfrak{D}/\text{Inn}\mathfrak{D}$, the conclusion holds. $\square$

3. 2-Local Derivations on the Mirror Heisenberg–Virasoro Algebra

Let L be an algebra. Δ is called a 2-local derivation of L if, for every $x, y \in L$, there exists a derivation $\Delta_{x,y}$ such that

$$\Delta_{x,y}(x) = \Delta(x), \Delta_{x,y}(y) = \Delta(y).$$

Let Δ be a 2-local derivation on the mirror Heisenberg–Virasoro algebra $\mathfrak{D}$. By Theorem 2, it can be written as

$$\Delta_{x,y} = \text{ad}\left(\sum_{i \in \mathbb{Z}} (a_i(x, y)d_i + b_i(x, y)h_{i+\frac{1}{2}})\right) + l_1(x, y)\mathbf{c} + l_2(x, y)\mathbf{l} + \alpha(x, y)D_1 + \beta(x, y)D_2. \quad (3)$$

Lemma 8. Let Δ be a 2-local derivation on the mirror Heisenberg–Virasoro algebra $\mathfrak{D}$. Then, for any fixed $x \in \mathfrak{D}$, we have the following:

(i) If $\Delta(d_i) = 0$, then

$$\Delta_{d_i, x} = \text{ad}(a_i(d_i, x)d_i + b_0(d_i, x)h_{\frac{1}{2}} + l_1(d_i, x)\mathbf{c} + l_2(d_i, x)\mathbf{l}) + \alpha(d_i, x)D_1 - \frac{1}{2}b_0(d_i, x)D_2.$$

(ii) If $\Delta(h_{\frac{1}{2}}) = 0$, then

$$\Delta_{h_{\frac{1}{2}}, x} = \text{ad}(a_0(h_{\frac{1}{2}}, x)d_0 + \sum_{i \neq -1} b_i(h_{\frac{1}{2}}, x)h_{i+\frac{1}{2}} + l_1(h_{\frac{1}{2}}, x)\mathbf{c} + l_2(h_{\frac{1}{2}}, x)\mathbf{l}) + \frac{1}{2}a_0(h_{\frac{1}{2}}, x)D_1 + \beta(h_{\frac{1}{2}}, x)D_2.$$

(iii) If $\Delta(h_{-\frac{1}{2}}) = 0$, then

$$\begin{aligned} \Delta_{h_{-\frac{1}{2}}, x} &= \text{ad}(a_0(h_{-\frac{1}{2}}, x)d_0 + \sum_{i \in \mathbb{Z}} b_i(h_{-\frac{1}{2}}, x)h_{i+\frac{1}{2}} + l_1(h_{-\frac{1}{2}}, x)\mathbf{c} + l_2(h_{-\frac{1}{2}}, x)\mathbf{l}) \\ &\quad - \frac{1}{2}a_0(h_{-\frac{1}{2}}, x)D_1 - \frac{1}{2}b_0(h_{-\frac{1}{2}}, x)D_2. \end{aligned}$$

(iv) If $\Delta(h_{i+\frac{1}{2}}) = 0$ for $i \neq 0, -1$, then

$$\begin{aligned} \Delta_{h_{i+\frac{1}{2}}, x} &= \text{ad}(a_0(h_{i+\frac{1}{2}}, x)d_0 + \sum_{j \neq -1-i} b_j(h_{i+\frac{1}{2}}, x)h_{j+\frac{1}{2}} + l_1(h_{i+\frac{1}{2}}, x)\mathbf{c} + l_2(h_{i+\frac{1}{2}}, x)\mathbf{l}) \\ &\quad + (i + \frac{1}{2})a_0(h_{i+\frac{1}{2}}, x)D_1 + \beta(h_{i+\frac{1}{2}}, x)D_2. \end{aligned}$$

Proof. (i) According to $\Delta(d_i) = 0$ and the Formula (3), we have

$$\begin{aligned} \Delta(d_i) &= \Delta_{d_i, x}(d_i) \\ &= \left[\sum_{j \in \mathbb{Z}} (a_j(d_i, x)d_j + b_j(d_i, x)h_{j+\frac{1}{2}}) + l_1(d_i, x)\mathbf{c} + l_2(d_i, x)\mathbf{l}, d_i \right] + \alpha(d_i, x)D_1(d_i) + \beta(d_i, x)D_2(d_i) \\ &= \sum_{j \in \mathbb{Z}} ((j-i)a_j(d_i, x)d_{i+j} + (j + \frac{1}{2})b_j(d_i, x)h_{i+j+\frac{1}{2}}) - \frac{i^3-i}{12}a_{-i}(d_i, x)\mathbf{c} + \beta(d_i, x)h_{i+\frac{1}{2}} = 0. \end{aligned}$$

Then, we have

$$a_j(d_i, x) = 0, \text{ for } j \neq i; b_j(d_i, x) = 0, \text{ for } j \neq 0; \beta(d_i, x) = -\frac{1}{2}b_0(d_i, x).$$

(ii) According to $\Delta(h_{\frac{1}{2}}) = 0$ and the Formula (3), we have

$$\begin{aligned} \Delta(h_{\frac{1}{2}}) &= \Delta_{h_{\frac{1}{2}}, x}(h_{\frac{1}{2}}) \\ &= \left[\sum_{j \in \mathbb{Z}} (a_j(h_{\frac{1}{2}}, x)d_j + b_j(h_{\frac{1}{2}}, x)h_{j+\frac{1}{2}}) + l_1(h_{\frac{1}{2}}, x)\mathbf{c} + l_2(h_{\frac{1}{2}}, x)\mathbf{l}, h_{\frac{1}{2}} \right] + \alpha(h_{\frac{1}{2}}, x)D_1(h_{\frac{1}{2}}) + \beta(h_{\frac{1}{2}}, x)D_2(h_{\frac{1}{2}}) \\ &= \sum_{j \in \mathbb{Z}} \left(-\frac{1}{2}a_j(h_{\frac{1}{2}}, x)h_{j+\frac{1}{2}} - \frac{1}{2}b_{-1}(h_{\frac{1}{2}}, x)\mathbf{l} + \alpha(h_{\frac{1}{2}}, x)h_{\frac{1}{2}} \right) = 0. \end{aligned}$$

Then, we have

$$a_j(h_{\frac{1}{2}}, x) = 0, \text{ for } j \neq 0; b_{-1}(h_{\frac{1}{2}}, x) = 0; \alpha(h_{\frac{1}{2}}, x) = \frac{1}{2}a_0(h_{\frac{1}{2}}, x).$$

(iii) By $\Delta(h_{-\frac{1}{2}}) = 0$ and the Formula (3), we have

$$\begin{aligned} \Delta(h_{-\frac{1}{2}}) &= \Delta_{h_{-\frac{1}{2}}, x}(h_{-\frac{1}{2}}) = \left[\sum_{j \in \mathbb{Z}} (a_j(h_{-\frac{1}{2}}, x)d_j + b_j(h_{-\frac{1}{2}}, x)h_{j+\frac{1}{2}}) + l_1(h_{-\frac{1}{2}}, x)\mathbf{c} + l_2(h_{-\frac{1}{2}}, x)\mathbf{l}, h_{-\frac{1}{2}} \right] \\ &\quad + \alpha(h_{-\frac{1}{2}}, x)D_1(h_{-\frac{1}{2}}) + \beta(h_{-\frac{1}{2}}, x)D_2(h_{-\frac{1}{2}}) \\ &= \sum_{j \in \mathbb{Z}} \left(\frac{1}{2}a_j(h_{-\frac{1}{2}}, x)h_{j-\frac{1}{2}} + \frac{1}{2}b_0(h_{-\frac{1}{2}}, x)\mathbf{l} + \alpha(h_{-\frac{1}{2}}, x)h_{-\frac{1}{2}} + \beta(h_{-\frac{1}{2}}, x)\mathbf{l} \right) = 0. \end{aligned}$$

Then, we have

$$a_j(h_{-\frac{1}{2}}, x) = 0, \text{ for } j \neq 0; \alpha(h_{-\frac{1}{2}}, x) = -\frac{1}{2}a_0(h_{-\frac{1}{2}}, x); \beta(h_{-\frac{1}{2}}, x) = -\frac{1}{2}b_0(h_{-\frac{1}{2}}, x).$$

(iv) By $\Delta(h_{i+\frac{1}{2}}) = 0$ and the Formula (3), we have

$$\begin{aligned} \Delta(h_{i+\frac{1}{2}}) &= \Delta_{h_{i+\frac{1}{2}}, x}(h_{i+\frac{1}{2}}) \\ &= \left[\sum_{j \in \mathbb{Z}} (a_j(h_{i+\frac{1}{2}}, x)d_j + b_j(h_{i+\frac{1}{2}}, x)h_{j+\frac{1}{2}}) + l_1(h_{i+\frac{1}{2}}, x)\mathbf{c} + l_2(h_{i+\frac{1}{2}}, x)\mathbf{l}, h_{i+\frac{1}{2}} \right] \\ &\quad + \alpha(h_{i+\frac{1}{2}}, x)D_1(h_{i+\frac{1}{2}}) + \beta(h_{i+\frac{1}{2}}, x)D_2(h_{i+\frac{1}{2}}) \\ &= \sum_{j \in \mathbb{Z}} \left(-\left(i + \frac{1}{2}\right)a_j(h_{i+\frac{1}{2}}, x)h_{j+i+\frac{1}{2}} + \left(-i - \frac{1}{2}\right)b_{-1-i}(h_{i+\frac{1}{2}}, x)\mathbf{l} + \alpha(h_{i+\frac{1}{2}}, x)h_{i+\frac{1}{2}} \right) = 0. \end{aligned}$$

Then, we have

$$b_j(h_{i+\frac{1}{2}}, x) = 0, \text{ for } j = -1 - i; a_j(h_{i+\frac{1}{2}}, x) = 0, \text{ for } j \neq 0; \alpha(h_{i+\frac{1}{2}}, x) = \left(i + \frac{1}{2}\right)a_0(h_{i+\frac{1}{2}}, x).$$

□

Lemma 9. Let Δ be a 2-local derivation on the mirror Heisenberg–Virasoro algebra $\mathfrak{D}$ such that $\Delta(d_0) = \Delta(d_1) = 0$. Then, $\Delta(d_i) = 0$.

Proof. Set $k = 0$ or 1 . By Lemma 8, suppose

$$\Delta_{d_k, x} = \text{ada}_k(d_k, x)d_k + b_0(d_k, x)h_{\frac{1}{2}} + l_1(d_k, x)\mathbf{c} + l_2(d_k, x)\mathbf{l} + \alpha(d_k, x)D_1 - \frac{1}{2}b_0(d_k, x)D_2.$$

Then,

$$\begin{aligned}\Delta(d_i) &= \Delta_{d_0, d_i}(d_i) \\ &= [a_0(d_0, d_i)d_0 + b_0(d_0, d_i)h_{\frac{1}{2}} + l_1(d_0, d_i)\mathbf{c} + l_2(d_0, d_i)\mathbf{l}, d_i] + \alpha(d_0, d_i)D_1(d_i) - \frac{1}{2}b_0(d_0, d_i)D_2(d_i) \\ &= -ia_0(d_0, d_i)d_i + \frac{1}{2}b_0(d_0, d_i)h_{i+\frac{1}{2}} - \frac{1}{2}b_0(d_0, d_i)h_{i+\frac{1}{2}} = -ia_0(d_0, d_i)d_i.\end{aligned}$$

We also have

$$\begin{aligned}\Delta(d_i) &= \Delta_{d_1, d_i}(d_i) \\ &= [a_1(d_1, d_i)d_1 + b_0(d_1, d_i)h_{\frac{1}{2}} + l_1(d_1, d_i)\mathbf{c} + l_2(d_1, d_i)\mathbf{l}, d_i] + \alpha(d_1, d_i)D_1(d_i) - \frac{1}{2}b_0(d_1, d_i)D_2(d_i) \\ &= (1-i)a_1(d_1, d_i)d_{i+1} + \frac{1}{2}b_0(d_1, d_i)h_{i+\frac{1}{2}} - \frac{1}{2}b_0(d_1, d_i)h_{i+\frac{1}{2}} = (1-i)a_1(d_1, d_i)d_{i+1}.\end{aligned}$$

Summarizing, we obtain $a_0(d_0, d_i) = a_1(d_1, d_i) = 0$. Therefore, $\Delta(d_i) = 0$. $\square$

Lemma 10. Let Δ be a 2-local derivation on the mirror Heisenberg–Virasoro algebra $\mathfrak{D}$ such that $\Delta(d_i) = 0$. Then, for any $x = \sum_{t \in \mathbb{Z}} (\alpha_t d_t + \beta_t h_{t+\frac{1}{2}}) + k_1 \mathbf{c} + k_2 \mathbf{l}$, it is true that

$$\Delta(x) = \lambda_x \left(\sum_{t \in \mathbb{Z}} \beta_t h_{t+\frac{1}{2}} + 2k_2 \mathbf{l} \right)$$

Proof. For $x = \sum_{t \in \mathbb{Z}} (\alpha_t d_t + \beta_t h_{t+\frac{1}{2}}) + k_1 \mathbf{c} + k_2 \mathbf{l} \in \mathfrak{D}$, since $\Delta(d_i) = 0$, by Lemma 8, we have

$$\begin{aligned}\Delta(x) &= \Delta_{d_0, x}(x) = (\text{ad}(a_0(d_0, x)d_0 + b_0(d_0, x)h_{\frac{1}{2}} + l_1(d_0, x)\mathbf{c} + l_2(d_0, x)\mathbf{l}) + \alpha(d_0, x)D_1 - \frac{1}{2}b_0(d_0, x)D_2)(x) \\ &= a_0(d_0, x) \left(\sum_{t \in \mathbb{Z}} -t\alpha_t d_t - (t + \frac{1}{2})\beta_t h_{t+\frac{1}{2}} \right) + b_0(d_0, x) \left(\sum_{t \in \mathbb{Z}} \frac{1}{2}\alpha_t h_{t+\frac{1}{2}} + \frac{1}{2}\beta_{-1}\mathbf{l} \right) \\ &\quad + \alpha(d_0, x) \left(\sum_{t \in \mathbb{Z}} \beta_t h_{t+\frac{1}{2}} + 2k_2 \mathbf{l} \right) - \frac{1}{2}b_0(d_0, x) \left(\sum_{t \in \mathbb{Z}} \alpha_t h_{t+\frac{1}{2}} + \beta_{-1}\mathbf{l} \right) \\ &= a_0(d_0, x) \left(\sum_{t \in \mathbb{Z}} -t\alpha_t d_t - (t + \frac{1}{2})\beta_t h_{t+\frac{1}{2}} \right) + \alpha(d_0, x) \left(\sum_{t \in \mathbb{Z}} \beta_t h_{t+\frac{1}{2}} + 2k_2 \mathbf{l} \right).\end{aligned}$$

Similarly,

$$\begin{aligned}\Delta(x) &= \Delta_{d_i, x}(x) = (\text{ad}(a_i(d_i, x)d_i + b_0(d_i, x)h_{\frac{1}{2}} + l_1(d_i, x)\mathbf{c} + l_2(d_i, x)\mathbf{l}) + \alpha(d_i, x)D_1 - \frac{1}{2}b_0(d_i, x)D_2)(x) \\ &= a_i(d_i, x) \left(\sum_{t \in \mathbb{Z}} (i-t)\alpha_t d_{i+t} - (t + \frac{1}{2})\beta_t h_{i+t+\frac{1}{2}} \right) + b_0(d_i, x) \left(\sum_{t \in \mathbb{Z}} \frac{1}{2}\alpha_t h_{t+\frac{1}{2}} + \frac{1}{2}\beta_{-1}\mathbf{l} \right) \\ &\quad + \alpha(d_i, x) \left(\sum_{t \in \mathbb{Z}} \beta_t h_{t+\frac{1}{2}} + 2k_2 \mathbf{l} \right) - \frac{1}{2}b_0(d_i, x) \left(\sum_{t \in \mathbb{Z}} \alpha_t h_{t+\frac{1}{2}} + \beta_{-1}\mathbf{l} \right) - \frac{i^3 - i}{12}a_{-i}(d_i, x)\mathbf{c} \\ &= a_i(d_i, x) \left(\sum_{t \in \mathbb{Z}} (i-t)\alpha_t d_{i+t} - (t + \frac{1}{2})\beta_t h_{i+t+\frac{1}{2}} \right) + \alpha(d_i, x) \left(\sum_{t \in \mathbb{Z}} \beta_t h_{t+\frac{1}{2}} + 2k_2 \mathbf{l} \right) - \frac{i^3 - i}{12}a_{-i}(d_i, x)\mathbf{c}.\end{aligned}$$

Comparing the two equations above, when neither α_t nor β_t is equal to 0, by taking different $i \in \mathbb{Z}$, we have

$$a_0(d_0, x) = a_i(d_i, x) = 0, \quad \alpha(d_0, x) = \alpha(d_i, x).$$

If $\alpha_t = 0, \beta_t \neq 0$, by taking different $i \in \mathbb{Z}$, then we have

$$a_0(d_0, x) = a_i(d_i, x) = 0, \quad \alpha(d_0, x) = \alpha(d_i, x).$$

If $\alpha_t \neq 0, \beta_t = 0$, then we have

$$a_0(d_0, x) = a_i(d_i, x) = 0, \quad \alpha(d_0, x) = \alpha(d_i, x).$$

If $\alpha_t = \beta_t = 0$, then we have

$$\alpha(d_0, x) = \alpha(d_i, x).$$

Thus, we have $\Delta(x) = \lambda_x(\sum_{t \in \mathbb{Z}} \beta_t h_{t+\frac{1}{2}} + 2k_2 \mathbf{1})$, where $\lambda_x = \alpha(d_i, x)$ are complex-valued numbers depending on x . $\square$

Lemma 11. Let Δ be a 2-local derivation on the mirror Heisenberg–Virasoro algebra $\mathfrak{D}$ such that $\Delta(d_0) = \Delta(d_1) = 0$ and there exists $\Delta(h_{t-\frac{1}{2}}) = 0$ for fixed $t \in \mathbb{Z} \setminus \{0, 1\}$. Then, $\Delta(x) = 0$ for $x \in \mathfrak{D}$.

Proof. We proceed in two steps:

(i) We prove for $x = d_{2t} + h_{t+\frac{1}{2}}, t \in \mathbb{Z} \setminus \{0, 1\}$. Let us consider

$$\begin{aligned} \Delta_{d_{2t}+h_{t+\frac{1}{2}}, y} &= \text{ad}(a_{2t}(d_{2t} + h_{t+\frac{1}{2}}, y)(d_{2t} + h_{t+\frac{1}{2}}) + b_0(d_{2t} + h_{t+\frac{1}{2}}, y)h_{\frac{1}{2}} \\ &\quad + b_{-t}(d_{2t} + h_{t+\frac{1}{2}}, y)h_{-t+\frac{1}{2}} + l_1(d_{2t} + h_{t+\frac{1}{2}}, y)\mathbf{c} + l_2(d_{2t} + h_{t+\frac{1}{2}}, y)\mathbf{l}) \\ &\quad + (t - \frac{1}{2})b_{-t}(d_{2t} + h_{t+\frac{1}{2}}, y)D_1 - \frac{1}{2}b_0(d_{2t} + h_{t+\frac{1}{2}}, y)D_2. \end{aligned}$$

On the one hand, according to Lemma 9,

$$\Delta(d_{2t} + h_{t+\frac{1}{2}}) = \lambda_{d_{2t}+h_{t+\frac{1}{2}}} h_{t+\frac{1}{2}},$$

and on the other hand, since $\Delta(h_{t-\frac{1}{2}}) = 0$, we have

$$\begin{aligned} \Delta(d_{2t} + h_{t+\frac{1}{2}}) &= \Delta_{h_{t-\frac{1}{2}}, d_{2t}+h_{t+\frac{1}{2}}}(d_{2t} + h_{t+\frac{1}{2}}) \\ &= (\text{ad}(a_0(h_{t-\frac{1}{2}}, d_{2t} + h_{t+\frac{1}{2}})d_0 + \sum_{j \neq -t} b_j(h_{t-\frac{1}{2}}, d_{2t} + h_{t+\frac{1}{2}})h_{j+\frac{1}{2}} \\ &\quad + l_1(h_{t-\frac{1}{2}}, d_{2t} + h_{t+\frac{1}{2}})\mathbf{c} + l_2(h_{t-\frac{1}{2}}, d_{2t} + h_{t+\frac{1}{2}})\mathbf{l}) \\ &\quad + (t - \frac{1}{2})a_0(h_{t-\frac{1}{2}}, d_{2t} + h_{t+\frac{1}{2}})D_1 + \beta(h_{t-\frac{1}{2}}, d_{2t} + h_{t+\frac{1}{2}})D_2)(d_{2t} + h_{t+\frac{1}{2}}) \\ &= a_0(h_{t-\frac{1}{2}}, d_{2t} + h_{t+\frac{1}{2}})(-2td_{2t} - (t + \frac{1}{2})h_{t+\frac{1}{2}}) \\ &\quad + \sum_{j \neq -t} b_j(h_{t-\frac{1}{2}}, d_{2t} + h_{t+\frac{1}{2}})(j + \frac{1}{2})h_{2t+j+\frac{1}{2}} + (-t - \frac{1}{2})b_{-1-t}(h_{t-\frac{1}{2}}, d_{2t} + h_{t+\frac{1}{2}})\mathbf{l} \\ &\quad + (t - \frac{1}{2})a_0(h_{t-\frac{1}{2}}, d_{2t} + h_{t+\frac{1}{2}})h_{t+\frac{1}{2}} + \beta(h_{t-\frac{1}{2}}, d_{2t} + h_{t+\frac{1}{2}})(h_{2t+\frac{1}{2}} + \delta_{t,-1}\mathbf{l}). \end{aligned}$$

Comparing the two equations above, we have

$$\begin{aligned} a_0(h_{t-\frac{1}{2}}, d_{2t} + h_{t+\frac{1}{2}}) &= 0, \\ b_{-1-t}(h_{t-\frac{1}{2}}, d_{2t} + h_{t+\frac{1}{2}}) &= 0, \text{ for } t \neq -1, \\ \beta(h_{t-\frac{1}{2}}, d_{2t} + h_{t+\frac{1}{2}}) &= -\frac{1}{2}b_0(h_{t-\frac{1}{2}}, d_{2t} + h_{t+\frac{1}{2}}). \end{aligned}$$

Therefore, $\lambda_{d_{2t}+h_{t+\frac{1}{2}}} = 0$. Furthermore, we have $\Delta(d_{2t} + h_{t+\frac{1}{2}}) = 0$.

By Lemma 3, set

$$\begin{aligned}
\Delta_{d_{2t}+h_{t+\frac{1}{2}},y} &= \text{ad}\left(\sum_{i \in \mathbb{Z}} a_i(d_{2t}+h_{t+\frac{1}{2}},y)d_i + b_i(d_{2t}+h_{t+\frac{1}{2}},y)h_{i+\frac{1}{2}} + l_1(d_{2t}+h_{t+\frac{1}{2}},y)\mathbf{c}\right. \\
&\quad \left.+ l_2(d_{2t}+h_{t+\frac{1}{2}},y)\mathbf{1}\right) + \alpha(d_{2t}+h_{t+\frac{1}{2}},y)D_1 + \beta(d_{2t}+h_{t+\frac{1}{2}},y)D_2 \\
\Delta(d_{2t}+h_{t+\frac{1}{2}}) &= \Delta_{d_{2t}+h_{t+\frac{1}{2}},y}(d_{2t}+h_{t+\frac{1}{2}}) \\
&= \left(\text{ad}\left(\sum_{i \in \mathbb{Z}} a_i(d_{2t}+h_{t+\frac{1}{2}},y)d_i + b_i(d_{2t}+h_{t+\frac{1}{2}},y)h_{i+\frac{1}{2}} + l_1(d_{2t}+h_{t+\frac{1}{2}},y)\mathbf{c}\right.\right. \\
&\quad \left.\left.+ l_2(d_{2t}+h_{t+\frac{1}{2}},y)\mathbf{1}\right) + \alpha(d_{2t}+h_{t+\frac{1}{2}},y)D_1 + \beta(d_{2t}+h_{t+\frac{1}{2}},y)D_2\right)(d_{2t}+h_{t+\frac{1}{2}}) \\
&= \sum_{i \in \mathbb{Z}} a_i(d_{2t}+h_{t+\frac{1}{2}},y)\left((i-2t)d_{2t+i} - \left(t+\frac{1}{2}\right)h_{i+t+\frac{1}{2}}\right) - \frac{4t^3-t}{6}a_{-2t}(d_{2t}+h_{t+\frac{1}{2}},y)\mathbf{c} \\
&\quad + \sum_{i \in \mathbb{Z}} b_i(d_{2t}+h_{t+\frac{1}{2}},y)\left(i+\frac{1}{2}\right)h_{2t+i+\frac{1}{2}} + \left(-t-\frac{1}{2}\right)b_{-1-t}(d_{2t}+h_{t+\frac{1}{2}},y)\mathbf{1} \\
&\quad + \alpha(h_{d_{2t}+h_{t+\frac{1}{2}}},y)h_{t+\frac{1}{2}} + \beta(d_{2t}+h_{t+\frac{1}{2}},y)(h_{2t+\frac{1}{2}} + \delta_{t,-1}\mathbf{1}) = 0.
\end{aligned}$$

So, we have

$$\begin{aligned}
a_i(d_{2t}+h_{t+\frac{1}{2}}) &= 0, \text{ for } i \neq 2t, \alpha(h_{d_{2t}+h_{t+\frac{1}{2}}},y) = \left(t-\frac{1}{2}\right)b_{-t}(d_{2t}+h_{t+\frac{1}{2}}), \\
\beta(d_{2t}+h_{t+\frac{1}{2}},y) &= -\frac{1}{2}b_0(d_{2t}+h_{t+\frac{1}{2}},y), a_{2t}(d_{2t}+h_{t+\frac{1}{2}},y) = b_t(d_{2t}+h_{t+\frac{1}{2}},y), \\
b_i(d_{2t}+h_{t+\frac{1}{2}},y) &= 0, \text{ for } i \neq 0, t, -t.
\end{aligned}$$

(ii) We now prove that $\Delta(x) = 0$ for any $x \in \mathfrak{D}$. If $x = \sum_{k \in \mathbb{Z}} \alpha_k d_k + \beta_k h_{k+\frac{1}{2}} + k_1 \mathbf{c} + k_2 \mathbf{1} \in \mathfrak{D}$, since $\Delta(d_0) = \Delta(d_1) = 0$, by Lemma 9, then $\Delta(d_i) = 0$, $i \in \mathbb{Z}$. According to Lemma 10, we obtain

$$\Delta(x) = \lambda_x \left(\sum_{k \in \mathbb{Z}} \beta_k h_{k+\frac{1}{2}} + 2k_2 \mathbf{1} \right)$$

and

$$\begin{aligned}
\Delta(x) &= \Delta_{d_{2t}+h_{t+\frac{1}{2}},x}(x) = \left(\text{ad}\left(a_{2t}(d_{2t}+h_{t+\frac{1}{2}},x)(d_{2t}+h_{t+\frac{1}{2}}) + b_0(d_{2t}+h_{t+\frac{1}{2}},x)h_{\frac{1}{2}} + b_{-t}(d_{2t}+h_{t+\frac{1}{2}},x)h_{-t+\frac{1}{2}}\right.\right. \\
&\quad \left.\left.+ l_1(d_{2t}+h_{t+\frac{1}{2}},x)\mathbf{c} + l_2(d_{2t}+h_{t+\frac{1}{2}},x)\mathbf{1}\right) + \left(t-\frac{1}{2}\right)b_{-t}(d_{2t}+h_{t+\frac{1}{2}},x)D_1 - \frac{1}{2}b_0(d_{2t}+h_{t+\frac{1}{2}},x)D_2\right)(x) = \\
&= a_{2t}(d_{2t}+h_{t+\frac{1}{2}},x)\left(\sum_{k \in \mathbb{Z}} (2t-k)\alpha_k d_{2t+k} - \frac{(2k+1)\beta_k}{2}h_{2t+k+\frac{1}{2}} + \frac{(2t+1)\alpha_k}{2}h_{k+t+\frac{1}{2}} + \frac{(2t+1)\beta_{-1-t}}{2}\mathbf{1} + \frac{(4t^3-t)\alpha_{-2t}}{6}\mathbf{c}\right) + \\
&\quad b_0(d_{2t}+h_{t+\frac{1}{2}},x)\left(\sum_{k \in \mathbb{Z}} \frac{1}{2}\alpha_k h_{k+\frac{1}{2}} + \frac{1}{2}\beta_{-1}\mathbf{1}\right) + b_{-t}(d_{2t}+h_{t+\frac{1}{2}},x)\left(\sum_{k \in \mathbb{Z}} \left(-t+\frac{1}{2}\right)\alpha_k h_{k-t+\frac{1}{2}} + \beta_{t-1}\mathbf{1}\right) + \\
&\quad \left(t-\frac{1}{2}\right)b_{-t}(d_{2t}+h_{t+\frac{1}{2}},x)\left(\sum_{k \in \mathbb{Z}} \beta_k h_{k+\frac{1}{2}} + 2k_2\mathbf{1}\right) - \frac{1}{2}b_0(d_{2t}+h_{t+\frac{1}{2}},x)\left(\sum_{k \in \mathbb{Z}} \alpha_k h_{k+\frac{1}{2}} + \beta_{-1}\mathbf{1}\right) = \\
&= a_{2t}(d_{2t}+h_{t+\frac{1}{2}},x)\left(\sum_{k \in \mathbb{Z}} (2t-k)\alpha_k d_{2t+k} - \left(k+\frac{1}{2}\right)\beta_k h_{2t+k+\frac{1}{2}} + \left(t+\frac{1}{2}\right)\alpha_k h_{k+t+\frac{1}{2}} + \beta_{-1-t}\left(t+\frac{1}{2}\right)\mathbf{1} + \alpha_{-2t}\frac{4t^3-t}{6}\mathbf{c}\right) + \\
&\quad b_{-t}(d_{2t}+h_{t+\frac{1}{2}},x)\left(\sum_{k \in \mathbb{Z}} \left(-t+\frac{1}{2}\right)\alpha_k h_{k-t+\frac{1}{2}} + \beta_{t-1}\mathbf{1}\right) + \left(t-\frac{1}{2}\right)b_{-t}(d_{2t}+h_{t+\frac{1}{2}},x)\left(\sum_{k \in \mathbb{Z}} \beta_k h_{k+\frac{1}{2}} + 2k_2\mathbf{1}\right).
\end{aligned}$$

Comparing the two equations above, since $t \neq 0$, we have

$$a_{2t}(d_{2t}+h_{t+\frac{1}{2}},x) = b_{-t}(d_{2t}+h_{t+\frac{1}{2}},x) = 0.$$

Therefore, $\lambda_x = 0$ and we have $\Delta(x) = 0$. $\square$

Theorem 3. Every 2-local derivation on the mirror Heisenberg–Virasoro algebra $\mathfrak{D}$ is a derivation.

Proof. Suppose Δ is a 2-local derivation on $\mathfrak{D}$. Then, there exists a derivation Δ_{d_0, d_1} such that

$$\Delta(d_0) = \Delta_{d_0, d_1}(d_0), \Delta(d_1) = \Delta_{d_0, d_1}(d_1).$$

Let $\Delta^* = \Delta - \Delta_{d_0, d_1}$. Then, $\Delta^*(d_0) = \Delta^*(d_1) = 0$. By Lemma 9, we have $\Delta^*(d_i) = 0$, $i = 1, 2$, and by Lemma 10, for fixed $t \in \mathbb{Z} \setminus \{0, 1\}$, we have

$$\Delta^*(h_{t-\frac{1}{2}}) = \lambda_{h_{t-\frac{1}{2}}} h_{t-\frac{1}{2}}.$$

Let $\Delta^{**} = \Delta^* - \lambda_{h_{t-\frac{1}{2}}} D_1$. Then, we have

$$\Delta^{**}(d_0) = \Delta^* d_0 - \lambda_{h_{t-\frac{1}{2}}} D_1(d_0) = 0,$$

$$\Delta^{**}(d_1) = \Delta^* d_1 - \lambda_{h_{t-\frac{1}{2}}} D_1(d_1) = 0,$$

$$\Delta^{**}(h_{t-\frac{1}{2}}) = \Delta^* h_{t-\frac{1}{2}} - \lambda_{h_{t-\frac{1}{2}}} D_1(h_{t-\frac{1}{2}}) = 0.$$

According to Lemma 11, $\Delta^{**} = 0$, that is, $\Delta = \Delta_{d_0, d_1} + \lambda_{h_{t-\frac{1}{2}}} D_1$. Therefore, Δ is a derivation. $\square$

4. $\frac{1}{2}$ -Derivations on the Mirror Heisenberg–Virasoro Algebra

The present section is dedicated to studying $\frac{1}{2}$ -derivations of the mirror Heisenberg–Virasoro algebra and some corollaries about transposed Poisson structures and local $\frac{1}{2}$ -derivations.

Let $(L, [\cdot, \cdot])$ be an algebra with multiplication $[\cdot, \cdot]$ and φ be a linear map. Then, φ is a $\frac{1}{2}$ -derivation if it satisfies

$$\varphi([x, y]) = \frac{1}{2}([\varphi(x), y] + [x, \varphi(y)]).$$

The main example of $\frac{1}{2}$ -derivations is the multiplication by an element from the ground field. Let us call such $\frac{1}{2}$ -derivations as *trivial $\frac{1}{2}$ -derivations*. As it follows from the following theorem, we are not interested in trivial $\frac{1}{2}$ -derivations.

Theorem 4. *There are no non-trivial $\frac{1}{2}$ -derivations of the mirror Heisenberg–Virasoro algebra $\mathfrak{D}$.*

Proof. Let us remember that $\mathfrak{D}$ is $\mathbb{Z}_2$ -grading. So, $\mathfrak{D}_0$ is generated by $\{d_m, c, 1\}$, and $\mathfrak{D}_1$ is generated by $\{h_{r+\frac{1}{2}}\}$. Let φ be a $\frac{1}{2}$ -derivation of $\mathfrak{D}$. Then, the even part φ_0 and the odd part φ_1 of φ are, respectively, an even $\frac{1}{2}$ -derivation and an odd $\frac{1}{2}$ -derivation of $\mathfrak{D}$.

It is easy to see that

- if $m \neq -n$, then $(m-n)\varphi_0(d_{m+n}) = \varphi_0([d_m, d_n]) = \frac{1}{2}([\varphi_0(d_m), d_n] + [d_m, \varphi_0(d_n)]) \subseteq \mathcal{V}$.
- $2n\varphi_0(d_0) + \frac{n^3-n}{12}\varphi_0(c) = \varphi_0([d_n, d_{-n}]) = \frac{1}{2}([\varphi_0(d_n), d_{-n}] + [d_n, \varphi_0(d_{-n})]) \subseteq \mathcal{V}$; hence, $\varphi_0(d_0), \varphi_0(c) \subseteq \mathcal{V}$.

This observation shows that $\varphi_0(\mathcal{V}) \subseteq \mathcal{V}$, and thanks to Theorem 26 [17], $\varphi_0|_{\mathcal{V}}$ is trivial, i.e., $\varphi_0(v) = \alpha^* v$ for each $v \in \mathcal{V}$. We also can suppose that $\alpha^* = 0$.

Let us say $\varphi_0(h_{n+\frac{1}{2}}) = \sum \beta_k^n h_{n+\frac{1}{2}+k}$; then,

$$\sum \beta_k^n h_{n+\frac{1}{2}+k} = -\frac{2}{2n+1} \varphi([d_0, h_{n+\frac{1}{2}}]) = -\frac{1}{2n+1} [d_0, \varphi_0(h_{n+\frac{1}{2}})] = \frac{n+\frac{1}{2}+k}{2n+1} \sum \beta_k^n h_{n+\frac{1}{2}+k},$$

which gives $\varphi_0 = 0$ and is trivial.

It is known the commutator of a $\frac{1}{2}$ -derivation and one derivation gives a new $\frac{1}{2}$ -derivation. Now, let ϕ_x be an inner odd derivation of $\mathfrak{D}$ (left multiplication of $x \in \mathfrak{D}_1$).

Then, $\varphi_1\phi_x - \varphi_1\phi_x$ is an even $\frac{1}{2}$ -derivation of $\mathfrak{D}$, which is trivial. Then, if $\varphi_1(\mathbf{c}) = \sum \gamma_k h_{k+\frac{1}{2}}$, we have

$$\alpha_r \mathbf{c} = (\varphi_1\phi_{h_{r+\frac{1}{2}}} - \varphi_1\phi_{h_{r+\frac{1}{2}}})(\mathbf{c}) = \varphi_1([h_{r+\frac{1}{2}}, \mathbf{c}]) - [h_{r+\frac{1}{2}}, \varphi_1(\mathbf{c})] = -[h_{r+\frac{1}{2}}, \sum \gamma_k h_{k+\frac{1}{2}}] \subseteq \langle \mathbf{1} \rangle,$$

which gives $\alpha_r = 0$ and $\varphi_1\phi_{h_{r+\frac{1}{2}}} = \phi_{h_{r+\frac{1}{2}}}\varphi_1$. Hence,

$$0 = \varphi_1([h_{r+\frac{1}{2}}, x]) - [h_{r+\frac{1}{2}}, \varphi_1(x)] = \frac{1}{2}([\varphi_1(h_{r+\frac{1}{2}}), x] + [h_{r+\frac{1}{2}}, \varphi_1(x)]) - [h_r, \varphi_1(x)] = \frac{1}{2}([\varphi_1(h_{r+\frac{1}{2}}), x] - [h_{r+\frac{1}{2}}, \varphi_1(x)]),$$

which gives $[\varphi_1(h_{r+\frac{1}{2}}), x] = [h_{r+\frac{1}{2}}, \varphi_1(x)]$ for every $x \in \mathfrak{D}$. Hence, if $x \in \mathfrak{D}_0$, then

$$\varphi_1(x) \in \mathfrak{D}_1 \text{ and } \varphi_1(h_{r+\frac{1}{2}}) \subseteq \langle \mathbf{c}, \mathbf{1} \rangle, \varphi_1(\mathfrak{D}_0) = 0.$$

From the last observation, we have

$$\varphi_1(h_{r+\frac{1}{2}}) = -2\varphi_1([d_r, h_{\frac{1}{2}}]) = -([\varphi_1(d_r), h_{\frac{1}{2}}] + [d_r, \varphi_1(h_{\frac{1}{2}})]) = 0.$$

The last observation shows that $\varphi_1 = 0$ and φ is trivial. $\square$

Let $\mathfrak{L}$ be a vector space equipped with two nonzero bilinear operations $\cdot$ and $[\cdot, \cdot]$. The triple $(\mathfrak{L}, \cdot, [\cdot, \cdot])$ is called a *transposed Poisson algebra* if $(\mathfrak{L}, \cdot)$ is a commutative associative algebra and $(\mathfrak{L}, [\cdot, \cdot])$ is a Lie algebra that satisfies the following compatibility condition

$$2z \cdot [x, y] = [z \cdot x, y] + [x, z \cdot y].$$

A transposed Poisson structure on a Lie algebra $(L, [\cdot, \cdot])$ is a commutative associative operation $\cdot$ on $\mathfrak{L}$, which makes $(L, \cdot, [\cdot, \cdot])$ a transposed Poisson algebra. The notion of transposed Poisson algebras was introduced in [18] (see also Section 7.3 [19] and [20] for recent results in this topic). Summarizing results from Theorem 4 and Theorem 26 [17], we have the following corollary.

Corollary 1. *There are no transposed Poisson structures defined on the mirror Heisenberg–Virasoro algebra.*

Following the ideas of local and 2-local maps, we define local and 2-local $\frac{1}{2}$ -derivations of an algebra L , namely as follows:

- A linear map ∇ is called a local $\frac{1}{2}$ -derivation of L if, for any element $x \in L$, there exists a $\frac{1}{2}$ -derivation $\varphi_x : L \rightarrow L$ such that $\nabla(x) = \varphi_x(x)$.
- A (not necessary linear) map Δ is called a 2-local derivation of L if, for any two elements $x, y \in L$, there exists an $\frac{1}{2}$ -derivation $\varphi_{x,y} : L \rightarrow L$ such that $\Delta(x) = \varphi_{x,y}(x)$, $\Delta(y) = \varphi_{x,y}(y)$.

It is easy to see the following corollary from Theorem 4.

Corollary 2. *Every local (and 2-local) $\frac{1}{2}$ -derivation structures on the mirror Heisenberg–Virasoro algebra is a $\frac{1}{2}$ -derivation.*

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References

1. Barron, K. On twisted modules for $N = 2$ supersymmetric vertex operator superalgebras. In *Lie Theory and Its Applications in Physics: IX International Workshop*; Springer: Tokyo, Japan, 2013; pp. 411–420.
2. Gao, D.; Ma, Y.; Zhao, K. Non-weight modules over the mirror Heisenberg-Virasoro algebra. *Sci. China Math.* **2022**, *65*, 2243–2254. [\[CrossRef\]](#)
3. Gao, D.F.; Zhao, K.M. Tensor product weight modules for the mirror Heisenberg-Virasoro algebra. *J. Pure Appl. Algebra* **2022**, *226*, 106929. [\[CrossRef\]](#)
4. Liu, D.; Pei, Y.F.; Xia, L.M.; Zhao, K.M. Irreducible modules over the mirror Heisenberg-Virasoro algebra. *Commun. Contem. Math.* **2022**, *24*, 2150026. [\[CrossRef\]](#)
5. Ayupov, S.; Arzikulov, F. 2-Local derivations on associative and Jordan matrix rings over commutative rings. *Linear Algebra Appl.* **2017**, *522*, 28–50. [\[CrossRef\]](#)
6. Ayupov, S.; Kudaybergenov, K. Local derivations on finite-dimensional Lie algebras. *Linear Algebra Appl.* **2016**, *493*, 381–398. [\[CrossRef\]](#)
7. Ayupov, S.; Elduque, A.; Kudaybergenov, K. Local derivations and automorphisms of Cayley algebras. *J. Pure Appl. Algebra* **2023**, *227*, 107277. [\[CrossRef\]](#)
8. Arzikulov, F.; Umrzaqov, N. Conservative algebras of 2-dimensional algebras, III. *Comm. Math.* **2021**, *29*, 255–267. [\[CrossRef\]](#)
9. Buzaglo, L. Derivations, extensions, and rigidity of subalgebras of the Witt algebra. *J. Algebra* **2024**. [\[CrossRef\]](#)
10. Ferreira, B.L.M.; Kaygorodov, I.; Kudaybergenov, K. Local and 2-local derivations of simple n -ary algebras. *Ric. Mat.* **2021**, 1–10. [\[CrossRef\]](#)
11. Kaygorodov, I.; Kudaybergenov, K.; Yuldashev, I. Local derivations of semisimple Leibniz algebras. *Comm. Math.* **2022**, *30*, 1–12. [\[CrossRef\]](#)
12. Ran, S.; Jiang, C.P. The derivation algebra and automorphism group of the twisted Heisenberg-Virasoro algebra. *Commun. Algebra* **2006**, *34*, 2547–2558.
13. Zhao, Y.; Cheng, S. 2-local derivations on the twisted Heisenberg-Virasoro algebra. *arXiv* **2021**, arXiv:2103.04228.
14. Zhao, Y.; Cheng, S. The derivation algebra and automorphism group of the mirror Heisenberg-Virasoro Algebra. *Math. Theory Appl.* **2022**, *42*, 36.
15. Tan, H.J.; Yao, Y.F.; Zhao, K.M. Simple restricted modules over the Heisenberg-Virasoro algebra as VOA modules. *arXiv* **2021**, arXiv:2110.05714.
16. Ayupov, S.; Yusupov, B. 2-local derivations on Virasoro algebras. *Bull. Natl. Univ. Uzb. Math. Nat. Sci.* **2019**, *2*, 217–230. [\[CrossRef\]](#)
17. Ferreira, B.L.M.; Kaygorodov, I.; Lopatkin, V. $\frac{1}{2}$ -derivations of Lie algebras and transposed Poisson algebras. *Rev. Real Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.* **2021**, *115*, 142. [\[CrossRef\]](#)
18. Bai, C.; Bai, R.; Guo, L.; Wu, Y. Transposed Poisson algebras, Novikov-Poisson algebras, and 3-Lie algebras. *J. Algebra* **2023**, *632*, 535–566. [\[CrossRef\]](#)
19. Kaygorodov, I. Non-associative algebraic structures: Classification and structure. *Comm. Math.* **2024**, *32*, 1–62. [\[CrossRef\]](#)
20. Sartayev, B. Some generalizations of the variety of transposed Poisson algebras. *Comm. Math.* **2024**, *32*, 55–62. [\[CrossRef\]](#)

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