mathematics

Article

Hyers–Ulam Stability of Caputo Fractional Stochastic Delay Differential Systems with Poisson Jumps

Zhenyu Bai [1] and Chuanzhi Bai [2,*]

[1] XJTLU Wisdom Lake Academy of Pharmacy, Xi'an Jiaotong-Liverpool University, Suzhou 215123, China; zhenyu.bai22@student.xjtlu.edu.cn
[2] Department of Mathematics, Huaiyin Normal University, Huaian 223300, China
* Correspondence: czbai@hytc.edu.cn

Abstract: In this paper, we explore the stability of a new class of Caputo-type fractional stochastic delay differential systems with Poisson jumps. We prove the Hyers–Ulam stability of the solution by utilizing a version of fixed point theorem, fractional calculus, Cauchy–Schwartz inequality, Jensen inequality, and some stochastic analysis techniques. Finally, an example is provided to illustrate the effectiveness of the results.

Keywords: stochastic fractional delay differential systems; Hyers–Ulam stability; fixed point theorem; stochastic calculus

MSC: 34A08; 34D20; 60H10

Citation: Bai, Z.; Bai, C. Hyers–Ulam Stability of Caputo Fractional Stochastic Delay Differential Systems with Poisson Jumps. *Mathematics* **2024**, *12*, 804. https://doi.org/10.3390/math12060804

Academic Editors: Faïçal Ndaïrou and Delfim F. M. Torres

Received: 5 February 2024
Revised: 28 February 2024
Accepted: 5 March 2024
Published: 8 March 2024

1. Introduction

In 1941, Hyers [1] gave the first positive answer to the question on the stability of group homomorphisms proposed by Ulam in 1940 [2]. Since then, the theory of Hyers–Ulam stability (HUS) has been gradually developed (see [3–8]). The theory of HUS opened a new research line in stability analysis.

In the past four decades, fractional differential equations have become more popular and important because they are more accurate and convenient than integer-order differential equations. Stability is a basic problem of fractional differential equations (FDEs). For recent results on the HUS of FDEs, we refer the reader to some works (see [9–15]).

Fractional stochastic differential equations can be used to model systems with memory and randomness, such as biological systems with fractional-order kinetics and stochastic effects, anomalous diffusion processes, etc. This provides a powerful framework for predicting the behavior of complex systems with memory and randomness. Recently, some authors extended the HUS problem from fractional differential equations (FDEs) to stochastic fractional differential equations. In [16], Ahmadova and Mahmudov established stability results in the Hyers–Ulam sense for nonlinear fractional stochastic neutral differential equations. Guo et al. [17] investigated the existence and Hyers–Ulam stability of solutions for impulsive Riemann–Liouville fractional stochastic differential equations with infinite delay. Mchiri et al. [18] investigated the Hyers–Ulam stability of a class of pantograph fractional stochastic differential equations. Very recently, Kahouli et al. [19] studied the Hyers–Ulam stability of a neutral fractional stochastic differential equation:

$$^{C}D^{\varpi_2}\zeta(\varrho) - {}^{C}D^{\varpi_1}h(\varrho,\zeta(\varrho)) = f(\varrho,\zeta(\varrho)) + g(\varrho,\zeta(\varrho))\frac{dW(\varrho)}{d\varrho}, \quad (0 \le \varrho \le \alpha), \qquad (1)$$

where $^{C}D^{\varpi_2}\zeta$ is the Caputo fractional derivative of order ϖ_2 of function ζ, initial condition $\zeta(0) = \omega$, $0 < \varpi_1 < \frac{1}{2}$ and $\frac{1}{2} + \varpi_1 < \varpi_2 < 1$.

Recently, Liu et al. [20] gave the exact solutions of a class of fractional delay differential equations. Li and Wang in [21] studied the existence and uniqueness of a class of Caputo fractional stochastic delay differential systems (FSDDSs). In [22], we extended the main results of [21]. Up to now, to the best of our knowledge, the HUS of solutions for fractional stochastic delay differential systems (FSDDSs) has not been investigated. Motivated by [19–22], in the present paper, we study the Hyers–Ulam stability for the following Caputo FSDDSs with Poisson jumps:

$$\begin{cases} ({}^{C}D_{0}^{\alpha}x)(t) = Ax(t) + Bx(t-\tau) + f(t, x(t), x(t-\tau)) + \sigma(t, x(t), x(t-\tau))\frac{dW(t)}{dt} \\ \qquad\qquad + \int_V g(t, x(t), x(t-\tau), v)\bar{N}(dt, dv), \quad t \in J, \\ x(t) = \phi(t), \quad x'(t) = \phi'(t), \quad -\tau \leq t \leq 0, \end{cases} \tag{2}$$

where ${}^{C}D_{0}^{\alpha}$ is the left Caputo fractional derivative with $1 < \alpha < 2$; $J = [0, T]$; $\tau > 0$ is a fixed delay time; $T = N\tau$ for a fixed $N \in \{1, 2, \ldots\}$; $A, B \in \mathbb{R}^{n\times n}$ are two constant matrices; the state vector $x \in \mathbb{R}^n$ is a stochastic process; $f : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ $\sigma : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$, and $g : J \times \mathbb{R}^n \times \mathbb{R}^n \times V \to \mathbb{R}^n$ are measurable continuous functions; and ϕ is an arbitrary twice continuously differentiable vector function that determines initial conditions. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space equipped with some filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual condition; $W(t)$ is an m-dimensional Brownian motion on the probability space $(\Omega, \mathcal{F}, P)$ adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$. Let $(V, \Phi, \lambda(dv))$ be a σ-finite measurable space. Given the stationary Poisson point process $(p_t)_{t\geq 0}$, which is defined on $(\Omega, \mathcal{F}, P)$ with values in V and with characteristic measure λ, we denote by $N(t, dv)$ the counting measure of p_t such that $\bar{N}(t, \Theta) := \mathbb{E}(N(t, \Theta)) = t\lambda(\Theta)$ for $\Theta \in \Phi$. Define $\bar{N}(t, dv) := N(t, dv) - t\lambda(dv)$ and the Poisson martingale measure generated by p_t.

The main contributions and highlights of this paper are as follows:

(i) With the aid of weighted distance, Itô's isometry formula, stochastic inequality, Cauchy–Schwartz inequality, and Banach fixed point theorem, the existence, uniqueness, and Hyers–Ulam stability of solutions for Caputo FSDDSs (2) are obtained.
(ii) The fractional calculus and stochastic calculus are effectively used to establish our results.
(iii) Our work in this paper is novel and more technical.

This paper is organized as follows. In Section 2, we give some definitions and preliminaries. In Section 3, we prove the existence, uniqueness, and HUS of solutions for Caputo FSDDSs (2) with Poisson jumps. In Section 4, an example is presented to illustrate our theoretical results. Finally, the paper is concluded in Section 5.

2. Preliminaries

Let $\Upsilon = \mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ denote the space of all $\mathcal{F}(t)$ measurable, mean square integrable functions $x : \Omega \to \mathbb{R}^n$ with $\|x(t)\|_{ms} := \sqrt{\sum_{i=1}^{n} \mathbb{E}(|x_i(t)|^2)}$, and $\|x\| = \sqrt{\sum_{i=1}^{n} |x_i|^2}$ and $\|A\| = \max_{1\leq j\leq n} \sum_{i=1}^{n} |a_{ij}|$ be the vector norm and matrix norm, respectively. A process $x : [-\tau, T] \to \mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ is said to be $\mathcal{F}(t)$-adapted if $x(t) \in \Upsilon$.

Definition 1 ([23]). *Let $\alpha > 0$ and f be an integrable function defined on $[a, b]$. The left Riemann–Liouville fractional integral operator of order α of a function f is defined by*

$${}_{a}I_{t}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \quad t > a. \tag{3}$$

Definition 2 ([23]). *Let $n - 1 < \alpha < n$ and $f \in C^n([a, b])$. The left Caputo fractional derivative of order α of a function f is defined by*

$${}_{a}^{C}D_{t}^{\alpha} f(t) = ({}_{a}I_{t}^{n-\alpha} f^{(n)})(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds, \quad t > a, \tag{4}$$

where $n = [\alpha] + 1$.

Definition 3 ([24])**.** *The coefficient matrices* $Q_k(s)$, $k = 0, 1, 2, \ldots$ *satisfy the following multivariate determining matrix equations:*

$$Q_0(s) = Q_k(-\tau) = \Theta, \quad Q_1(0) = E, \quad k = 0, 1, 2, \cdots, \quad s = 0, \tau, 2\tau, \cdots,$$

$$Q_{k+1}(s) = AQ_k(s) + BQ_k(s - \tau), \quad k = 0, 1, 2, \cdots, \quad s = 0, \tau, 2\tau, \cdots,$$

where E *is an identity matrix and* Θ *is a zero matrix.*

Definition 4 ([20])**.** *The matrix function* $C^{A,B}_{\tau,\alpha}(t) : \mathbb{R} \to \mathbb{R}^n$*, defined by*

$$C^{A,B}_{\tau,\alpha}(t) := \begin{cases} \Theta, & t \in [-\tau, 0), \\ E, & t = 0, \\ E + \sum\limits_{i=1}^{\infty} \frac{Q_{i+1}(0)}{\Gamma(i\alpha+1)} t^{i\alpha}, & 0 < t \leq \tau, \\ \sum\limits_{i=0}^{\infty} \frac{Q_{i+1}(0)}{\Gamma(i\alpha+1)} t^{i\alpha} + \sum\limits_{i=1}^{\infty} \frac{Q_{i+1}(\tau)}{\Gamma(i\alpha+1)} (t-\tau)^{i\alpha} & \\ \quad + \cdots + \sum\limits_{i=p}^{\infty} \frac{Q_{i+1}(p\tau)}{\Gamma(i\alpha+1)} (t - p\tau)^{i\alpha}, & p\tau < t \leq (p+1)\tau. \end{cases} \tag{5}$$

is called the generalized cosine-type delay Mittag–Leffler matrix function, where $p = 0, 1, \ldots$.

Definition 5 ([20])**.** *The matrix function* $S^{A,B}_{\tau,\alpha}(t) : \mathbb{R} \to \mathbb{R}^n$*, defined by*

$$S^{A,B}_{\tau,\alpha}(t) := \begin{cases} \Theta, & t \in [-\tau, 0), \\ tE, & t = 0, \\ tE + \sum\limits_{i=1}^{\infty} \frac{Q_{i+1}(0)}{\Gamma(i\alpha+2)} t^{i\alpha+1}, & 0 < t \leq \tau, \\ \sum\limits_{i=0}^{\infty} \frac{Q_{i+1}(0)}{\Gamma(i\alpha+2)} t^{i\alpha+1} + \sum\limits_{i=1}^{\infty} \frac{Q_{i+1}(\tau)}{\Gamma(i\alpha+2)} (t-\tau)^{i\alpha+1} & \\ \quad + \cdots + \sum\limits_{i=p}^{\infty} \frac{Q_{i+1}(p\tau)}{\Gamma(i\alpha+2)} (t - p\tau)^{i\alpha+1}, & p\tau < t \leq (p+1)\tau, \end{cases} \tag{6}$$

is called the generalized sine-type delay Mittag–Leffler matrix function, where $p = 0, 1, \ldots$.

From Theorem 1 in [20], we can easily obtain the following definition:

Definition 6. *An* $\mathbb{R}^n$*-value stochastic process* $\{x(t) : -\tau \leq t \leq T\}$ *is called a solution of (2) if* $x(t)$ *satisfies the integral equation of the following form:*

$$x(t) = \begin{cases} C^{A,B}_{\tau,\alpha}(t+\tau)\phi(-\tau) + S^{A,B}_{\tau,\alpha}(t+\tau)\phi'(-\tau) \\ \quad + \int_{-\tau}^{0} S^{A,B}_{\tau,\alpha}(t-s)[\phi'' - A(D^{2-\alpha}_{-\tau^+}\phi)(s)]ds \\ \quad + \int_0^t D_0^{2-\alpha} S^{A,B}_{\tau,\alpha}(t-s) f(s, x(s), x(s-\tau))ds \\ \quad + \int_0^t D_0^{2-\alpha} S^{A,B}_{\tau,\alpha}(t-s)\sigma(s, x(s), x(s-\tau))dW(s) \\ \quad + \int_0^t D_0^{2-\alpha} S^{A,B}_{\tau,\alpha}(t-s) \int_V g(s, x(s), x(s-\tau), v)\tilde{N}(ds, dv), \quad t \in J, \\ \phi(t), \quad t \in [-\tau, 0], \end{cases} \tag{7}$$

where $x(t)$ *is* $\mathcal{F}(t)$*-adapted and* $\mathbb{E}(\int_{-\tau}^{T} \|x(t)\|^2 dt) < \infty$.

Lemma 1. *For any* $t \geq 0$, $1 < \alpha < 2$*, we have*

$$\|C^{A,B}_{\tau,\alpha}(t)\| \leq E_{\alpha,1}((\|A\| + \|B\|)t^{\alpha}), \tag{8}$$

where $E_{\alpha,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}$, $z \in \mathbb{R}$ *is the Mittag–Leffler function.*

Proof. For $p\tau < t \le (p+1)\tau$, $p = 0, 1, \ldots$, by (5) and Definition 3, one has

$$\begin{aligned}
\|C_{\tau,\alpha}^{A,B}(t)\| &\le \sum_{i=0}^{\infty} \frac{\|A\|^i}{\Gamma(i\alpha+1)} t^{i\alpha} + \sum_{i=1}^{\infty} \begin{pmatrix} i \\ 1 \end{pmatrix} \frac{\|A\|^{i-1}\|B\|}{\Gamma(i\alpha+1)} (t-\tau)^{i\alpha} + \cdots \\
&\quad + \sum_{i=p}^{\infty} \begin{pmatrix} i \\ p \end{pmatrix} \frac{\|A\|^{i-p}\|B\|^p}{\Gamma(i\alpha+1)} (t-p\tau)^{i\alpha} \\
&\le \sum_{i=0}^{\infty} \frac{\|A\|^i}{\Gamma(i\alpha+1)} t^{i\alpha} + \sum_{i=1}^{\infty} \begin{pmatrix} i \\ 1 \end{pmatrix} \frac{\|A\|^{i-1}\|B\|}{\Gamma(i\alpha+1)} t^{i\alpha} + \cdots + \sum_{i=p}^{\infty} \begin{pmatrix} i \\ p \end{pmatrix} \frac{\|A\|^{i-p}\|B\|^p}{\Gamma(i\alpha+1)} t^{i\alpha} \\
&= 1 + \frac{\|A\| + \|B\|}{\Gamma(\alpha+1)} t^{\alpha} + \frac{\|A\|^2 + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \|A\|\|B\| + \|B\|^2}{\Gamma(2\alpha+1)} t^{2\alpha} + \cdots \\
&\quad + \frac{\|A\|^p + \begin{pmatrix} p \\ 1 \end{pmatrix} \|A\|^{p-1}\|B\| + \cdots + \|B\|^p}{\Gamma(p\alpha+1)} t^{p\alpha} \\
&\le \sum_{i=0}^{\infty} \frac{(\|A\| + \|B\|)^i}{\Gamma(i\alpha+1)} t^{i\alpha} = E_{\alpha,1}((\|A\| + \|B\|)t^{\alpha}).
\end{aligned}$$

□

Lemma 2. *For any* $t \ge 0$, $1 < \alpha < 2$, *we have*

$$\|S_{\tau,\alpha}^{A,B}(t)\| \le tE_{\alpha,2}((\|A\| + \|B\|)t^{\alpha}), \tag{9}$$

where $E_{\alpha,2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+2)}$, $z \in \mathbb{R}$ *is the Mittag–Leffler function.*

Proof. For $p\tau < t \le (p+1)\tau$, $p = 0, 1, \ldots$, by (6) and Definition 3, we have

$$\begin{aligned}
\|S_{\tau,\alpha}^{A,B}(t)\| &\le \sum_{i=0}^{\infty} \frac{\|A\|^i}{\Gamma(i\alpha+2)} t^{i\alpha+1} + \sum_{i=1}^{\infty} \begin{pmatrix} i \\ 1 \end{pmatrix} \frac{\|A\|^{i-1}\|B\|}{\Gamma(i\alpha+2)} (t-\tau)^{i\alpha+1} + \cdots \\
&\quad + \sum_{i=p}^{\infty} \begin{pmatrix} i \\ p \end{pmatrix} \frac{\|A\|^{i-p}\|B\|^p}{\Gamma(i\alpha+2)} (t-p\tau)^{i\alpha+1} \\
&\le \sum_{i=0}^{\infty} \frac{\|A\|^i}{\Gamma(i\alpha+2)} t^{i\alpha+1} + \sum_{i=1}^{\infty} \begin{pmatrix} i \\ 1 \end{pmatrix} \frac{\|A\|^{i-1}\|B\|}{\Gamma(i\alpha+2)} t^{i\alpha+1} + \cdots \\
&\quad + \sum_{i=p}^{\infty} \begin{pmatrix} i \\ p \end{pmatrix} \frac{\|A\|^{i-p}\|B\|^p}{\Gamma(i\alpha+2)} t^{i\alpha+1} \\
&= t + \frac{\|A\| + \|B\|}{\Gamma(\alpha+2)} t^{\alpha+1} \\
&\quad + \frac{\|A\|^2 + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \|A\|\|B\| + \|B\|^2}{\Gamma(2\alpha+2)} t^{2\alpha+1} + \cdots \\
&\quad + \frac{\|A\|^p + \begin{pmatrix} p \\ 1 \end{pmatrix} \|A\|^{p-1}\|B\| + \cdots + \|B\|^p}{\Gamma(p\alpha+2)} t^{p\alpha+1} \\
&\le \sum_{i=0}^{\infty} \frac{(\|A\| + \|B\|)^i}{\Gamma(i\alpha+2)} t^{i\alpha+1} \\
&= tE_{\alpha,2}((\|A\| + \|B\|)t^{\alpha}).
\end{aligned}$$

□

Lemma 3. *For any $t \geq 0$, $1 < \alpha < 2$, we have*

$$\left\| \frac{d^2}{dt^2} S^{A,B}_{\tau,\alpha}(t) \right\| \leq (\|A\| + \|B\|) t^{\alpha-1} E_{\alpha,\alpha}((\|A\| + \|B\|) t^{\alpha}), \tag{10}$$

and

$$\left\| D_0^{2-\alpha} S^{A,B}_{\tau,\alpha}(t-s) \right\| \leq \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} (\|A\| + \|B\|) E_{\alpha,\alpha}((\|A\| + \|B\|)(t-s)^{\alpha})(t-s)^{2\alpha-1}, \tag{11}$$

where $E_{\alpha,\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\alpha)}$, $z \in \mathbb{R}$ is the Mittag–Leffler function.

Proof. For $p\tau < t \leq (p+1)\tau$, $p = 0, 1, \ldots$, by (6), one has

$$\frac{d^2}{dt^2} S^{A,B}_{\tau,\alpha}(t) := \begin{cases} \Theta, & t \in [-\tau, 0], \\ \sum\limits_{i=1}^{\infty} \frac{Q_{i+1}(0)}{\Gamma(i\alpha)} t^{i\alpha-1}, & 0 < t \leq \tau, \\ \sum\limits_{i=1}^{\infty} \frac{Q_{i+1}(0)}{\Gamma(i\alpha)} t^{i\alpha-1} + \sum\limits_{i=1}^{\infty} \frac{Q_{i+1}(\tau)}{\Gamma(i\alpha)} (t-\tau)^{i\alpha-1} & \\ \quad + \cdots + \sum\limits_{i=p}^{\infty} \frac{Q_{i+1}(p\tau)}{\Gamma(i\alpha)} (t-p\tau)^{i\alpha-1}, & p\tau < t \leq (p+1)\tau. \end{cases} \tag{12}$$

Thus, from (12) and Definition 3, we obtain

$$\begin{aligned}
\left\| \frac{d^2}{dt^2} S^{A,B}_{\tau,\alpha}(t) \right\| &\leq \sum_{i=1}^{\infty} \frac{\|A\|^i}{\Gamma(i\alpha)} t^{i\alpha-1} + \sum_{i=1}^{\infty} \begin{pmatrix} i \\ 1 \end{pmatrix} \frac{\|A\|^{i-1}\|B\|}{\Gamma(i\alpha)} (t-\tau)^{i\alpha-1} + \cdots \\
&\quad + \sum_{i=p}^{\infty} \begin{pmatrix} i \\ p \end{pmatrix} \frac{\|A\|^{i-p}\|B\|^p}{\Gamma(i\alpha)} (t-p\tau)^{i\alpha-1} \\
&\leq \sum_{i=1}^{\infty} \frac{\|A\|^i}{\Gamma(i\alpha)} t^{i\alpha-1} + \sum_{i=1}^{\infty} \begin{pmatrix} i \\ 1 \end{pmatrix} \frac{\|A\|^{i-1}\|B\|}{\Gamma(i\alpha)} t^{i\alpha-1} + \cdots \\
&\quad + \sum_{i=p}^{\infty} \begin{pmatrix} i \\ p \end{pmatrix} \frac{\|A\|^{i-p}\|B\|^p}{\Gamma(i\alpha)} t^{i\alpha-1} \\
&= \frac{\|A\| + \|B\|}{\Gamma(\alpha)} t^{\alpha-1} \\
&\quad + \frac{\|A\|^2 + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \|A\|\|B\| + \|B\|^2}{\Gamma(2\alpha)} t^{2\alpha-1} + \cdots \\
&\quad + \frac{\|A\|^p + \begin{pmatrix} p \\ 1 \end{pmatrix} \|A\|^{p-1}\|B\| + \cdots + \|B\|^p}{\Gamma(p\alpha)} t^{p\alpha-1} \\
&\leq \sum_{i=0}^{\infty} \frac{(\|A\| + \|B\|)^i}{\Gamma(i\alpha)} t^{i\alpha-1} \\
&= (\|A\| + \|B\|) t^{\alpha-1} E_{\alpha,\alpha}((\|A\| + \|B\|) t^{\alpha}).
\end{aligned}$$

Moreover, by (10), we obtain

$$\begin{aligned}\left\|D_0^{2-\alpha}S_{\tau,\alpha}^{A,B}(t-s)\right\| &= \left\|\frac{1}{\Gamma(\alpha)}\int_0^{t-s}(t-s-u)^{\alpha-1}\frac{d^2}{du^2}S_{\tau,\alpha}^{A,B}(u)du\right\| \\ &\le \frac{1}{\Gamma(\alpha)}\int_0^{t-s}(t-s-u)^{\alpha-1}\left\|\frac{d^2}{du^2}S_{\tau,\alpha}^{A,B}(u)\right\|du \\ &\le \frac{1}{\Gamma(\alpha)}\int_0^{t-s}(t-s-u)^{\alpha-1}\cdot(\|A\|+\|B\|)u^{\alpha-1}E_{\alpha,\alpha}((\|A\|+\|B\|)u^{\alpha})du \\ &\le \frac{1}{\Gamma(\alpha)}(\|A\|+\|B\|)E_{\alpha,\alpha}((\|A\|+\|B\|)(t-s)^{\alpha})\int_0^{t-s}(t-s-u)^{\alpha-1}\cdot u^{\alpha-1}du \\ &= \frac{\Gamma(\alpha)}{\Gamma(2\alpha)}(\|A\|+\|B\|)E_{\alpha,\alpha}((\|A\|+\|B\|)(t-s)^{\alpha})(t-s)^{2\alpha-1}.\end{aligned}$$
□

Lemma 4 ([25,26])**.** *Let $\phi : R_+ \times V \to R^n$ and assume that*

$$\int_0^t\int_V|\phi(s,v)|^p\lambda(dv)ds < \infty, \quad p \ge 2.$$

Then, there exists $D_p > 0$ such that

$$\begin{aligned}&\mathbb{E}\left(\sup_{0\le t\le u}\left|\int_0^t\int_V\phi(s,v)\bar{N}(ds,dv)\right|^p\right) \\ &\quad\le D_p\left\{\mathbb{E}\left(\int_0^u\int_V|\phi(s,v)|^2\lambda(dv)ds\right)^{\frac{p}{2}}+\mathbb{E}\left(\int_0^u\int_V|\phi(s,v)|^p\lambda(dv)ds\right)\right\}.\end{aligned} \tag{13}$$

Lemma 5. *For any $\alpha \in (1,2)$ and $\mu > 0$, one has*

$$\int_0^t(t-s)^{4\alpha-2}E_{4\alpha-1,1}(\mu s^{4\alpha-1})ds \le \frac{\Gamma(4\alpha-1)}{\mu}E_{4\alpha-1,1}(\mu t^{4\alpha-1}), \tag{14}$$

where $\Gamma(\alpha) := \int_0^{+\infty}s^{\alpha-1}e^{-s}ds$ is the Gamma function.

Proof. Let $\mu > 0$ be arbitrary. Consider the corresponding linear Caputo fractional differential equation of the following form:

$${}^{C}D_{0^+}^{4\alpha-1}x(t) = \mu x(t). \tag{15}$$

From [27], it is easy to know that the Mittag–Leffler function $E_{4\alpha-1,1}(\mu t^{4\alpha-1})$ is a solution of (15). So, the following equality holds:

$$E_{4\alpha-1,1}(\mu t^{4\alpha-1}) = 1+\frac{\mu}{\Gamma(4\alpha-1)}\int_0^t(t-s)^{4\alpha-2}E_{4\alpha-1,1}(\mu s^{4\alpha-1})ds,$$

which completes the proof. □

Lemma 6 ([28])**.** *Assume that (X,d) is a complete metric space and $Q : X \to X$ is a contraction (with $\nu \in (0,1)$). Furthermore, let $x \in X$, $\epsilon > 0$ and $d(x,Q(x)) \le \epsilon$. Then, there exists a unique $y \in X$ that satisfies $y = Q(y)$. Moreover, we have*

$$d(x,y) \le \frac{\epsilon}{1-\nu}.$$

To study the qualitative properties of the solution for (2), we impose the following conditions on the data of the problem:

Hypothesis 1 (H1). *For any $x_1, x_2, y_1, y_2 \in R^n$ and $t \in J$, there exists a constant $C > 0$ such that*

$$\|f(t,x_1,y_1) - f(t,x_2,y_2)\|^2 \vee \|\sigma(t,x_1,y_1) - \sigma(t,x_2,y_2)\|^2 \vee \int_V \|g(t,x_1,y_1,v) - g(t,x_2,y_2,v)\|^2 \lambda(dv) \le C^2(\|x_1 - x_2\|^2 + \|y_1 - y_2\|^2),$$

where $\|\cdot\|$ is the norm of $\mathbb{R}^n$ and $x \vee y = \max\{x, y\}$.

Hypothesis 2 (H2). *Let $\sigma(\cdot,0,0)$ and $g(\cdot,0,0,0)$ be essentially bounded, i.e.,*

$$\|\sigma(\cdot,0,0)\|_\infty := ess \sup_{t\in[0,T]} \|\sigma(t,0,0)\| < +\infty, \quad \|g(\cdot,0,0,0)\|_\infty := ess \sup_{t\in[0,T]} \|g(t,0,0,0)\| < +\infty,$$

and $f(\cdot,0,0)$ be $\mathbb{L}^2$-integrable, i.e.,

$$\|f\|_{\mathbb{L}^p} = \int_0^T \|f(t,0,0)\|^2 dt < +\infty.$$

3. Existence and Uniqueness Result

Let $\mathbb{H}^2([0,T])$ be the space of all the processes x which are measurable, $\mathcal{F}(t)$-adapted, and satisfy that $\|x\|_{\mathbb{H}^2} := \sup_{0\le t\le T} \sqrt{\sum_{i=1}^{n} \mathbb{E}(|x_i(t)|^2)} < \infty$. Obviously, $(\mathbb{H}^2([0,T]), \|\cdot\|_{\mathbb{H}^2})$ is a Banach space [27].

Now, we state the Hyers–Ulam stability concepts for (2). Let $\varepsilon > 0$. We consider (2) with inequality

$$\mathbb{E}\bigg\|({}^C D_0^\alpha y)(t) - Ay(t) - By(t-\tau) - f(t,y(t),y(t-\tau)) - \sigma(t,y(t),y(t-\tau))\frac{dW(t)}{dt} - \int_V g(t,y(t),y(t-\tau),v)\bar{N}(dt,dv)\bigg\|^2 \le \varepsilon, \quad t \in J, \tag{16}$$

with $y(t) = \phi(t), \quad y'(t) = \phi'(t), \quad -\tau \le t \le 0$.

Definition 7. *The FSDDSs (2) is Hyers–Ulam-stable if there is a constant $c > 0$ such that, for each $\varepsilon > 0$ and for each solution $y \in \mathbb{H}^2([0,T],\mathbb{R}^n)$ of inequality (16), there exists a solution $x \in \mathbb{H}^2([0,T],\mathbb{R}^n)$ of (2), with $x(t) = \phi(t)$ and $x'(t) = \phi'(t)$ for $-\tau \le t \le 0$, which satisfies*

$$\mathbb{E}\|y(t) - x(t)\|^2 \le c\varepsilon, \quad t \in J.$$

We define an operator $\mathcal{T} : \mathbb{H}^2([0,T],\mathbb{R}^n) \to \mathbb{H}^2([0,T],\mathbb{R}^n)$ as follows :

$$\begin{aligned}
(\mathcal{T}x)(t) = {} & C^{A,B}_{\tau,\alpha}(t+\tau)\phi(-\tau) + S^{A,B}_{\tau,\alpha}(t+\tau)\phi'(-\tau) \\
& + \int_{-\tau}^0 S^{A,B}_{\tau,\alpha}(t-s)[\phi'' - A(D^{2-\alpha}_{-\tau^+}\phi)(s)]ds \\
& + \int_0^t D_0^{2-\alpha} S^{A,B}_{\tau,\alpha}(t-s) f(s,x(s),x(s-\tau))ds \\
& + \int_0^t D_0^{2-\alpha} S^{A,B}_{\tau,\alpha}(t-s)\sigma(s,x(s),x(s-\tau))dW(s) \\
& + \int_0^t D_0^{2-\alpha} S^{A,B}_{\tau,\alpha}(t-s)\int_V g(s,x(s),x(s-\tau),v)\bar{N}(ds,dv), \quad t \in J.
\end{aligned} \tag{17}$$

From Theorem 1 in [20], it is easy to know that the fixed point of operator $\mathcal{T}$ is a solution of (2).

Lemma 7. *Suppose (H1) and (H2) hold. Then, the operator $\mathcal{T}$ is well defined.*

Proof. Let $\gamma = \|A\| + \|B\|$. For any $x \in \mathbb{H}^2([0,T])$, by (17) and the elementary inequality

$$\left\|\sum_{i=1}^{m} a_i\right\|^2 \le m \sum_{i=1}^{m} \|a_i\|^2, \quad a_i \in \mathbb{R}^n,\ i = 1,2,\ldots,m, \tag{18}$$

we have

$$\begin{aligned}
\|(\mathcal{T}x)(t)\|_{ms}^2 \le{}& 6\mathbb{E}(\|C_{\tau,\alpha}^{A,B}(t+\tau)\phi(-\tau)\|^2) + 6\mathbb{E}(\|S_{\tau,\alpha}^{A,B}(t+\tau)\phi'(-\tau)\|^2) \\
&+ 6\mathbb{E}\left(\left\|\int_{-\tau}^{0} S_{\tau,\alpha}^{A,B}(t-s)[\phi'' - A(D_{-\tau^+}^{2-\alpha}\phi)(s)]ds\right\|^2\right) \\
&+ 6\mathbb{E}\left(\left\|\int_0^t D_0^{2-\alpha} S_{\tau,\alpha}^{A,B}(t-s) f(s,x(s),x(s-\tau))ds\right\|^2\right) \\
&+ 6\mathbb{E}\left(\left\|\int_0^t D_0^{2-\alpha} S_{\tau,\alpha}^{A,B}(t-s)\sigma(s,x(s),x(s-\tau))dW(s)\right\|^2\right) \\
&+ 6\mathbb{E}\left(\left\|\int_0^t D_0^{2-\alpha} S_{\tau,\alpha}^{A,B}(t-s)\int_V g(s,x(s),x(s-\tau),v)\bar{N}(ds,dv)\right\|^2\right) \\
:={}& I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\end{aligned} \tag{19}$$

For I_1, from Lemma 1, one has

$$\begin{aligned}
I_1 &= 6\mathbb{E}(\|C_{\tau,\alpha}^{A,B}(t+\tau)\phi(-\tau)\|^2) \le 6\mathbb{E}(\|C_{\tau,\alpha}^{A,B}(t+\tau)\|^2\|\phi(-\tau)\|^2) \\
&\le 6\|\phi(-\tau)\|^2(E_{\alpha,1}(\gamma(T+\tau)^\alpha))^2.
\end{aligned} \tag{20}$$

For I_2, by Lemma 2, one has

$$\begin{aligned}
I_2 &= 6\mathbb{E}(\|S_{\tau,\alpha}^{A,B}(t+\tau)\phi'(-\tau)\|^2) \le 6\mathbb{E}(\|S_{\tau,\alpha}^{A,B}(t+\tau)\|^2\|\phi'(-\tau)\|^2) \\
&\le 6\|\phi'(-\tau)\|^2(T+\tau)^2(E_{\alpha,2}(\gamma(T+\tau)^\alpha))^2.
\end{aligned} \tag{21}$$

For I_3, by using Lemma 2 and the Cauchy–Schwartz inequality, we obtain

$$\begin{aligned}
I_3 &= 6\mathbb{E}\left(\left\|\int_{-\tau}^{0} S_{\tau,\alpha}^{A,B}(t-s)[\phi'' - A(D_{-\tau^+}^{2-\alpha}\phi)(s)]ds\right\|^2\right) \\
&\le 6\int_{-\tau}^{0}\|S_{\tau,\alpha}^{A,B}(t-s)\|^2 ds \cdot \mathbb{E}\left(\int_{-\tau}^{0}\|\phi'' - A(D_{-\tau^+}^{2-\alpha}\phi)(s)\|^2 ds\right) \\
&\le 2(T+\tau)^3\Xi(E_{\alpha,2}(\gamma(T+\tau)^\alpha)))^2,
\end{aligned} \tag{22}$$

where $\Xi = \int_{-\tau}^{0}\|\phi'' - A(D_{-\tau^+}^{2-\alpha}\phi)(s)\|^2 ds < \infty$.

For I_4, applying (H1), (H2), the Cauchy–Schwartz inequality, the Jensen inequality, and Lemma 3, one has

$$\begin{aligned}
I_4 &= 6\mathbb{E}\left(\left\|\int_0^t D_0^{2-\alpha} S_{\tau,\alpha}^{A,B}(t-s)f(s,x(s),x(s-\tau))ds\right\|^2\right) \\
&\le 6\left(\int_0^t \|D_0^{2-\alpha} S_{\tau,\alpha}^{A,B}(t-s)\|^2 ds\right)\cdot\mathbb{E}\left(\int_0^t \|f(s,x(s),x(s-\tau))-f(s,0,0)+f(s,0,0)\|^2 ds\right) \\
&\le 6\int_0^t \frac{\gamma^2\Gamma^2(\alpha)}{\Gamma^2(2\alpha)} E_{\alpha,\alpha}(\gamma(t-s)^\alpha)^2 (t-s)^{2(2\alpha-1)} ds \\
&\quad \cdot 2\mathbb{E}\left(\int_0^t \|f(s,x(s),x(s-\tau))-f(s,0,0)\|^2 ds + \int_0^t \|f(s,0,0)\|^2 ds\right) \\
&\le \frac{12\gamma^2\Gamma^2(\alpha)}{(4\alpha-1)\Gamma^2(2\alpha)} T^{4\alpha-1} E_{\alpha,\alpha}(\gamma T^\alpha)^2 \mathbb{E}\left(\int_0^t C^2(\|x(s)\|^2+\|x(s-\tau)\|^2)ds + \int_0^t \|f(s,0,0)\|^2 ds\right) \\
&\le \frac{12\gamma^2\Gamma^2(\alpha)}{(4\alpha-1)\Gamma^2(2\alpha)} T^{4\alpha-1} E_{\alpha,\alpha}(\gamma T^\alpha)^2 \Big(TC^2(2\|x\|_{\mathbb{H}^2}^2+\|\phi\|^2)+\|f\|_{\mathbb{L}^2}\Big),
\end{aligned} \tag{23}$$

since

$$\begin{aligned}
\sup_{0\le t\le T} \mathbb{E}\|x(t-\tau)\|^2 &\le \max\left\{\sup_{-\tau\le t\le 0}\mathbb{E}\|\phi(t)\|^2,\ \sup_{0\le t\le T}\mathbb{E}\|x(t)\|^2\right\} \\
&= \max\left\{\|\phi\|^2,\ \|x\|_{\mathbb{H}^2}^2\right\} \le \|\phi\|^2 + \|x\|_{\mathbb{H}^2}^2.
\end{aligned}$$

For I_5, by using (H1), (H2), Ito's isometry, Lemma 3, and the Jensen inequality, we have

$$\begin{aligned}
I_5 &= 6\mathbb{E}\left(\left\|\int_0^t D_0^{2-\alpha} S_{\tau,\alpha}^{A,B}(t-s)\sigma(s,x(s),x(s-\tau))dW(s)\right\|^2\right) \\
&= 6\mathbb{E}\left(\int_0^t \|D_0^{2-\alpha} S_{\tau,\alpha}^{A,B}(t-s)\|^2 \|\sigma(s,x(s),x(s-\tau))\|^2 ds\right) \\
&\le 6\mathbb{E}\left(\int_0^t \frac{\gamma^2\Gamma^2(\alpha)}{\Gamma^2(2\alpha)} E_{\alpha,\alpha}(\gamma(t-s)^\alpha)^2 (t-s)^{2(2\alpha-1)} \|\sigma(s,x(s),x(s-\tau))\|^2 ds\right) \\
&\le \frac{12\gamma^2\Gamma^2(\alpha)}{\Gamma^2(2\alpha)} E_{\alpha,\alpha}(\gamma T^\alpha)^2 \mathbb{E}\left(\int_0^t (t-s)^{4\alpha-2}[C^2(\|x(s)\|^2+\|x(s-\tau)\|^2)+\|\sigma(s,0,0)\|^2]ds\right) \\
&\le \frac{12\gamma^2\Gamma^2(\alpha)}{(4\alpha-1)\Gamma^2(2\alpha)} T^{4\alpha-1} E_{\alpha,\alpha}(\gamma T^\alpha)^2 (C^2(2\|x\|_{\mathbb{H}^2}^2+\|\phi\|^2)+\|\sigma(\cdot,0,0)\|_\infty^2).
\end{aligned} \tag{24}$$

For I_6, by using (H1), (H2), Lemmas 3 and 4, and the Jensen inequality, we obtain

$$\begin{aligned}
I_6 &= 6\mathbb{E}\left(\left\|\int_0^t\int_V D_0^{2-\alpha} S_{\tau,\alpha}^{A,B}(t-s)g(s,x(s),x(s-\tau),v)\bar{N}(ds,dv)\right\|^2\right) \\
&\le 12D_2\mathbb{E}\left(\int_0^t\int_V \|D_0^{2-\alpha} S_{\tau,\alpha}^{A,B}(t-s)\|^2 \|g(s,x(s),x(s-\tau),v)\|^2 \lambda(dv)ds\right) \\
&\le 12D_2\mathbb{E}\left(\int_0^t \frac{\gamma^2\Gamma^2(\alpha)}{\Gamma^2(2\alpha)} E_{\alpha,\alpha}(\gamma(t-s)^\alpha)^2 (t-s)^{2(2\alpha-1)} \int_V \|g(s,x(s),x(s-\tau),v)\|^2 \lambda(dv)ds\right) \\
&\le \frac{24D_2\gamma^2\Gamma^2(\alpha)}{\Gamma^2(2\alpha)} E_{\alpha,\alpha}(\gamma T^\alpha)^2 \mathbb{E}\left(\int_0^t (t-s)^{4\alpha-2}[C^2(\|x(s)\|^2+\|x(s-\tau)\|^2)+\int_V \|g(s,0,0,0)\|^2\lambda(dv)]ds\right) \\
&\le \frac{12D_2\gamma^2\Gamma^2(\alpha)}{(4\alpha-1)\Gamma^2(2\alpha)} T^{4\alpha-1} E_{\alpha,\alpha}(\gamma T^\alpha)^2 (C^2(2\|x\|_{\mathbb{H}^2}^2+\|\phi\|^2)+\lambda(V)\|g(\cdot,0,0,0)\|_\infty^2).
\end{aligned} \tag{25}$$

Submitting (20)–(25) into (19) implies that $\|\mathcal{T}x\|_{\mathbb{H}^2} < \infty$. Thus, the operator $\mathcal{T}$ is well defined. □

Theorem 1. *Assume that* $(H1)$ *and* $(H2)$ *hold. Then, FSDDSs (2) is Hyers–Ulam stable.*

Proof. On the space $\mathbb{H}^2([0,T])$, for a constant $\mu > 0$, we define a metric $d_\mu : \mathbb{H}^2([0,T]) \times \mathbb{H}^2([0,T]) \to \mathbb{R}_+$ as below:

$$d_\mu(x,y) = \sqrt{\sup_{t\in[0,T]} \frac{\mathbb{E}\|x(t)-y(t)\|^2}{E_{4\alpha-1,1}(\mu t^{4\alpha-1})}}, \quad \forall x,y \in \mathbb{H}^2([0,T]). \tag{26}$$

By Lemma 7, $\mathcal{T}$ is well defined. Next, we will check that $\mathcal{T}$ is a contraction operator for some $\mu > 0$.

For each $x, y \in \mathbb{H}^2([0,T])$. From (17) and (18), we have

$$\begin{aligned}
&\mathbb{E}(\|\mathcal{T}x(t) - \mathcal{T}y(t)\|^2) \\
&\le 3\mathbb{E}\left(\left\|\int_0^t D_0^{2-\alpha} S_{\tau,\alpha}^{A,B}(t-s)(f(s,x(s),x(s-\tau)) - f(s,y(s),y(s-\tau)))ds\right\|^2\right) \\
&\quad + 3\mathbb{E}\left(\left\|\int_0^t D_0^{2-\alpha} S_{\tau,\alpha}^{A,B}(t-s)(\sigma(s,x(s),x(s-\tau)) - \sigma(s,y(s),y(s-\tau)))dW(s)\right\|^2\right) \\
&\quad + 3\mathbb{E}\left(\left\|\int_0^t D_0^{2-\alpha} S_{\tau,\alpha}^{A,B}(t-s)\int_V (g(s,x(s),x(s-\tau),v) - g(s,y(s),y(s-\tau),v))\bar{N}(ds,dv)\right\|^2\right) \\
&= 3(J_1 + J_2 + J_3).
\end{aligned} \tag{27}$$

For J_1, by using the Cauchy–Schwartz inequality, (H1), and Lemma 3, we obtain

$$\begin{aligned}
&\mathbb{E}\left(\left\|\int_0^t D_0^{2-\alpha} S_{\tau,\alpha}^{A,B}(t-s)(f(s,x(s),x(s-\tau)) - f(s,y(s),y(s-\tau)))ds\right\|^2\right) \\
&\le \mathbb{E}\left(\left(\int_0^t 1^2 ds\right) \cdot \int_0^t \|D_0^{2-\alpha} S_{\tau,\alpha}^{A,B}(t-s)\|^2 \|f(s,x(s),x(s-\tau)) - f(s,y(s),y(s-\tau))\|^2\right)ds \\
&\le \frac{\gamma^2\Gamma^2(\alpha)}{\Gamma^2(2\alpha)} T E_{\alpha,\alpha}(\gamma T^\alpha)^2 \int_0^t (t-s)^{2(2\alpha-1)} \mathbb{E}\left(\|f(s,x(s),x(s-\tau)) - f(s,y(s),y(s-\tau))\|^2\right)ds \\
&\le \frac{\gamma^2\Gamma^2(\alpha)}{\Gamma^2(2\alpha)} T E_{\alpha,\alpha}(\gamma T^\alpha)^2 C^2 \int_0^t (t-s)^{4\alpha-2} \mathbb{E}(\|x(s)-y(s)\|^2 + \|x(s-\tau)-y(s-\tau)\|^2)ds.
\end{aligned} \tag{28}$$

For J_2, similar to the proof of (24), one has

$$\begin{aligned}
&\mathbb{E}\left(\left\|\int_0^t D_0^{2-\alpha} S_{\tau,\alpha}^{A,B}(t-s)(\sigma(s,x(s),x(s-\tau)) - \sigma(s,y(s),y(s-\tau)))dW(s)\right\|^2\right) \\
&= \mathbb{E}\left(\int_0^t \|D_0^{2-\alpha} S_{\tau,\alpha}^{A,B}(t-s)\|^2 \|\sigma(s,x(s),x(s-\tau)) - \sigma(s,y(s),y(s-\tau))\|^2 ds\right) \\
&\le \frac{\gamma^2\Gamma^2(\alpha)}{\Gamma^2(2\alpha)} E_{\alpha,\alpha}(\gamma T^\alpha)^2 \int_0^t (t-s)^{2(2\alpha-1)} \mathbb{E}\left(\|\sigma(s,x(s),x(s-\tau)) - \sigma(s,y(s),y(s-\tau))\|^2\right)ds \\
&\le \frac{\gamma^2\Gamma^2(\alpha)}{\Gamma^2(2\alpha)} E_{\alpha,\alpha}(\gamma T^\alpha)^2 C^2 \int_0^t (t-s)^{4\alpha-2} \mathbb{E}(\|x(s)-y(s)\|^2 + \|x(s-\tau)-y(s-\tau)\|^2)ds.
\end{aligned} \tag{29}$$

For J_3, similar to the proof of (25), we obtain

$$\begin{aligned}
&\mathbb{E}\left(\left\|\int_0^t D_0^{2-\alpha} S_{\tau,\alpha}^{A,B}(t-s)\int_V (g(s,x(s),x(s-\tau),v) - g(s,y(s),y(s-\tau),v))\bar{N}(ds,dv)\right\|^2\right) \\
&\le 2D_2\mathbb{E}\left(\int_0^t \|D_0^{2-\alpha} S_{\tau,\alpha}^{A,B}(t-s)\|^2 \int_V \|g(s,x(s),x(s-\tau),v) - g(s,y(s),y(s-\tau),v)\|^2 \lambda(dv)ds\right) \\
&\le \frac{2D_2\gamma^2\Gamma^2(\alpha)}{\Gamma^2(2\alpha)} E_{\alpha,\alpha}(\gamma T^\alpha)^2 C^2 \int_0^t (t-s)^{4\alpha-2} \mathbb{E}(\|x(s)-y(s)\|^2 + \|x(s-\tau)-y(s-\tau)\|^2)ds.
\end{aligned} \tag{30}$$

For each $x, y \in \mathbb{H}^p([0,T])$, from (27)–(30), we have

$$\begin{aligned}
&\mathbb{E}(\|\mathcal{T}x(t) - \mathcal{T}y(t)\|^2) \\
&\quad \le \omega \int_0^t (t-s)^{4\alpha-2}\mathbb{E}(\|x(s)-y(s)\|^2 + \|x(s-\tau) - y(s-\tau\|^2)ds,
\end{aligned} \tag{31}$$

where

$$\omega := \frac{3\gamma^2\Gamma^2(\alpha)}{\Gamma^2(2\alpha)} E_{\alpha,\alpha}(\gamma T^\alpha)^2 C^2(T + 2D_2 + 1).$$

For $t > \tau$, one has

$$\begin{aligned}
&\int_0^t (t-s)^{4\alpha-2}\mathbb{E}(\|x(s-\tau) - y(s-\tau)\|^2)ds \\
&\quad = \int_0^\tau + \int_\tau^t (t-s)^{4\alpha-2}\mathbb{E}(\|x(s-\tau) - y(s-\tau)\|^2)ds \\
&\quad = \int_\tau^t (t-s)^{4\alpha-2}\mathbb{E}(\|x(s-\tau) - y(s-\tau)\|^2)ds \\
&\quad = \int_0^{t-\tau} (t-\tau-u)^{4\alpha-2}\mathbb{E}(\|x(u) - y(u)\|^2)du \\
&\quad \le \int_0^t (t-u)^{4\alpha-2}\mathbb{E}(\|x(u) - y(u)\|^2)du.
\end{aligned} \tag{32}$$

From Lemma 5, combining (31) and (32), for each $t \in [0,T]$, we obtain

$$\begin{aligned}
\mathbb{E}(\|\mathcal{T}x(t) - \mathcal{T}y(t)\|^2) &\le 2\omega \int_0^t (t-s)^{4\alpha-2}\mathbb{E}(\|x(s) - y(s)\|^2)ds \\
&= 2\omega \int_0^t (t-s)^{4\alpha-2}\frac{\mathbb{E}(\|x(s)-y(s)\|^2)}{E_{4\alpha-1,1}(\mu s^{4\alpha-1})} E_{4\alpha-1,1}(\mu s^{4\alpha-1})ds \\
&\le 2\omega d_\mu^2(x,y) \int_0^t (t-s)^{4\alpha-2} E_{4\alpha-1,1}(\mu s^{4\alpha-1})ds \\
&\le \frac{2\omega\Gamma(4\alpha-1)}{\mu} d_\mu^2(x,y) E_{4\alpha-1,1}(\mu t^{4\alpha-1}),
\end{aligned} \tag{33}$$

which implies that

$$d_\mu(\mathcal{T}x, \mathcal{T}y) \le \rho d_\mu(x,y), \tag{34}$$

where $\rho = \sqrt{\frac{2\omega\Gamma(4\alpha-1)}{\mu}}$. Hence, $\mathcal{T}$ is a contraction mapping on $\mathbb{H}^2([0,T])$ for some $\mu > 2\omega\Gamma(4\alpha-1)$.

Let

$$\begin{aligned}
({}^C D_0^\alpha y)(t) &= Ay(t) + By(t-\tau) + f(t, y(t), y(t-\tau)) \\
&\quad + \sigma(t, y(t), y(t-\tau))\frac{dW(t)}{dt} + \int_V g(t, y(t), y(t-\tau), v)\bar{N}(dt, dv) + h(t).
\end{aligned} \tag{35}$$

From (16), one has

$$\mathbb{E}(\|h(t)\|^2) \le \varepsilon, \quad t \in J. \tag{36}$$

By (17), (35), and Theorem 1 in [20], we obtain

$$y(t) = \mathcal{T}y(t) + \int_0^t D_0^{2-\alpha} S_{\tau,\alpha}^{A,B}(t-s)h(s)ds. \tag{37}$$

Thus,

$$\begin{aligned}\frac{\mathbb{E}\|y(t)-\mathcal{T}y(t)\|^2}{E_{4\alpha-1,1}(\mu t^{4\alpha-1})} &\leq \frac{\int_0^t 1^2 ds\cdot\int_0^t \|D_0^{2-\alpha}S_{\tau,\alpha}^{A,B}(t-s)\|^2\mathbb{E}(h^2(s))ds}{E_{4\alpha-1,1}(\mu t^{4\alpha-1})} \\ &\leq \frac{t\int_0^t \frac{\gamma^2\Gamma^2(\alpha)}{\Gamma^2(2\alpha)}E_{\alpha,\alpha}(\gamma(t-s)^\alpha)^2(t-s)^{4\alpha-2}\mathbb{E}(h^2(s))ds}{E_{4\alpha-1,1}(\mu t^{4\alpha-1})} \\ &\leq \frac{\frac{\gamma^2\Gamma^2(\alpha)}{(4\alpha-1)\Gamma^2(2\alpha)}E_{\alpha,\alpha}(\gamma t^\alpha)^2 t^{4\alpha}}{E_{4\alpha-1,1}(\mu t^{4\alpha-1})}\varepsilon \\ &\leq \frac{\frac{\gamma^2\Gamma^2(\alpha)}{(4\alpha-1)\Gamma^2(2\alpha)}E_{\alpha,\alpha}(\gamma t^\alpha)^2 t^{4\alpha}}{\frac{\mu t^{4\alpha-1}}{\Gamma(4\alpha)}}\varepsilon \\ &\leq \frac{\frac{\gamma^2\Gamma^2(\alpha)}{(4\alpha-1)\Gamma^2(2\alpha)}\Gamma(4\alpha)E_{\alpha,\alpha}(\gamma T^\alpha)^2 T}{\mu}\varepsilon, \quad t\in J.\end{aligned} \tag{38}$$

For some $\mu > \frac{\gamma^2\Gamma^2(\alpha)}{\Gamma^2(2\alpha)}\Gamma(4\alpha-1)E_{\alpha,\alpha}(\gamma T^\alpha)^2 T$, we obtain

$$\frac{\mathbb{E}\|y(t)-\mathcal{T}y(t)\|^2}{E_{4\alpha-1,1}(\mu t^{4\alpha-1})} \leq \varepsilon, \tag{39}$$

for all $t \in J$, which implies that

$$d_\mu(y,\mathcal{T}y) \leq \sqrt{\varepsilon}. \tag{40}$$

From Lemma 6, there exists a unique solution $z \in \mathbb{H}^2([0,T])$ such that

$$d_\mu(y,z) \leq \frac{\sqrt{\varepsilon}}{1-\rho}. \tag{41}$$

Consequently, $\forall t \in J$. We have

$$\mathbb{E}\|y(t)-z(t)\|^2 \leq \frac{E_{4\alpha-1,1}(\mu T^{4\alpha-1})}{(1-\rho)^2}\varepsilon. \tag{42}$$

Thus, (2) is HUS. The proof of this theorem is complete. □

4. An Example

Example 1. *Consider the following Caputo fractional stochastic delay differential system (FSDDSs) with Poisson jumps:*

$$\begin{cases} ({}^C D_0^{1.8}x)(t) = Ax(t) + Bx(t-0.3) + f(t,x(t),x(t-0.3)) + \sigma(t,x(t),x(t-0.3))\frac{dW(t)}{dt} \\ \qquad\qquad + \int_V g(t,x(t),x(t-0.3),v)\tilde{N}(dt,dv), \quad t\in J, \\ x(t)=\phi(t), \quad x'(t)=\phi'(t), \quad -0.3\leq t\leq 0, \end{cases} \tag{43}$$

where $\alpha = 1.8$; $\tau = 0.3$; $J = [0,6]$; $x(t) = (x_1(t), x_2(t))^T$;

$$A = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.6 \end{pmatrix}; \quad B = \begin{pmatrix} 0.7 & 0 \\ 0 & 0.3 \end{pmatrix}; \quad \phi(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix};$$

$$f(t,x(t),x(t-0.3)) = \begin{pmatrix} \sin^2(x_1(t)) + \arctan(x_1(t-0.3)) + 1 \\ \cos^2(x_2(t)) + \arctan(x_2(t-0.3)) + \sqrt{t} \end{pmatrix};$$

$$\sigma(t,x(t),x(t-0.4)) = \begin{pmatrix} \frac{t}{2}\arctan(x_1(t)) + \frac{t}{2}\sin(x_1(t-0.3)) + \frac{t}{3} \\ 2e^{-t}\sin(x_2(t)) + 2e^{-t}\arctan(x_2(t-0.3)) + 1 \end{pmatrix};$$

and

$$g(t,x(t),x(t-0.4),v) = \begin{pmatrix} \frac{1}{2}(1+x_1(t)) \\ e^{-t}(1+\sin(x_2(t-0.3))) \end{pmatrix}.$$

Let $(t,x,y) \in [0,6] \times \mathbb{R}^2 \times \mathbb{R}^2$. *Then,*

$$\begin{aligned} &\|f(t,x(t),x(t-0.3)) - f(t,y(t),y(t-0.3))\| \\ &\quad \le 2|x_1(t)-y_1(t)| + |x_1(t-0.3)-y_1(t-0.3)| + 2|x_2(t)-y_2(t)| + |x_2(t-0.3)-y_2(t-0.3)| \\ &\quad \le 2(\|x(t)-y(t)\| + \|x(t-0.3)-y(t-0.3)\|), \end{aligned}$$

$$\begin{aligned} &\|\sigma(t,x(t),x(t-0.3)) - \sigma(t,y(t),y(t-0.3))\| \\ &\quad \le 3|x_1(t)-y_1(t)| + 3|x_1(t-0.3)-y_1(t-0.3)| + 2|x_2(t)-y_2(t)| + 2|x_2(t-0.3)-y_2(t-0.3)| \\ &\quad \le 3(\|x(t)-y(t)\| + \|x(t-0.3)-y(t-0.3)\|), \end{aligned}$$

and

$$\begin{aligned} &\|g(t,x(t),x(t-0.3),v) - g(t,y(t),y(t-0.3),v)\| \\ &\quad \le \frac{1}{2}|x_1(t)-y_1(t)| + |x_2(t-0.3)-y_2(t-0.3)| \\ &\quad \le \frac{1}{2}(\|x(t)-y(t)\| + \|x(t-0.3)-y(t-0.3)\|). \end{aligned}$$

Thus, assumption (H1) is fulfilled. Moreover, one has

$$\|f(\cdot,0,0)\|_{L^2}^2 = \int_0^T \|f(s,0,0)\|^2 ds = \int_0^6 (1+\sqrt{s})^2 ds = 24 + 8\sqrt{6},$$

$$\|\sigma(\cdot,0,0)\|_\infty = ess \sup_{s\in[0,6]} \|\sigma(s,0,0)\| = 3, \quad \|g(\cdot,0,0)\|_\infty = ess \sup_{s\in[0,6]} \|g(s,0,0)\| = \frac{3}{2}.$$

So, assumption (H2) holds true. Thus, applying Theorem 1, FSDDSs (43) is HUS on $[0,6]$.

5. Conclusions

In this paper, our main target is to provide general results on the stability analysis of nonlinear Caputo-type fractional stochastic delay differential systems (FSDDSs) with Poisson jumps. Compared with the existing research, the system we are studying is more generalized because it has not only the stochastic term, but also Poisson jumps and the delay term with respect to the Caputo fractional derivative. By using fractional calculus, the stochastic analysis method, fixed point theorem, and appropriate hypotheses on nonlinear terms, the Hyers–Ulam stability for FSDDSs has been proved. Finally, an illustrative example is given to verify the obtained theoretical results. In future work, we intend to consider the Hyers–Ulam stability problems for an impulsive Caputo-type fractional fuzzy stochastic differential system with delay.

Author Contributions: Conceptualization, Z.B.; methodology, Z.B. and C.B.; formal analysis, Z.B.; investigation, C.B.; writing—original draft preparation, Z.B.; writing—review and editing, C.B. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the Natural Science Foundation of China (11571136).

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Acknowledgments: The authors thank the reviewers for their valuable comments and suggestions that have improved the quality of our manuscript.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Hyers, D.H. On the stability of linear functional equation. *Proc. Natl. Acad. Sci. USA* **1941**, *27*, 222–224. [CrossRef]
2. Ulam, S.M. *A Collection of Mathematical Problems*; Wiley: New York, NY, USA, 1960.
3. Rassias, T.M. On the stability of the linear mapping in Banach spaces. *Proc. Am. Math. Soc.* **1978**, *72*, 297–300. [CrossRef]
4. Hyers, D.H.; Isac, G.; Rassias, T.M. *Stability of Functional Equations in Several Variables*; Birkhauser: Boston, MA, USA, 1998.
5. Jung, S.M. *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis*; Springer: New York, NY, USA, 2011.
6. Brzdek, J.; Popa, D.; Rasa, I.; Xu, B. *Ulam Stability of Operators*; Academic Press: Cambridge, MA, USA, 2018.
7. Xu, B.; Brzdek, J.; Zhang, W. Fixed point results and the Hyers-Ulam stability of linear equations of higher orders. *Pac. J. Math.* **2015**, *273*, 483–498. [CrossRef]
8. Fukutaka, R.; Onitsuka, M. Best constant in Hyers-Ulam stability of first-order homogeneous linear differential equations with a periodic coefficient. *J. Math. Anal. Appl.* **2019**, *473*, 1432–1446. [CrossRef]
9. Wang, J.; Zhou, Y. Mittag-Leffler-Ulam stabilities of fractional evolution equations. *Appl. Math. Lett.* **2012**, *25*, 723–728. [CrossRef]
10. Wang, J.; Li, X. A uniform method to Hyers-Ulam stability for some linear fractional equations. *Mediterr. J. Math.* **2016**, *13*, 625–635. [CrossRef]
11. Dai, Q.; Gao, R.; Li, Z.; Wang, C. Stability of Ulam-Hyers and Ulam-Hyers-Rassias for a class of fractional differential equation. *Adv. Differ. Equ.* **2020**, *2020*, 103. [CrossRef]
12. Rizwan, R.; Zada, A. Existence theory and Ulam's stabilities of fractional Langevin equation. *Qualit. Theory Dyn. Syst.* **2021**, *20*, 57. [CrossRef]
13. Luo, D.; Wang, X. Caraballo, T.; Zhu, Q. Ulam-Hyers stability of Caputo-type fractional fuzzy stochastic differential equations with delay. *Commun. Nonlinear Sci. Numer. Simul.* **2023**, *121*, 107229. [CrossRef]
14. Liu, K.; Wang, J.; Zhou, Y.; O'Regan, D. Ulam-Hyers stability and existence of solution for fractional differential equation with Mittag-Leffler kernal. *Chaos Solitons Fractals* **2020**, *132*, 109534. [CrossRef]
15. Benchaib, A.; Salim, A.; Abbas, S.; Benchohra, M. New stability results for abstract fractional differential equations with delay and non-instantaneous impulses. *Mathematics* **2023**, *11*, 3490. [CrossRef]
16. Ahmadova, A.; Mahmudov, N.I. Ulam-Hyers stability of Caputo type fractional stochastic neutral differential equations. *Stat. Probab. Lett.* **2021**, *168*, 108949. [CrossRef]
17. Guo, Y.; Shu, X.; Li, Y.; Xu, T. The existence and Hyers-Ulam stability of solution for an impulsive Riemann-Liouville fractional neutral functional stochastic differential equation with infinite delay of order $1 < \beta < 2$. *Bound. Value Probl.* **2019**, *2019*, 59.
18. Mchiri, L.; Makhlouf, A.B.; Rguigui, H. Ulam-Hyers stability of pantograph fractional stochastic differential equations. *Math. Methods Appl. Sci.* **2023**, *46*, 4134–4144. [CrossRef]
19. Kahouli, O.; Albadran, S.; Elleuch, Z.; Bouteraa, Y.; Makhlouf, A.B. Stability results for neutral fractional stochastic differential equations. *AIMS Math.* **2024**, *9*, 3253–3263. [CrossRef]
20. Liu, L.; Dong, Q.; Li, G. Exact solutions of fractional oscillation systems with pure delay. *Fract. Calc. Appl. Anal.* **2022**, *25*, 1688–1712. [CrossRef]
21. Li, M.; Wang, J. The existence and averaging principle for Caputo fractional stochastic delay differential systems. *Fract. Calc. Appl. Anal.* **2023**, *26*, 893–912. [CrossRef]
22. Bai, Z.; Bai, C. The existence and averaging principle for Caputo fractional stochastic delay differential systems with Poisson jumps. *Axioms* **2024**, *13*, 68. [CrossRef]
23. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006.
24. Mahmudov, N.I. Delayed perturbation of Mittag-Leffler functions and their applications to fractional linear delay differential equations. *Math. Meth. Appl. Sci.* **2019**, *42*, 5489–5497. [CrossRef]
25. Applebaum, D. *Lévy Process and Stochastic Calculus*; Cambridge University Press: Cambridge, UK, 2009.
26. Kunita, H. Stochastic differential equations based on Lévy processes and stochastic flows of diffeomorphisms. In *Real and Stochastic Analysis*; New Perspectives, Birkhauser: Basel, Switzerland, 2004; pp. 305–373.
27. Son, D.; Huong, P.; Kloeden, P.; Tuan, H. Asymptotic separation between solutions of Caputo fractional stochastic differential equations. *Stoch. Anal. Appl.* **2018**, *36*, 654–664. [CrossRef]
28. Diaz, J.B.; Margolis, B. A fixed point theorem of the alternative, for contractions on a generalized complete metric space. *Bull. Amer. Math. Soc.* **1968**, *74*, 305–309. [CrossRef]