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Hyperstability for a Generalized Class of Pexiderized Functional Equations on Monoids via Páles' Approach

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Abstract: In this paper, we deduce some hyperstability results for a generalized class of homogeneous Pexiderized functional equations, expressed as $\sum_{\rho \in \Gamma} f(x\rho.y) = \ell f(x) + \ell g(y)$, $x, y \in M$, which is inspired by the concept of Ulam stability. Indeed, we prove that function f that approximately satisfies an equation can, under certain conditions, be considered an exact solution. Domain M is a monoid (semigroup with a neutral element), Γ is a finite subgroup of the automorphisms group of M , ℓ is the cardinality of Γ , and $f, g : M \rightarrow G$ such that $(G, +)$ denotes an ℓ -cancellative commutative group. We also examine the hyperstability of the given equation in its inhomogeneous version $\sum_{\rho \in \Gamma} f(x\rho.y) = \ell f(x) + \ell g(y) + \psi(x, y)$, $x, y \in M$, where $\psi : M \times M \rightarrow G$. Additionally, we apply the main results to elucidate the hyperstability of various functional equations with involutions.

Keywords: stability; hyperstability; Pexiderized functional equations; involutions; monoids**MSC:** 39B82; 39B52

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1. Introduction

The stability theory of functional equations is a branch of mathematics that deals with the study of the behavior of solutions to functional equations under perturbations. Stability theory has its roots in the work of Ulam [1] (p. 63), who in 1940 posed the following question:

Ulam's question. Let $(E, .)$ be a group and let $(F, ., d)$ be a metric group with the metric d . Given $\delta > 0$, does there exist $\varepsilon > 0$ such that if mapping $f : E \rightarrow F$ satisfies the inequality $d(f(xy), f(x)f(y)) < \varepsilon$ for all $x, y \in E$, then there exists a homomorphism $h : E \rightarrow F$ with $d(f(x), h(x)) < \delta$ for all $x \in E$?

In 1941, the study of the stability theory of functional equations was initiated by the work of Hyers. In [2], he provided an affirmative partial answer to Ulam's question for the additive functional equation when E and F are Banach spaces. Hyers' result was generalized in 1950 by Aoki [3], Bourgin [4], and Rassias [5] for additive mappings and linear mappings, respectively, by considering an unbounded Cauchy difference. In 1994, Găvruta [6] provided a further generalization of Rassias et al.'s result in which he replaced the bound by a general control positive function $\varphi(x, y)$ for the existence of unique linear mapping.

During that time, a distinct form of stability, known as hyperstability, was established. The hyperstability of a functional equation requires that any mapping satisfying the equation approximately (in some sense) must be a real solution to it. While the term "hyperstability" was coined in 2001 by Maksa [7], the earliest discovery of hyperstability seems to date back to 1949, as introduced by Bourgin in [8]. One of the most famous methods for proving the hyperstability of functional equations is the fixed point approach,

which appeared in the past two decades through the contributions of J. Brzdęk and K. Ciepliński [9] to the fixed point theorem, and then other authors. We refer, for example, to [10–16]. In addition, there are dozens of published papers that are concerned with the hyperstability of functional equations; see, for instance, [17–21].

The simplest form of the Pexiderized functional equation can be expressed as $h(x + y) = f(x) + g(y)$, where f, g , and h are functions of a given domain and range and x and y are variables. This form is a generalization of the Cauchy functional equation $f(x + y) = f(x) + f(y)$. The term “Pexiderized” comes from the fact that this form was first introduced by J. V. Pexider in 1903 [22].

In this paper, we deduce some hyperstability results for a generalized class of homogeneous Pexiderized functional equations, expressed as follows:

$$\sum_{\rho \in \Gamma} f(x\rho.y) = \ell f(x) + \ell g(y), \quad x, y \in M, \quad (1)$$

for functions $f, g : M \rightarrow G$ where domain M is a monoid (a semigroup with a neutral element), Γ is a finite subgroup of the automorphisms group of M , ℓ is the cardinality of Γ (i.e., $\ell = \text{card}(\Gamma)$), and $(G, +)$ denotes an ℓ -cancellative commutative group. In 2012, Łukasik [23] established the complete solution for Equation (1) when M is a commutative semigroup. The proof of our main results is based on the development of Pales’ method in [24]. The idea behind this method is to derive an identity for the two-variable function obtained by taking the difference of the left and right sides of the equation under consideration. This method was used, for the first time, in [7], then in [25–27]. In addition, the main results allow for the conclusion of many results in the hyperstability of the functional equations with involutions. This paper is designed as follows: In Section 2, we deduce the hyperstability results for Equation (1) and its inhomogeneous version

$$\sum_{\rho \in \Gamma} \ell f(x\rho.y) = \ell f(x) + \ell g(y) + \psi(x, y), \quad x, y \in M, \quad (2)$$

where $\psi : M \times M \rightarrow G$. Several results concerning the hyperstability of functional equations with involutions are explained in Section 3 of this paper.

2. The Main Results

This section is devoted to deducing the main hyperstability results of Equation (1) and its inhomogeneous version (2). From now on, we suppose that $(M, *)$ is a monoid (a semigroup with a neutral element e), Γ is a finite subgroup of the automorphisms group of M , and $(G, +, d)$ is an invariant metric ℓ -cancellative commutative group with metric $d : G \times G \rightarrow [0, \infty)$. ℓ -cancellative means that $\ell u = \ell v$ implies $u = v$, where $u, v \in G$ and $\ell \in \mathbb{N}$. The invariant metric d means that

$$d(u + w, v + w) = d(u, v), \quad u, v, w \in G.$$

We write xy instead of $x * y$ and $\rho.x := \rho(x)$ for all $x, y \in M$, unless we mention otherwise. From the basic properties of the metric, we have

$$d(u_1 + u_2, v_1 + v_2) \leq d(u_1, v_1) + d(u_2, v_2), \quad u_1, u_2, v_1, v_2 \in G.$$

Hence, by mathematical induction on $n \in \mathbb{N}$ with $n \geq 2$, it is easy to show that

$$d\left(\sum_{i=1}^n u_i, \sum_{i=1}^n v_i\right) \leq \sum_{i=1}^n d(u_i, v_i),$$

for all $u_1, \dots, u_n, v_1, \dots, v_n \in G$.

In the next theorem, we prove the hyperstability results for Equation (1).

Theorem 1. Consider the existence of sequence $\{a_n\}_{n \in \mathbb{N}}$ in M that meets the following conditions:

$$\lim_{n \rightarrow \infty} \inf \varphi(x, y\rho.a_n) = \lim_{n \rightarrow \infty} \inf \varphi(x, ya_n) = 0, \quad (3)$$

and

$$\lim_{n \rightarrow \infty} \inf \varphi(a_n\rho.x, y) = \lim_{n \rightarrow \infty} \inf \varphi(a_nx, y) = 0, \quad (4)$$

for all $x, y \in M$ and all $\rho \in \Gamma$, where $\varphi : M \times M \rightarrow [0, \infty)$. Assume that $f, g : M \rightarrow G$ satisfy the following functional inequality:

$$d\left(\sum_{\rho \in \Gamma} f(x\rho.y), \ell f(x) + \ell g(y)\right) \leq \varphi(x, y), \quad x, y \in M. \quad (5)$$

Then, we have

1. f is a solution of equation

$$\sum_{\rho \in \Gamma} f(x\rho.y) + \ell f(e) = \ell f(x) + \sum_{\rho \in \Gamma} f(\rho.y), \quad x, y \in M. \quad (6)$$

2. g is a solution of equation

$$\sum_{\rho \in \Gamma} g(x\rho.y) = \ell g(x) + \ell g(y), \quad x, y \in M. \quad (7)$$

Proof. We consider function $D : M \times M \rightarrow [0, \infty)$ that is defined by

$$D(x, y) := d\left(\sum_{\rho \in \Gamma} f(x\rho.y), \ell f(x) + \ell g(y)\right),$$

for all $x, y \in M$. Inequality (5) can be written as

$$D(x, y) \leq \varphi(x, y), \quad x, y \in M. \quad (8)$$

(1) For every $x, y, z \in M$, we have

$$\begin{aligned} & d\left(\sum_{\rho \in \Gamma} \ell f(x\rho.y) + \ell^2 f(e), \ell^2 f(x) + \sum_{\rho \in \Gamma} \ell f(\rho.y)\right) = \\ & d\left(\sum_{\rho \in \Gamma} \ell f(x\rho.y) + \ell^2 f(e) + \sum_{\rho \in \Gamma} \sum_{\eta \in \Gamma} g(z) + \sum_{\rho \in \Gamma} \sum_{\eta \in \Gamma} f(x\rho.y\eta.z) + \sum_{\rho \in \Gamma} \sum_{\eta \in \Gamma} g(y\eta.z) + \sum_{\rho \in \Gamma} \sum_{\eta \in \Gamma} f(\rho.y\eta.z) \right. \\ & \quad \left. , \ell^2 f(x) + \sum_{\rho \in \Gamma} \ell f(\rho.y) + \sum_{\rho \in \Gamma} \sum_{\eta \in \Gamma} g(z) + \sum_{\rho \in \Gamma} \sum_{\eta \in \Gamma} f(x\rho.y\eta.z) + \sum_{\rho \in \Gamma} \sum_{\eta \in \Gamma} g(y\eta.z) + \sum_{\rho \in \Gamma} \sum_{\eta \in \Gamma} f(\rho.y\eta.z)\right) \\ & \leq d\left(\sum_{\rho \in \Gamma} \ell f(x\rho.y) + \sum_{\rho \in \Gamma} \ell g(z), \sum_{\rho \in \Gamma} \sum_{\eta \in \Gamma} f(x\rho.y\eta.z)\right) + d\left(\sum_{\rho \in \Gamma} \sum_{\eta \in \Gamma} f(x\rho.y\eta.z), \ell^2 f(x) + \sum_{\rho \in \Gamma} \sum_{\eta \in \Gamma} g(y\eta.z)\right) \\ & \quad + d\left(\sum_{\rho \in \Gamma} \sum_{\eta \in \Gamma} f(\rho.y\eta.z), \sum_{\rho \in \Gamma} \ell f(\rho.y) + \sum_{\rho \in \Gamma} \ell g(z)\right) + d\left(\sum_{\rho \in \Gamma} \sum_{\eta \in \Gamma} f(\rho.y\eta.z), \sum_{\rho \in \Gamma} \ell g(y.\rho.z) + \sum_{\rho \in \Gamma} \ell f(e)\right) \\ & \leq \sum_{\rho \in \Gamma} d\left(\ell f(x\rho.y) + \ell g(z), \sum_{\eta \in \Gamma} f(x\rho.y\eta.z)\right) + \sum_{\eta \in \Gamma} d\left(\sum_{\rho \in \Gamma} f(x\rho.y\eta.z), \ell f(x) + g(y\eta.z)\right) \\ & \quad + \sum_{\rho \in \Gamma} d\left(\sum_{\eta \in \Gamma} f(\rho.y\eta.z), \ell f(\rho.y) + \ell g(z)\right) + \sum_{\rho \in \Gamma} d\left(\sum_{\eta \in \Gamma} f(\eta.(y\rho.z)), \ell g(y\rho.z) + \ell f(e)\right). \end{aligned}$$

Therefore,

$$d\left(\sum_{\rho \in \Gamma} \ell f(x\rho.y) + \ell^2 f(e), \ell^2 f(x) + \sum_{\rho \in \Gamma} \ell f(\rho.y)\right) \leq \sum_{\rho \in \Gamma} D(x\rho.y, z) + \sum_{\eta \in \Gamma} D(x, y\eta.z) \\ + \sum_{\rho \in \Gamma} D(\rho.y, z) + \sum_{\rho \in \Gamma} D(e, y.\rho z), \quad (9)$$

for all $x, y, z \in M$. Next, we assume that there is a sequence, $\{a_n\}_{n \in \mathbb{N}}$, of elements of M that satisfies (3) and (4). Letting $x, y, t \in M$ be fixed and replacing z by ta_n in (9), we obtain

$$d\left(\sum_{\rho \in \Gamma} \ell f(x\rho.y) + \ell^2 f(e), \ell^2 f(x) + \sum_{\rho \in \Gamma} \ell f(\rho.y)\right) \leq \sum_{\rho \in \Gamma} D(x\rho.y, ta_n) + \sum_{\eta \in \Gamma} D(x, y\eta.ta_n) \\ + \sum_{\rho \in \Gamma} D(\rho.y, ta_n) + \sum_{\rho \in \Gamma} D(e, y.\rho ta_n), \quad (10)$$

for all $n \in \mathbb{N}$. Using (8), we notice that

$$D(x\rho.y, ta_n) \leq \varphi(x\rho.y, ta_n), \quad D(x, y\rho.ta_n) \leq \varphi(x, y\rho.ta_n), \quad \text{and} \quad D(\rho.y, ta_n) \leq \varphi(\rho.y, ta_n),$$

for all $n \in \mathbb{N}$, and all $\rho \in \Gamma$. In view of (3) and (4), we have

$$\lim_{n \rightarrow \infty} \inf D(x\rho.y, ta_n) = 0, \quad \lim_{n \rightarrow \infty} \inf D(x, y\rho.ta_n) = 0, \quad \text{and} \quad \lim_{n \rightarrow \infty} \inf D(\rho.y, ta_n) = 0. \quad (11)$$

Taking $n \rightarrow \infty$ in (10) and applying (11), we deduce that

$$d\left(\sum_{\rho \in \Gamma} \ell f(x\rho.y) + \ell^2 f(e), \ell^2 f(x) + \sum_{\rho \in \Gamma} \ell f(\rho.y)\right) = 0,$$

for all $x, y, z \in M$. Therefore,

$$\sum_{\rho \in \Gamma} \ell f(x\rho.y) + \ell^2 f(e) = \ell^2 f(x) + \sum_{\rho \in \Gamma} \ell f(\rho.y),$$

for all $x, y \in M$, which implies that

$$\ell\left(\sum_{\rho \in \Gamma} f(x\rho.y) + \ell f(e)\right) = \ell\left(\ell f(x) + \sum_{\rho \in \Gamma} f(\rho.y)\right),$$

for all $x, y \in M$. Based on the fact that G is an ℓ -cancellative commutative group, we obtain

$$\sum_{\rho \in \Gamma} f(x\rho.y) + \ell f(e) = \ell f(x) + \sum_{\rho \in \Gamma} f(\rho.y),$$

for all $x, y \in M$.

(2) For each $x, y, z \in M$, we notice that

$$\begin{aligned}
d\left(\sum_{\rho \in \Gamma} \ell g(x\rho.y), \ell^2 g(x) + \ell^2 g(y)\right) &= d\left(\sum_{\rho \in \Gamma} \ell g(x\rho.y) + \ell^2 f(z) + \sum_{\rho \in \Gamma} \sum_{\eta \in \Gamma} f(z\eta.x\rho.y) + \sum_{\rho \in \Gamma} \sum_{\eta \in \Gamma} f(z\eta.x) \right. \\
&\quad \left. , \ell^2 g(x) + \ell^2 g(y) + \ell^2 f(z) + \sum_{\rho \in \Gamma} \sum_{\eta \in \Gamma} f(z\eta.x\rho.y) + \sum_{\rho \in \Gamma} \sum_{\eta \in \Gamma} f(z\eta.x)\right) \\
&\leq d\left(\sum_{\rho \in \Gamma} \ell g(x\rho.y) + \ell^2 f(z), \sum_{\rho \in \Gamma} \sum_{\eta \in \Gamma} f(z\eta.x\rho.y)\right) \\
&\quad + d\left(\sum_{\rho \in \Gamma} \sum_{\eta \in \Gamma} f(z\eta.x\rho.y), \sum_{\rho \in \Gamma} \sum_{\eta \in \Gamma} f(z\eta.x) + \ell^2 g(y)\right) \\
&\quad + d\left(\sum_{\rho \in \Gamma} \sum_{\eta \in \Gamma} f(z\eta.x), \sum_{\rho \in \Gamma} \ell f(y) + \sum_{\rho \in \Gamma} \ell g(z)\right) \\
&\leq \sum_{\rho \in \Gamma} d\left(\ell g(x\rho.y) + \ell f(z), \sum_{\eta \in \Gamma} f(z\eta.x\rho.y)\right) \\
&\quad + \sum_{\eta \in \Gamma} d\left(\sum_{\rho \in \Gamma} f(z\eta.x\rho.y), \ell f(z\eta.x) + \ell g(y)\right) \\
&\quad + \sum_{\rho \in \Gamma} d\left(\sum_{\eta \in \Gamma} f(z\eta.x), \ell f(z) + \ell g(x)\right).
\end{aligned}$$

Hence,

$$d\left(\sum_{\rho \in \Gamma} \ell g(x\rho.y), \ell^2 g(x) + \ell^2 g(y)\right) \leq \sum_{\rho \in \Gamma} D(z, x\rho.y) + \sum_{\rho \in \Gamma} D(z\rho.x, y) + \sum_{\rho \in \Gamma} D(z, x), \quad (12)$$

for all $x, y, z \in M$. Assuming the existence of sequence $\{a_n\}_{n \in \mathbb{N}}$ of elements in M that satisfies (3) and (4), and letting $x, y, s \in M$ be fixed, we can replace z by sa_n in (12) to obtain the following inequality:

$$d\left(\sum_{\rho \in \Gamma} \ell g(x\rho.y), \ell^2 g(x) + \ell^2 g(y)\right) \leq \sum_{\rho \in \Gamma} D(sa_n, x\rho.y) + \sum_{\rho \in \Gamma} D(sa_n\rho.x, y) + \sum_{\rho \in \Gamma} D(sa_n, x), \quad (13)$$

for all $n \in \mathbb{N}$. Using (8), we obtain

$$D(sa_n, x\rho.y) \leq \varphi(sa_n, x\rho.y), \quad D(sa_n\rho.x, y) \leq \varphi(sa_n\rho.x, y), \quad \text{and} \quad D(sa_n, x) \leq \varphi(sa_n, x),$$

for all $n \in \mathbb{N}$, and all $\rho \in \Gamma$. Taking into account (3) and (4), we find

$$\lim_{n \rightarrow \infty} \inf D(sa_n, x\rho.y) = 0, \quad \lim_{n \rightarrow \infty} \inf D(sa_n\rho.x, y) = 0, \quad \text{and} \quad \lim_{n \rightarrow \infty} \inf D(sa_n, x) = 0. \quad (14)$$

Letting $n \rightarrow \infty$ in (13) and using (14), we deduce that

$$d\left(\sum_{\rho \in \Gamma} \ell g(x\rho.y), \ell^2 g(x) + \ell^2 g(y)\right) = 0,$$

for all $x, y \in M$. Then,

$$\ell \sum_{\rho \in \Gamma} g(x\rho.y) = \ell \left(\ell g(x) + \ell g(y) \right),$$

for all $x, y \in M$. Keeping in mind the ℓ -cancellativity of G , g is a solution to (7). $\square$

Letting $\ell g(y) \equiv \sum_{\rho \in \Gamma} f(\rho.y)$ for all $y \in M$, we obtain the special case of Equation (1) as follows:

$$\sum_{\rho \in \Gamma} f(x\rho.y) = \ell f(x) + \sum_{\rho \in \Gamma} f(\rho.y), \quad x, y \in M. \quad (15)$$

As a direct consequence, we conclude the next corollary that corresponds to the hyperstability results of Equation (15), which was studied in [27].

Corollary 1. Consider the existence of sequence $\{a_n\}_{n \in \mathbb{N}}$ in M that meets the following condition:

$$\lim_{n \rightarrow \infty} \inf \varphi(x, y\rho.a_n) = \lim_{n \rightarrow \infty} \inf \varphi(x, ya_n) = 0,$$

for all $x, y \in M$ and all $\rho \in \Gamma$, where $\varphi : M \times M \rightarrow [0, \infty)$. Suppose that $f : M \rightarrow G$ fulfills inequality

$$d\left(\sum_{\rho \in \Gamma} f(x\rho.y), \ell f(x) + \sum_{\rho \in \Gamma} f(\rho.y)\right) \leq \varphi(x, y), \quad x, y \in M.$$

Then, f is a solution of equation

$$\sum_{\rho \in \Gamma} f(x\rho.y) + \ell f(e) = \ell f(x) + \sum_{\rho \in \Gamma} f(\rho.y), \quad x, y \in M. \quad (16)$$

Also, f is a solution to Equation (15) if and only if $f(e) = 0$.

Proof. Taking $\ell g(y) \equiv \sum_{\rho \in \Gamma} f(\rho.y)$ for all $y \in M$ in Theorem 1 implies that f is a solution to Equation (16). From the ℓ -cancellativity of G , we determine that f is a solution to Equation (15) if and only if $f(e) = 0$. $\square$

Based on Theorem 1, we discuss the hyperstability of the inhomogeneous functional Equation (2).

Theorem 2. Let $\varphi : M \times M \rightarrow [0, \infty)$ be a function and consider the existence of sequence $\{a_n\}_{n \in \mathbb{N}}$ in M such that (3) and (4) hold. Let $f, g : M \rightarrow G$ and $\psi : M \times M \rightarrow G$ be functions such that

$$d\left(\sum_{\rho \in \Gamma} f(x\rho.y), \ell f(x) + \ell g(y) + \psi(x, y)\right) \leq \varphi(x, y), \quad x, y \in M.$$

Suppose that (2) admits solutions $f_0, g_0 : M \rightarrow G$. Then,

1. f is a solution of equation

$$\sum_{\rho \in \Gamma} f(x\rho.y) + \ell f(e) = \ell f(x) + \sum_{\rho \in \Gamma} f(\rho.y) + \psi(x, y), \quad x, y \in M. \quad (17)$$

2. g is a solution of equation

$$\sum_{\rho \in \Gamma} g(x\rho.y) = \ell g(x) + \ell g(y) + \psi(x, y), \quad x, y \in M. \quad (18)$$

Proof. We let $f_1(x) := f(x) - f_0(x)$, and $g_1(x) = g(x) - g_0(x)$ for all $x \in M$. Then,

$$\begin{aligned}
& d\left(\sum_{\rho \in \Gamma} f_1(x\rho.y), \ell f_1(x) + \ell g_1(y)\right) \\
&= d\left(\sum_{\rho \in \Gamma} f(x\rho.y) - \sum_{\rho \in \Gamma} f_0(x\rho.y), \ell f(x) + \ell g(y) + \psi(x, y) - \ell f_0(x) - \ell g_0(y) - \psi(x, y)\right) \\
&\leq d\left(\sum_{\rho \in \Gamma} f(x\rho.y), \ell f(x) + \ell g(y) + \psi(x, y)\right) \\
&\quad + d\left(-\sum_{\rho \in \Gamma} f_0(x\rho.y), -\ell f_0(x) - \ell g_0(y) - \psi(x, y)\right) \\
&= d\left(\sum_{\rho \in \Gamma} f(x\rho.y), \ell f(x) + \ell g(y) + \psi(x, y)\right) \\
&\leq \varphi(x, y), \quad x, y \in M.
\end{aligned}$$

As a result, the conditions of Theorem 1 are satisfied when f and g are replaced by f_1 and g_1 . Therefore,

$$\begin{aligned}
& \sum_{\rho \in \Gamma} f(x\rho.y) + \ell f(e) - \ell f(x) - \sum_{\rho \in \Gamma} f(\rho.y) - \psi(x, y) = \sum_{\rho \in \Gamma} f_1(x\rho.y) + \ell f_1(e) - \ell f_1(x) - \sum_{\rho \in \Gamma} f_1(\rho.y) \\
&\quad + \sum_{\rho \in \Gamma} f_0(x\rho.y) + \ell f_0(e) - \ell f_0(x) - \sum_{\rho \in \Gamma} f_0(\rho.y) - \psi(x, y) \\
&= 0,
\end{aligned}$$

for all $x, y \in M$. Moreover,

$$\begin{aligned}
& \sum_{\rho \in \Gamma} g(x\rho.y) - \ell g(x) - \ell g(y) - \psi(x, y) = \sum_{\rho \in \Gamma} g_1(x\rho.y) - \ell g_1(x) - \ell g_1(y) + \sum_{\rho \in \Gamma} g_0(x\rho.y) \\
&\quad - \ell g_0(x) - \ell g_0(y) - \psi(x, y) = 0,
\end{aligned}$$

for all $x, y \in M$. It means that f is a solution to Equation (17) and g is a solution of (18). $\square$

In the next corollary, we present the hyperstability of the inhomogeneous version of Equation (15) that is given as

$$\sum_{\rho \in \Gamma} f(x\rho.y) = \ell f(x) + \sum_{\rho \in \Gamma} f(\rho.y) + \psi(x, y), \quad x, y \in M, \quad (19)$$

where $\psi : M \times M \rightarrow G$.

Corollary 2. Let $\varphi : M \times M \rightarrow [0, \infty)$ be a function and consider the existence of sequence $\{a_n\}_{n \in \mathbb{N}}$ in M such that (3) and (4) hold. Let $f : M \rightarrow G$ and $\psi : M \times M \rightarrow G$ be functions such that

$$d\left(\sum_{\rho \in \Gamma} f(x\rho.y), \ell f(x) + \sum_{\rho \in \Gamma} f(\rho.y) + \psi(x, y)\right) \leq \varphi(x, y), \quad x, y \in M.$$

Assume that functional Equation (19) admits solutions $f_0, g_0 : M \rightarrow G$. Then,

1. f is a solution of equation

$$\sum_{\rho \in \Gamma} f(x\rho.y) + \ell f(e) = \ell f(x) + \sum_{\rho \in \Gamma} f(\rho.y) + \psi(x, y), \quad x, y \in M. \quad (20)$$

2. f is a solution to Equation (19) if and only if $f(e) = 0$.

Proof. Taking $\ell g(y) \equiv \sum_{\rho \in \Gamma} f(\rho \cdot y)$ for all $y \in M$ in Theorem 2 implies that f is a solution to Equation (20). From the ℓ -cancellativity of G , we determine that f is a solution to Equation (19) if and only if $f(e) = 0$. $\square$

3. Applications on Functional Equations with Involutions

Automorphism $\sigma : M \rightarrow M$ is said to be involution if $\sigma \circ \sigma = id_M$. The stability of functional equations with involutions was first studied by Bouikhalene et al. [28] in 2007, when they proved the Hyers–Ulam stability of the quadratic functional equation in normed spaces. In 2008, Jung and Lee [29] applied the fixed point method to prove the stability of quadratic functional equations with involutions for a large class of functions. The study of functional equations with involutions has continued to attract the attention of numerous researchers, and the results have been applied in a wide range of mathematical fields, e.g., [30–35]. The hyperstability study of this type of functional equations began in 2016 when Almahalebi [25] investigated the hyperstability of σ -Drygas functional equations with an involution. In 2018, EL-Fassi et al. [26] studied the hyperstability of Pexiderized quadratic functional equations with involutions in semigroups with a neutral element. Additionally, EL Ghali et al. [36] investigated the hyperstability of a Jensen functional equation in non-Archimedean 2-Banach spaces with an involutorial homomorphism.

In this section, we apply our main results to prove the hyperstability of some functional equations with involutions $\sigma, \tau : M \rightarrow M$.

In 2021, Akkaoui et al. [37] described solutions $f, h : M \rightarrow G$ of the following Pexiderized (σ, τ) -quadratic functional equation:

$$f(x\sigma(y)) + f(x\tau(y)) = 2f(x) + h(y), \quad x, y \in M, \quad (21)$$

in the situation where M is a commutative semigroup, G is a 2-torsion free commutative group, and $\sigma, \tau : M \rightarrow M$ are endomorphisms. The following result is concerning for the hyperstability of Equation (21).

Corollary 3. Let $\varphi : M \times M \rightarrow [0, \infty)$ be a function and consider the existence of sequence $\{a_n\}_{n \in \mathbb{N}}$ in M that meets the following conditions:

$$\lim_{n \rightarrow \infty} \inf \varphi(x, ya_n) = \lim_{n \rightarrow \infty} \inf \varphi(x, y\sigma(a_n)) = \lim_{n \rightarrow \infty} \inf \varphi(x, y\tau(a_n)) = 0 \quad (22)$$

and

$$\lim_{n \rightarrow \infty} \inf \varphi(a_n x, y) = \lim_{n \rightarrow \infty} \inf \varphi(a_n \sigma(x), y) = \lim_{n \rightarrow \infty} \inf \varphi(a_n \tau(x), y) = 0, \quad (23)$$

for all $x, y \in M$. Assume that $f, h : M \rightarrow G$ satisfy inequality

$$d(f(x\sigma(y)) + f(x\tau(y)), 2f(x) + h(y)) \leq \varphi(x, y), \quad x, y \in M,$$

for all $x, y \in M$. Then,

1. f is a solution of equation

$$f(x\sigma(y)) + f(x\tau(y)) + 2f(e) = 2f(x) + f(\sigma(y)) + f(\tau(y)), \quad x, y \in M$$

2. h is a solution of equation

$$h(x\sigma(y)) + h(x\tau(y)) = 2h(x) + 2h(y), \quad x, y \in M.$$

Proof. Putting $\Gamma = \{\sigma, \tau\}$ in Theorem 1, we obtain $\ell = 2$. If we take $2g(y) \equiv h(y)$ for all $y \in M$, then we notice that Conditions (3) and (4) hold, which means that the result is achieved. $\square$

The hyperstability of the inhomogeneous version of Equation (21) is given in the following corollary.

Corollary 4. Let $\varphi : M \times M \rightarrow [0, \infty)$ be a function and consider the existence of sequence $\{a_n\}_{n \in \mathbb{N}}$ in M such that Conditions (22) and (23) hold. Let $f, h : M \rightarrow G$ and $\psi : M \times M \rightarrow G$ be functions such that

$$d(f(x\sigma(y)) + f(x\tau(y)), 2f(x) + h(y) + \psi(x, y)) \leq \varphi(x, y), \quad x, y \in M.$$

Suppose that the following equation,

$$f(x\sigma(y)) + f(x\tau(y)) = 2f(x) + h(y) + \psi(x, y),$$

admits solutions $f_0, h_0 : M \rightarrow G$. Then,

1. f is a solution of the following equation:

$$f(x\sigma(y)) + f(x\tau(y)) + 2f(e) = 2f(x) + f(\sigma(y)) + f(\tau(y)) + \psi(x, y), \quad x, y \in M.$$

2. h is a solution of equation

$$h(x\sigma(y)) + h(x\tau(y)) = 2h(x) + 2h(y) + \psi(x, y), \quad x, y \in M.$$

Proof. We just apply Theorem 2 and use the same substitutions in the proof of Corollary 3. $\square$

In 2018, EL-Fassi et al. [26] discussed the hyperstability of the following Pexiderized σ -quadratic functional equation:

$$f(xy) + f(x\sigma(y)) = 2f(x) + h(y), \quad x, y \in M, \quad (24)$$

in the case where G is a semigroup with a neutral element and where $\sigma : M \rightarrow M$ is an endomorphism and an involution. In the following two corollaries, we find the same results [26], but as special cases of Theorems 1 and 2 by taking $\Gamma = \{id_M, \sigma\}$ and $2g(y) \equiv h(y)$ for all $y \in M$.

Corollary 5. Let $\varphi : M \times M \rightarrow [0, \infty)$ be a function and consider the existence of sequence $\{a_n\}_{n \in \mathbb{N}}$ in M that meets the conditions

$$\lim_{n \rightarrow \infty} \inf \varphi(x, ya_n) = \lim_{n \rightarrow \infty} \inf \varphi(x, y\sigma(a_n)) = 0 \quad (25)$$

and

$$\lim_{n \rightarrow \infty} \inf \varphi(a_n x, y) = 0, \quad (26)$$

for all $x, y \in M$. Suppose that $f, h : M \rightarrow G$ satisfy inequality

$$d(f(xy) + f(x\sigma(y)), 2f(x) + h(y)) \leq \varphi(x, y),$$

for all $x, y \in M$. Then,

1. f is a solution of equation

$$f(xy) + f(x\sigma(y)) + 2f(e) = 2f(x) + f(y) + f(\sigma(y)), \quad x, y \in M.$$

2. h is a solution of equation

$$h(xy) + h(x\sigma(y)) = 2h(x) + 2h(y), \quad x, y \in M.$$

Corollary 6. Let $\varphi : M \times M \rightarrow [0, \infty)$ be a function and consider the existence of sequence $\{a_n\}_{n \in \mathbb{N}}$ in M such that Conditions (25) and (26) hold. Let $f, h : M \rightarrow G$ and $\psi : M \times M \rightarrow G$ be functions such that

$$d(f(xy) + f(x\sigma(y)), 2f(x) + h(y) + \psi(x, y)) \leq \varphi(x, y), \quad x, y \in M.$$

Suppose that functional equation

$$f(xy) + f(x\sigma(y)) = 2f(x) + h(y) + \psi(x, y)$$

admits solutions $f_0, h_0 : M \rightarrow G$. Then,

1. f is a solution of equation

$$f(xy) + f(x\sigma(y)) + 2f(e) = 2f(x) + f(y) + f(\sigma(y)) + \psi(x, y), \quad x, y \in M.$$

2. h is a solution of equation

$$h(xy) + h(x\sigma(y)) = 2h(x) + 2h(y) + \psi(x, y), \quad x, y \in M.$$

The (σ, τ) -quadratic function equation is given as

$$f(x\sigma(y)) + f(x\tau(y)) = 2f(x) + 2f(y), \quad x, y \in M. \quad (27)$$

Solutions $f : M \rightarrow G$ of Equation (27) are described in [38]. In the following two corollaries, it is enough to put $\Gamma = \{\sigma, \tau\}$ and $g(y) \equiv f(y)$ for all $y \in M$ to prove the hyperstability of Equation (27) and its inhomogeneous version as special cases of Theorem 1 and Theorem 2, respectively.

Corollary 7. Let $\varphi : M \times M \rightarrow [0, \infty)$ be a function and consider the existence of sequence $\{a_n\}_{n \in \mathbb{N}}$ in M that meets the following conditions:

$$\liminf_{n \rightarrow \infty} \varphi(x, ya_n) = \liminf_{n \rightarrow \infty} \varphi(x, y\sigma(a_n)) = \liminf_{n \rightarrow \infty} \varphi(x, y\tau(a_n)) = 0 \quad (28)$$

and

$$\liminf_{n \rightarrow \infty} \varphi(a_n x, y) = \liminf_{n \rightarrow \infty} \varphi(a_n \sigma(x), y) = \liminf_{n \rightarrow \infty} \varphi(a_n \tau(x), y) = 0, \quad (29)$$

for all $x, y \in M$. Assume that $f : M \rightarrow G$ satisfies inequality

$$d(f(x\sigma(y)) + f(x\tau(y)), 2f(x) + 2f(y)) \leq \varphi(x, y),$$

for all $x, y \in M$. Then, f is a solution to Equation (27).

Corollary 8. Let $\varphi : M \times M \rightarrow [0, \infty)$ be a function and consider the existence of sequence $\{a_n\}_{n \in \mathbb{N}}$ in M such that Conditions (28) and (29) hold. Let $f : M \rightarrow G$ and $\psi : M \times M \rightarrow G$ be functions such that

$$d(f(x\sigma(y)) + f(x\tau(y)), 2f(x) + 2f(y) + \psi(x, y)) \leq \varphi(x, y), \quad x, y \in M.$$

Suppose that functional equation

$$f(x\sigma(y)) + f(x\tau(y)) = 2f(x) + 2f(y) + \psi(x, y)$$

admits a solution $f_0 : M \rightarrow G$. Then,

1. f is a solution of equation

$$f(x\sigma(y)) + f(x\tau(y)) + 2f(e) = 2f(x) + 2f(y) + \psi(x, y), \quad x, y \in M.$$

2. f is a solution of equation

$$f(x\sigma(y)) + f(x\tau(y)) = 2f(x) + 2f(y) + \psi(x, y), \quad x, y \in M,$$

if and only if $f(e) = 0$.

When $\Gamma = \{id_M, \sigma\}$ and $g(y) \equiv f(y)$ for all $y \in M$, Equation (1) is just the following σ -quadratic functional equation:

$$f(xy) + f(x\sigma(y)) = 2f(x) + 2f(y), \quad x, y \in M. \quad (30)$$

All solutions $f : M \rightarrow G$ of Equation (30) are described in [39]. The following results concern the hyperstability of Equation (30) and its inhomogeneous version as direct consequences of Theorems 1 and 2.

Corollary 9. Let $\varphi : M \times M \rightarrow [0, \infty)$ be a function and consider the existence of sequence $\{a_n\}_{n \in \mathbb{N}}$ in M that meets the following conditions:

$$\lim_{n \rightarrow \infty} \inf \varphi(x, ya_n) = \lim_{n \rightarrow \infty} \inf \varphi(x, y\sigma(a_n)) = 0 \quad (31)$$

and

$$\lim_{n \rightarrow \infty} \inf \varphi(a_n x, y) = \lim_{n \rightarrow \infty} \inf \varphi(a_n \sigma(x), y) = 0, \quad (32)$$

for all $x, y \in M$. Assume that $f : M \rightarrow G$ satisfies inequality

$$d(f(xy) + f(x\sigma(y)), 2f(x) + 2f(y)) \leq \varphi(x, y),$$

for all $x, y \in M$. Then, f is a solution to Equation (30).

Corollary 10. Let $\varphi : M \times M \rightarrow [0, \infty)$ be a function and consider the existence of sequence $\{a_n\}_{n \in \mathbb{N}}$ in M such that Conditions (31) and (32) hold. Let $f : M \rightarrow G$ and $\psi : M \times M \rightarrow G$ be functions such that

$$d(f(xy) + f(x\sigma(y)), 2f(x) + 2f(y) + \psi(x, y)) \leq \varphi(x, y), \quad x, y \in M.$$

Assume that functional equation

$$f(xy) + f(x\sigma(y)) = 2f(x) + 2f(y) + \psi(x, y)$$

admits solution $f_0 : M \rightarrow G$. Then,

1. f is a solution of equation

$$f(xy) + f(x\sigma(y)) + 2f(e) = 2f(x) + 2f(y) + \psi(x, y), \quad x, y \in M.$$

2. f is a solution of equation

$$f(xy) + f(x\sigma(y)) = 2f(x) + 2f(y) + \psi(x, y), \quad x, y \in M,$$

if and only if $f(e) = 0$.

In 2021, Akkaoui et al. [37] described solutions $f : M \rightarrow G$ of the following (σ, τ) -Drygas functional equation:

$$f(x\sigma(y)) + f(x\tau(y)) = 2f(x) + f(\sigma(y)) + f(\tau(y)), \quad x, y \in M. \quad (33)$$

In the following two corollaries, we obtain the hyperstability of Equation (33) and its inhomogeneous version.

Corollary 11. Let $\varphi : M \times M \rightarrow [0, \infty)$ be a function and consider the existence of sequence $\{a_n\}_{n \in \mathbb{N}}$ in M that meets the following conditions:

$$\lim_{n \rightarrow \infty} \inf \varphi(x, ya_n) = \lim_{n \rightarrow \infty} \inf \varphi(x, y\sigma(a_n)) = \lim_{n \rightarrow \infty} \inf \varphi(x, y\tau(a_n)) = 0 \quad (34)$$

and

$$\lim_{n \rightarrow \infty} \inf \varphi(a_n x, y) = \lim_{n \rightarrow \infty} \inf \varphi(a_n \sigma(x), y) = \lim_{n \rightarrow \infty} \inf \varphi(a_n \tau(x), y) = 0, \quad (35)$$

for all $x, y \in M$. Assume that $f : M \rightarrow G$ satisfies inequality

$$d\left(f(x\sigma(y)) + f(x\tau(y)), 2f(x) + f(\sigma(y)) + f(\tau(y))\right) \leq \varphi(x, y), \quad x, y \in M.$$

Then,

1. f is a solution to equation

$$f(x\sigma(y)) + f(x\tau(y)) + 2f(e) = 2f(x) + f(\sigma(y)) + f(\tau(y)), \quad x, y \in M. \quad (36)$$

2. f is a solution to Equation (33) if and only if $f(e) = 0$.

Proof. Taking $\Gamma = \{\sigma, \tau\}$ with $2g(y) \equiv f(\sigma(y)) + f(\tau(y))$ in Theorem 1 implies that f is a solution to Equation (36). Also, because of the two-cancellativity of G , we determine directly that f is a solution to Equation (33) if and only if $f(e) = 0$. $\square$

Using a similar method, we can prove the following corollary.

Corollary 12. Let $\varphi : M \times M \rightarrow [0, \infty)$ be a function and consider the existence of sequence $\{a_n\}_{n \in \mathbb{N}}$ in M such that Conditions (34) and (35) hold. Let $f : M \rightarrow G$ and $\psi : M \times M \rightarrow G$ be functions such that

$$d\left(f(x\sigma(y)) + f(x\tau(y)), 2f(x) + f(\sigma(y)) + f(\tau(y)) + \psi(x, y)\right) \leq \varphi(x, y), \quad x, y \in M.$$

Suppose that functional equation

$$f(x\sigma(y)) + f(x\tau(y)) = 2f(x) + f(\sigma(y)) + f(\tau(y)) + \psi(x, y)$$

admits solution $f_0 : M \rightarrow G$. Then,

1. f is a solution of equation

$$f(x\sigma(y)) + f(x\tau(y)) + 2f(e) = 2f(x) + f(\sigma(y)) + f(\tau(y)) + \psi(x, y), \quad x, y \in M.$$

2. f is a solution of equation

$$f(xy) + f(x\tau(y)) = 2f(x) + f(\sigma(y)) + f(\tau(y)) + \psi(x, y), \quad x, y \in M,$$

if and only if $f(e) = 0$.

When $\Gamma = \{id_M, \sigma\}$ and $2g(y) \equiv f(y) + f(\sigma(y))$, we notice that Equation (1) becomes the following σ -Drygas functional equation:

$$f(xy) + f(x\sigma(y)) = 2f(x) + f(y) + f(\sigma(y)), \quad x, y \in M. \quad (37)$$

All solutions $f : M \rightarrow G$ of Equation (37) are described in [39]. Also, the hyperstability of Equation (37) is proven in [25,26].

In direct consequence of Theorem 1, the result on the hyperstability of Equation (37) follows.

Corollary 13. Let $\varphi : M \times M \rightarrow [0, \infty)$ be a function and consider the existence of sequence $\{a_n\}_{n \in \mathbb{N}}$ in M that meets the following conditions:

$$\lim_{n \rightarrow \infty} \inf \varphi(x, ya_n) = \lim_{n \rightarrow \infty} \inf \varphi(x, y\sigma(a_n)) = 0 \quad (38)$$

and

$$\lim_{n \rightarrow \infty} \inf \varphi(a_n x, y) = \lim_{n \rightarrow \infty} \inf \varphi(a_n \sigma(x), y) = 0, \quad (39)$$

for all $x, y \in M$. Suppose that $f : M \rightarrow G$ satisfies inequality

$$d(f(xy) + f(x\sigma(y)), 2f(x) + f(y) + f(\sigma(y))) \leq \varphi(x, y), \quad x, y \in M.$$

Then,

1. f is a solution of equation

$$f(xy) + f(x\sigma(y)) + 2f(e) = 2f(x) + f(y) + f(\sigma(y)), \quad x, y \in M.$$

2. f is a solution to Equation (37) if and only if $f(e) = 0$.

Corollary 14. Let $\varphi : M \times M \rightarrow [0, \infty)$ be a function and consider the existence of sequence $\{a_n\}_{n \in \mathbb{N}}$ in M such that Conditions (38) and (39) hold. Let $f : M \rightarrow G$ and $\psi : M \times M \rightarrow G$ be functions such that

$$d(f(xy) + f(x\sigma(y)), 2f(x) + f(y) + f(\sigma(y)) + \psi(x, y)) \leq \varphi(x, y), \quad x, y \in M.$$

Suppose that functional equation

$$f(xy) + f(x\sigma(y)) = 2f(x) + f(y) + f(\sigma(y)) + \psi(x, y)$$

admits solution $f_0 : M \rightarrow G$. Then,

1. f is a solution of equation

$$f(xy) + f(x\sigma(y)) + 2f(e) = 2f(x) + f(y) + f(\sigma(y)) + \psi(x, y), \quad x, y \in M.$$

2. f is a solution of equation

$$f(xy) + f(x\sigma(y)) = 2f(x) + f(y) + f(\sigma(y)) + \psi(x, y), \quad x, y \in M,$$

if and only if $f(e) = 0$.

4. Conclusions

Throughout this paper, we demonstrate the hyperstability of an equation characterized by its generality. Additionally, we derive the hyperstability of a number of functional equations with involutions simultaneously as special cases and direct consequences of the main results presented in this paper.

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References

1. Ulam, S.M. *A Collection of Mathematical Problems*; Interscience: New York, NY, USA, 1960.
2. Hyers, D.H. On the stability of the linear functional equation. *Proc. Nat. Acad. Sci. USA* **1941**, *27*, 222–224. [\[CrossRef\]](#)
3. Aoki, T. On the stability of the linear transformation in Banach spaces. *J. Math. Soc. Jpn.* **1950**, *2*, 64–66. [\[CrossRef\]](#)
4. Bourgin, D.G. Classes of transformations and bordering transformations. *Bull. Am. Math. Soc.* **1951**, *57*, 223–237. [\[CrossRef\]](#)
5. Rassias, T.M. On the stability of linear mapping in Banach spaces. *Proc. Am. Math. Soc.* **1978**, *72*, 297–300. [\[CrossRef\]](#)
6. Găvruta, P. A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings. *J. Math. Anal. Appl.* **1994**, *184*, 431–436. [\[CrossRef\]](#)
7. Maksa, G.; Páles, Z. Hyperstability of a class of linear functional equations. *Acta Math. Acad. Paedag. Nyíregyháziensis* **2001**, *17*, 107–112.
8. Bourgin, D.G. Approximately isometric and multiplicative transformations on continuous function rings. *Duke Math. J.* **1949**, *16*, 385–397. [\[CrossRef\]](#)
9. Brzdęk, J.; Ciepliński, K. A fixed point approach to the stability of functional equations in non-Archimedean metric spaces. *Nonlinear Anal.* **2011**, *74*, 6861–6867. [\[CrossRef\]](#)
10. Brzdęk, J.; Chudziak, J.; Páles, Z. A fixed point approach to stability of functional equations. *Nonlinear Anal.* **2011**, *74*, 6728–6732. [\[CrossRef\]](#)
11. Brzdęk, J. Stability of additivity and fixed point methods. *Fixed Point Theory Appl.* **2013**, *1*, 285. [\[CrossRef\]](#)
12. Brzdęk, J.; Ciepliński, K. On a fixed point theorem in 2-Banach spaces and some of its applications. *Acta Math. Sci.* **2018**, *38*, 377–744. [\[CrossRef\]](#)
13. Brzdęk, J.; Ciepliński, K. A fixed point theorem in n -Banach spaces and Ulam stability. *J. Math. Anal. Appl.* **2019**, *470*, 632–646. [\[CrossRef\]](#)
14. Almahalebi, M.; Chahbi, A. Hyperstability of the Jensen functional equation in ultrametric spaces. *Aequationes Math.* **2017**, *91*, 647–661. [\[CrossRef\]](#)
15. Almahalebi, M.; Chahbi, A. Approximate solution of p -radical functional equation in 2-Banach spaces. *Acta Math. Sci.* **2019**, *39*, 551–566. [\[CrossRef\]](#)
16. Almahalebi, M.; AL Ali, S.; Hryrou, M.; Elkettani, Y. A fixed point theorem in ultrametric n -Banach spaces and hyperstability results. *Fixed Point Theory* **2023**, *24*, 433–458.
17. Bahyrycz, A.; Brzdęk, J.; El-Hady, E.-S.; Leśniak, Z. On Ulam stability of functional equations in 2-normed spaces—A survey. *Symmetry* **2021**, *13*, 2200. [\[CrossRef\]](#)
18. El-Hady, E.-S.; Brzdęk, J. On Ulam stability of functional equations in 2-normed spaces—A survey II. *Symmetry* **2022**, *14*, 1365. [\[CrossRef\]](#)
19. Chang, I.-S.; Lee, Y.-H.; Roh, J. Representation and Stability of General Nonic Functional Equation. *Mathematics* **2023**, *11*, 3173. [\[CrossRef\]](#)
20. Hayati, B.; Khodaei, H. On triple θ -centralizers. *Int. J. Nonlinear Anal. Appl.* **2024**, *15*, 9–16.
21. Khodaei, H. Asymptotic Behavior of n -Jordan Homomorphisms. *Mediterr. J. Math.* **2020**, *17*, 143. [\[CrossRef\]](#)
22. Pexider, J.V. Notiz über Funktional theoreme. *Monatsh. Math. Phys.* **1903**, *14*, 293–301. [\[CrossRef\]](#)
23. Łukasik, R. Some generalization of Cauchy's and the quadratic functional equations. *Aequationes Math.* **2012**, *3*, 75–86. [\[CrossRef\]](#)
24. Páles, Z. Bounded solutions and stability of functional equations for two variable functions. *Results Math.* **1994**, *26*, 360–365. [\[CrossRef\]](#)
25. Almahalebi, M. On the hyperstability of σ -Drygas functional equation on semigroups. *Aequationes Math.* **2016**, *90*, 849–857. [\[CrossRef\]](#)
26. El-Fassi, I.; Brzdęk, J. On the hyperstability of a pexiderised σ -quadratic functional equation on semigroups. *Bull. Aust. Math. Soc.* **2018**, *97*, 459–470. [\[CrossRef\]](#)
27. Almahalebi, M.; EL Ghali, R.; Kabbaj, S. On the hyperstability of the generalized class of Drygas functional equations on semigroups. *Aequationes Math.* **2021**, *95*, 667–676. [\[CrossRef\]](#)
28. Bouikhalene, B.; Elqorachi, E.; Rassias, T.M. On the generalized Hyers-Ulam stability of the quadratic functional equation with a general involution. *Nonlinear Funct. Anal. Appl.* **2007**, *12*, 247–262.
29. Jung, S.-M.; Lee, Z.-H. A fixed point approach to the stability of quadratic functional equation with involution. *Fixed Point Theory App.* **2008**, *2008*, 732086. [\[CrossRef\]](#)
30. Pourpasha, M.M.; Rassias, J.M.; Saadati, R.; Vaezpour, S.M. A fixed point approach to the stability of Pexider quadratic functional equation with involution. *J. Inequal. Appl.* **2010**, *2010*, 839639. [\[CrossRef\]](#)
31. Kim, C.I.; Han, G.J. Stability of Drygas type functional equations with involution in non-Archimedean Banach spaces by fixed point method. *J. Appl. Math. Inform.* **2016**, *34*, 509–517. [\[CrossRef\]](#)
32. Chang, J.; Choi, C.K.; Kim, J.; Sahoo, P.K. Stability of the cosine-sine functional equation with involution. *Adv. Oper. Theory.* **2017**, *2*, 531–546.

33. EL-Fassi, I.; Chahbi, A.; Kabbaj, S. On Approximately of a σ -Quadratic Functional Equation on a Set of Measure Zero. *Complex Anal. Oper. Theory* **2018**, *12*, 111–121. [[CrossRef](#)]
34. Kangn, D.; Kim, H.B. Generalized stability of cubic functional equations with an automorphism on a quasi- β normed space. *J. Comput. Anal. Appl.* **2019**, *27*, 235–246.
35. Nuino, A.; Almahalebi, M.; Charifi, A. Measure Zero Stability Problem for Drygas Functional Equation with Complex Involution. In *Frontiers in Functional Equations and Analytic Inequalities*; Anastassiou, G., Rassias, J., Eds.; Springer: Berlin/Heidelberg, Germany, 2019.
36. EL Ghali, R.; Kabbaj, S. Some Hyperstability Results in Non-Archimedean 2-Banach Space for a σ -Jensen Functional Equation. In *Approximation and Computation in Science and Engineering*; Daras, N.J., Rassias, T.M., Eds.; Springer Optimization and Its Applications; Springer: Berlin/Heidelberg, Germany, 2022; Volume 180.
37. Akkaoui, A.; Fadli, B.; El Fatini, M. The Drygas Functional Equation on Abelian Semigroups with Endomorphisms. *Results Math.* **2021**, *76*, 42. [[CrossRef](#)]
38. Fadli, B.; Chahbi, A.; El-Fassi, I.; Kabbaj, S. On Jensen's and the quadratic functional Equations with involutions. *Proyecciones (Antofagasta)* **2016**, *35*, 213–223. [[CrossRef](#)]
39. Stetkær, H. Functional equations on abelian groups with involution. *Aequationes Math.* **1997**, *54*, 144–172. [[CrossRef](#)]

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