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Simple Moment Generating Function Optimisation Technique to Design Optimum Electronic Filter for Underwater Wireless Optical Communication Receiver

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Abstract: This paper introduces a new simple moment-generating function (MGF) design modelling method to conclude an optimum filter to maximize the Q-factor and increase the link communication span. This approach mitigates the pulse temporal dispersion, particularly the underwater wireless optical communication (UWOC) systems. Hence, some form of equalizing filter design is highly desirable. The model solution environment includes a Double Gamma Function (DGF) water channel impulse response, intersymbol interference (ISI), stochastic Poisson process, and additive Gaussian thermal noise (AGTN). The optimal filters exhibit temporal profiles comparable to those derived by published works based on complex Chernoff Bound (CB) and Modified Chernoff Bound (MCB) methods. The results show the impact of the optimum filter at a signal level and optical receiver level utilizing Eye-Diagrams and BER vs. Q-Factor, respectively. The computation involves four different UWOC propagation channel models for Coastal and Harbor waters. One of the main conclusions indicates that the optimum filter manages the temporal dispersion due to the ISI impairment correctly. Also, the proposed optimum filter reduces eye-opening and the corresponding Q-Factor by less than 15% for a five-times increase in pulse width for the same transmitted optical power level.

Keywords: MGF; UWOC; receiver filter design; eye diagram; Q-Factor; BER

MSC: 94-10



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1. Introduction

Many research groups worldwide, such as [1–7] work on different aspects of underwater wireless optical communications (UWOC). Their efforts include advanced simulation platforms and the design of receiver units for the UWOC Systems (Figure 1). One of the milestones in these projects is formulating a valid, operable, and consistent model for the water channel impulse response (CIR) function $h_c(t)$ that accounts for underwater channels and the associated optical propagation impairments (absorption, single/multiple scattering, and scintillation). These impairments cause signal degradations, which are manifested as serious ISI noise. This causes severe limitations for the UWOC system [8] in terms of bit rate and communication linkspan. To overcome such challenges, many research groups reduced the link length to around 50 m [8] and the transmission rate to a low of a few Mbps [8].

Nevertheless, none of [1–7] provides an approach to derive a UWOC receiver filter (Figure 1) to enhance the receiver unit performance in the presence of ISI without restoring to reducing the bit rate or link span. The studies in [1–7] presented UWOC optical system performance calculations in terms of Bit Error Rate (BER), assuming that the impulse

response of the receiver photodetector and its backend electronic circuitry (pre-amplifier, amplifier, and filter/equalizer) can be represented as a delta functions. Such an unrealistic assumption implies that the actual performance computation is at the input of the receiver photodetector rather than at the input of the receiver decision unit (Figure 1). Also, the work in [1–7] does not include any attempt to derive an optimum filter which requires an optimum Signal to Noise Ratio (SNR) or Q-Factor to secure an optimum BER for maximum channel link span.

There is a large volume of work [9–17] based on complex CB and MCB to handle receiver design optimization for optical fiber communication. However, the main elements of receiver optimization methodology are the filter impulse response $h_f(t)$, the photodetector internal gain (g), and the threshold for the decision unit F_D (Figure 1). In [9], the author discussed the receiver design based on directly minimizing the BER. In [10–16], the authors developed clever approaches to formulate a functional filter (E_{filter}) by applying upper CB or MCB on the BER for binary optical fiber systems. Typically, the derived E_{filter} encompasses the receiver filter function $h_f(t)$, its derivative $h'_f(t)$, and F_D . This implies that the function $h_f(t)$ must be continuous, and its 1st and 2nd derivatives must exist. Consequently, E_{filter} satisfies the following variational calculus relation [18]:

$$I = \int_{t_1}^{t_2} E_{filter}(t, h_c(t), h_f(t), h'_f(t), F_D) dt. \quad (1)$$

Based on these mathematical foundations, the authors in [10–16] considered E_{filter} as an input for an Euler–Lagrange operator (J) = $(\frac{\partial}{\partial h} - \frac{d}{dt} \frac{\partial}{\partial h'})$ [18] to obtain an extremal (minimum or maximum) value of E_{filter} by setting $J(E_{filter}) = 0$. The extremal conditions are: for all $t \in (t_1, t_2)$, if $\frac{\partial^2}{\partial h'^2} E_{filter} > 0$, then the optimum $h_f(t)$ minimizes E_{filter} , if $\frac{\partial^2}{\partial h'^2} E_{filter} < 0$ then $h(t)$ maximizes E_{filter} , and if $\frac{\partial^2}{\partial h'^2} E_{filter} = 0$, then E_{filter} degenerates [18].

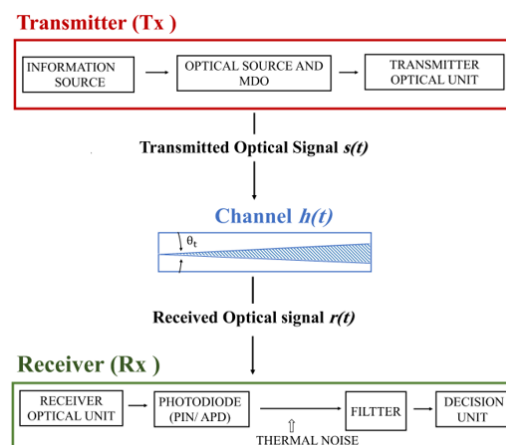


Figure 1. Typical Optical Communication System [8].

The authors [10–17] considered realistic noise environment scopes, including the ISI and AGTN. Analytical solutions were obtained for simple scenarios (e.g., without AGTN, with white ADTN, and without ISI). For complex noise environments, they obtained numerical solutions for closed-form filter equations. In [15,16], the authors extended the receiver approach to include the m-ary optical receiver system and coherent homodyne detection optical receiver [16]. This extension adds credibility to the CB optimization methodology as a reliable receiver design methodology. The teams [10–17] assumed that the channel impulse response $h_c(t)$ is a delta function for optical fiber, which is not the case for UWOC. While the CB approach [10–16] could be applied to the UWOC system, it includes complicated steps in deriving the CB filter equation, optimization, and implementations for linear detection systems. Also, this approach minimizes the chance of obtaining a

closed-form E_{filter} . Against such a background, we propose a simple and effective MGF receiver filter modelling methodology in this paper. This new approach can benefit the CB method while avoiding its pitfalls. The work [17] derived a Bayes optimal receiver using the Karhunen–Loève expansion and upper CB for system performance. This work does not offer an easier way, but it handles the aspects of filter optimization using different mathematical criteria.

Figure 2 shows the main four underwater channel models for $h_c(t)$ [8]. All underwater CIR models are rooted in the primary Double Gamma Function (DGF) channel model. These CIR models differ based on the scope of propagation impairments for the underlying water type [8]. The primary UWOC channel impairments are absorption, scattering, and scintillation. The last two have pronounced impacts on the temporal and spatial profile of the transmitted optical pulse, causing ISI and power distribution distortion on the surface of the photodetector [8]. In this communication, we utilize a DGF model to describe the temporal profile of the $h_c(t)$. The DGF water channel model provides enough realistic scope of the UWOC channel deficiencies. Also, the DGF model parameters for Coastal and Harbor waters are available [8]. Figure 2 shows the temporal profiles of the UWOC propagation channel models for Coastal and Harbor waters.

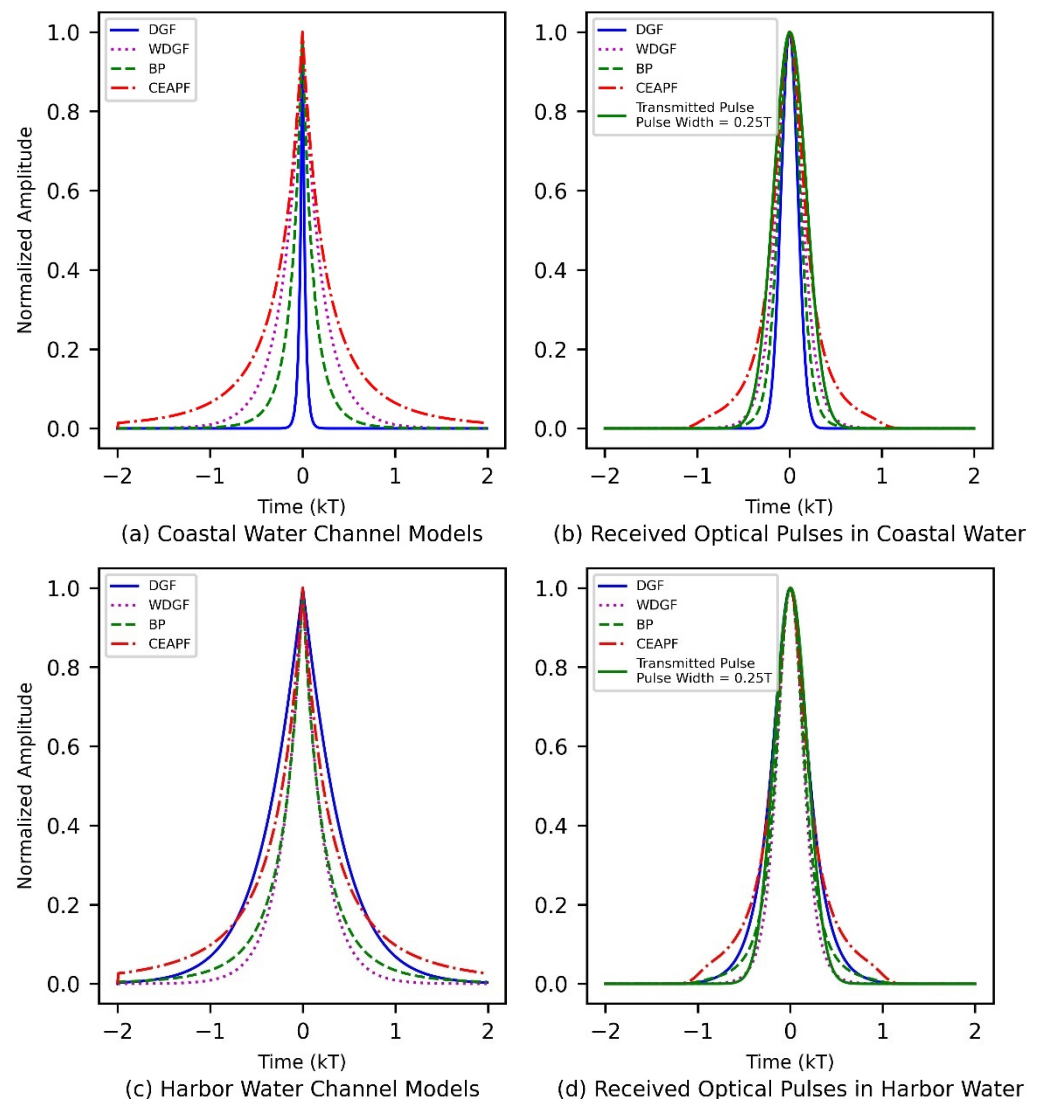


Figure 2. Water channel impulse responses $h_c(t)$ of four channel models {DGF, WDGF, BP, CEAPF} and the corresponding received optical pulses $r_{opt}(t)$ for a transmitted optical pulse $s(t)$ with pulse width = 0.25, Bit rate: 200 Mbit, water link span: 100 m.

The work in [8] provides a good survey for the UWOC research activities. Yet, UWOC technology, from a communication engineering perspective, is an evolving field with ongoing research and advancements. The summary of the main directions of explorations and accomplishments that made notable contributions in the state-of-the-art UWOC systems are (I) Modulation techniques and Wavelength selection: Researchers in [19,20] have been developing efficient modulation techniques for underwater optical channels. These techniques aim to address challenges such as signal attenuation, scattering, and absorption in water. Also, choosing appropriate wavelengths for communication is crucial in underwater environments. The researchers have explored different parts of the electromagnetic spectrum to find wavelengths that offer better UWOC transmission characteristics [21], (II) Multiple Input Multiple Output (MIMO) Transmission [21]: MIMO systems have been investigated to enhance the capacity and reliability of UWOC links by utilizing multiple transmitters and receivers simultaneously, (III) Advanced Signal Processing [20,22] and Energy-Efficient Communication Protocols [23]: Signal processing algorithms have been developed to mitigate the impact of underwater channel distortions at a limited scale, including methods for equalization, error correction, and adaptive modulation constructed on the advancement in optical fiber communication. Also, developing energy-efficient communication protocols [23] is essential for UWOC systems, especially those relying on battery-powered devices. Researchers are exploring ways to optimize energy consumption in UWOC systems, (IV) Experimental Deployments and Field Trials [19–23]: Field trials and experimental deployments have been conducted to test the performance of UWOC systems in real-world environments, such as underwater environments. These trials help validate theoretical models and identify practical challenges, and (V) Integration with Other Sensor Networks [24]: Integration of UWOC with other underwater sensor networks, such as acoustic or radio-frequency systems, has been explored to create more robust and versatile communication networks for underwater applications.

It is helpful to state that the optical signal (bit stream) is converted into the electrical domain at the photodetector output, which represents the current due to the shot noise and Additive Gaussian Thermal Noise (AGTN) originating from pre-amplifier and amplifier at the detector backend. These electronic components are added to form the input to the electronic filter $h_f(t)$ (Figure 3), which acts as a pulse-shaping unit whose optimum bandwidth needs to be concluded [25]. The last task is the main objective of this paper.

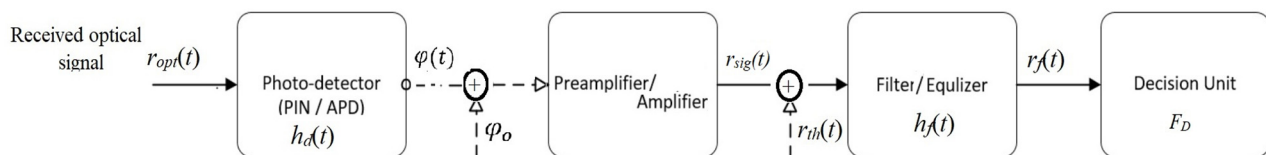


Figure 3. Typical Direct Detection Optical Receiver Model.

This work presents the required steps to develop a new mathematical optical filter design optimization technique using a UWOC optical receiver as a case study. Section 1 briefly surveys the optical design efforts in UWOC and optical fiber systems to render this possible. Also, it identifies the gaps related to UWOC filter design efforts. Section 2 shows the necessary tooling of the mathematical steps to formulate the framework of the optical receiver design model. Section 3 presents the formulation of the optical receiver filter equation (ORFE). This section uses three use cases with ascending noise complexity to demonstrate the evolution of the ORFE from a simple scenario to a complex one. Section 4, titled “Results and discussions” covers the following topics: (A) MGF model validation, (B) software implementation notes, and (C) computation data visualization and optimum filter characteristics. Finally, Section 5 presents the main conclusion and future work. It is essential to mention here that this work covers the temporal profile of the optical received signal and the resultant electronic signal at the input of the decision unit.

2. Optical Receiver Mathematical Design Model

However, the derivation in this paper involves the optical receiver elements and signal plus noise components which are shown in Figure 3.

In a typical optical digital communication system, the transmitted signal $s(t)$ can be depicted as:

$$s(t) = \sum_{k=-\infty}^{\infty} a_k h_p(t - kT). \quad (2)$$

where T is the signalling period. If τ is the timespan of each bit within a symbol. For binary (OOK) signal format, $T = \tau$, so $1/T$ is the bit rate. The a_k is the energy received in the k^{th} symbol, whereas for binary system $a_k \in \{0, 1\}$, the $h_p(t)$ represents a transmitted optical pulse. $s(t)$ suffers temporal distortions while propagating through a medium channel (air, fiber optic, or water) depending on the characteristic impulse response of the channel $h_c(t)$. Hence, the received optical signal $r_{opt}(t)$, which describes the temporal impact of $h_c(t)$ on $s(t)$ signal, is given as (see Figure 2):

$$r_{opt}(t) = s(t) * h_c(t), \quad (3a)$$

$$= \sum_{k=-\infty}^{\infty} a_k h_p(t - kT) * h_c(t), \quad (3b)$$

$$= a_0 h_p(t) * h_c(t) + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} a_k h_p(t - kT) * h_c(t), \quad (3c)$$

where $*$ denotes the convolution operation. The $r_{opt}(t)$ is the received optical signal which provides the optical input to the photodetector. The first term of Equation (3c) is depicted in Figure 2b,d. We can notice that $h_c(t)$ causes a temporal broadening in $s(t)$ (which means narrowing the $r_{opt}(t)$ compared with the bandwidth of $s(t)$). In this work, we are considering a binary direct detection optical receiver system (Rx) depicted in Figure 3. Without losing generality, we will assume that the Rx uses a PIN photodetector when the internal gain (g) equals one. The photodetector converts the input photons of $r_{opt}(t)$ into photoelectrons that form an output electronic signal $r_{sig}(t)$ [26].

$$r_{sig}(t) = \sum_{j=1}^{N(t)} g_j h_d(t - t_j), \quad (4)$$

$\{t_j\}$ denotes the photoelectron emission times. g_j is photodetector avalanche gain. In this work, we are assuming the use of a PIN detector where $g_j = 1 \forall j$. The technological advancement in photodetector (PIN and APD) manufacturing permits us to assume that $h_d(t)$ is a delta function $\delta(t)$ (i.e., vast bandwidth). In Figure 3, the amplifier's role elevates the signal strength. However, from the conceptual perspective, the equalizer is expected to compensate for any temporal changes caused by the Rx front end. Alternatively, the "optimum" filter should be able to do that explicitly. From Equation (4), we can conclude the filter output as follows:

$$r_f(t) = r_{sig}(t) * h_f(t) + r_{th}(t), \quad (5a)$$

$$= \sum_{j=1}^{N(t)} g_j h_d(t - t_j) * h_f(t) + r_{th}(t), \quad (5b)$$

where we can assume safely that $h_d(t) = \delta(t)$. The first term represents the signal component $r_{sig}(t)$, and the second term is the AGTN $r_{th}(t)$. In Equations (4) and (5), the $\{t_j\}$ is a set of photoelectrons' arrival times that follows Poisson statistics governed by $N(t)$. $N(t)$ represents the stochastic counting process of the generated photoelectrons during $\{t_0 = 0, t\}$. This counting process is inhomogeneous [27,28] with the time-varying rate intensity of $\{a_k\}$ in Equation (2). This makes $r_{sig}(t)$ a doubly stochastic marked and filtered Poisson process (DSPP) [27]. $N(t)$ is directly proportional to the received optical intensity $\lambda(t)$ [16,27] which can be expressed as follows:

$$\lambda(t) = \varphi(t) + \varphi_{dc}, \quad (6a)$$

where φ_{dc} is the photoelectron count due to the detector's dark current. It is a stationary background noise with a flat power spectrum N_{ob} . Hence, the optical receiver of bandwidth BW_{Rx} collects a noise power $= N_{ob} BW_{Rx}$ [27]. $\varphi(t) = (\eta/h\nu) r_{opt}(t)$ is the intensity function of the photoelectron count rate due to the received optical signal $r_{opt}(t)$. From Equation (3), we get

$$\lambda(t) = \frac{\eta}{h\nu} \sum_{k=-\infty}^{\infty} a_k h_p(t - kT) * h_c(t) + \varphi_{dc}, \quad (6b)$$

$$= \frac{\eta}{h\nu} a_0 h_p * h_c + \frac{\eta}{h\nu} \sum_{\substack{k = -k_L \\ k \neq 0}}^{k_L} a_k h_p(kT) * h_c + \varphi_{dc}, \quad (6c)$$

where $h_{(.)} \equiv h_{(.)}(t)$, η is the detector quantum efficiency for a given wavelength ($1/\nu$) with a value limitation of $\eta \leq 1$ and $h\nu$ is the energy of one photon. The first term in Equation (6c) depicts the power contribution due to the 0th position. The second term in Equation (6c) represents the ISI impairment, excluding the 0th position. k_L is the left and right limit positions for the ISI bit-train. In the rest of this paper, we will focus on Equation (6) in developing the equalizer/filter equation. The $r_f(t)$ is an electronic signal at the input of the receiver decision unit. Hence, $r_{sig}(t)$ and $r_f(t)$ are the foundations for formulating the receiver filter/equalizer equation. To complete the mathematical description of the Rx system in Figure 3, we need to add a parameter that controls the operation of the decision unit. This parameter is the decision threshold (F_D). F_D is highly expected to depend on $r_{opt}(t)$ and $r_f(t)$.

3. Optical Receiver Filter Equation–MGF Optimisation Procedure

In this section, we are presenting the derivation of the receiver filter $h_f(t)$ equation based on the MGF of $r_{sig}(t)$. We should note here that the output of $h_f(t)$ constitutes the input to the decision unit (Figure 3). For clarification purposes, the mathematical manipulations are mapped to three use cases with ascending noise complexity. The first use case (I) handles the signal $r_{sig}(t)$ without ISI degradation and additive thermal Gaussian noise (ATGN). However, the computation covers the status (1) without ISI (which resembles the path-loss channel) and (2) with ISI. The second use case (II) covers the signal $r_{sig}(t)$ scenarios with white AGTN and with and without ISI. The third use case (III) deals with $r_{sig}(t)$, which includes colored AGTN and ISI. The derivations of the three use cases are based on (a) the r_{sig} includes the ISI term expressed in Equation (6) and (b) the modulation format is binary OOK, which covers the return-to-zero (RTZ) and non-return-to-zero (NRZ). During the derivation, we use two types of MGFs: ordinary MGF and central MGF. Such naming is based on [28], and it helps distinguish the formulation attributes of the underlying MGF.

However, the statistical entities of the MGF Design model can be defined as follows: (I) the system, (II) the system's attributes, (III) the system structures, and (IV) the connected sequence of underlying processes within the system. Hence, our MGF optimization model encapsulates the following: (I) the system is the Optical Receiver System (Rx), (II) the system's attributes include $\lambda(t)$ in Equation (6), $h_c(t)$ given in [8], $r_{th}(t)$ which is the AGTN in Equation (8a), and $h_f(t)$ which is the solution under consideration, (III) the system structure layout is shown in Figure 3, and (IV) the connected sequence of processes are: (IV-I) The convolution process of the transmitted digital OOK optical signal $s(t)$ in Equation (2) with the propagation channel $h_c(t)$ given in [8]. The outcome of this convolution is the received optical signal $\lambda(t)$ in Equation (6c), (IV-II) the photodetector counting process which yields $N(t)$ of photoelectrons that form the electronic signal $r_{sig}(t)$ in Equation (4), where the photoelectron count $N(t)$ is governed by incident photon intensity $\lambda(t)$, (IV-III) at the backend of the photodetector an electronic pre-amplifier/amplifier process that causes the addition of the AGTN in Equation (8a), and (IV-IV) the electronic equalization/filtering process with an impulse response function $h_f(t)$ that represents the solution of Equation (19), which we are trying to optimize.

While the authors in [10–16] used CB to conclude the receiver functional ψ , we use a new Rx model to create the ψ based on an MGF, which describes $r_f(t)$. This model

includes $r_{opt}(t)$ as an input, $r_f(t)$ as output, and F_D as a selection reference value. Hence, the central MGF can be used by considering F_D as a reference value for $\lambda(t)$ defined in Equation (6). However, the statistical entities MGF or LMGF can be used to describe the system's attributes, structures, and the connected sequence of underlying processes.

The filter output signal $r_f(t)$ is sampled at regular time intervals, and each resulting sample value $r^*(kT)$ is compared with a fixed F_D . Based on a built-in hypothesis role $\{H_i\}$ where $i = \{0, 1\}$, the receiver decision unit concludes a decision related to the input $r_f(t)$ being above F_D or below F_D .

The pivotal variables in the MGF optimization procedure are (a) the received optical intensity $\lambda(t)$ given in Equation (6) and (b) the electronic filter $h_f(t)$ shown in Figure 3. We should remember that $\lambda(t)$ formula in Equation (6) includes the convolution of the transmitted optical pulse $h_p(t)$ and water channel impulse response $h_c(t)$. Regarding the $h_c(t)$, the diagrams in Figure 2a,c reveal the UWOC channel model temporal characteristics for Coastal and Harbour waters, respectively. The impact of $h_c(t)$ on a transmitted optical pulse with 0.25 pulse width at half maxima in Coastal and Harbour waters is shown in Figure 2b,d, respectively.

3.1. MGF and LMGF for Optical Receiver Signal

The MGF uniquely determines the probability density function (PDF) of the underlying random variable by using the MGF's first and second derivatives as the mean and variance of the underlying random variable [28]. After applying the definition of the ordinary MGF [28] on $r_f(t)$ signal component only in Equation (5) and taking into consideration the DSPP photoelectron generation process, we obtain the following [15,16]:

$$M_{r_{f(m)}}(s) = \text{Exp} \left[\int_{-\infty}^{\infty} \lambda_m(t) (\text{Exp}[sh_f(\tau - t)] - 1) d\tau \right], \quad (7a)$$

$$L_{r_{f(m)}}(s) = \int_{-\infty}^{\infty} \lambda_m(t) (\text{Exp}[sh_f(\tau - t)] - 1) d\tau, \quad (7b)$$

where $m = \{0, 1\}$ and based on Equation (6), $\lambda_1(t) = \varphi_1(t) + \varphi_{dc}$ is for the ON bit and $\lambda_0(t) = \varphi_0(t) + \varphi_{dc}$ is for the OFF bit. $s \in \{-1, 1\}$ is the transformation parameter [28]. For the sake of simplicity, we will drop the infinity notation in the limits of the integral. In a typical optical receiver, the colored ATGN variance [10]:

$$\sigma^2 = \frac{N_0}{2} \int \left(h_f^2(t) + \frac{h_f'^2(t)}{w_0^2} \right) dt. \quad (8a)$$

Then, the MGF and its corresponding LMGF for the AGTN noise component $r_{th}(t)$ in Equation (5) takes the form [10]:

$$M_{r_{th}}(s) = \text{Exp} \left[\frac{s^2 N_0}{4} \int \left(h_f^2(t) + \frac{h_f'^2(t)}{w_0^2} \right) dt \right], \quad (8b)$$

$$L_{r_{th}}(s) = \frac{s^2 N_0}{4} \int \left(h_f^2(t) + \frac{h_f'^2(t)}{w_0^2} \right) dt, \quad (8c)$$

$h_f'(t)$ is the first derivative of $h_f(t)$, w_0 is the central noise power spectral frequency, and N_0 is a Gaussian process's signal-sided thermal noise spectral density. The first term of Equation (8) represents the white AGTN. Since $r_{sig}(t)$ and $r_{th}(t)$ are statistically independent components of $r_f(t)$, then the overall MGF and LMGF are:

$$M_{r_f}(s) = M_{r_{sig}}(s) \cdot M_{r_{th}}(s), \quad (9a)$$

$$L_{r_f}(s) = L_{r_{sig}}(s) + L_{r_{th}}(s), \quad (9b)$$

where $L_{(.)}(s)$ is the Log Moment Generating Function (LMGF) of $(.)$. Hence, Equation (9) models the values taken by the filter output at the sampling time, i.e., at the input of the decision unit in Figure 3. In the rest of this paper, we will drop the notation of (s) inside the MGF and LMGF. In Equation (5), if we set $h_f(t) \equiv sh_f(t - \tau)$ and $\lambda_m(t) \equiv \varphi_m(t) + \varphi_{dc}$, we can write the LMGF for $r_{sig}(t)$ for OFF ($m = 0$) and ON ($m = 1$) bits as follows:

$$L_{r_{f(m)}} = (-1)^m \int \lambda_m(t) (\text{Exp} [(-1)^m h_f(t)] - 1) dt + \frac{N_0}{4} \int \left(h_f^2(t) + \frac{h_f'^2(t)}{w_0^2} \right) dt, \quad (10)$$

where $m \in \{0, 1\}$.

3.2. MGF-Based Model for Time-Invariant Rx System

To develop an MGF model for the Rx system, it would be a good idea to evolve it through three optical receiver operating environments. The first Rx model environment includes the main signal $r_{sig}(t)$ without AGTN noise (use case I). The second Rx model environment incorporates the main signal $r_{sig}(t)$ with white AGTN noise (use case II). The third Rx model environment covers $r_{sig}(t)$ with colored AGTN noise (use case III).

3.2.1. Use Case I: $r_{sig}(t)$ without AGTN Noise and ISI

This use case involves a linear time-invariant Rx without AGTN noise. This means we are considering a path-loss channel, which implies that we are considering just the absorption aspect of the seawater impairments. In this work, we assume that the optical receiver under consideration is deploying a PIN photodetector (i.e., the internal gain $g = 1$). The LMGF in Equation (10) for such a scenario reduces to the following [11,15]:

$$L_{r_{f(m)}} = (-1)^m \int \lambda_m(t) (\text{Exp} [h_f(t)] - 1) dt. \quad (11a)$$

Equation (11a) is the first term of Equation (10), which represents $r_{sig}(t)$, which explicitly includes the ISI in addition to the 0th transmitted bit. A linear Rx implicitly is an invariant system with respect to the insertion of a time shift to $\lambda_m(t)$ or $h_f(t)$. This means that:

$$\int \lambda_0(t \pm \Delta) (\text{Exp} [h_f(t)] - 1) dt = \int \lambda_0(t) (\text{Exp} [h_f(t \pm \Delta)] - 1) dt, \text{ For } m = 0, \quad (11b)$$

$$\int \lambda_1(t \pm \Delta) (\text{Exp} [-h_f(t)] - 1) dt = \int \lambda_1(t) (\text{Exp} [-h_f(t \pm \Delta)] - 1) dt, \text{ For } m = 1. \quad (11c)$$

If we take a very small Δ , then it will be a valid approximation to take the first two terms of Taylor's series of $\lambda(t \pm \Delta)$ and $h_f(t \pm \Delta)$. From Equation (10), using the notation of $m \in \{0, 1\}$, we can rewrite the relationships in Equation (11) as follows:

$$\int \lambda_m' (\text{Exp} [(-1)^m h_f] - 1) dt + (-1)^m \int \lambda_m h_f' \text{Exp} [(-1)^m h_f] dt = 0, \quad (12a)$$

or in a more compact (shorthand) form

$$\int d [\lambda_m (\text{Exp} [(-1)^m h_f] - 1)] dt = 0, \quad (12b)$$

where $\lambda_m \equiv \lambda_m(t)$, $\lambda_m' \equiv \lambda_m'(t)$, $h_f \equiv h_f(t)$, and $h_f' \equiv h_f'(t)$. The above equation shows that the system constraint could be applied directly to the integrand of the LMGF. This

is possible because $L_{rf(m)}$ is a continuous function. By applying Equation (12) for each bit value, i.e., $m = \{0, 1\}$, and execute the integrating in Equation (12), then we obtain:

$$\lambda_1 (\text{Exp}[-h_f] - 1) + C_1 = 0, \text{ For ON state} \quad (13a)$$

$$\lambda_0 (\text{Exp}[h_f] - 1) + C_0 = 0, \text{ For ON state} \quad (13b)$$

where C_m is the constant of integration. The boundary values of $\lambda_m(t = \pm\infty) \sim \varphi_{dc}$ for both bit states and the boundary values for $h_f(t = \pm\infty) \sim 0$. Therefore, we can safely say that $C_1 = C_2 = 0$. Equation (13) describes two independent states in one time-invariant system. Hence, we can assemble both Equations (13a) and (13b) to describe the states of the underlying overall Rx system. Therefore, we can rewrite Equation (13) as follows:

$$\lambda_0 \text{Exp}[2h_f] - (\lambda_1 + \lambda_0) \text{Exp}[h_f] + \lambda_1 = 0. \quad (14)$$

The above quadratic equation describes the Rx system model without AGTN noise. This model equation includes the following operating attributes: λ_m through Equation (6) includes transmitted pulse shape h_p , propagation channel impulse response h_c , photodetector quantum efficiency (η), and photodetector impulse response h_d , also Equation (14) includes the filter impulse response h_f . In Equation (14), the decision threshold F_D is seemingly missing. To include F_D in the Rx model, we need to use the central moment [28] definition of r_f in Equation (7) instead of the ordinary moment. Including F_D requires the replacement of λ_m in Equation (7) with $(\lambda_m + (-1)^m F_D)$. Equation (14) has an analytical solution which is given as follows:

$$h_f = \ln \left[\frac{\lambda_1}{\lambda_0} \right]. \quad (15)$$

Equation (15) represents a Maximum Likelihood Filter (MLF) when considering the signal component without the thermal noise. This is the same equation that [10,11] derived using CB and MCB, respectively, without the AGTN noise.

3.2.2. Use Case II: $r_{\text{sig}}(t)$ with White AGTN Noise and ISI

For a linear time-invariant Rx with PIN photodetector and a white AGTN noise, the LMGF in Equation (10) takes the following form [11,15]:

$$L_{r_f(m)} = (-1)^m \int \lambda_m (\text{Exp}[h_f] - 1) dt + \frac{N_0}{4} \int h_f^2 dt. \quad (16)$$

Then, once we apply the same steps from Equations (11)–(13), the filter equation in Equation (14) changes to the following form:

$$\lambda_0 \text{Exp}[h_f] - \lambda_1 \text{Exp}[-h_f] + \frac{N_0}{2} h_f^2 + (\lambda_1 - \lambda_0) = 0. \quad (17)$$

Equation (17) accounts for the white AGTN noise alongside the main signal r_{sig} . It is possible to conclude a reliable analytical solution of Equation (17) by taking the first three terms of the exponential power series:

$$h_f = \frac{2(\lambda_1 + \lambda_0)}{(\lambda_1 - \lambda_0 - N_0)}. \quad (18)$$

3.2.3. Use Case III: $r_{\text{sig}}(t)$ with Colored AGTN Noise and ISI

However, let us try one step further, considering a r_{sig} with a colored AGTN. For this use case, we need to use the full LMGF in Equation (10). By applying the same steps

from Equations (11)–(13), the equalizer/filter equation in Equation (17) changes to the following form:

$$-\lambda_1 \text{Exp}[-h_f] + \lambda_0 \text{Exp}[h_f] - (\lambda_1 - \lambda_0) + \frac{N_0}{2} \left(h_f^2 + \frac{h_f'^2}{w_0^2} \right) = 0. \quad (19)$$

The solution for Equation (19) is a nonlinear differential equation with a numerical solution corresponding to the filter impulse response. It includes all the essential signal processes encountered in typical optical communications receivers: quantum noise, dark current noise, colored AGTN noise, and ISI. Equation (19) can be tailored to any particular use case depending on the specified optical communication attributes: ν , $h_p(t)$, transmitted pulse width, the set of amplitudes associated with the binary OOK, $h_c(t)$, η , φ_{dc} , N_0 , and w_0 . Equation (19) can be reduced to Equation (14) by setting N_0 to zero. Also, it reduces to Equation (17) once we set h_f' to zero.

3.3. Optimum Filter Impulse Response Function

We should notice that the analytical solutions in Equations (15) and (17) are not optimum. The same is true for Equation (19) numerical solution. To get the optimum solutions, we need to apply the J operator on Equations (14), (17) and (19). The J operator execution transforms these equations to the following optimum filter equations:

$$2\lambda_0 \text{Exp}[h_f] - (\lambda_1 + \lambda_0) = 0, \text{ NO AGTN} \quad (20)$$

$$\lambda_1 \text{Exp}[-h_f] - \lambda_0 \text{Exp}[h_f] = 4N_0 h_f, \text{ White AGTN} \quad (21)$$

$$\lambda_1 \text{Exp}[-h_f] + \lambda_0 \text{Exp}[h_f] = \frac{N_0 h_f''}{w_0^2} - N_0 h_f, \text{ Colored AGTN.} \quad (22)$$

Equation (20) is the filter equation without AGTN has the following analytical optimum solution:

$$h_f = \ln \left[\frac{(\lambda_1 + \lambda_0)}{2(\lambda_0)} \right]. \quad (23)$$

Equations (21) and (22) need a numerical optimum solution. In summary, the equations from Equations (14)–(19) are the general solution of $h_f(t)$, while the equations from Equations (20)–(23) provide the optimum solution of $h_f(t)$. It is worth noting that all these equations show that $h_f(t)$, as a general solution or optimum, depends on the $\lambda(t)$ given in Equation (6). Equation (6b) reveals that $\lambda(t)$ embeds the dependency on the transmitted optical pulse $h_p(t)$ and the propagation channel impulse response $h_c(t)$. Equation (6c) exposes the fact that the ISI is rooted in the digital stream of bits around the zero-position bit. Hence, the ISI in the UWOC optical receiver will have two sources: (1) the digital signal itself and (2) the scattering and scintillation impairments that cause the stretching of the optical pulse tails. This tail elongation is not related to the pulse center shift.

4. Results and Discussion

The discussion covers the following topics: A—MGF model validation, B—software implementation notes, and C—computation data visualization. The discussion analysis in this section takes into consideration three concluding observations from Figure 2. These observations are: (1) the water channel impulse response $h_c(t)$ based on the models DGF, WDGF, BP, and CEAPF has a bandwidth narrowing impact on the transmitted signal $s(t)$, (2) the bandwidth narrowing of the four models nearly the same at Full Width at Half Maxima (FWHM), and (3) the bandwidth narrowing in general for Harbour water is more significant than that of the Coastal water.

The computation in this paper considers the propagation channel impulse response based on the DGF model only. The reason behind such a step can be justified as follows: the

channel models' data visualisation in Figure 2 for Coastal and Harbor seawaters helps shed more light on the justification of using just the DGF channel model. Figure 2a,c show the channel models for Coastal and Harbor waters, respectively. From these figures, it is easy to notice that all seawater channel models have the same temporal profile, practically a DGF profile, but with different temporal broadenings (i.e., different frequency bandwidths). To support this observation, we can use the statistical techniques of transforming the random variable [21–Chapter 2] on the WDGF, CEAPF, and BP to produce the DGF distribution.

Including the DGF impulse response in our computation makes the comparison with the receiver performance in [10–16] unfruitful. This is because the works in [10–16] considered the propagation channel impulse function a delta function. However, when relevant, we mentioned the other aspects of comparisons in this section.

4.1. MGF Model Validation

In terms of mathematical formulation, first, we must emphasise that any model should preserve the dynamics of the statistical processes and maintain their impact boundaries. In our case, the statistical process is the photodetector DSPP process, which has been depicted in Equations (5) and (6). This is what [10–16] based their design optimisation approaches on. Based on this mathematical fabric, the Rx model should produce a causal equaliser/filter unit that considers the DSPP. The DSPP is described in terms of MGF and LMFG through Equations (7)–(10). We need to verify that our proposed MGF model for the Rx system is reliable in this sense. We must apply our model for a photodetector with a direct convolutional process, not a transformational-convolutional DSPP process [27]. Also, we utilise the use case II of a r_{sig} signal with a white AGTN noise. For such a configuration, the LMFG takes the following form [10,15]:

$$L_{r_f(m)} = (-1)^m \int \lambda_m h_f dt + \frac{N_0}{4} \int h_f^2 dt. \quad (24)$$

Now we apply the same steps from Equations (12)–(14), then the equalizer/filter equation in Equation (14) takes the following form:

$$\lambda_0 - \lambda_1 + 2N_0 h_f = 0. \quad (25)$$

The general analytical solution of Equation (25) is:

$$h_f = \frac{\lambda_1 - \lambda_0}{2N_0}. \quad (26)$$

Equation (26) represents a whitened matched filter for a deterministic signal corrupted by a white AGTN noise. While Equation (26) differs from the DSPP-based Equation (18) above, it is the same as that published by [10,13] using CB and MCB, respectively. This observation means that CB and MCB approach optimization steps inadvertently masked the built-in DSPP in the optical receiver design, which is assumed by [10,13]. In other words, CB and MCB do not produce a causal filter unit. On the contrary, our model does, which makes the MGF Model more reliable than the CB and MCB method. If we consider RTZ OOK, Equation (17) takes the same format reported in [13] for the same use case. However, for the use case I of the r_{sig} without AGTN, the MCB filter impulse response in [13] is similar to Equation (15) above. For the use case III r_{sig} signal with a colored AGTN noise, Equation (22) agrees with that published by [13,15].

Second, regarding optimisation criteria, the CB and MCB optimization in [10–16] is based on minimizing the BER for the decision unit. Our model's optimization is based on directly maximizing the received processed signal energy presence around the pulse center to elevate the signal-to-noise ratio (SNR) at decision time. The two approaches are essentially the same due to the probability of error association with SNR through the error function (ERF). The outcome SNR is shown in the eye diagram of each use case's optimum filter out r_f .

When comparing our proposed optimum filter with the raised cosine filter (RCF), which is a reference industry standard, it is essential to mention that RCF access bandwidth (BW) is proportional to the roll-off factor (β). This means that we need a lower β at the run time to reduce the access BW. This reduction in β could mean less ability for the filter to reduce the ISI at high and moderate bit rates. At a low bit rate, the RCF is fine. This is inherently linked to the RCF's undershoot cycles being tied up to the bit period boundaries, causing unnecessary higher flooring in the eye diagram. Our optimum filter for ISI use-cases shown in Figures 4f and 5f for Coastal, Figures 6f and 7f for Harbour, shows practically no floor for the Coastal scenarios and limited floor depth for the Harbour scenarios. The reason behind this is that for the corresponding ISI use cases in Figures 4b–7b, we can notice that the undershoot cycle crossing is not tied up to the bit boundaries like the RCF. Such behavior means our proposed filter is optimized when necessary and is not fixed to fixed bit boundaries.

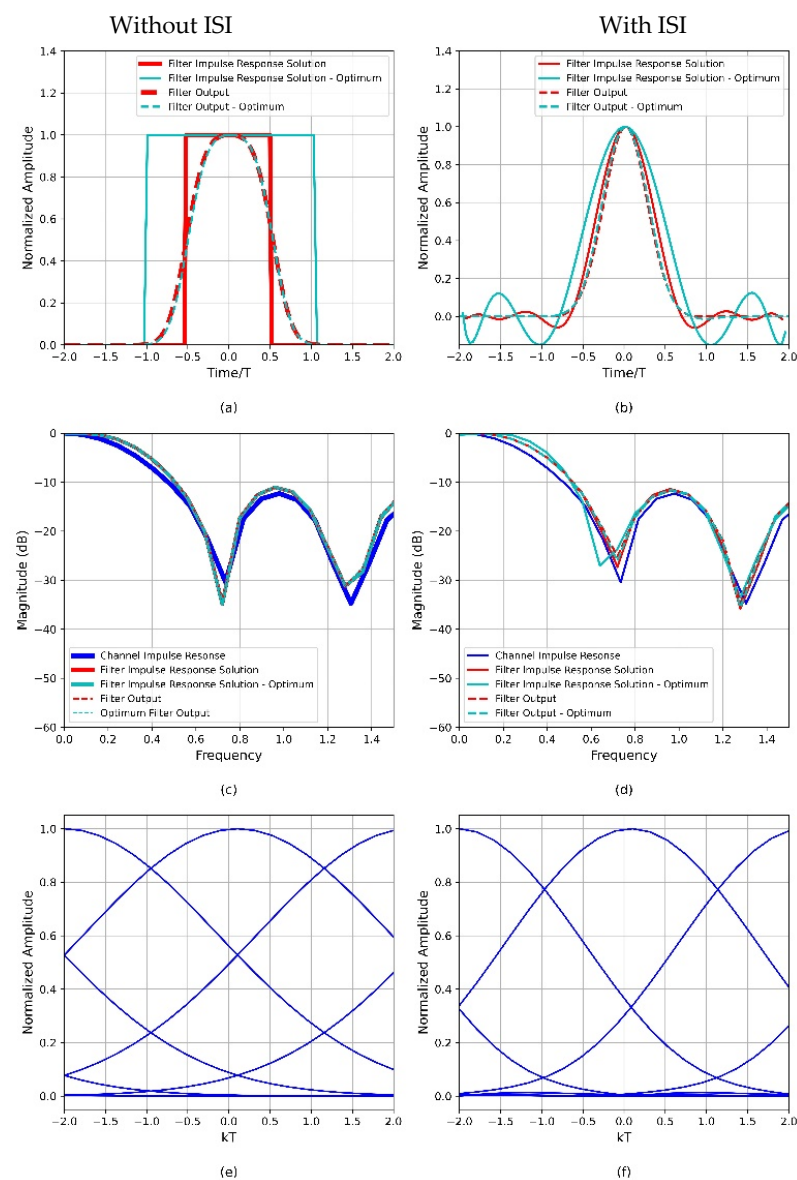


Figure 4. Use Case I: $r_{sig}(t)$ without AGTN. The analytical solution of Equation (15) for Coastal water (without and with ISI). (a) optimum $h_f(t)$ solution, (b) the corresponding outputs $r_f(t)$, (c) the frequency response for the three $h_f(t)$ solutions, (d) the frequency response for $h_c(t)$, optimum $h_f(t)$, & $r_f(t)$, (e) the eye diagram for the Gaussian transmitted optical pulse $h_t(t)$ with pw = 0.25, and (f) the eye diagram for the filter output $r_f(t)$.

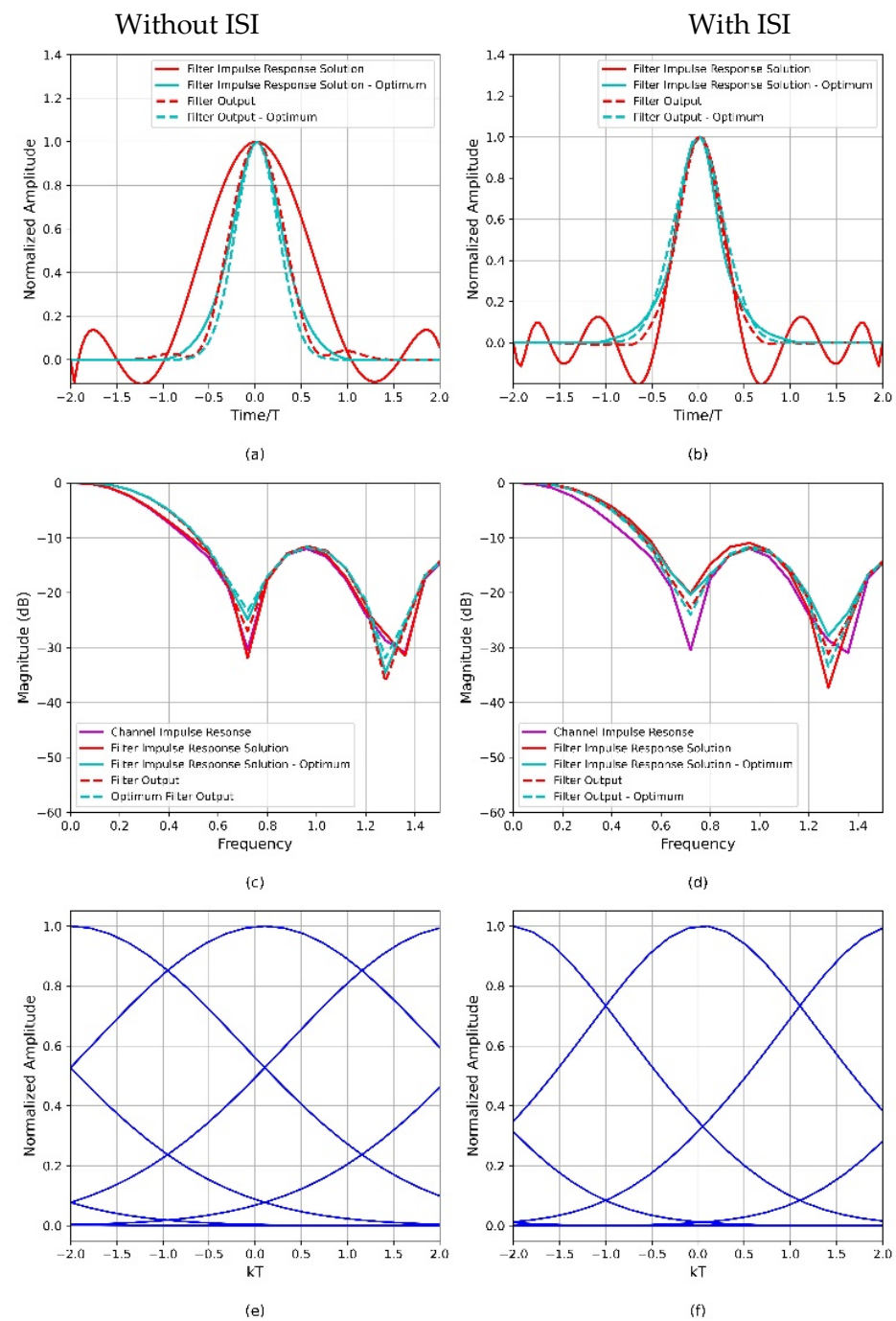


Figure 5. Use Case II: $r_{sig}(t)$ with white AGTN The analytical solution of Equation (18) for Coastal water (without and with ISI). (a) optimum $h_f(t)$ solution, (b) the corresponding outputs $r_f(t)$, (c) the frequency response for the three $h_f(t)$ solutions, (d) the frequency response for $h_c(t)$, optimum $h_f(t)$, & $r_f(t)$, (e) the eye diagram for the Gaussian transmitted optical pulse $h_i(t)$ with $\text{pw} = 0.25$, and (f) the eye diagram for the filter output $r_f(t)$.

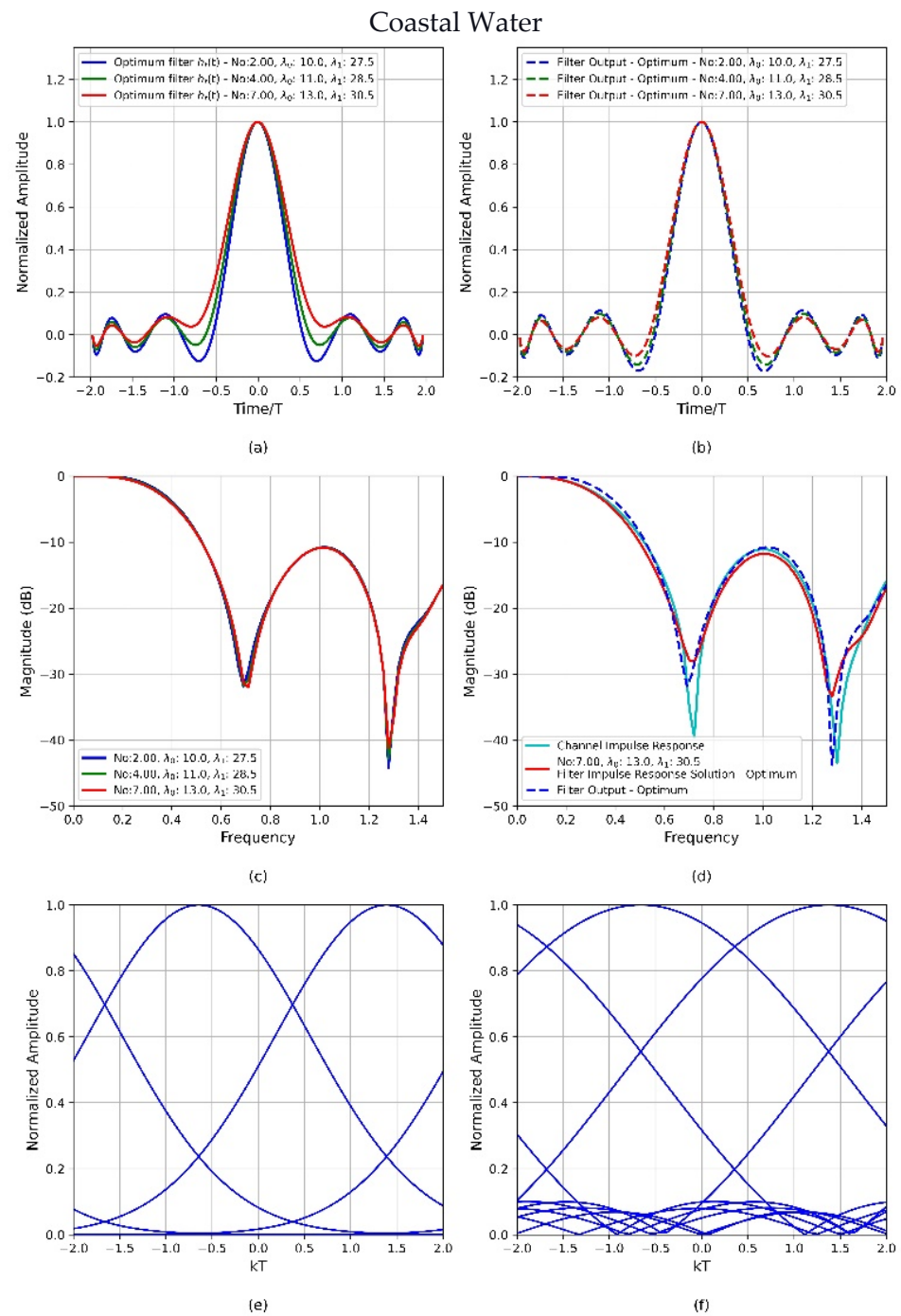


Figure 6. Use Case III: $r_{sig}(t)$ with colored AGTN and ISI-[Equation (22) Numerical solution for Coastal water]. (a) three optimum $h_f(t)$ solution scenarios with different $\{N_0, a_0, a_1\}$ value sets, (b) the corresponding outputs $r_f(t)$, (c) the frequency response for the three $h_f(t)$ solutions, (d) the frequency response for $h_c(t)$, optimum $h_f(t)$, & $r_f(t)$, (e) the eye diagram for the transmitted optical pulse $h_t(t)$ with $pw = 0.25$, and (f) the eye diagram for the filter output $r_f(t)$.

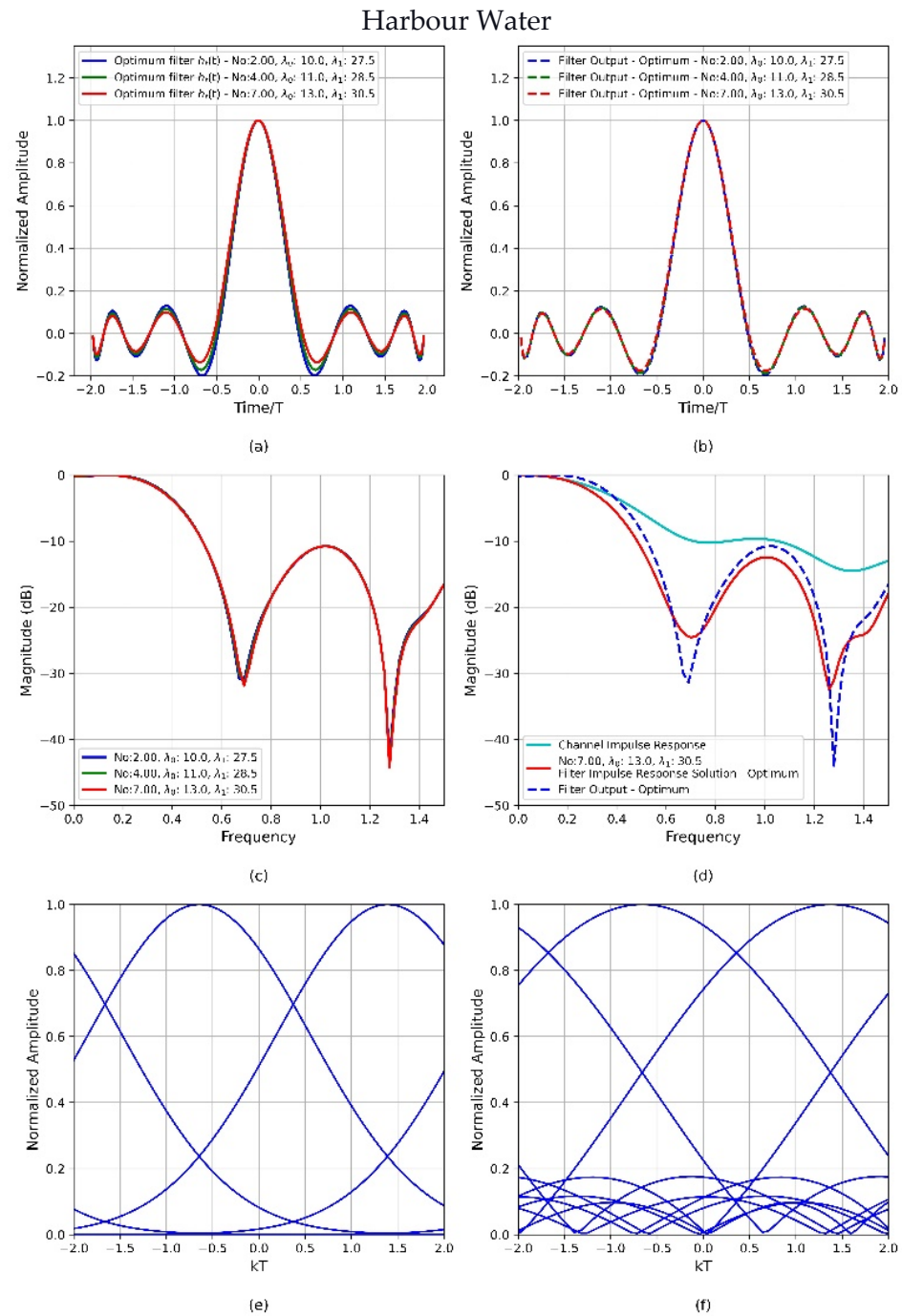


Figure 7. Use Case III: $r_{sig}(t)$ with colored AGTN and ISI-[Equation (22) Numerical solution for Harbour water]. (a) three optimum $h_f(t)$ solution scenarios with different $\{N_0, a_0, a_1\}$ value sets, (b) the corresponding outputs $r_f(t)$, (c) the frequency response for the three $h_f(t)$ solutions, (d) the frequency response for $h_c(t)$, optimum $h_f(t)$, & $r_f(t)$, (e) the eye diagram for the transmitted optical pulse $h_t(t)$ with $pw = 0.25$, and (f) the eye diagram for the filter output $r_f(t)$.

4.2. Software Implementation Notes

All numerical computation tools are based on a set of Python libraries. *Odeint*, *interpolate*, *fftpack*, *fftshift*, *splrep*, *splev*, *symbols*, *Eq*, *solve*, and *findpeaks*.

The most challenging element in deploying *Odeint* and *interpolate* libraries is the internal curve smoothing we don't control. Hence, we found that using external smoothing (*splrep*, *splev*) to limit the impact of the internal curve smoothing. It is essential to recognize that not any set of values for $\{N_0, a_0, a_1\}$ produces a realizable filter solution h_f (general and

optimum). Table 1 exhibits a working map of the three use cases, associated equations, and diagrams. Therefore, we will not repeat the description of the sub-figures and equations in this section. Regarding the numerical solution of Equation (22), the *Odeint* Python library has 25 parameters, including the *args()* and the *func*. The function is preferred to be a monotonically non-decreasing or non-increasing function. Therefore, we used half of the intended array, and later, we completed the solution using a bespoke data symmetry method. This helped us to cut the processing time to half.

Table 1. Mapping the use cases to the figures and equations.

Use Case	Water Type	Equation			ISI		Computation Attributes $\{N_0, a_0, a_1, pw\}$
		General	Optimum	Filter	Without (Signal Only = Path Loss Channel)	With	
III (Figure 4)	Coastal	(15)	(20)	(15)	{a, b, c}	{d, e, f}	{0.0, 3.0, 13., 0.25}
III (Figure 5)	Coastal	(17)	(21)	(18, 23)	{a, b, c}	{d, e, f}	{2.75, 10, 20, 0.25}
III (Figure 6)	Coastal, Harbour	(19)	(22)			{a, b, c, d, e, f}	Several sets in Figure 6

The other aspect of using the *Odeint* library is the tolerances. We noticed that tightening the tolerances will not necessarily make the numerical half-solution monotonic. There is no way to guarantee that. If we are approaching a steady state monotonically, tightening tolerances will limit false oscillations (expressly, it will limit their amplitude, but not their total variation). However, it did not eliminate them for sure. Consequently, we noticed amplified undershoots for the $h_f(t)$ solution in Figure 6a,d.

As part of the MGF model reliability check, we worked on finding valid and safe solutions because they provide realizable and causal filters. We decided to find general solutions besides the corresponding optimum ones to bridge the gap between valid and safe solutions. In this manner, we can examine the new MGF designing model frame of reliability by changing the configuration solution set defined by $\{N_0, a_0, a_1, pw\}$ plus the ISI depth.

4.3. Computation Data Visualisation

Table 1 maps the h_f solutions to the curve sets with the underlying configurations. The computations for the three use cases were performed for h_f with pulse rms widths (pw) {0.25, 0.35} with different sets of binary OOK amplitudes $\{a_0, a_1\}$ and different AGTN noise power N_0 . The implementation of the use cases covers r_{sig} without and with ISI in two sets of curves within the set figure. The h_c was computed based on the DGF channel impulse response for Coastal waters [6] (see Figure 2). Due to the paper size limitation, we are presenting only results for the transmitted $pw = 0.25$. The eye diagrams show that the SNR suffers when ISI is part of r_{sig} for all use cases. This outcome is highly expected. The discussions above show that the new MGF design modelling works well in balancing the various impairments (AGTN and ISI). The resulting optimal optical receiver filter h_f provides improved power performance. Figure 2 shows the received optical pulse r_{opt} due to the convolution of the transmitted optical pulse h_p (pulse width = 0.25) and the DGF underwater channel response function h_c .

Figure 4 presents the computation results of use case I, which includes DGF for Coastal water. This use case covers two scenarios: (a) without ISI, which represents a path-loss-oriented propagation channel, and (b) with ISI, which represents limited scattering and potential scintillation impairments. Figure 4a demonstrates that the normalized curves of h_f (of Equations (15) and (23)) without ISI overlap. Their corresponding outputs r_f do the same. Figure 4d shows that the curves of h_f and its corresponding r_f exhibit undershoot due to the ISI inclusion to the main body of r_{sig} (see Equation (6c)). The undershoots in our results are relatively more than that reported by [12–16]. Such curve tails could be due to the inflated internal smoothing of the Python libraries. However, it is interesting to note that, although not explicitly constrained by the J optimization step, near-zero ISI results support the zero-forcing inclination used in many practical receivers [9], like RC filters.

Figure 5 depicts the computation results of use case II, which includes DGF for Coastal water. This use case covers two scenarios: (a) without ISI, which represents a path-loss-oriented propagation channel, and (b) with ISI, which represents limited scattering and potential scintillation impairments. Here, we should keep in mind that the AWTN noise is due to the electronic circuitry of the photodetector backend (see Figure 3). Figure 5a without ISI indicates that the normalized curve of h_f of Equation (18) has a distinctive profile from that of Equation (21). While the optimum solution shows no undershoot, the general solution exhibits a small false one. We believe this is due to unnecessary internal curve smoothing, allowing a distortion at the tail-end of h_f . The optimum r_f profile looks as expected, while the general r_f exhibits small, false undershoot. Figure 5b with ISI indicates that h_f of Equation (18) has a distinctive undershoot at both ends of h_f curve while the optimum h_f shows a very small undershoot. Their corresponding outputs r_f mirror the same difference too.

By observing the frequency bandwidth (FBW) profiles of h_f (general and optimum) in Figure 4b,e and Figure 5b,e of use cases I and II, respectively, we can conclude that the FBWs are nearly the same for each scenario. It is worth stating that the FBW narrowing in Figure 4c (without ISI and AGTN) and Figure 5c (without ISI and with white AGTN) are due to seawater channel h_c . At the same time, FBW narrowing in CB and MCB wouldn't happen because [10–16] assumed delta impulse response for the fiber channel. In Figures 4d and 5d the FBW narrowing is due to the signal ISI (the second term in Equation (6c)) and seawater channel h_c , while the FBW narrowing in [10–16] is due to signal ISI only.

The eye diagrams in Figure 4e,f and Figure 5e,f of use cases I and II, respectively, are associated with the filter impulse function h_f (general and optimum). The vertical and horizontal opening of the eye diagram for the electronic signal in use case I (without ISI and WTGN) in Figure 4e is relatively equal to that for use case II (without ISI and with WTGN) in Figure 5e. This is a clear indication that the optimum filter for use case II plays the role of equalization to maintain the signal tail boundaries and peak at their designated points. Figure 5e (use case II) shows a slight reduction rather than a considerable reduction in the vertical and horizontal eye-opening when we compare it to Figure 4f (without WTGN). The work in [10–16] had not addressed this aspect of the filter, so we couldn't compare it.

The first observation we can conclude from Figures 6a and 7a is that the undershoot of the optimum h_f for Harbor water is more profound than that of Coastal water. The optimum h_f protects against the more dispersive channel. Also, it is easy to observe from Figures 6d and 7d that the -3 dB FBW for the Coastal and Harbor waters are practically equal. This can be attributed to the undershoot depth being deeper for Harbor water than for Coastal water. Such a conclusion cannot be drawn in the CB and MCB filter case because [10–16] assumed the delta function for the fiber channel.

Figure 6 shows the numerical computation outcome of use case III for Coastal water, which only includes the ISI scenario alongside the colored AGTN. Figure 6a shows a set of three optimum solutions of h_f of Equation (22) for different $\{N_0, a_0, a_1\}$ value sets. The undershoot decreases with N_0 , because the AGTN is an additive element to the ISI signal component. However, this does not impact the filter bandwidth, as shown in (6c). This is one of the indications that the filter h_f is doing equalization. The corresponding outputs r_f in Figure 6b mirror the exact behavior of h_f . Figure 6f shows the limited impact on the eye-opening compared to Figure 6e, which exhibits the eye diagram of the transmitted optical pulse h_t . Hence, Figure 6f shows the filter's compelling performance in combating the ISI, which is partly a footprint of the scattering and scintillation impairments.

The electrotonic filter h_f can reshape the pulses (i.e., equalization) when its bandwidth is smaller than the transmission bit rate [25] and relatively smaller than the water channel h_c . This is depicted in Figure 6c,d for Coastal water and Figure 7c,d for Harbor water. Such a conclusion cannot be made for CB and MCB due to the nullification of the propagation channel in the design steps in [10–16].

Additionally, the optimum filter h_f for Harbor water in Figure 7d has a bandwidth narrower than the channel impulse response h_c , making h_f a bandpass filter for the ISI signal component. The eye diagram in Figure 7f for Harbor water reveals that the ISI component level is higher than that for Coastal water for the same transmitted optical pulse h_t with a pulse width (pw) equals to $0.25/T$. We conducted several runs for the Coastal water scenarios to quantify the ISI degradation to the signal. We noticed that h_t with $\text{pw} = 0.5/T$ in Coastal water gets nearly the same filter output r_f for Harbor water with $\text{pw} = 0.25$. This reflects that even a narrow pw might not be enough to limit the impact of ISI in Harbor waters.

One of the techniques to measure the system performance is provided by the extent of the eye-opening (O_{eye}) in the eye diagram, which is typically affected by the dispersive and nonlinear effects accumulated inside the propagation channel [25]. Hence, to further examine the performance of the optimum filter unit h_f , which is the solution of Equation (22), we used four water channel models of h_c {DGF, WDGF, BP, CEAPF} shown in Figure 2, at different transmitted pw values of h_t {0.1, 0.3, 0.5} to produce the eye diagrams of the r_f , which is the output of the optimum filter h_f (see Figure 3). The assessment indicator of the r_f eye diagrams is based on the O_{eye} which is marked in red in Figure 8 (for Coastal water) and Figure 9 (for Harbor water). The importance of O_{eye} is that its value is proportional to the Q-Factor. The Q-Factor determines the BER of the receiver system [25,29–32].

We can draw three observations from Figures 8 and 9. First, O_{eye} decreases with increasing the pw; second, within the same water channel, the h_c models predict identical values of O_{eye} at different pw values; and third is that the O_{eye} reduction in Harbor water is relatively smaller than that of the Coastal water. Regarding the first observation, optimum filter h_f reduces O_{eye} by nearly 10–15% for a five-times increase in pw. In other words, we can conclude that the optimum h_f manages the power dispersion correctly. The changes in pw values can be considered a pulse broadening simulation due to ISI impairment sources like scattering and scintillation. Hence, we can conclude that the proposed optimum filter h_f efficiently deals with the ISI.

Figure 10 shows the BER performance at the optical receiver system level, and the optimum filter h_f is part of it. Figure 10a is for Coastal water, and Figure 10b is for Harbor water. It is easy to observe that using the same transmitted power across different transmitted pw values, the Q-Factor decreases in the same manner that BER declines (i.e., better performance) with increasing power.

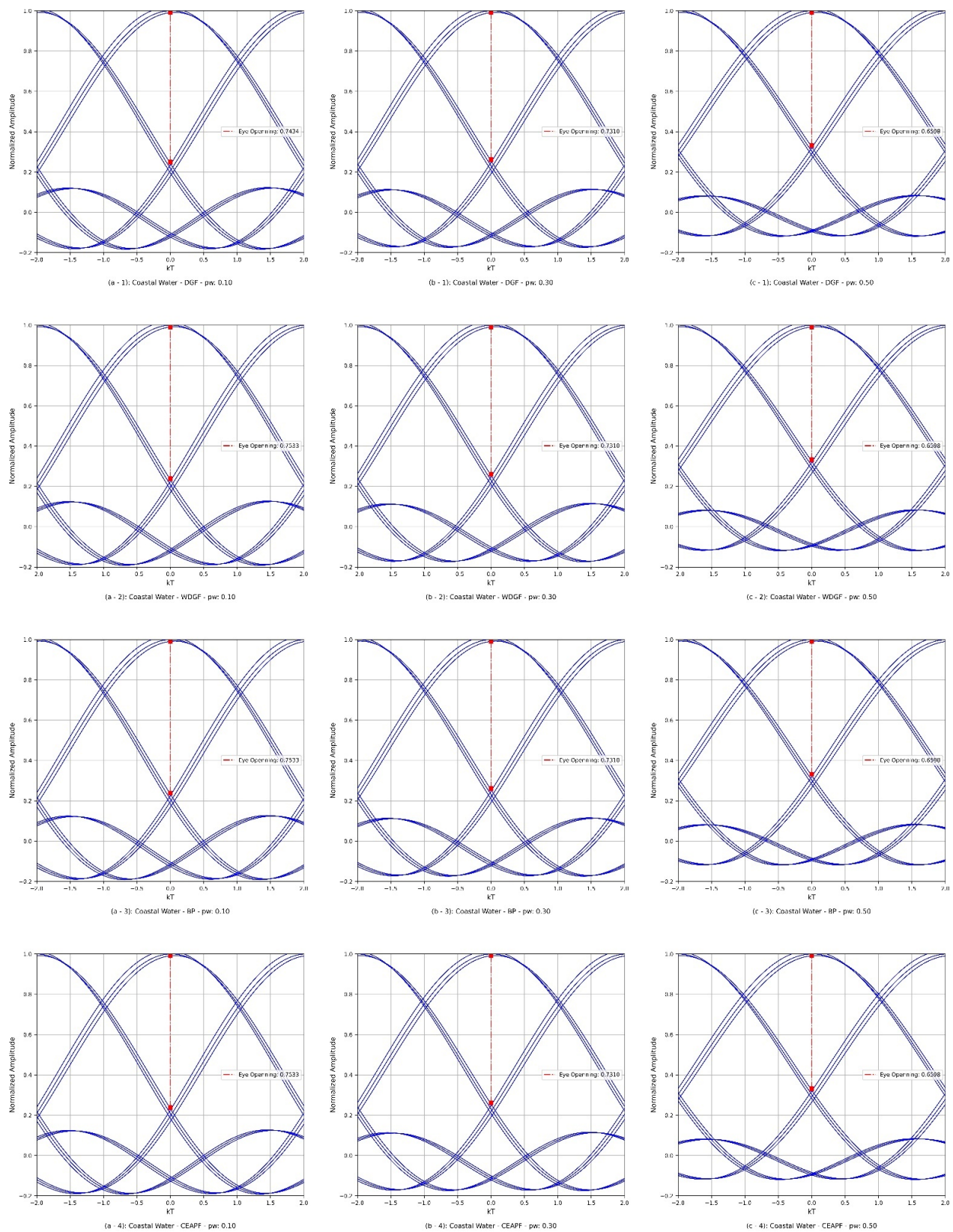


Figure 8. Use Case III: Eye-opening (in red) for Coastal Water. The eye diagrams of the filter output $r_f(t)$ (Equation (5a)) for $s(t)$ (Equation (2)) pulse-width (pw) set {0.1, 0.3, 0.5} for four propagation channel models of $h_c(t)$ [DGF, WDGF, BP, CEAPF]. Bit rate: 200 Mbit, water link span: 100 m.

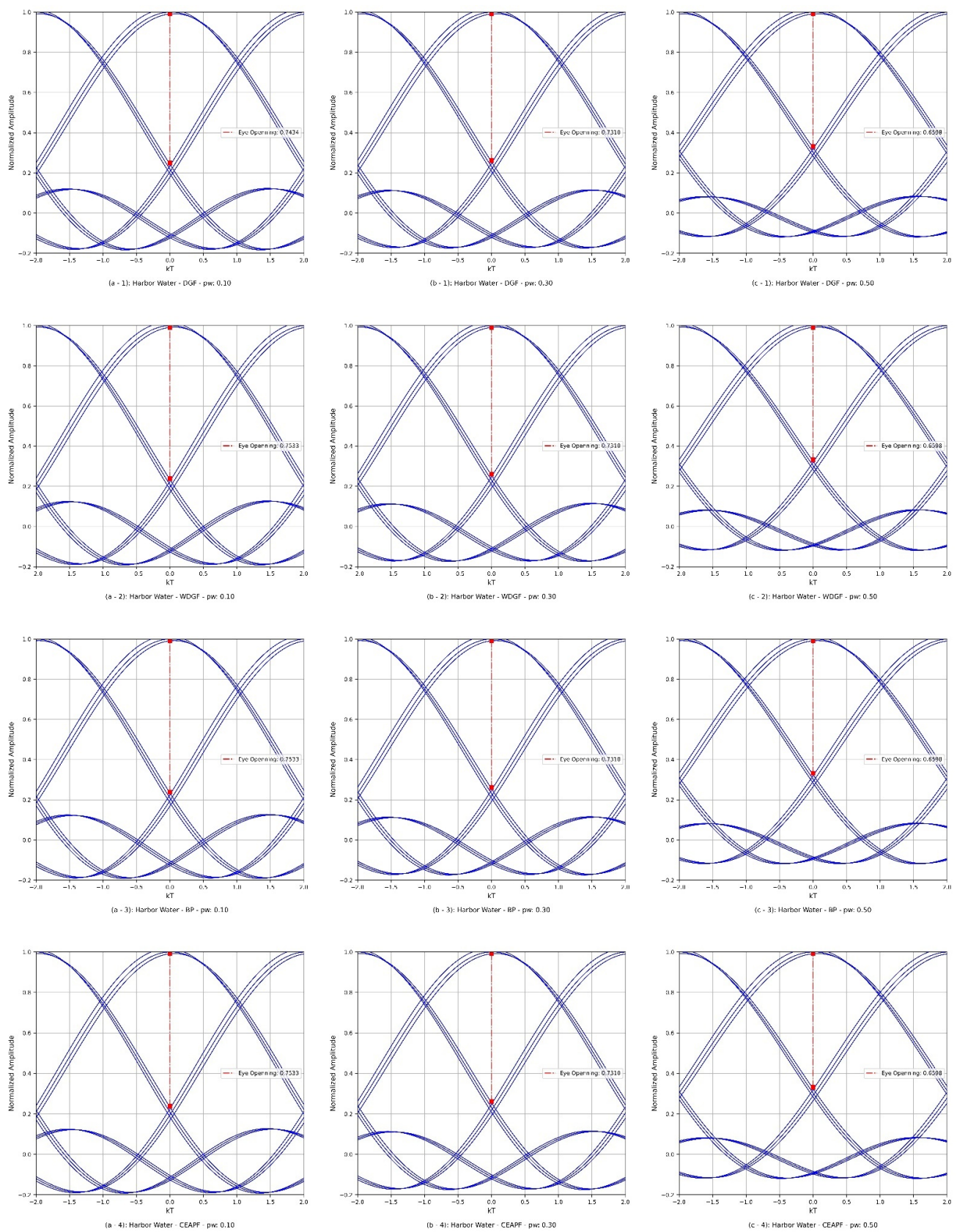


Figure 9. Use Case III: Eye-opening (in red) for Harbor Water. The eye diagrams of the filter output $r_f(t)$ (Equation (5a)) for $s(t)$ (Equation (2)) pulse-width (pw) set {0.1, 0.3, 0.5} for four propagation channel models of $h_c(t)$ {DGF, WDFG, BP, CEAPF} bit rate: 200 Mbit, water link span: 100 m.

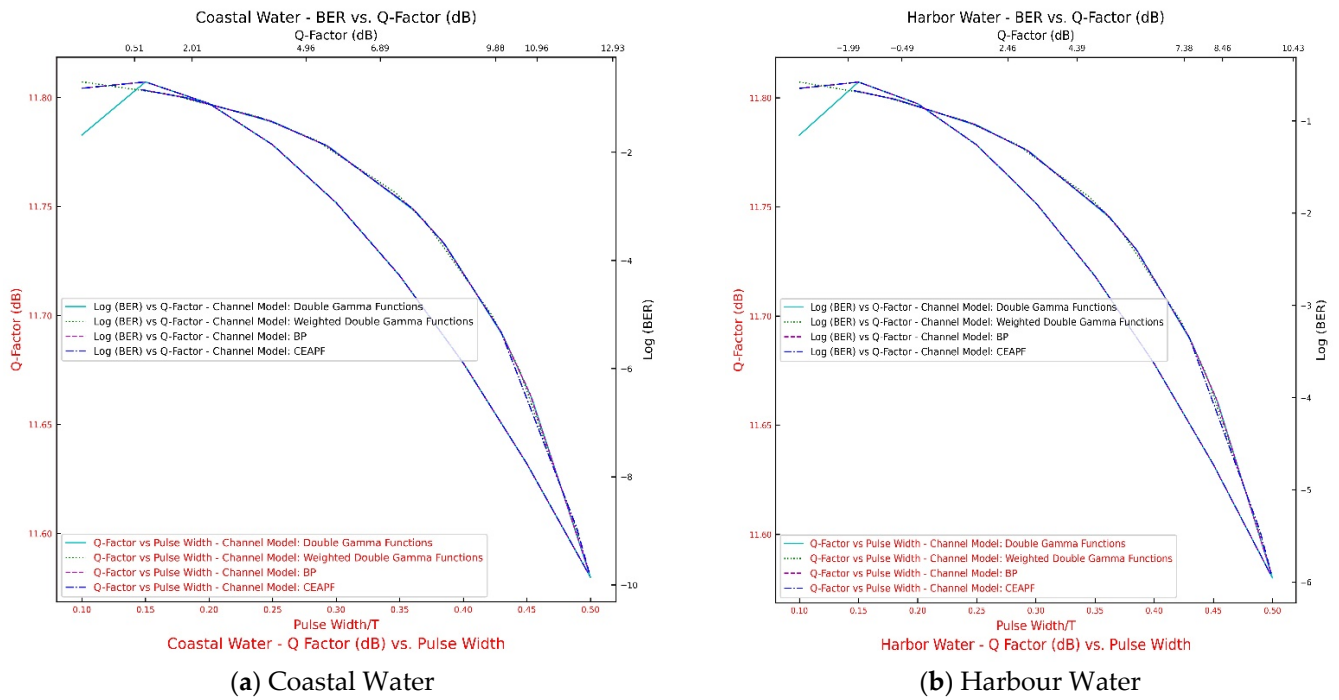


Figure 10. Use Case III: The impact of the optimum filter on the Optical receiver system performance in terms of the Q-Factor ($\propto$ SNR) and log(BER).

We used BER vs. Q-Factor and Q-factor vs. Pulse Width to complement the eye diagrams instead of BER vs. OSNR because, in this paper, we are not discussing the UWOC link power budget. Also, the Q-factor, as discussed in [25,30–32], is a suitable measure to investigate eye diagram behaviors. However, the Q-factor is related to the OSNR by the following equation:

$$Q_{\text{Factor}} = \text{OSNR} + 10 \log \frac{B_o}{B_e}. \quad (27)$$

In the above equation, B_o is the optical bandwidth of the photodetector (end device) and B_e is the electronic bandwidth of the receiver electronic filter. Hence, we can say that Q is somewhat proportional to the OSNR. There is OSNR and Q-factor Interdependency, depending on whether the system is single-wavelength or multiwavelength. In a single wavelength system (which is the assumption in our paper), the OSNR and Q-factor have a direct relationship. Increasing the OSNR improves the Q-factor, resulting in better signal quality.

Conversely, decreasing the OSNR degrades the Q-factor, leading to a higher BER and poor signal quality. But this is not completely true for a multiwavelength system. The interdependence of OSNR and Q-factor is more complex.

5. Conclusions and Future Work

This research defines the steps to develop a mathematical optical receiver design optimization technique. This approach involves applying the time-invariant feature of the optical receiver system to formulate an overall MGF design model. Hence, this model describes the underlying system signal lifecycle dynamics, which includes system attributes and processes. Consequently, such a design model aims to optimize the underlying system performance to overcome the limitations imposed by the existing signal impairments due to various potential processes.

Our work demonstrates how the MGF modelling design furnishes a closed-form optical receiver filter equation (ORFE). The ORFE equation includes a transmitted optical signal $s(t)$, propagation channel impulse response $h_c(t)$, received optical pulse $\lambda(t)$, photodetector efficiency μ , and photoelectron generation processes g , which equals one for PIN

photodetector, ISI, and AGTN noise associated with the amplifier circuitry. Also, this work explains the necessary steps to optimize the solution of ORFE. ORFE incorporates both the optical and receiver electronic domain boundaries of the UWOC system. The optical domain includes the optical attributes set $\{s(t), h_c(t), \lambda(t)\}$ and the associated optical process set {optical digital signal modulation, propagation channel impulse response} of UWOC. The electronic domain contains the photoelectron conversion process and its attribute set $\{\mu, g, N(t)\}$, the photodetector backend electronic circuitry process $\{\sigma_{2nd}\}$, and the equalization and filtering process $\{h_f(t)\}$.

To show the reliability of our modelling design, we deployed three use cases covering simple and complex scenarios of linear optical receivers for the UWOC system. The validity of the proposed new optical receiver modelling design was based on results comparison with closely related published MCB works. Through this, we also identified one potential weakness of MCB optimization. In conclusion, our MGF modelling methodology is more reliable than the CB and MCB design approaches.

Our new MGF design modelling approach preserves the limits between the general solution and optimum space due to the J optimization. So, their undershoots, depths, and variations are different rather than uniform for CB and MCB filters [10,16]. Also, it is worth noting that the analytical solution of use case II for the white AGTN noise exhibits undershoot when including ISI. Such behavior indicates that the undershoot is an intrinsic behavior of the solution rather than curve interpolation by the curve fitting library. Here, we do not mean that the numerical solution will not be subjected to changes due to possible higher fitting polynomial values.

From Figures 9 and 10, we can conclude that the proposed optimum filter h_f reduces O_{eye} by nearly 10–15% for five times increase in optical pulse width (pw). This outcome indicates that the optimum h_f manages the power dispersion correctly. Similarly, suppose the changes in pw values are mapped as the transmitted pulse broadening due to ISI impairment sources like scattering and scintillation. In that case, we can conclude that the proposed optimum filter h_f efficiently deals with the seawater impairments projected by the seawater channel model.

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