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Heat-Semigroup-Based Besov Capacity on Dirichlet Spaces and Its Applications

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Abstract: In this paper, we investigate the Besov space and the Besov capacity and obtain several important capacitary inequalities in a strictly local Dirichlet space, which satisfies the doubling condition and the weak Bakry–Émery condition. It is worth noting that the capacitary inequalities in this paper are proved if the Dirichlet space supports the weak $(1,2)$-Poincaré inequality, which is weaker than the weak $(1,1)$-Poincaré inequality investigated in the previous references. Moreover, we first consider the strong subadditivity and its equality condition for the Besov capacity in metric space.

Keywords: Besov space; Dirichlet space; heat kernel; capacity; Sobolev inequality

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1. Introduction

The Besov space is an important class of function spaces in the fields of geometry, harmonic analysis, and PDEs, and it has a close relationship with the Sobolev space and the bounded variation space (BV space). Thus, more and more scholars are turning their attention to the Besov space due to its wide range of applications. Mainly trace and extension result in Besov spaces which are defined on Ahlfors Q-regular subsets of $\mathbb{R}^n$ were investigated by Jonsson and Wallin in [1]. Xiao in [2] investigated the embeddings of homogeneous Besov spaces. Recently, the authors of [3,4] studied the quasi-conformal mappings in metric spaces and geometric group theory via relevant theories of Besov spaces. In addition, the Bourgain–Brezis–Mironescu formula characterizes the relationship between the normal Sobolev norm $\|\cdot\|_{W^{1,p}}$ and the limit of the fractional Sobolev norm $\|\cdot\|_{W^{\alpha,p}}$ when $\alpha \to 1^-$. However, we can replace the fractional Sobolev norm with the Besov norm in the Bourgain–Brezis–Mironescu formula, which can make the proof process simpler (cf. [5]). The Besov space in Euclidean space has been studied quite thoroughly. More scholars began to investigate theories of Besov spaces in other settings. In particular, as a hot topic, many papers consider Besov spaces, Sobolev spaces, and BV spaces in Dirichlet space under some extra assumptions, which are typical examples of metric spaces (cf. [6,7]). They prove some of their results in Dirichlet space, which satisfies the weak $(1,2)$-Poincaré inequality, where the condition is weaker than the weak $(1,1)$-Poincaré inequality.

The capacities originally came from the field of electrostatics in physics and they play crucial roles in studying the pointwise behavior of a Sobolev function. As of now, the notion of capacity has been applied in the fields of geometric measure theory and PDEs and it has been associated with various function spaces. Netrusov in [8,9], Adams and Xiao in [10], and Adams and Hurri-Syrjänen in [11] studied capacities related to Besov spaces. Bourdon in [3] investigated the Besov B_p-capacity in metric space $\mathbb{X}$ under assumptions that metric space is compact and Ahlfors Q-regular for $Q > 1$, where the authors dealt with functions from $A_p(\mathbb{X})$, the algebra of continuous functions that are in $B_p(\mathbb{X})$. The authors of [3,4,12] showed that the algebra $A_p(\mathbb{X})$ does not separate the points of $\mathbb{X}$ if $1 \le p \le Q < \infty$. Furthermore, in [13], Besov p-capacities and their relationship with the Hausdorff measure in Ahlfors regular metric spaces of dimension Q were considered.

Motivated by the previous results on Besov spaces and Besov capacities, we investigate these topics in Dirichlet space and obtain some important inequalities. It should be noted that our results extend the Besov capacity to more general settings under weaker conditions. Moreover, since Dirichlet space covers complete Riemannian manifolds with non-negative Ricci curvature and Carnot groups, our results pave the way for the continuing research on these topics in more complex space. Moreover, the Besov capacity in this paper can be applied to characterize the quasicontinuous representative of the Besov function, and capacitary inequalities may have potential applications for obtaining norms of embedding operators and bilateral estimates of eigenvalues for boundary value problems and so on. These applications will be investigated in our future research.

In [7], if a weak Bakry–Émery estimate, which is defined as in (5), is satisfied and the volume growth condition, $\mu(B(x,r)) \ge Cr^Q$, $r > 0$ for some $Q > 0$, is true, the following Sobolev inequality in Dirichlet space is obtained by Alonso-Ruiz, Baudoin, Chen, Rogers, Shanmugalingam and Teplyaev: for any $u \in BV(\mathbb{X})$,

$$\|u\|_{L^{\frac{Q}{Q-1}}(\mathbb{X})} \le C|Du|(\mathbb{X}), \tag{1}$$

and the isoperimetric inequality for the finite perimeter set $K \subset \mathbb{X}$,

$$[\mu(K)]^{\frac{Q}{Q-1}} \le CP(K,\mathbb{X}),$$

where $|Du|(\mathbb{X})$ is the total variation (cf. [7] Definition 3.1) of u and $P(K,\mathbb{X}) := |D1_K|(\mathbb{X})$ denotes the perimeter measure of K.

Let $(\mathbb{X},\mu,\mathcal{E},\mathcal{F} = \mathrm{dom}(\mathcal{E}))$ be a strictly local (see Definition 2) Dirichlet space with a compact metric measure μ satisfying a weak Bakry–Émery estimate and the same volume growth condition $\mu(B(x,r)) \ge Cr^Q$. Based on the previous results in [7] mentioned above, the goal of this paper is to establish further characterizations of Besov capacities in Dirichlet space $(\mathbb{X},\mu,\mathcal{E},\mathcal{F} = \mathrm{dom}(\mathcal{E}))$, where metric space $\mathbb{X}$ is compact, the metric measure μ is a doubling Radon measure, and a strictly local Dirichlet form $(\mathcal{E},\mathcal{F} = \mathrm{dom}(\mathcal{E}))$ is defined on $L^2(\mathbb{X})$. We denote the Besov capacity of arbitrary set $E \subset \mathbb{X}$ as $\mathrm{cap}_p^\alpha(E)$. If the assumption that the weak Bakry–Émery condition holds true is added, then the Sobolev inequality (1) is split into

$$\begin{cases} \| f \|_{L^q(\mathbb{X},\mu)} \lesssim \left\{ \int_0^\infty \left[\mathrm{cap}_p^\alpha\left(\left\{x \in \mathbb{X} : \ |f(x)| \ge t\right\}\right)\right]^{\frac{q}{p}} dt^q \right\}^{1/q}, \ f \in L^q(\mathbb{X},d\mu), \\ \left[\int_0^\infty \left(\mathrm{cap}_p^\alpha\left(\left\{x \in \mathbb{X} : |f(x)| \ge t\right\}\right)\right)^{\frac{q}{p}} dt^q \right]^{1/q} \lesssim \| f \|_{p,\alpha}, \ f \in \mathbf{B}^{p,\alpha}(\mathbb{X}), \end{cases}$$

where $q = \frac{p\beta}{\beta - p\alpha}$ with β appearing in (7), $L^q(\mathbb{X},d\mu)$ denotes the class of all L^q integrable functions, and $\mathbf{B}^{p,\alpha}(\mathbb{X})$ is defined in Definition 7. These results generalize the similar counterparts obtained in [14] in Euclidean space and in [15] on the Grushin plane, respectively.

It should be pointed out that, in previous references (for example, cf. [16–18]) for the research on the capacity in metric space $\mathbb{X}$, the base space needs to satisfy the weak $(1,1)$-Poincaré inequality: there are constants $\lambda \ge 1$ and $C_P > 0$ such that

$$\frac{1}{\mu(B(x,r))} \int_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \le C_P \frac{r}{\mu(B(x,\lambda r))} \int_{B(x,\lambda r))} |\nabla u| \, d\mu \tag{2}$$

for any ball $B(x,r) \subseteq \mathbb{X}$, where $u_{B(x,r)}$ denotes the average integral of f on the ball $B(x,r)$. Specifically,

$$u_{B(x,r)} := \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u(x) d\mu(x).$$

In the paper [7] mentioned above, the relative isoperimetric inequality on $(\mathbb{X}, \mu, \mathcal{E}, \mathcal{F} = \mathrm{dom}(\mathcal{E}))$ can be deducedfrom (2), which plays an important role in the field of harmonic analysis and geometric measure theory. Throughout this paper, we assume that $(\mathbb{X}, \mu, \mathcal{E}, \mathcal{F} = \mathrm{dom}(\mathcal{E}))$ supports the weak $(1,2)$-Poincaré inequality, i.e.

$$\frac{1}{\mu(B(x,r))} \int_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \leq C_P \, r \left(\frac{1}{\mu(B(x,\lambda r))} \int_{B(x,\lambda r)} |\nabla u|^2 \, d\mu \right)^{1/2}. \tag{3}$$

From Definition 4 below, it is easy to prove that (3) is weaker than (2) by the Hölder inequality.

Under the Bakry–Émery condition $BE(K,\infty)$, the De Giorgi characterization of the total variation of the BV function is obtained in [19] in the metric setting. The definition of the $BE(K,\infty)$ condition is formulated in [19–21] and stated as follows.

Definition 1 (Bakry–Émery condition)**.** *The Dirichlet form $(\mathcal{E}, \mathcal{F} = dom(\mathcal{E}))$ is said to satisfy the $BE(K,\infty)$ condition with $K \in \mathbb{R}$ if*

$$\frac{1}{2} \int_{\mathbb{X}} |\nabla f|^2 A\varphi d\mu - \int_{\mathbb{X}} \varphi \nabla f \cdot \nabla A f d\mu \geq K \int_{\mathbb{X}} \varphi |\nabla f|^2 d\mu$$

holds true for every $f \in dom(A)$ such that $Af \in \mathcal{F}$, where $\varphi \in dom(A) \cap L^\infty(\mathbb{X})$ is a nonnegative function satisfying $A\varphi \in L^\infty(\mathbb{X})$ and $(\mathcal{E}, \mathcal{F} = dom(\mathcal{E}))$ is given as in Section 2 in relation to the Cheeger differentiable structure.

We list the structure of this paper as follows. The basic properties of strictly local Dirichlet spaces and our standing assumptions are given in Section 2. In Section 3 we investigate some basic results for heat-semigroup-based Besov spaces in Dirichlet space under assumptions given in Section 2. Section 4 gives some basic properties for Besov capacities and proves the capacitary strong type estimate in Dirichlet space. In Section 5, we finalize the article, state the main results, and propose the next research plan.

Throughout this article, c and C are used to denote the positive constants, which are independent of main parameters and may be different at each occurrence. We use the symbol $\lesssim$ (respectively $\gtrsim$) between two nonnegative expressions u, v to indicate that there is a constant $C > 0$ such that $u \leq Cv$ (respectively, $u \geq Cv$). We also use the symbol $\approx$ between two positive expressions u, v to indicate that it satisfies $u \gtrsim v$ and $u \lesssim v$.

2. Standing Assumptions

Next, we begin to list some basic concepts of Dirichlet spaces and introduce assumptions that are needed to state our results. Most of the basic knowledge of Dirichlet spaces is introduced in [6,7].

Let $\mathbf{X}$ be a locally compact metric space and equip a Radon measure μ which is supported on $\mathbf{X}$. The Dirichlet form $(\mathcal{E}, \mathcal{F} = \mathrm{dom}(\mathcal{E}))$ on $\mathbf{X}$ is a densely defined, closed, symmetric, and Markovian form on $L^2(\mathbf{X})$.

We denote $C_c(\mathbf{X})$ as the vector space of all continuous functions with compact support in $\mathbf{X}$ and denote $C_0(\mathbf{X})$ as closure of $C_c(\mathbf{X})$ in terms of the supremum norm. A core for $(\mathbf{X}, \mu, \mathcal{E}, \mathcal{F} = \mathrm{dom}(\mathcal{E}))$ is a subset $\mathcal{C}$ of $C_c(\mathbf{X}) \cap \mathcal{F}$, which is dense in $C_c(\mathbf{X})$ in the supremum norm and dense in $\mathcal{F}$ in the norm

$$\left(\|u\|^2_{L^2(\mathbf{X}, d\mu)} + \mathcal{E}(u,u) \right)^{1/2}.$$

If it admits a core, we call the Dirichlet form $(\mathcal{E}, \mathcal{F} = \mathrm{dom}(\mathcal{E}))$ regular. In addition, if for any two compactly supported functions $u, v \in \mathcal{F}$ such that u is constant in a neighborhood of the support of v, we have $\mathcal{E}(u,v) = 0$, then we call the Dirichlet form *strongly local*.

Assume $(\mathcal{E}, \mathcal{F} = \mathrm{dom}(\mathcal{E}))$ is regular, and then for every $u, v \in \mathcal{F} \cap L^\infty(\mathbf{X})$, the authors in [6,7] define the energy measure $\Gamma(u,v)$ via the formula

$$\int_{\mathbf{X}} \varphi \, d\Gamma(u,v) = \frac{1}{2}[\mathcal{E}(\varphi u, v) + \mathcal{E}(\varphi v, u) - \mathcal{E}(\varphi, uv)], \quad \varphi \in \mathcal{F} \cap C_c(\mathbf{X}).$$

With respect to the Dirichlet form $(\mathcal{E}, \mathcal{F} = \mathrm{dom}(\mathcal{E}))$, the following *intrinsic metric* $d_{\mathcal{E}}$ on $\mathbf{X}$ is defined by

$$d_{\mathcal{E}}(x,y) = \sup\left\{u(x) - u(y) \,:\, u \in \mathcal{F} \cap C_0(\mathbf{X}) \text{ and } d\Gamma(u,u) \le d\mu\right\},$$

where the condition $d\Gamma(u,u) \le d\mu$ means that $\Gamma(u,u)$ is absolutely continuous with respect to μ, whose Radon–Nikodym derivative is bounded by 1.

The associated infinitesimal generator A acts on a dense subspace $\mathrm{dom}(A)$ of $\mathcal{F}$ so that for each $f \in \mathrm{dom}(A)$ and for every $g \in \mathcal{F}$,

$$\int_{\mathbf{X}} f A g d\mu = -\mathcal{E}(f,g).$$

The operator A is dissipative in the sense that

$$\int_{\mathbf{X}} f A f d\mu = -\mathcal{E}(f,f) \le 0$$

and is merely the Laplacian Δ when $\mathbf{X} = \mathbb{R}^n$.

Definition 2 (Definition 2.2 in [7])**.** *If $d_{\mathcal{E}}$ is a metric on $\mathbf{X}$ and the topology generated by $d_{\mathcal{E}}$ is same as the topology on $\mathbf{X}$, then the strongly local regular Dirichlet space $(\mathbf{X}, \mu, \mathcal{E}, \mathcal{F} = dom(\mathcal{E}))$ is strictly local.*

In the remaining sections, the assumption that the Dirichlet space $(\mathbf{X}, \mu, \mathcal{E}, \mathcal{F} = \mathrm{dom}(\mathcal{E}))$ is strictly local is always true. Hence, the Dirichlet form $(\mathcal{E}, \mathcal{F} = \mathrm{dom}(\mathcal{E}))$ is strongly local and regular. What is more, the metric $d_{\mathcal{E}}$ is equipped on $(\mathbf{X}, \mu, \mathcal{E}, \mathcal{F} = \mathrm{dom}(\mathcal{E}))$, which induces the topology on $\mathbf{X}$. At this time, $(\mathbf{X}, d_{\mathcal{E}})$ is complete. If $\Gamma(u,u)$ is absolutely continuous for the measure μ, similar to locally Lipschitz functions, we know that $|\nabla u|$ is the square root of its Radon–Nikodym derivative, which implies $\Gamma(u,u) = |\nabla u|^2 d\mu$.

In addition, we assume that the strictly local Dirichlet space $(\mathbf{X}, \mu, \mathcal{E}, \mathcal{F} = \mathrm{dom}(\mathcal{E}))$ satisfies the following assumptions.

Assumption 1 (The doubling condition)**.** *A Radon measure μ satisfies that the volume of every ball $B(x,r)$ in $(\mathbf{X}, \mu, \mathcal{E}, \mathcal{F} = dom(\mathcal{E}))$ is doubling throughout this paper and we give its definition as follows.*

Definition 3 (Definition 2.6 in [7])**.** *The strictly local Dirichlet space $(\mathbf{X}, \mu, \mathcal{E}, \mathcal{F} = dom(\mathcal{E}))$ is said to satisfy the volume doubling property if there is a constant $C_D > 0$ such that*

$$\mu(B(x,2r)) \le C_D \, \mu(B(x,r)).$$

hold true for every $B(x,r) \subset \mathbf{X}$ with $r > 0$.

The above doubling condition could imply that there are constants $C_Q > 0$ and $Q > 0$ depending only on C_D such that for all $0 < r < R$,

$$\mu(B(x,R)) \le C_Q (R/r)^Q \mu(B(x,r)) \tag{4}$$

hold on for every $B(x,r) \subset \mathbf{X}$. Notice that $(\mathbf{X}, \mu, \mathcal{E}, \mathcal{F} = \mathrm{dom}(\mathcal{E}))$ satisfies the volume doubling property if and only if it satisfies (4) for some $Q > 0$, where $Q \approx \log_2 C_D$. Moreover, if (4) holds true for Q, then it holds true for each $Q' > Q$. Hence, we shall assume that $Q \geq 2$ in our setting without loss of generality.

Assumption 2 (The Poincaré inequality)**.** *The strictly local Dirichlet space $(\mathbf{X}, \mu, \mathcal{E}, \mathcal{F} = dom(\mathcal{E}))$ also needs to support the following weak $(1,p)$-Poincaré inequality, which is given as follows.*

Definition 4 (Definition 2.12 in [7])**.** *The strictly local Dirichlet space $(\mathbf{X}, \mu, \mathcal{E}, \mathcal{F} = dom(\mathcal{E}))$ supports a weak $(1,p)$-Poincaré inequality with $1 \leq p < \infty$ if there are constants $C_P > 0$ and $\lambda \geq 1$ such that*

$$\frac{1}{\mu(B(x,r))} \int_{B(x,r)} |u - u_B(x,r)| \, d\mu \leq C_P \, r \left(\frac{1}{\mu(B(x,\lambda r))} \int_{B(x,\lambda r)} |\nabla u|^p \, d\mu \right)^{1/p}.$$

where $u \in \mathcal{F}$ and $B(x,r)$ is a ball in $(\mathbf{X}, \mu, \mathcal{E}, \mathcal{F} = dom(\mathcal{E}))$ (with respect to the metric $d_{\mathcal{E}}$) .

It follows from [7] and references therein that there are some examples of strictly local Dirichlet spaces $(\mathbf{X}, \mu, \mathcal{E}, \mathcal{F} = \mathrm{dom}(\mathcal{E}))$ that satisfy the doubling condition and the weak $(1,2)$-Poincaré inequality, including Carnot groups, metric graphs with bounded geometry, complete Riemannian manifolds with non-negative Ricci curvature and other complete sub-Riemannian manifolds satisfying a generalized curvature dimension inequality.

Assumption 3 (The curvature condition)**.** *The weak Bakry–Émery curvature condition is an additional requirement for obtaining capacitary inequalities on the strictly local Dirichlet spaces.*

Definition 5 (Definition 2.13 in [7])**.** *The strictly local Dirichlet space $(\mathbf{X}, \mu, \mathcal{E}, \mathcal{F} = dom(\mathcal{E}))$ is said to satisfy a weak Bakry–Émery curvature condition if*

$$\| |\nabla \mathcal{P}_t u| \|^2_{L^\infty(\mathbf{X})} \leq \frac{C}{t} \|u\|^2_{L^\infty(\mathbf{X})} \tag{5}$$

holds true for every $t > 0$, where $\{\mathcal{P}_t\}_{t \in [0,\infty)}$ is the semigroup of contractions on $L^2(\mathbf{X}, \mu)$ generated by A on the Dirichlet space $(\mathbf{X}, \mu, \mathcal{E}, \mathcal{F} = dom(\mathcal{E}))$.

The semigroup $\{\mathcal{P}_t\}_{t \in [0,\infty)}$ is also called the heat semigroup on $(\mathbf{X}, \mu, \mathcal{E}, \mathcal{F} = \mathrm{dom}(\mathcal{E}))$ and we can obtain properties of its corresponding heat kernel $p_t(x,y)$ on $[0,\infty) \times \mathbf{X} \times \mathbf{X}$ from ([19] Remark 2.1). From [7] and references therein, the weak Bakry–Émery curvature condition is valid in the following settings, for example, complete Riemannian manifolds with non-negative Ricci curvature, Carnot groups, complete sub-Riemannian manifolds with generalized non-negative Ricci curvature, and so on.

We will also need a stronger condition than (5) for solving some more difficult problems, such as De Giorgi characterizations (cf. [22] Theorem 2.12).

Definition 6 (Definition 2.15 in [7])**.** *The strictly local Dirichlet metric space $(\mathbf{X}, \mu, \mathcal{E}, \mathcal{F} = dom(\mathcal{E}))$ is said to satisfy a quasi-Bakry–Émery curvature condition, if there is a constant $C > 0$ such that*

$$|\nabla P_t u| \leq C P_t |\nabla u|, \;\; \mu - a.e. \tag{6}$$

for every $u \in \mathcal{F}$ and $t \geq 0$.

Via the proof of ([23] Theorem 3.3), it is obvious that the weak Bakry–Émery curvature condition (5) could be implied by the quasi-Bakry–Émery curvature condition. The authors in [19] obtained these characterizations under the Bakry–Émery condition in the metric setting, which is also denoted by the $BE(K,\infty)$ condition (refer to Definition 1).

Following from ([19] Proposition 6.2), we conclude that the weak $(1,1)$-Poincaré inequality is equivalent to the $BE(K,\infty)$ condition. However, the direct connection between quasi-Bakry–Émery curvature condition and the $BE(K,\infty)$ condition on the strictly local Dirichlet space cannot be obtained by the previous arguments.

3. Heat-Semigroup-Based Besov Spaces

The strictly local Dirichlet space $(\mathbf{X}, \mu, \mathcal{E}, \mathcal{F} = \mathrm{dom}(\mathcal{E}))$ is symmetric, as mentioned in Section 2. Let the metric space $\mathbf{X}$ be equipped with the Radon measure μ and a closed Markovian bilinear form $\mathcal{E}$ on $L^2(\mathbf{X}, \mu)$, where $\mathcal{F}$ denotes the collection of all functions $f \in L^2(\mathbf{X}, \mu)$ with $\mathcal{E}(u,u)$ finite. The readers can refer to theories of Dirichlet forms from the book [24] and references therein. Moreover, the heat semigroup $\{\mathcal{P}_t\}_{t\in[0,\infty)}$ is called the Markovian semigroup associated with $(\mathbf{X}, \mu, \mathcal{E}, \mathcal{F} = \mathrm{dom}(\mathcal{E}))$. Throughout this section, the measurable heat kernel function is assumed to satisfy

$$p_t(x,y) \leq Ct^{\beta} \quad \mu \times \mu - a.e. \tag{7}$$

for every $(t,x,y) \in (0,\infty) \times \mathbf{X} \times \mathbb{X}$, where $C > 0$ and $\beta > 0$.

Next, we will recall the theories for Besov spaces on the strictly local Dirichlet space established in [7] and investigate several important properties of the Besov space on the strictly local Dirichlet space $(\mathbf{X}, \mu, \mathcal{E}, \mathcal{F} = \mathrm{dom}(\mathcal{E}))$. In [6,7], the definition of the Besov seminorm based on the heat semigroup is given as follows.

Definition 7. *Suppose the Dirichlet space $(\mathbf{X}, \mu, \mathcal{E}, \mathcal{F} = dom(\mathcal{E}))$ is strictly local. Let $p \geq 1$ and $\alpha \geq 0$. For $u \in L^p(\mathbf{X}, \mu)$, the Besov seminorm of u is defined as:*

$$\|u\|_{p,\alpha} = \sup_{t>0} t^{-\alpha} \left(\int_{\mathbf{X}} \mathcal{P}_t(|u - u(y)|^p)(y) d\mu(y) \right)^{1/p}$$

and the Besov function space on the strictly local Dirichlet space is defined as:

$$\mathbf{B}^{p,\alpha}(\mathbf{X}) = \{u \in L^p(\mathbf{X}, \mu) \,:\, \|u\|_{p,\alpha} < +\infty\},$$

where the norm of u on $\mathbf{B}^{p,\alpha}(\mathbf{X})$ is defined as:

$$\|u\|_{\mathbf{B}^{p,\alpha}(\mathbf{X})} = \|u\|_{L^p(\mathbf{X},\mu)} + \|u\|_{p,\alpha}.$$

If $\mathcal{P}_t$ admits a heat kernel $p_t(x,y)$, then by the definition of Markovian semigroup $\mathcal{P}_t$, we have

$$\int_{\mathbf{X}} \mathcal{P}_t(|u - u(y)|^p)(y) d\mu(y) = \int_{\mathbf{X}} \int_{\mathbf{X}} |u(x) - u(y)|^p p_t(x,y) d\mu(x) d\mu(y). \tag{8}$$

It should be noted that even if $\mathcal{P}_t$ does not have a heat kernel, the Besov seminorm $\|\cdot\|_{p,\alpha}$ can be well defined.

Strum [25,26] investigated that the volume doubling property together with the weak $(1,2)$-Poincaré inequality are equivalent to the fact that the Markovian semigroup $\mathcal{P}_t$ admits a heat kernel $p_t(x,y)$ on $\mathbf{X} \times \mathbf{X}$ with $t > 0$ and the heat kernel satisfies the following fact: there are constants $c > 1$ and $C > 1$ such that

$$\frac{1}{C} \frac{e^{-\frac{cd(x,y)^2}{t}}}{\sqrt{\mu(B(x,\sqrt{t}))\mu(B(y,\sqrt{t}))}} \leq p_t(x,y) \leq C \frac{e^{-\frac{c^{-1}d(x,y)^2}{t}}}{\sqrt{\mu(B(x,\sqrt{t}))\mu(B(y,\sqrt{t}))}}$$

holds true for any $(x,y) \in \mathbf{X} \times \mathbf{X}$.

Following from ([6] Proposition 4.14) and ([6] Corollary 4.16), we know that the heat-semigroup-based Besov space $\mathbf{B}^{p,\alpha}(\mathbf{X})$ is a Banach space for $p \geq 1$ and that it is a reflexive space for $p > 1$.

In ([7] Theorem 4.4), the relationship between $\mathbf{B}^{p,\alpha}(\mathbf{X})$ and $BV(\mathbf{X})$ is implied, which is stated as the following proposition.

Proposition 1. *Suppose the strictly local Dirichlet space $(\mathbf{X}, \mu, \mathcal{E}, \mathcal{F} = dom(\mathcal{E}))$ satisfies the weak Bakry–Émery condition (5), then we have $\mathbf{B}^{p,1/2}(\mathbf{X}) = BV(\mathbf{X})$ with comparable seminorms.*

In order to reveal the relationship between $\mathbf{B}^{p,\alpha}(\mathbf{X})$ and the Sobolev space, we recall its definition given in [7]. The Sobolev space $W^{1,p}(\mathbf{X})$ with $p \geq 1$ can be defined as follows:

$$W^{1,p}(\mathbf{X}) := \{f \in L^p(\mathbf{X}, d\mu) \bigcap \mathcal{F}_{loc}(\mathbf{X}) : \Gamma(f,f) \ll f, |\nabla f| \in L^p(\mathbf{X}, d\mu)\},$$

where

$$\mathcal{F}_{loc}(\mathbf{X}) := \{f \in L^2_{loc}(\mathbf{X}) : \forall \Omega \subseteq \mathbf{X} \text{ is compact, } \exists g \in \mathcal{F} \text{ such that } f = g \mid_{\Omega} a.e.\}.$$

It follows from ([7] Section 4.4) that the following proposition is valid.

Proposition 2. *If the strictly local Dirichlet space $(\mathbf{X}, \mu, \mathcal{E}, \mathcal{F} = dom(\mathcal{E}))$ satisfies the quasi-Bakry–Émery condition (6), then we have $\mathbf{B}^{p,1/2}(\mathbf{X}) = W^{1,p}(\mathbf{X})$ with comparable norms for every $p > 1$.*

The following definition of the Besov seminorm is obtained from [27], which is regards as a generalization of $\|\cdot\|_{p,\alpha}$. For $0 \leq \alpha < \infty$, $1 \leq p < \infty$ and $p < q \leq \infty$, let $B^{\alpha}_{p,q}(\mathbf{X})$ consist of functions $f \in L^p(\mathbf{X}, d\mu)$, and then the Besov seminorm is defined as:

$$\|f\|_{B^{\alpha}_{p,q}(\mathbf{X})} := \left(\int_0^{\infty}\left(\int_{\mathbf{X}}\int_{B(x,t)} \frac{|f(x)-f(y)|^p}{t^{\alpha p}\mu(B(x,t))} d\mu(y)d\mu(x)\right)^{\frac{q}{p}} \frac{dt}{t}\right)^{1/q} < \infty$$

if $q < \infty$;

$$\|f\|_{B^{\alpha}_{p,\infty}(\mathbf{X})} := \sup_{t>0}\left(\int_{\mathbf{X}}\int_{B(x,t)} \frac{|f(x)-f(y)|^p}{t^{\alpha p}\mu(B(x,t))} d\mu(y)d\mu(x)\right)^{1/p} < \infty$$

if $q = \infty$.

In addition, for $\Omega \subset \mathbf{X}$ and $0 < \alpha \leq 1$, we define α-Lipschitz function as

$$lip^{\alpha}(\Omega) := \{f : \Omega \to \mathbb{R} : |f(x) - f(y)| < Cd(x,y)^{\alpha}, \forall x, y \in \Omega\}.$$

In order to reveal the relationship between $B^{\alpha}_{p,\infty}(\mathbf{X})$ and $\mathbf{B}^{p,\alpha}(\mathbf{X})$, it follows from ([7] Section 4.1) that the following proposition is valid.

Proposition 3. *For $1 \leq p < \infty$ and $0 < \alpha < \infty$ we have*

$$B^{2\alpha}_{p,\infty}(\mathbf{X}) = \mathbf{B}^{p,\alpha}(\mathbf{X})$$

with equivalent seminorms

$$C_1\|f\|_{p,\alpha} \leq \|f\|_{B^{2\alpha}_{p,\infty}(\mathbf{X})} \leq C_2\|f\|_{p,\alpha},$$

where constant C_1 and C_2 depend on the volume doubling constant C_D.

We use ([28] Lemma 2.2) to obtain the following lemma.

Lemma 1. *Suppose the Dirichlet space $(\mathbf{X}, \mu, \mathcal{E}, \mathcal{F} = dom(\mathcal{E}))$ is strictly local. When $f_j \to f$ in $\mathbf{B}^{p,\alpha}(\mathbf{X})$ and $g_j \to g$ in $\mathbf{B}^{p,\alpha}(\mathbf{X})$, we deduce that $\min(f_j, g_j) \to \min(f,g)$ in $\mathbf{B}^{p,\alpha}(\mathbf{X})$.*

Proof. Let $f^+(x) = \max(f(x), 0)$ and $f^-(x) = \min(f(x), 0)$. Since $\min(f, g) = g + (f - g)^+$, we only need to prove that if $f_j \to f$ in $\mathbf{B}^{p,\alpha}(\mathbf{X})$, then $f_j^+ \to f^+$ in $\mathbf{B}^{p,\alpha}(\mathbf{X})$. Due to (8) and

$$|f^+(x) - f^+(y)| \le |f(x) - f(y)|$$

for every $x, y \in \mathbf{X}$, we have that $f^+ \in \mathbf{B}^{p,\alpha}(\mathbf{X})$ whenever $f \in \mathbf{B}^{p,\alpha}(\mathbf{X})$. Since

$$|f_j^+(x) - f^+(x)| \le |f_j(x) - f(x)| \tag{9}$$

for every $x \in \mathbf{X}$, it is clear that $f_j^+ \to f^+$ in $L^p(\mathbf{X}, d\mu)$. For every $x, y \in \mathbf{X}$, via the triangle inequality and (9),

$$\begin{aligned} |(f_j^+(x) - f^+(x)) - (f_j^+(y) - f^+(y))| &\le |f_j^+(x) - f^+(x)| + |f_j^+(y) - f^+(y)| \\ &\le |f_j(x) - f(x)| + |f_j(y) - f(y)| \\ &\le |(f_j(x) - f(x)) - (f_j(y) - f(y))| + 2|f_j(y) - f(y)|. \end{aligned}$$

Substituting the above formula into (8) and using the Hölder inequality deduces

$$\begin{aligned} &\left(\int_{\mathbf{X}}\int_{\mathbf{X}} |(f_j^+(x) - f^+(x)) - (f_j^+(y) - f^+(y))|^p p_t(x,y) d\mu(x) d\mu(y)\right)^{\frac{1}{p}} \\ \le &\left(\int_{\mathbf{X}}\int_{\mathbf{X}} |(f_j(x) - f(x)) - (f_j(y) - f(y))|^p p_t(x,y) d\mu(x) d\mu(y)\right)^{\frac{1}{p}} \\ +2&\left(\int_{\mathbf{X}}\int_{\mathbf{X}} |f_j(y) - f(y)|^p p_t(x,y) d\mu(x) d\mu(y)\right)^{\frac{1}{p}} \\ = &\left(\int_{\mathbf{X}}\int_{\mathbf{X}} |(f_j(x) - f(x)) - (f_j(y) - f(y))|^p p_t(x,y) d\mu(x) d\mu(y)\right)^{\frac{1}{p}} \\ +2&\left(\int_{\mathbf{X}} |f_j(y) - f(y)|^p d\mu(y)\right)^{\frac{1}{p}}. \end{aligned}$$

Since $f_j \to f$ in $\mathbf{B}^{p,\alpha}(\mathbf{X})$, via letting $j \to \infty$, we have

$$\int_{\mathbf{X}}\int_{\mathbf{X}} |(f_j(x) - f(x)) - (f_j(y) - f(y))|^p p_t(x,y) d\mu(x) d\mu(y) \to 0,$$

and

$$\int_{\mathbf{X}} |f_j(y) - f(y)|^p d\mu(y) \to 0.$$

Then,

$$\int_{\mathbf{X}}\int_{\mathbf{X}} |(f_j^+(x) - f^+(x)) - (f_j^+(y) - f^+(y))|^p p_t(x,y) d\mu(x) d\mu(y) \to 0,$$

which, together with the fact that $f_j^+ \to f^+$ in $L^p(\mathbf{X}, d\mu)$, deduces $f_j^+ \to f^+$ in $\mathbf{B}^{p,\alpha}(\mathbf{X})$ and our claim is proved. □

Lemma 2. *Suppose the Dirichlet space $(\mathbf{X}, \mu, \mathcal{E}, \mathcal{F} = dom(\mathcal{E}))$ is strictly local. Let $p \ge 1$ and $0 < \alpha \le 1/2$.*

(i) *If $f, g \in \mathbf{B}^{p,\alpha}(\mathbf{X})$, then $\min(f, g)$ and $\max(f, g)$ belong to $\mathbf{B}^{p,\alpha}(\mathbf{X})$ and*

$$\|\max(f,g)\|_{p,\alpha} + \|\min(f,g)\|_{p,\alpha} \le \|f\|_{p,\alpha} + \|g\|_{p,\alpha}. \tag{10}$$

(ii) *If $f \in \mathbf{B}^{p,\alpha}(\mathbf{X})$ is nonnegative, then there is a sequence of nonnegative functions $\varphi_j \in lip_0^{2\alpha}(\mathbf{X})$ converging to f in $\mathbf{B}^{p,\alpha}(\mathbf{X})$, where $lip_0^{2\alpha}(\mathbf{X}) = lip^{2\alpha}(\mathbf{X}) \cap C_0(\mathbf{X})$.*

Proof. (i) Denote by $\phi := \min(f, g)$ and $\psi := \max(f, g)$. It is obvious that $\phi, \psi \in L^p(\mathbf{X})$. Let $\Omega_1 = \{x \in \mathbf{X} : f(x) \leq g(x)\}$ and $\Omega_2 = \{x \in \mathbf{X} : f(x) > g(x)\}$. By computation, we use (8) to obtain

$$\begin{aligned}\int_{\mathbf{X}}\int_{\mathbf{X}}|\phi(x)-\phi(y)|^p p_t(x,y)d\mu(x)d\mu(y) &= \int_{\Omega_1}\int_{\Omega_1}|f(x)-f(y)|^p p_t(x,y)d\mu(x)d\mu(y)\\ &+ \int_{\Omega_1}\int_{\Omega_2}|g(x)-f(y)|^p p_t(x,y)d\mu(x)d\mu(y)\\ &+ \int_{\Omega_2}\int_{\Omega_1}|f(x)-g(y)|^p p_t(x,y)d\mu(x)d\mu(y)\\ &+ \int_{\Omega_2}\int_{\Omega_2}|g(x)-g(y)|^p p_t(x,y)d\mu(x)d\mu(y)\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbf{X}}\int_{\mathbf{X}}|\psi(x)-\psi(y)|^p p_t(x,y)d\mu(x)d\mu(y) &= \int_{\Omega_1}\int_{\Omega_1}|g(x)-g(y)|^p p_t(x,y)d\mu(x)d\mu(y)\\ &+ \int_{\Omega_1}\int_{\Omega_2}|f(x)-g(y)|^p p_t(x,y)d\mu(x)d\mu(y)\\ &+ \int_{\Omega_2}\int_{\Omega_1}|g(x)-f(y)|^p p_t(x,y)d\mu(x)d\mu(y)\\ &+ \int_{\Omega_2}\int_{\Omega_2}|f(x)-f(y)|^p p_t(x,y)d\mu(x)d\mu(y).\end{aligned}$$

Via Definition 7, we have

$$\begin{aligned}&\|\phi\|_{p,\alpha} + \|\psi\|_{p,\alpha}\\ =\ & \sup_{t>0} t^{-\alpha p}\Big(\int_{\mathbf{X}}\int_{\mathbf{X}}|\phi(x)-\phi(y)|^p p_t(x,y)d\mu(x)d\mu(y)\Big)\\ +\ & \sup_{t>0} t^{-\alpha p}\Big(\int_{\mathbf{X}}\int_{\mathbf{X}}|\psi(x)-\psi(y)|^p p_t(x,y)d\mu(x)d\mu(y)\Big)\\ =\ & \sup_{t>0} t^{-\alpha p}\Big(\int_{\mathbf{X}}\int_{\mathbf{X}}|\phi(x)-\phi(y)|^p p_t(x,y)d\mu(x)d\mu(y) + \int_{\mathbf{X}}\int_{\mathbf{X}}|\psi(x)-\psi(y)|^p p_t(x,y)d\mu(x)d\mu(y)\Big)\\ =\ & \sup_{t>0} t^{-\alpha p}\Big\{\Big(\int_{\Omega_1}\int_{\Omega_1}|f(x)-f(y)|^p p_t(x,y)d\mu(x)d\mu(y)\\ +\ & \int_{\Omega_1}\int_{\Omega_1}|g(x)-g(y)|^p p_t(x,y)d\mu(x)d\mu(y)\Big)\\ +\ & \Big(\int_{\Omega_2}\int_{\Omega_2}|f(x)-f(y)|^p p_t(x,y)d\mu(x)d\mu(y) + \int_{\Omega_2}\int_{\Omega_2}|g(x)-g(y)|^p p_t(x,y)d\mu(x)d\mu(y)\Big)\\ +\ & \int_{\Omega_1}\int_{\Omega_2}\big(|g(x)-f(y)|^p + |f(x)-g(y)|^p\big)p_t(x,y)d\mu(x)d\mu(y)\\ +\ & \int_{\Omega_2}\int_{\Omega_1}\big(|f(x)-g(y)|^p + |g(x)-f(y)|^p\big)p_t(x,y)d\mu(x)d\mu(y)\Big\}.\end{aligned}$$

We use ([29] Lemma 2.2) to obtain

$$|g(x)-f(y)|^p + |f(x)-g(y)|^p \leq |f(x)-f(y)|^p + |g(x)-g(y)|^p \tag{11}$$

if $x \in \Omega_1$ and $y \in \Omega_2$ and

$$|f(x)-g(y)|^p + |g(x)-f(y)|^p \leq |f(x)-f(y)|^p + |g(x)-g(y)|^p \tag{12}$$

if $x \in \Omega_2$ and $y \in \Omega_1$. Combining (11), (12) and (13), we conclude that

$$\begin{aligned}\|\phi\|_{p,\alpha} + \|\psi\|_{p,\alpha} &\leq \sup_{t>0} t^{-\alpha p}\Big(\int_{\mathbf{X}}\int_{\mathbf{X}} |f(x)-f(y)|^p p_t(x,y)d\mu(x)d\mu(y)\Big) \\ &+ \sup_{t>0} t^{-\alpha p}\Big(\int_{\mathbf{X}}\int_{\mathbf{X}} |g(x)-g(y)|^p p_t(x,y)d\mu(x)d\mu(y)\Big) \\ &= \|f\|_{p,\alpha} + \|g\|_{p,\alpha}.\end{aligned}$$

which deduces that (10) holds true.

(ii) Due to Lemma 1, it is easy to prove that f^+ is in $\mathbf{B}^{p,\alpha}(\mathbf{X})$ whenever f is in $lip_0^{2\alpha}(\mathbf{X})$. But this is immediate, since $f^+ \in lip_0^{2\alpha}(\mathbf{X})$ whenever $f \in lip_0^{2\alpha}(\mathbf{X})$. This completes the proof. □

Lemma 3. *Assume the 2α-Lipschitz function φ with compact support is defined on the strictly local Dirichlet space $(\mathbf{X}, \mu, \mathcal{E}, \mathcal{F} = dom(\mathcal{E}))$. For any $f \in \mathbf{B}^{p,\alpha}(\mathbf{X})$, we have $f\varphi \in \mathbf{B}^{p,\alpha}(\mathbf{X})$ with*

$$\|f\phi\|_{\mathbf{B}^{p,\alpha}(\mathbf{X})} \leq C\|f\|_{\mathbf{B}^{p,\alpha}(\mathbf{X})},$$

where the constant C is dependent of p, α, C_D, the 2α-Lipschitz constant of φ and the diameter of $\mathrm{supp}(\phi)$.

Proof. Without loss of generality, we assume that $\phi \not\equiv 0$ and denote R as the diameter of $\mathrm{supp}(\phi)$. Choosing $x_0 \in \mathrm{supp}(\phi)$, it is obvious that $\mathrm{supp}(\phi) \subset \overline{B(x_0, R)}$, where $\overline{B(x_0, R)}$ is the closure of the ball $B(x_0, R)$. Via the definition of the 2α-Lipschitz function, there exists a constant $K > 0$ such that $|\phi(x) - \phi(y)| \leq Kd(x,y)^{2\alpha}$ for every $x, y \in \mathbf{X}$. Note that $\|\phi\|_{L^\infty(\mathbf{X})} \leq (2R)^{2\alpha}K$. We also notice that

$$\|f\phi\|_{L^p(\mathbf{X},d\mu)} \leq \|f\|_{L^p(\mathbf{X},d\mu)}\|\phi\|_{L^\infty(\mathbf{X})},$$

hence $f\phi \in L^p(\mathbf{X}, d\mu)$. For every $x, y \in \mathbf{X}$ we have

$$|f(x)\phi(x) - f(y)\phi(y)| \leq |f(x) - f(y)||\phi(y)| + |f(x)||\phi(x) - \phi(y)|.$$

Observe that

$$\begin{aligned}&\int_{\mathbf{X}}\int_{B(x,t)} \frac{|f(x)\phi(x) - f(y)\phi(y)|^p}{t^{2\alpha p}\mu(B(x,t))} d\mu(y)d\mu(x) \\ &\leq \int_{\mathbf{X}}\int_{B(x,t)} \frac{2^p|f(x) - f(y)|^p|\phi(y)|^p}{t^{2\alpha p}\mu(B(x,t))} d\mu(y)d\mu(x) \\ &\quad + \int_{\mathbf{X}}\int_{B(x,t)} \frac{2^p|f(x)|^p|\phi(x) - \phi(y)|^p}{t^{2\alpha p}\mu(B(x,t))} d\mu(y)d\mu(x) \\ &= I_1 + I_2.\end{aligned}$$

On the one hand,

$$\sup_{t>0} I_1 \leq 2^p\|\phi\|^p_{L^\infty(\mathbf{X})}\|f\|^p_{B^{2\alpha}_{p,\infty}(\mathbf{X})}. \tag{13}$$

On the other hand, since ϕ is 2α-Lipschitz function with constant K, we have

$$\begin{aligned}I_2 &\leq 2^p \int_{\mathbf{X}}\int_{B(x,t)} \frac{L^p d(x,y)^{2\alpha p}|f(x)|^p}{t^{2\alpha p}\mu(B(x,t))} d\mu(y)d\mu(x) \\ &= (2K)^p \int_{\mathbf{X}} |f(x)|^p \int_{B(x,t)} \frac{d(x,y)^{2\alpha p}}{t^{2\alpha p}\mu(B(x,t))} d\mu(y)d\mu(x) \\ &\leq (2K)^p \int_{\mathbf{X}} |f(x)|^p d\mu(x) = (2K)^p\|f\|^p_{L^p(\mathbf{X},d\mu)}.\end{aligned}$$

Via (13) we have

$$\|f\varphi\|_{\mathbf{B}^{p,\alpha}(\mathbf{X})} \leq \sup_{t>0}(I_1+I_2)^{\frac{1}{p}} \leq C\left(\|f\|^p_{B^{2\alpha}_{p,\infty}(\mathbf{X})} + \|f\|^p_{L^p(\mathbf{X},d\mu)}\right)^{\frac{1}{p}} \leq C\|f\|_{\mathbf{B}^{p,\alpha}(\mathbf{X})},$$

where the constant C is dependent of C_D, p, α, the 2α-Lipschitz constant of ϕ and the diameter of $\mathrm{supp}(\phi)$, which completes the proof. □

Lemma 4. *Assume that the 2α-Lipschitz compactly supported function ϕ is defined on the strictly local Dirichlet space $(\mathbf{X}, \mu, \mathcal{E}, \mathcal{F} = dom(\mathcal{E}))$. Suppose f_k is a sequence in $\mathbf{B}^{p,\alpha}(\mathbf{X})$ and $f_k \to f$ in $\mathbf{B}^{p,\alpha}(\mathbf{X})$. Then we have $f_k\phi \to f\phi$ in $\mathbf{B}^{p,\alpha}(\mathbf{X})$.*

Proof. For every $k \geq 1$, we obtain that $f_k\phi \in \mathbf{B}^{p,\alpha}(\mathbf{X})$ and $f\phi \in \mathbf{B}^{p,\alpha}(\mathbf{X})$ via Lemma 3. Moreover,

$$\|f_k\phi - f\phi\|_{\mathbf{B}^{p,\alpha}(\mathbf{X})} \leq C\|f_k - f\|_{\mathbf{B}^{p,\alpha}(\mathbf{X})}$$

holds for every $k \geq 1$, and since $f_k \to f$ in $\mathbf{B}^{p,\alpha}(\mathbf{X})$, it follows that $f_k\phi \to f\phi$ in $\mathbf{B}^{p,\alpha}(\mathbf{X})$. Then this completes the proof. □

Remark 1. *Assume that Ω and $\widetilde{\Omega}$ are two bounded and open subsets of $(\mathbf{X}, \mu, \mathcal{E}, \mathcal{F} = dom(\mathcal{E}))$ with $\Omega \subset\subset \widetilde{\Omega}$. Suppose that φ is a function in $lip_0^{2\alpha}(\widetilde{\Omega})$ with 2α-Lipschitz constant $\frac{C(C_D)}{\mathrm{dist}(\Omega,\mathbf{X}\setminus\widetilde{\Omega})^{2\alpha}}$ such that $0 \leq \phi \leq 1$ and $\phi = 1$ in Ω. By an argument similar to the one from Lemma 3, one can show that $f\phi \in \mathbf{B}^{p,\alpha}(\mathbf{X})$ whenever $f \in \mathbf{B}^{p,\alpha}(\mathbf{X})$. Moreover, in this case*

$$\|f\phi\|_{\mathbf{B}^{p,\alpha}(\mathbf{X})} \leq C\|f\|_{\mathbf{B}^{p,\alpha}(\mathbf{X})}$$

is valid for all $f \in \mathbf{B}^{p,\alpha}(\mathbf{X})$ and we can choose the constant $C > 0$ independent of C_D, p, α, $\mathrm{dist}(\Omega, \mathbf{X} \setminus \widetilde{\Omega})$ and the diameter of $\widetilde{\Omega}$.

Remark 2. *For every f, g are bounded functions in $\mathbf{B}^{p,\alpha}(\mathbf{X})$, it is easy to prove that $fg \in \mathbf{B}^{p,\alpha}(\mathbf{X})$, which can be obtained by the definition of Besov norm on the Dirichlet space. Moreover,*

$$\|fg\|_{L^p(\mathbf{X},d\mu)} \leq \min\Big\{\|f\|_{L^p(\mathbf{X},d\mu)}\|g\|_{L^\infty(\mathbf{X})}, \|g\|_{L^p(\mathbf{X},d\mu)}\|f\|_{L^\infty(\mathbf{X})}\Big\}$$

and

$$\|fg\|_{p,\alpha} \leq \|f\|_{p,\alpha}\|g\|_{L^\infty(\mathbf{X})} + \|g\|_{p,\alpha}\|f\|_{L^\infty(\mathbf{X})}.$$

4. Besov Capacities and Capacitary Inequalities

On metric spaces, some basic properties and related results of the Besov capacity are investigated in [6,13]. In this section, several properties of Besov capacities and relative capacitary inequalities on the Dirichlet space are proved. We first recall the definition of the Besov capacity on the Dirichlet space in [6]. Let $p \geq 1$ and $0 < \alpha < \beta$ with β appearing in (7). For a measurable set $E \subseteq \mathbf{X}$, the Besov capacity is defined as

$$\mathrm{cap}_p^\alpha(E) = \inf\{\|u\|^p_{p,\alpha} : u \in \mathcal{A}(E)\},$$

where $\mathcal{A}(E) := \{u \in \mathbf{B}^{p,\alpha}(\mathbf{X}), 1_E \leq u \leq 1\}$.

A capacity is a monotone and subadditive set function, i.e.,

$$\mathrm{cap}_p^\alpha(E_1 \cup E_2) \leq \mathrm{cap}_p^\alpha(E_1) + \mathrm{cap}_p^\alpha(E_2).$$

holds for every measurable set $E_1, E_2 \in \mathbf{X}$. Theorem 1 indicates that this is true for the heat-semigroup-based Besov capacity. As we know, few scholars have investigated the subadditivity and equality conditions for Besov capacity in metric space (cf. [30] for the Euclidean case). It should be noted that the subadditivity for Besov capacity is a kind of geometric property similar to the perimeter (cf. [31] Proposition 4.7) but measures generally do not satisfy this property.

Theorem 1. *Suppose* $(\mathbf{X}, \mu, \mathcal{E}, \mathcal{F} = dom(\mathcal{E}))$ *is a Dirichlet space. For any* $E, M, E_i, M_i \subset \mathbf{X}$ *with* $i \in \mathbb{N}$*, the heat-semigroup-based Besov capacity* $\mathrm{cap}_p^\alpha(\cdot)$ *enjoys properties as follows:*

(i) $\mathrm{cap}_p^\alpha(\emptyset) = 0$.

(ii) *Monotonicity: if* $F_1 \subset F_2$*, then* $\mathrm{cap}_p^\alpha(F_1) \le \mathrm{cap}_p^\alpha(F_2)$.

(iii) $\mathrm{cap}_p^\alpha(F) = \inf\Big\{\mathrm{cap}_p^\alpha(U) : F \subset U \subset \mathbf{X}, U \text{ open}\Big\}$.

(iv) *Downward monotone convergence: if* M_i *is a decreasing sequence of compact subsets of* $\mathbf{X}$ *with* $M = \bigcap_{i=1}^\infty M_i$*, then*

$$\mathrm{cap}_p^\alpha(M) = \lim_{i\to\infty} \mathrm{cap}_p^\alpha(M_i). \tag{14}$$

(v) *Upward monotone convergence: if* $M_1 \subset M_2 \subset \cdots \subset M = \bigcup_{i=1}^\infty M_i$*, then (14) holds true.*

(vi) *Subadditivity: if* $M = \bigcup_{i=1}^\infty M_i$*, then*

$$\mathrm{cap}_p^\alpha(M) \le \sum_{i=1}^\infty \mathrm{cap}_p^\alpha(M_i).$$

(vii) *Convexity or strong subadditivity: if* E_i *and* E_2 *are compact subsets of* $\mathbf{X}$*, then*

$$\mathrm{cap}_p^\alpha(E_1 \cup E_2) + \mathrm{cap}_p^\alpha(E_1 \cap E_2) \le \mathrm{cap}_p^\alpha(E_1) + \mathrm{cap}_p^\alpha(E_2). \tag{15}$$

Moreover, equality holds if $\mathrm{cap}_p^\alpha(E_1 \backslash E_1 \cap E_2) = 0$ *or* $\mathrm{cap}_p^\alpha(E_2 \backslash E_2 \cap E_1) = 0$.

Proof. (i), (ii) and (iii) can be obtained by the definition of Besov capacity in Dirichlet space. The proofs of (iv), (v) and (vi) are similar to the proof of ([28] Theorem 3.1) and we omit the details. It is enough to prove (vii). Without loss of generality, let

$$\mathrm{cap}_p^\alpha(E_1) + \mathrm{cap}_p^\alpha(E_2) < \infty.$$

For any $\varepsilon > 0$, there exist two functions $\phi \in \mathcal{A}(E_1)$ and $\psi \in \mathcal{A}(E_2)$ so that

$$\|\phi\|_{p,\alpha}^p < \mathrm{cap}_p^\alpha(E_1) + \frac{\varepsilon}{2}, \ \|\psi\|_{p,\alpha}^p < \mathrm{cap}_p^\alpha(E_2) + \frac{\varepsilon}{2}.$$

Let

$$\varphi_1 = \max\{\phi, \psi\} \ \& \ \varphi_2 = \min\{\phi, \psi\}.$$

It follows from Lemma 2 that

$$\varphi_1 \in \mathcal{A}(E_1 \cup E_2) \ \& \ \varphi_2 \in \mathcal{A}(E_1 \cap E_2).$$

Then,

$$\begin{aligned} & \mathrm{cap}_p^\alpha(E_1 \cup E_2) + \mathrm{cap}_p^\alpha(E_1 \cap E_2) \\ \le\ & \|\varphi_1\|_{p,\alpha}^p + \|\varphi_2\|_{p,\alpha}^p \\ =\ & \|\phi\|_{p,\alpha}^p + \|\psi\|_{p,\alpha}^p \\ \le\ & \mathrm{cap}_p^\alpha(E_1) + \mathrm{cap}_p^\alpha(E_2) + \varepsilon. \end{aligned}$$

Finally, via (15), we know that we only need to prove its converse inequality to complete the proof. Suppose $\mathrm{cap}_p^\alpha(E_1 \backslash E_1 \cap E_2) = 0$. Since $E_1 = (E_1 \backslash E_1 \cap E_2) \cup (E_1 \cap E_2)$, using (vi) we have

$$\mathrm{cap}_p^\alpha(E_1) \le \mathrm{cap}_p^\alpha((E_1 \backslash E_1 \cap E_2)) + \mathrm{cap}_p^\alpha(E_1 \cap E_2) = \mathrm{cap}_p^\alpha(E_1 \cap E_2).$$

(ii) implies that

$$\mathrm{cap}_p^\alpha(E_2) \le \mathrm{cap}_p^\alpha(E_1 \cup E_2).$$

Therefore, we have

$$\mathrm{cap}_p^\alpha(E_1) + \mathrm{cap}_p^\alpha(E_2) \le \mathrm{cap}_p^\alpha(E_1 \cup E_2) + \mathrm{cap}_p^\alpha(E_1 \cap E_2).$$

Another case can be similarly proved. This completes the of the assertion (vii). □

We are in a position to improve and split the Sobolev inequality established in [6].

Theorem 2. *Suppose* $(\mathbf{X}, \mu, \mathcal{E}, \mathcal{F} = dom(\mathcal{E}))$ *is a Dirichlet space. Let* $0 < \alpha < \beta$ *and* $1 \le p < \frac{\beta}{\alpha}$ *with* $q = \frac{p\beta}{\beta - p\alpha}$. *The following statements are equivalent:*

(i) *For each* L^q *integrable function* f *with compact support in* $\mathbf{X}$*,*

$$\| u \|_{L^q(\mathbf{X},\mu)} \lesssim \left\{ \int_0^\infty \left[\mathrm{cap}_p^\alpha\left(\left\{ x \in \mathbf{X} : \ |u(x)| \ge t \right\} \right) \right]^{\frac{q}{p}} dt^q \right\}^{1/q}; \tag{16}$$

(ii) $\mu(K) \lesssim (\mathrm{cap}_p^\alpha(K))^{\frac{q}{p}}$ *for each compact set* $K \subset \mathbb{X}$.

Moreover, (i) *and* (ii) *are valid.*

Proof. Firstly, we conclude that (i) holds true via ([6] Corollary 6.4). The equivalence between (i) and (ii) implies that (ii) also holds true.

(ii) $\Rightarrow$ (i) : Since (ii) is valid, then

$$\begin{aligned} \int_{\mathbf{X}} |u(x)|^q d\mu(x) &= \int_0^\infty \mu\left\{ x \in \mathbf{X} : |u(x)| \ge t \right\} dt^q \\ &\le \int_0^\infty \left[\mathrm{cap}_p^\alpha\left(\left\{ x \in \mathbf{X} : \ |u(x)| \ge t \right\} \right) \right]^{\frac{q}{p}} dt^q, \end{aligned}$$

which gives

$$\| u \|_{L^q(\mathbf{X},\mu)} \lesssim \left[\int_0^\infty \left(\mathrm{cap}_p^\alpha(\{ x \in \mathbf{X} : \ |u(x)| \ge t \}) \right)^{\frac{q}{p}} dt^q \right]^{1/q},$$

and (i) is valid.

(i) $\Rightarrow$ (ii) : Given a compact set $K \subseteq \mathbf{X}$, let $u = 1_K$. Then $\| u \|_{L^q(\mathbf{X},\mu)} = \mu(K)^{1/q}$ and

$$\{ x \in \mathbb{X} : \ |u(x)| \ge t \} = \begin{cases} K, & \text{if } \ t \in (0,1], \\ \emptyset, & \text{if } \ t \in (1,\infty). \end{cases}$$

Hence,

$$\begin{aligned} \mu(K)^{1/q} &\le \left[\int_0^\infty \left(\mathrm{cap}_p^\alpha(\{ x \in \mathbf{X} : \ |u(x)| \ge t \}) \right)^{\frac{q}{p}} dt^q \right]^{1/q} \\ &= \left[\int_0^1 \left(\mathrm{cap}_p^\alpha(\{ x \in \mathbf{X} : \ |u(x)| \ge t \}) \right)^{\frac{q}{p}} dt^q \right]^{1/q} \\ &= \left[\mathrm{cap}_p^\alpha(K) \right]^{\frac{1}{p}}, \end{aligned}$$

which derives that (i) implies (ii). □

In order to obtain the capacitary strong type estimate for $\mathbf{B}^{p,\alpha}(\mathbf{X})$, we need to add an additional assumption, which is introduced in ([6] Definition 6.7). The capacitary strong type estimate for the Besov space in the Euclidean setting has been investigated by Wu, Adams and Xiao (cf. [10,32]).

Definition 8. *We say that the Dirichlet space* $(\mathbf{X}, \mu, \mathcal{E}, \mathcal{F} = dom(\mathcal{E}))$ *has the property* $(P_{p,\alpha})$ *if there is a constant* $C > 0$ *such that*

$$\| u \|_{p,\alpha} \lesssim \liminf_{t \to 0} t^{-\alpha} \Big(\int_{\mathbf{X}} \int_{\mathbf{X}} |u(x) - u(y)|^p p_t(x,y) d\mu(x) d\mu(y) \Big)^{\frac{1}{p}}$$

hold on for every $u \in \mathbf{B}^{p,\alpha}(\mathbf{X})$.

If the weak Bakry–Émery curvature estimating is assumed in the strictly local Dirichlet space $(\mathbf{X}, \mu, \mathcal{E}, \mathcal{F} = \mathrm{dom}(\mathcal{E}))$ and the coefficient α is the Besov critical exponent of $(\mathbf{X}, \mu, \mathcal{E}, \mathcal{F} = \mathrm{dom}(\mathcal{E}))$ in the sense of L^p, the property $(P_{p,\alpha})$ is satisfied. See [6,7] for the details.

Theorem 3. *Assume that the Dirichlet space* $(\mathbf{X}, \mu, \mathcal{E}, \mathcal{F} = dom(\mathcal{E}))$ *satisfies the property* $(P_{p,\alpha})$ *is satisfied. Then,*

$$\Big[\int_0^\infty \Big(\mathrm{cap}_p^\alpha \Big(\Big\{ x \in \mathbf{X} : |f(x)| \geq t \Big\} \Big) \Big)^{\frac{q}{p}} dt^q \Big]^{1/q} \lesssim \| f \|_{p,\alpha} \tag{17}$$

hold on for any function $f \in \mathbf{B}^{p,\alpha}(\mathbf{X})$.

Proof. Let $f \in \mathbf{B}^{p,\alpha}(\mathbf{X})$ and $f \geq 0$. In the following, we set

$$E_t(f) := \Big\{ x \in \mathbf{X} : \ f(x) \geq t \Big\},$$

and denote $E_t(f)$ by E_t for simplicity. Now, we begin to prove the inequality (17). Note that $\frac{q}{p} \geq 1$. Via the monotonicity of $\mathrm{cap}_p^\alpha(\cdot)$, we deduce

$$\begin{aligned}
\int_0^\infty \Big[\mathrm{cap}_p^\alpha(E_t(f)) \Big]^{\frac{q}{p}} dt^q &= \sum_{k=-\infty}^{+\infty} \int_{2^k}^{2^{k+1}} \Big[\mathrm{cap}_p^\alpha(E_t(f)) \Big]^{\frac{q}{p}} dt^q \\
&\leq \sum_{k=-\infty}^{+\infty} \Big[\mathrm{cap}_p^\alpha(E_{2^k}(f)) \Big]^{\frac{q}{p}} [2^{(k+1)q} - 2^{kq}] \\
&= (2^q - 1) \sum_{k=-\infty}^{+\infty} 2^{kq} \Big[\mathrm{cap}_p^\alpha(E_{2^k}(f)) \Big]^{\frac{q}{p}} \\
&\leq (2^q - 1) \Big[\sum_{k=-\infty}^{+\infty} 2^{kp} \mathrm{cap}_p^\alpha(E_{2^k}(f)) \Big]^{\frac{q}{p}}.
\end{aligned}$$

Below, we estimate the series $\sum_{k=-\infty}^{+\infty} 2^{kp} \mathrm{cap}_p^\alpha(E_{2^k}(f))$. Denote $f_k = (f - 2^k)^+ \wedge 2^k, k \in \mathbb{Z}$. Since $2^k f_k \geq 1$ on $E_{2^k}(f)$, we have

$$\mathrm{cap}_p^\alpha(E_{2^k}(f)) \leq 2^{-kp} \|f_k\|_{p,\alpha}^p.$$

Thus,

$$\sum_{k=-\infty}^{+\infty} 2^{kp} \mathrm{cap}_p^\alpha(E_{2^k}(f)) \leq \sum_{k=-\infty}^{+\infty} \|f_k\|_{p,\alpha}^p \lesssim \|f\|_{p,\alpha}^p,$$

where we have used Lemma 6.10 in [6]. Then (17) holds true for nonnegative function $f \in \mathbf{B}^{p,\alpha}(\mathbf{X})$. We can similarly conclude it is valid for general function $f \in \mathbf{B}^{p,\alpha}(\mathbf{X})$. □

Remark 3. *It should be noted that (16) and (17) implies that*

$$\| f \|_{L^q(\mathbf{X},\mu)} \lesssim \|f\|_{p,\alpha}$$

holds true for all $f \in \mathbf{B}^{p,\alpha}(\mathbf{X})$, *which is exactly the strong Sobolev inequality in ([6] Theorem 6.9).*

Finally, the following trace inequalities for the Besov space are obtained by the capacitary inequalities in Theorem 3.

Theorem 4. *Assume that* $\alpha \in (0,1)$ *and* $1 \le p < q < \infty$ *and the Dirichlet space* $(\mathbf{X}, \mu, \mathcal{E}, \mathcal{F} = dom(\mathcal{E}))$ *with a nonnegative Radon measure* μ *is strictly local. Then, the following facts are equivalent:*

(i) For all sets $K \subset \mathbf{X}$, $\mu(K) \lesssim (\mathrm{cap}_p^\alpha(K))^{\frac{q}{p}}$.

(ii) For all functions $u \in \mathbf{B}^{p,\alpha}(\mathbf{X})$,

$$\Big(\int_{\mathbf{X}} |u|^q d\mu\Big)^{1/q} \lesssim \|u\|_{p,\alpha}.$$

Proof. Suppose that (i) is valid. Note that

$$\big(\sum_i |a_i|\big)^\beta \le \sum_i |a_i|^\beta \text{ with } \beta \in (0,1]. \tag{18}$$

For all functions $u \in \mathbf{B}^{p,\alpha}(\mathbf{X})$, it follows from Theorem 3 and the inequality (18) that

$$\begin{aligned}
\int_{\mathbf{X}} |u|^q d\mu &= \int_0^\infty \mu(\{x \in \mathbf{X} : |u(x)| > t\}) dt^q \\
&\approx \sum_{k\in\mathbb{Z}} 2^{kq} \mu(\{x \in \mathbf{X} : |u(x)| > 2^k\}) \\
&\lesssim \Bigg(\sum_{k\in\mathbb{Z}} 2^{kq} \Big(\mu(\{x \in \mathbf{X} : |u(x)| > 2^k\})\Big)^{p/q}\Bigg)^{\frac{q}{p}} \\
&\lesssim \Bigg(\sum_{j\in\mathbb{Z}} 2^{kp} \mathrm{Cap}_p^\alpha(\{x \in \mathbf{X} : |u(x)| > 2^k\})\Bigg)^{\frac{q}{p}} \\
&\lesssim \Bigg(\int_0^\infty \mathrm{Cap}_p^\alpha(\{x \in \mathbf{X} : |u(x)| > t\}) dt^p\Bigg)^{\frac{q}{p}} \\
&\lesssim \|u\|_{p,\alpha}^q,
\end{aligned}$$

whence (i) derives (ii).

On the contrary, if $K \subset \mathbf{X}$, then

$$\mu(K) \le \int_E |u|^q d\mu \lesssim \|u\|_{p,\alpha}^q$$

is valid for any $u \in \mathbf{B}^{p,\alpha}(\mathbf{X})$ with $u \ge 1$ on K. By the definition of $\mathrm{Cap}_p^\alpha(\cdot)$, we have

$$\mu(K) \lesssim (\mathrm{Cap}_p^\alpha(K))^{\frac{q}{p}},$$

which proves (i). □

5. Conclusions

As mentioned in the Introduction, one of the main motivations of the present paper was to study the Besov space (heat-semigroup-based) and the corresponding capacity in a Dirichlet context. Under assumptions that the strictly local Dirichlet space satisfies the doubling condition, the weak $(1,2)$-Poincaré inequality and the weak Bakry–Émery curvature condition, the properties of Besov capacity are proved via the proof method of ([28] Theorem 3.1). As the generalization of the Sobolev inequality established in [6], Theorem 2 can be regarded as combination of Theorem 6.1, Theorem 6.3 in [6] and the

Besov capacity theory. Similarly, the Besov norm in the inequality of ([6] Theorem 6.9) can be replaced with Besov capacity, which improves the Besov theory of the strictly local Dirichlet space. In [13], The authors investigate the metric space $(\mathbf{X}, d, \mu)$, which is proper and unbounded; moreover, it is also a Ahlfors Q-regular space for some $Q > 1$. Specifically, there exists a constant $C > 1$ such that

$$C^{-1}r^Q \leq \mu(B(x,r)) \leq Cr^Q$$

holds for each $B(x,r) \subset \mathbf{X}$ with $r > 0$. It should be noted that the Dirichlet space in our paper does not satisfy the above conditions. Therefore, the equivalence problem between Besov capacities and Hausdorff measures is still unsolved. We plan to investigate this question in our Dirichlet setting.

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