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Numerical Simulation for Solitary Waves of the Generalized Zakharov Equation Based on the Lattice Boltzmann Method

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Abstract: The generalized Zakharov equation is a widely used and crucial model in plasma physics, which helps to understand wave particle interactions and nonlinear wave propagation in plasma. The solitary wave solution of this equation provides insights into phenomena such as electron and ion acoustic waves, as well as magnetic field disturbances in plasma. The numerical simulation of solitary wave solutions to the generalized Zakharov equation is an interesting problem worth studying. This is crucial for plasma-based technology, as well as for understanding nonlinear wave propagation in plasma physics and other fields. In this study, a numerical investigation of the generalized Zakharov equation using the lattice Boltzmann method has been conducted. The lattice Boltzmann method is a new modeling and simulating method at the mesoscale. A lattice Boltzmann model was constructed by performing Taylor expansion, Chapman–Enskog expansion, and time multiscale expansion on the lattice Boltzmann equation. By defining the moments of the equilibrium distribution function appropriately, the macroscopic equation can be restored. Furthermore, the numerical experiments for the equation are carried out with the parameter lattice size $m = 100$, time step $\Delta t = 0.001$, and space step size $\Delta x = 0.4$. The solitary wave solution of the equation is numerically simulated. Numerical results under different parameter values are compared, and the convergence and effectiveness of the model are numerically verified. It is obtained that the model is convergent in time and space, and the convergence orders are all 2.24881. The effectiveness of our model was also verified by comparing the numerical results of different numerical methods. The lattice Boltzmann method demonstrates advantages in both accuracy and CPU time. The results indicate that the lattice Boltzmann method is a good tool for computing the generalized Zakharov equation.

Keywords: lattice Boltzmann method; the generalized Zakharov equation; solitary wave; numerical simulation

MSC: 37M05



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1. Introduction

The generalized Zakharov equation is an important realistic model in plasma. Plasmas, which are highly ionized gases, naturally occur in phenomena such as solar flares, auroras, and stellar interiors. The generalized Zakharov equation, crucial for understanding their behavior, captures nonlinear wave–particle interactions in these environments. The generalized Zakharov equation has demonstrated its utility in a diverse array of physical problems, encompassing scenarios like the interaction of intramolecular vibrations leading to the emergence of Davydov solitons with acoustic disturbances, as well as the interplay between high- and low-frequency gravity disturbances within an atmospheric context, among others.

The important application of generalized Zakharov systems in physical problems has aroused research interest in them. Studying the generalized Zakharov equation can provide

a unified framework for describing different physical systems. In addition, the Zakharov equation provides a rich mathematical structure, posing challenges and incentives for research in fields such as partial differential equations, nonlinear dynamics, and applied mathematics. Therefore, studying the generalized Zakharov equation is of great significance not only in physics but also in mathematics.

Due to the strong nonlinearity of the generalized Zakharov equation, solving it poses great challenges. Therefore, solving the generalized Zakharov equation has become an interesting and meaningful topic. Previous research has mainly focused on the analysis and numerical methods for solving the generalized Zakharov equation. Analysis methods typically involve perturbation techniques or asymptotic expansions to gain a deeper understanding of the behavior of solutions in extreme cases [1,2], as well as obtaining accurate solitary wave solutions for equations [3,4]. However, the applicability of these methods is often limited and may not fully reflect the complexity of the equations. Furthermore, obtaining analytical solutions for nonlinear partial differential equations is a challenging problem, and therefore numerical research has demonstrated its importance as a powerful tool for simulating generalized Zakharov equations. Finite difference, finite element, and spectral methods have been widely used to approximate solutions and study wave propagation dynamics [5,6]. Although these numerical techniques have achieved considerable success, they often require significant computational resources and may still face challenges in simulating certain aspects of equation behavior.

Recently, the lattice Boltzmann method has become a promising alternative to simulating complex fluid dynamics. This method was originally developed for fluid flow simulation and is now applicable to solving a wide range of problems, including those involving the generalized Zakharov equation. The Lattice Boltzmann method has several advantages, including the ability to handle complex geometric shapes and boundaries, as well as inherent parallelism, making it suitable for high-performance computing.

Despite these advances, further exploration and optimization of numerical methods for simulating the generalized Zakharov equation are still needed. This study aims to contribute to this work by introducing a new lattice Boltzmann model, which has a higher accuracy and efficiency compared to previous methods. By leveraging the advantages of the lattice Boltzmann method and addressing its limitations, we hope to provide a more robust and reliable tool for simulating complex wave phenomena controlled by the generalized Zakharov equation.

This study has reference significance not only for the systems described by the generalized Zakharov equation itself, but also for other nonlinear systems that need to be solved. In the future, this research can also be considered for further application in other fields [7].

In this work, we will apply the lattice Boltzmann method to investigate the solitary wave solution of the generalized Zakharov equation. The lattice Boltzmann method is a discrete mesoscopic model developed based on the fundamental principles of molecular dynamic theory and non-equilibrium statistical physics, in which particle velocity and space–time are discretized [8,9]. The lattice Boltzmann method is a new modeling and simulation method for complex systems, and it originated from the lattice gas automata. For some multiscale (different Knudsen number) behaviors of the complex fluid systems, the lattice Boltzmann method can be adaptive in physical description. In the lattice Boltzmann method, the fluid is abstracted as mesoscopic particles and these particles follow simple rules to collide and move on the regular lattice. The macroscopic motion characteristics of the fluid are statistically obtained through the motion of particles. In the traditional numerical method, the macroscopic variables are solved by the discretization of macroscopic equations. Unlike traditional methods, the lattice Boltzmann method obtains macroscopic variables by defining the moments of the equilibrium distribution function without the need to solve complex partial differential equations, which simplifies the problem. The macroscopic motion characteristics of the fluid can be obtained through the statistics of the particles' motion. These particle properties make the lattice Boltzmann method have a number of unique advantages that the conventional numerical methods do not have,

such as having an essentially parallel nature, clear physical images, and easy boundary handling. Specifically, the single relaxation lattice Boltzmann model raises the operation efficiency [10–13]. The lattice Boltzmann method has shown advantages in accuracy compared to traditional methods in solving nonlinear partial differential equations. Therefore, in this work, we choose to use the lattice Boltzmann method to study the generalized Zakharov equation. The lattice Boltzmann method has achieved success in the field of fluid dynamics nowadays and the field of nonlinear partial differential equations [14–20]. In particular, some research results on the lattice Boltzmann method for plasma solitary wave have been obtained [21,22].

The generalized Zakharov equation is a highly nonlinear and complex partial differential equation that involves multiple physical field interactions. This equation also has multiscale characteristics, requiring numerical methods to efficiently handle different scales, and it may also contain special physical properties. Traditional numerical methods may face challenges, such as loss of accuracy, when dealing with these characteristics. The lattice Boltzmann method, based on micro particle motion and collision simulation, can naturally handle nonlinear effects. Due to its parallelism and locality, it can efficiently handle multiscale problems while maintaining accuracy, thus better balancing computational efficiency and accuracy. The lattice Boltzmann method can conveniently simulate the special physical effects of equations by flexibly setting collision operators and boundary conditions, and more accurately describe complex physical processes and reflect physical phenomena. Therefore, the lattice Boltzmann method is more suitable for solving the generalized Zakharov equation.

This study demonstrates the flexibility and universality of the lattice Boltzmann method in simulating complex physical processes, making new contributions to plasma physics and solving nonlinear equations. By using the lattice Boltzmann method to numerically simulate the solitary wave solutions of the generalized Zakharov equation, not only have the challenges of traditional methods in dealing with highly nonlinear and multiscale problems been successfully overcome, but also a more intuitive, efficient, and accurate simulation tool has been provided, promoting the development of plasma physics. At the same time, this method also enriches the numerical method library for solving nonlinear equations, providing new ideas for solving similar equations.

In this work, we construct a new lattice Boltzmann model for the generalized Zakharov equation in Section 2 and simulate the solitary wave solution in Section 3. We conclude with some discussions in Section 4.

2. Lattice Boltzmann Model

The generalized Zakharov equation containing the self-generated magnetic field is usually in the following dimensionless form:

$$iq_t + q_{xx} + |q|^2 q = qr, \quad (1)$$

$$r_{tt} - r_{xx} = (|q|^2)_{xx}. \quad (2)$$

where q represents the envelope of the high-frequency electric field, which is a complex value, and before dimensionlessness, its measuring unit is V/m; the variable r is real, which represents the plasma density in equilibrium state, and its measuring unit is $1/\text{m}^3$ before dimensionlessness; i is an imaginary unit and $i = \sqrt{-1}$; t represents a time variable; and x is a spatial variable. When the term $|q|^2 q$ is ignored in (1), the Equations (1) and (2) are transformed into the Zakharov equation derived by Zakharov firstly [1]. The Zakharov equation ignores the influence of the magnetic field generated in the plasma, is a simplified model of strong Langmuir turbulence. It models the interaction between low-frequency ion-acoustic wave and high-frequency Langmuir waves in an unmagnetized plasma. In order to describe a more realistic system, more elements such as the magnetic field effect should be considered to generalize the Zakharov equation, so that the generalized Zakharov equation is obtained. The generalized Zakharov equation is an extension of the Zakharov equation.

2.1. Lattice Boltzmann Equation

We use the D1Q3 model to discretize one-dimensional space; the corresponding velocity is $\mathbf{e}_\alpha = (c, -c, 0)$, and $\alpha = 1, 2, 3$ corresponds to three directions. Let $f_\alpha(\mathbf{x}, t)$ represent the distribution function of one particle at position $\mathbf{x}$, time t with particle velocity $\mathbf{e}_\alpha$. Particles can only move along the grid lines towards neighboring grid points or remain stationary at the original grid points. Let $f_\alpha^{eq}(\mathbf{x}, t)$ be the equilibrium distribution function, which characterizes the one particle distribution when the system is in the state of equilibrium. Assuming that $f_\alpha^{eq}(\mathbf{x}, t)$ and $f_\alpha(\mathbf{x}, t)$ satisfy the conservation condition,

$$\sum_\alpha f(\mathbf{x}, t) = \sum_\alpha f_\alpha^{eq}(\mathbf{x}, t). \quad (3)$$

Then $f_\alpha(\mathbf{x}, t)$ evolves according to the following lattice Boltzmann equation:

$$f_\alpha(\mathbf{x} + \mathbf{e}_\alpha, t + 1) - f_\alpha(\mathbf{x}, t) = -\frac{1}{\tau}[f_\alpha(\mathbf{x}, t) - f_\alpha^{eq}(\mathbf{x}, t)] + \Omega_\alpha(\mathbf{x}, t). \quad (4)$$

where τ is the single relaxation time, which represents the average time interval between two collisions of particles. Ω_α is an additional term. Applying Chapman–Enskog expansion and other techniques to the equation, we can obtain a series of partial differential equations at different time scales; please refer to the appendix for details.

2.2. Recovery of the Generalized Zakharov Equation

We define q as

$$iq(\mathbf{x}, t) = \sum_\alpha f_\alpha(\mathbf{x}, t), \quad (5)$$

According to conservation conditions (3), it can be concluded that

$$iq(\mathbf{x}, t) = \sum_\alpha f_\alpha^{eq}(\mathbf{x}, t), \quad (6)$$

Summing the Equation (A7) with respect to α , and combining equation (A15), then the conservation laws in time scale t_0 can be obtained as

$$\frac{\partial iq(\mathbf{x}, t)}{\partial t_0} + \frac{\partial m^0}{\partial x} = 0. \quad (7)$$

Let

$$m^0 = \sum_\alpha f_\alpha^{(0)} e_\alpha = 0, \quad (8)$$

$$\pi^0 = \sum_\alpha f_\alpha^{(0)} e_\alpha^2 = \lambda q, \quad (9)$$

where the parameter λ is to be determined.

According to Equations (7)–(9), there are

$$\frac{\partial q}{\partial t_0} = 0, \quad (10)$$

$$\frac{\partial \pi^0}{\partial t_0} = 0. \quad (11)$$

Summing (A7) + $\varepsilon \times$ (A8) over α , yields

$$i \frac{\partial q}{\partial t} + \varepsilon C_2 \frac{\partial^2}{\partial x^2} \lambda q = \varepsilon \sum_\alpha \Omega_\alpha^{(2)} + O(\varepsilon^2). \quad (12)$$

Equation (12) is an approximate formula of the generalized Zakharov Equation (1). When $\lambda = \frac{\beta}{\varepsilon C_2}$ and $\varepsilon \sum_{\alpha} \Omega_{\alpha}^{(2)} = -(\gamma|q|^2 q - qr)$, the truncation error is $O(\varepsilon^2)$. If $\Omega_{\alpha}^{(2)}$ is assumed independent of α , then

$$\Omega_{\alpha}^{(2)} = \Omega^{(2)} = -\frac{\gamma|q|^2 q - qr}{\varepsilon(b+1)}, \quad (13)$$

In addition, the value of variable r in Equation (2) is obtained by using the follow difference scheme:

$$r(x, t + \Delta t) = [k^2 r(x + \Delta x, t) - 2k^2 r(x, t) + k^2 r(x - \Delta x, t) + |q(x + \Delta x, t)|^2 - 2|q(x, t)|^2 + |q(x - \Delta x, t)|^2] \frac{(\Delta t)^2}{(\Delta x)^2} + [|q(x + \Delta x, t)|^2 - 2|q(x, t)|^2 + |q(x - \Delta x, t)|^2] \frac{(\Delta t)^2}{(\Delta x)^2} + 2r(x, t) - r(x, t - \Delta t) \quad (14)$$

2.3. The Truncation Error of the Model

Summing (A7) + $\varepsilon \times$ (A8) + $\varepsilon^2 \times$ (A9) + $\varepsilon^3 \times$ (A10) over α , yields

$$i \frac{\partial q}{\partial t} + \frac{\partial^2 \beta q}{\partial x^2} + \gamma|q|^2 q - qr = E_2 + E_3 + O(\varepsilon^4). \quad (15)$$

where E_2 and E_3 represent the second-order and third-order errors, respectively. Combining Equations (6), (8) and (9), we obtain

$$f_{\alpha}^{(0)} = \begin{cases} \frac{\lambda q}{2c^2}, & \alpha = 1, 2, \\ iq - \frac{\lambda q}{c^2}, & \alpha = 0. \end{cases} \quad (16)$$

According to (16), the third and fourth moments are

$$P^0 = \sum_{\alpha} f_{\alpha}^{(0)} e_{\alpha}^3 = 0, \quad (17)$$

$$Q^0 = \sum_{\alpha} f_{\alpha}^{(0)} e_{\alpha}^4 = \lambda c^2 q. \quad (18)$$

The error terms

$$E_2 = 0, \quad (19)$$

$$E_3 = \frac{\varepsilon^2 C_4 c^2 + \varepsilon(3C_3 - C_2^2)i}{C_2} \frac{\partial^4 q}{\partial x^4} + \varepsilon \left(\frac{3C_3 i}{C_2} - C_2 i \right) \frac{\partial^2}{\partial x^2} (|q|^2 q - qr) - \frac{1}{2} \varepsilon i (r + |q|^2) \frac{\partial^2 q}{\partial x^2} - \frac{\varepsilon i}{2} q (r - |q|^2)^2 - \varepsilon q \operatorname{Re} \left(i \bar{q} \frac{\partial^2 q}{\partial x^2} \right). \quad (20)$$

Thus, the macroscopic Equation (1) can be recovered as

$$i \frac{\partial q}{\partial t} + \frac{\partial^2 \beta q}{\partial x^2} + \gamma|q|^2 q - qr = O(\varepsilon). \quad (21)$$

This means that the formula error generated by the numerical solution calculated using this method is approximately $O(\varepsilon)$.

3. Numerical Example

A numerical example of the generalized Zakharov equation is presented in this section. The initial and boundary conditions are as follows:

$$q(x, 0) = \sqrt{\frac{15}{7}} \operatorname{sech} x \exp(2ix), \quad (22)$$

$$r(x, 0) = \frac{1}{7} \operatorname{sech}^2 x. \quad (23)$$

$$q(x_L, t) = \sqrt{\frac{15}{7}} \operatorname{sech}(x_L - 4t) \exp[i(2x_L - 3t)], \quad (24)$$

$$r(x_L, t) = \frac{1}{7} \operatorname{sech}^2(x_L - 4t), \quad (25)$$

$$q(x_R, t) = \sqrt{\frac{15}{7}} \operatorname{sech}(x_R - 4t) \exp[i(2x_R - 3t)], \quad (26)$$

$$r(x_R, t) = \frac{1}{7} \operatorname{sech}^2(x_R - 4t). \quad (27)$$

The numerical results obtained from our study are presented in Figures 1–7. Figures 1 and 2 provide a direct comparison between the lattice Boltzmann solution and the exact solution, highlighting their similarities and differences. Figure 3 displays error curves for $|q|$ and r , ranging from $t = 0$ to $t = 4$. These curves characterize the deviation between the lattice Boltzmann solution and the exact solution, offering a quantitative measure of the accuracy of our approach. The error selected here is $Er = |q^N - q^E|$ and $Er = |r^N - r^E|$, where q^N and r^N represent the numerical solutions, q^E and r^E represent the exact solutions. Figure 4 shows the curve of error for $t = 1$. The results presented in these figures demonstrate that the lattice Boltzmann solution is in good agreement with the exact solution. This consistency validates the accuracy and reliability of our lattice Boltzmann model in simulating the phenomena governed by the generalized Zakharov equation. The exact solution mentioned here is as follows [4]:

$$q(x, t) = \sqrt{\frac{15}{7}} \operatorname{sech}(x - 4t) \exp[i(2x - 3t)], \quad (28)$$

$$r(x, t) = \frac{1}{7} \operatorname{sech}^2(x - 4t). \quad (29)$$

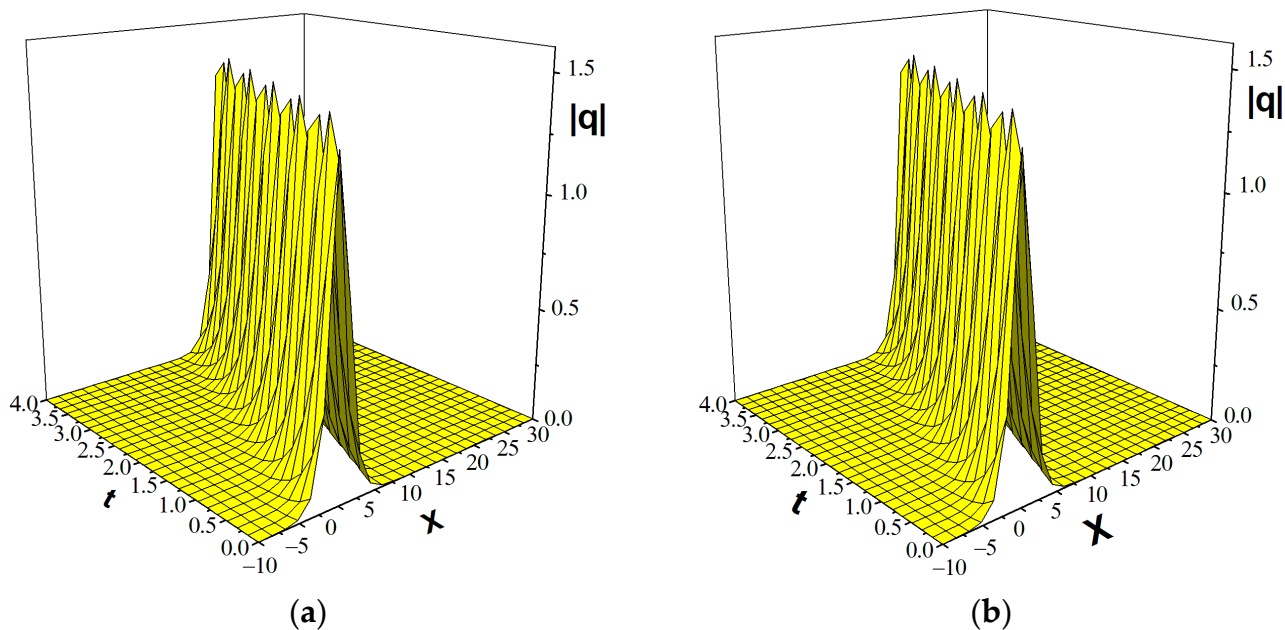


Figure 1. Solitary wave solution $|q|$ in case lattice size $m = 100$, $\Delta t = 0.001$, $\Delta x = 0.4$, $c = \Delta x / \Delta t$, $\tau = 0.985$. (a) Lattice Boltzmann solution; (b) exact solution.

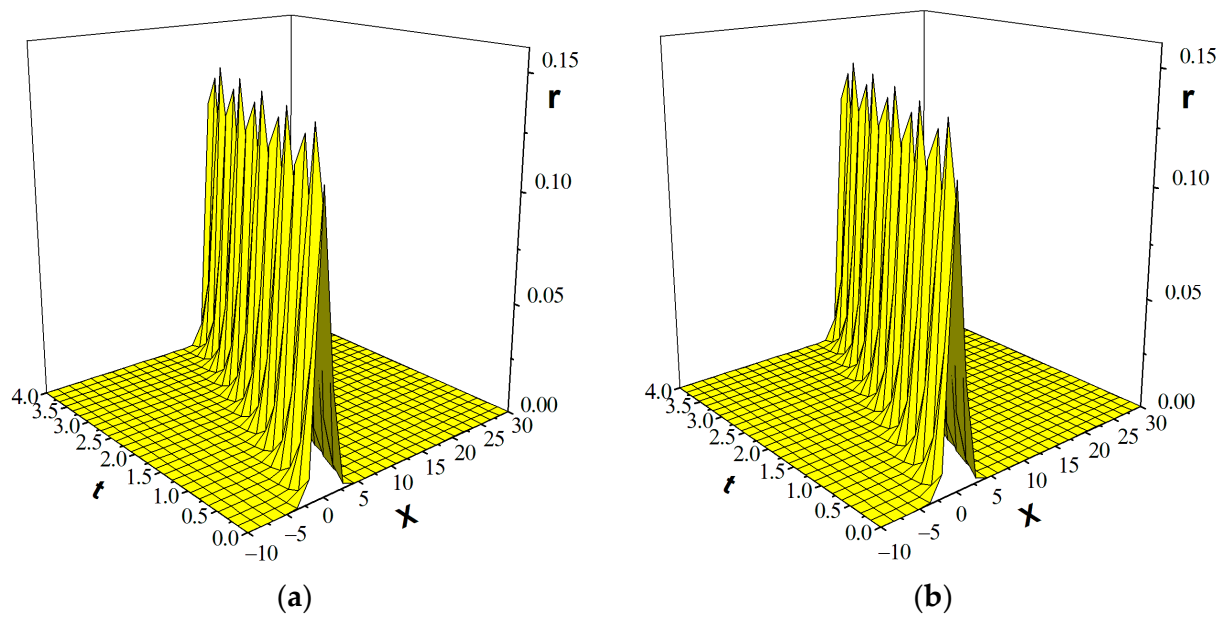


Figure 2. Solitary wave solution r . (a) Lattice Boltzmann solution; (b) exact solution.

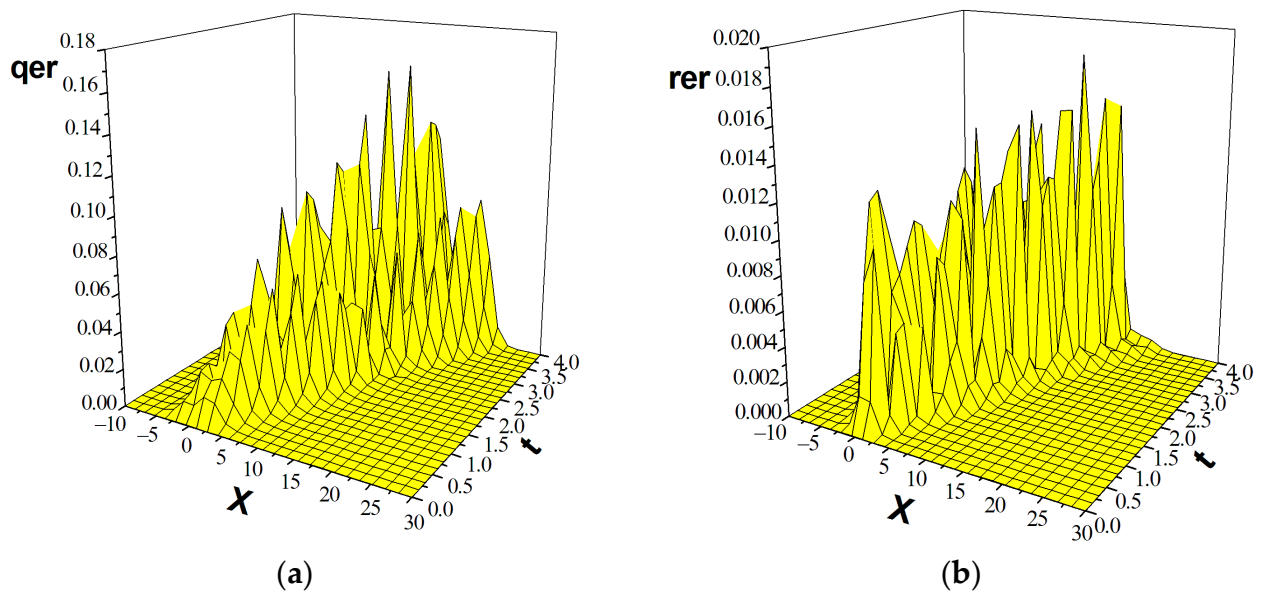


Figure 3. Error curve. (a) Error of $|q|$; (b) error of r .

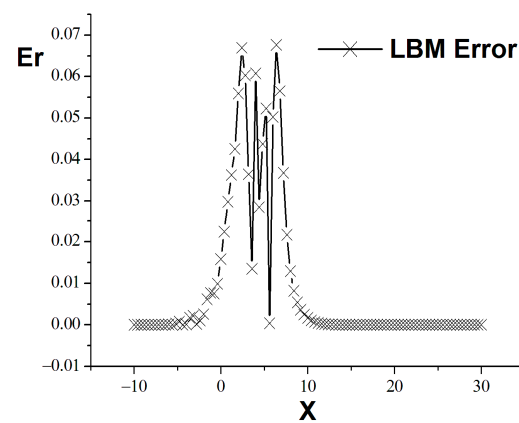


Figure 4. Error curve of q at $t = 1$.

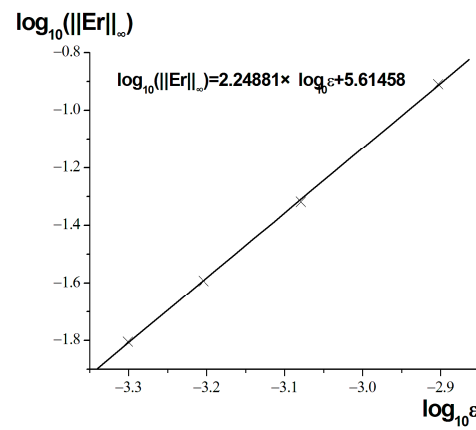


Figure 5. The curve of $\log_{10}(\|Er\|_{\infty})$ versus $\log_{10} \epsilon$, $t = 1$.

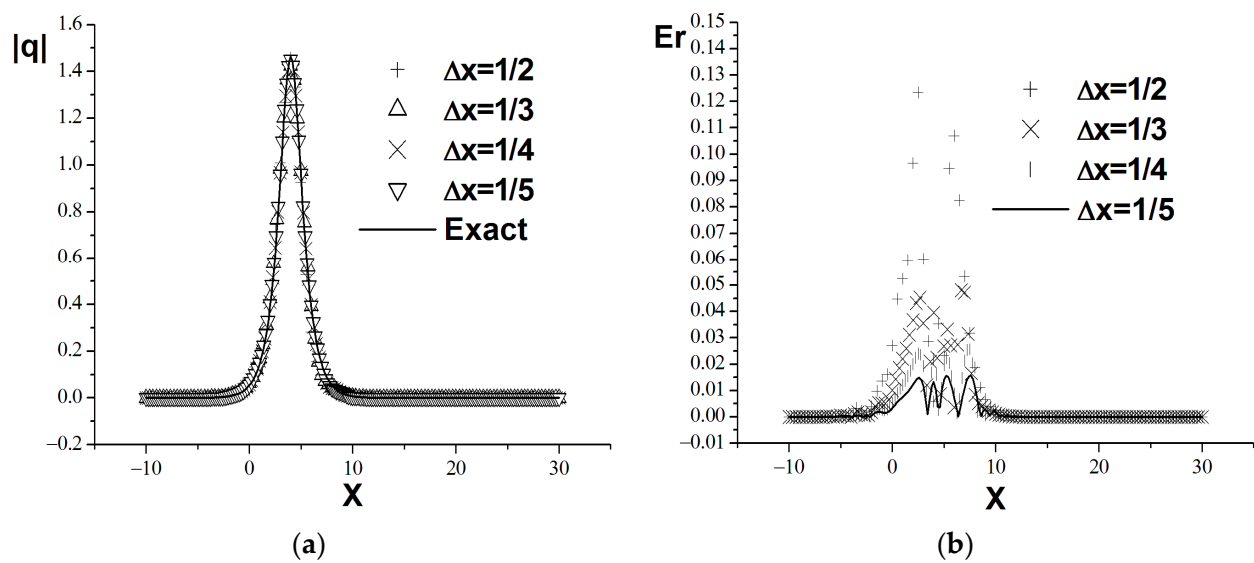


Figure 6. The lattice Boltzmann result with different Δx for $c = 400.0$, $t = 1$. (a) Lattice Boltzmann solution; (b) error of $|q|$.

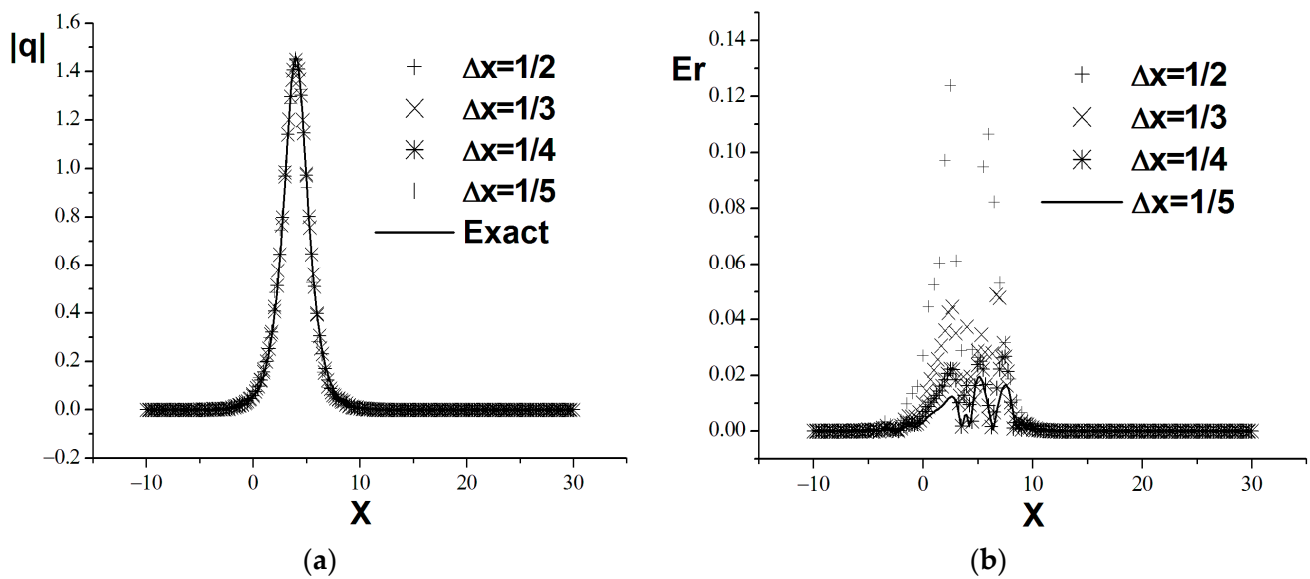


Figure 7. The lattice Boltzmann result with different Δx for $\Delta t = 0.001$, $t = 1$. (a) Lattice Boltzmann solution; (b) error of $|q|$.

To further evaluate the performance of our model, we introduce the parameter $\|Er\|_\infty = \max_i \{Er\}$ in Figure 5. This figure depicts the relationship between $\log_{10}(\|Er\|_\infty)$ and $\log_{10} \varepsilon$; the fitting line is $\log_{10}(\|Er\|_\infty) = 2.24881 \times \log_{10} \varepsilon + 5.61458$ and the slope is 2.24881. These findings indicate the convergence of our model and $\|Er\|_\infty$ is about $\varepsilon^{2.24881}$. Due to $\Delta x = c\Delta t = c\varepsilon$, it can be obtained that the model is convergent in time and space, and the convergence orders are all 2.24881. Figure 5 also indicates that $\|Er\|_\infty$ and ε have a dependency relationship, providing deeper insights into the model's behavior.

To investigate the impact of the space step, we present numerical results with various space steps in Figures 6 and 7. Figure 6 displays the results for different space steps with a fixed parameter c , while Figure 7 shows the results for a fixed time step with $\Delta x = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$. These figures allow us to assess the sensitivity of our model to changes in the space step. The comparisons of $\|Er\|_\infty$ with various space steps are also given in Tables 1 and 2. The results clearly demonstrate that as the space step decreases, the value of $\|Er\|_\infty$ also decreases, further supporting the convergence of our model. Moreover, we investigate the effect of varying the parameter on the model's performance. Through numerous numerical experiments, we observe that for a specific range of $t = 1$, the optimal value of τ lies around 0.985. At this point, $\|Er\|_\infty$ attains its minimum value. When the value of τ is below this optimal range, $\|Er\|_\infty$ increases with a decrease in τ . Conversely, when τ exceeds the optimal range, $\|Er\|_\infty$ increases with an increase in τ . These findings provide valuable insights into the optimal parameter selection for our lattice Boltzmann model.

Table 1. The comparison of $\|Er\|_\infty$ with different Δx at $t = 1$ for $c = 400.0$.

Space Step Δx	$\ Er\ _\infty$
1/2	1.233034×10^{-1}
1/3	4.804083×10^{-2}
1/4	2.561515×10^{-2}
1/5	1.568317×10^{-2}

Table 2. The comparison of $\|Er\|_\infty$ with different Δx at $t = 1$ for $\Delta t = 0.001$.

Space Step Δx	$\ Er\ _\infty$
1/2	1.238311×10^{-1}
1/3	4.894853×10^{-2}
1/4	2.668033×10^{-2}
1/5	1.950953×10^{-2}

We also use the fourth-order Runge–Kutta scheme to solve the generalized Zakharov Equations (1) and (2). In this method, Equations (1) and (2) are first written in the following form:

$$\frac{\partial q}{\partial t} = -\frac{1}{i} \left(\beta \frac{\partial^2 q}{\partial x^2} + \gamma |q|^2 q - qr \right) = f(t_n, q_j^n), \quad (30)$$

$$\frac{\partial}{\partial t} \left(\frac{\partial r}{\partial t} \right) = \frac{\partial^2 |q|^2}{\partial x^2} + k^2 \frac{\partial^2 r}{\partial x^2} = g(t_n, r_j^n). \quad (31)$$

The computing formula for the Runge–Kutta scheme is as follows:

$$q_j^{n+1} = q_j^n + \frac{\Delta t}{6} (fk_1 + 2fk_2 + 2fk_3 + fk_4), \quad (32)$$

$$r_j^{n+1} = 2r_j^n - r_j^{n-1} + \frac{(\Delta t)^2}{6} (gk_1 + 2gk_2 + 2gk_3 + gk_4), \quad (33)$$

$$f(t_n, q_j^n) = -\frac{1}{i} \left[\beta \frac{q_{j+1}^n - 2q_j^n + q_{j-1}^n}{(\Delta x)^2} + \gamma |q_j^n|^2 q_j^n - q_j^n r_j^n \right], \quad (34)$$

$$g(t_n, r_j^n) = \frac{|q_{j+1}^n|^2 - 2|q_j^n|^2 + |q_{j-1}^n|^2}{(\Delta x)^2} + k^2 \frac{r_{j+1}^n - 2r_j^n + r_{j-1}^n}{(\Delta x)^2}, \quad (35)$$

$$fk_1 = f(t_n, q_j^n), \quad (36)$$

$$fk_2 = f(t_n + \frac{\Delta t}{2}, q_j^n + \frac{\Delta t}{2}fk_1), \quad (37)$$

$$fk_3 = f(t_n + \frac{\Delta t}{2}, q_j^n + \frac{\Delta t}{2}fk_2), \quad (38)$$

$$fk_4 = f(t_n + \Delta t, q_j^n + \Delta t fk_3), \quad (39)$$

$$gk_1 = g(t_n, r_j^n), \quad (40)$$

$$gk_2 = g(t_n + \frac{\Delta t}{2}, r_j^n + \frac{\Delta t}{2}gk_1), \quad (41)$$

$$gk_3 = g(t_n + \frac{\Delta t}{2}, r_j^n + \frac{\Delta t}{2}gk_2), \quad (42)$$

$$gk_4 = g(t_n + \Delta t, r_j^n + \Delta t gk_3). \quad (43)$$

The comparison of the results of the two schemes is presented in Figure 8, and Tables 3 and 4. Based on Figure 8 and the tables, it is evident that lattice Boltzmann method exhibits excellent accuracy and efficiency in terms of computational accuracy and CPU time. This advantage is crucial for accurate and timely solutions, especially for complex problems like the generalized Zakharov equation. The results indicate that the constructed lattice Boltzmann model is effective in computing the generalized Zakharov equation. This effectiveness of the lattice Boltzmann method lies in its ability to accurately and effectively capture complex dynamics. Its discreteness allows for high-precision modeling of complex systems, while its computational efficiency ensures the practical applicability of large-scale simulations.

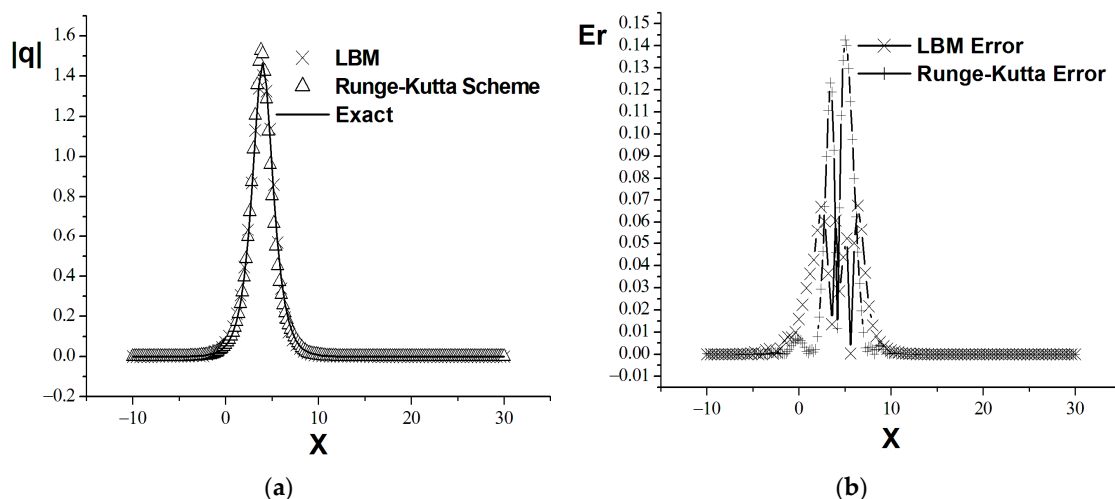


Figure 8. The comparison of the results of the two schemes, $t = 1$. (a) Solitary wave of $|q|$; (b) error of $|q|$.

Table 3. The comparison of the $\|Er\|_\infty$ of the two schemes.

Time	Runge–Kutta Scheme	Lattice Boltzmann Method
0.5	7.406551×10^{-2}	4.150305×10^{-2}
1.0	1.425483×10^{-1}	6.759164×10^{-2}
1.5	2.054748×10^{-1}	9.711214×10^{-2}
2.0	2.627019×10^{-1}	1.202797×10^{-1}

Table 4. The comparison of the CPU time(s) of the two schemes.

Time	Runge–Kutta Scheme	Lattice Boltzmann Method
0.5	0.312	0.203
1.0	0.546	0.328
1.5	0.796	0.468
2.0	1.123	0.577

4. Conclusions

In this work, we successfully developed a lattice Boltzmann model that not only outperformed some traditional numerical methods in accuracy, but also demonstrated excellent CPU time consumption. This achievement validates the effectiveness of our method, establishing it as a powerful tool for simulating complex wave phenomena controlled by the generalized Zakharov equation. Through rigorous numerical simulations, we have successfully demonstrated the proficiency of the model in simulating the propagation of solitary wave solutions, further confirming the feasibility and accuracy of the lattice Boltzmann method in analyzing these solutions.

The novelty and achievements of this study lie in several crucial aspects.

Firstly, the lattice Boltzmann method is applied to the generalized Zakharov equation, which is significantly different from traditional numerical methods. The lattice Boltzmann method was originally developed for fluid dynamics simulation and has successfully adapted to the challenges posed by the generalized Zakharov equation, which describes the interaction between plasma waves and ion sound waves. This new method provides a new perspective for simulating complex wave phenomena in plasma physics.

Secondly, this study demonstrates the effectiveness of the lattice Boltzmann method in simulating solitary waves, which is crucial for understanding various plasma processes. As shown by the numerical results, the accuracy and efficiency of the lattice Boltzmann method in capturing the complex dynamics of these waves represent a significant achievement. This not only verifies the feasibility of simulating the generalized Zakharov equation using the lattice Boltzmann method, but also makes it a promising tool for future plasma physics research.

In addition, this study has contributed to a wider range of numerical methods for partial differential equations. The successful application of the lattice Boltzmann method in the generalized Zakharov equation demonstrates the potential of this method in solving a series of complex problems. This achievement not only expands the toolbox available to researchers, but also opens up new avenues for exploring new numerical techniques.

In summary, this study represents an important step forward in the field of plasma physics simulation. It proves the novelty of applying lattice Boltzmann method to this problem and achieves accurate and effective simulation of solitary waves, thus contributing to the development of numerical methods for partial differential equations.

Looking ahead, we must continuously improve and optimize our model to achieve higher accuracy in simulating the generalized Zakharov equation. This pursuit will not only bring more accurate simulations, but also provide deeper insights into the dynamics of solitary wave solutions. We are confident that our methods and findings will lay a solid foundation for future research in this dynamic and exciting field, paving the way for more innovation and breakthrough discoveries.

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Appendix A

Let us define the Knudsen number ε as the ratio between the mean free path l and the characteristic length L . Taking ε to be equal to the time step Δt , the lattice Boltzmann Equation (4) in physical units is expressed as Equation (A1).

$$f_\alpha(\mathbf{x} + \varepsilon \mathbf{e}_\alpha, t + \varepsilon) - f_\alpha(\mathbf{x}, t) = -\frac{1}{\tau} [f_\alpha(\mathbf{x}, t) - f_\alpha^{eq}(\mathbf{x}, t)] + \Omega_\alpha(\mathbf{x}, t). \quad (\text{A1})$$

In Equation (A1), it is assumed that the Knudsen number ε is small, and the source term $\Omega_\alpha(\mathbf{x}, t)$ is also assumed to be a second-order term written as

$$\Omega_\alpha(\mathbf{x}, t) = \varepsilon^2 \Omega_\alpha^{(2)}(\mathbf{x}, t). \quad (\text{A2})$$

Performing a Taylor expansion on the left-hand side of Equation (A1), keeping terms up to $O(\varepsilon^4)$, yields

$$f_\alpha(\mathbf{x} + \varepsilon \mathbf{e}_\alpha, t + \varepsilon) - f_\alpha(\mathbf{x}, t) = \sum_{n=1}^3 \frac{\varepsilon^n}{n!} \left(\frac{\partial}{\partial t} + \mathbf{e}_\alpha \frac{\partial}{\partial \mathbf{x}} \right)^n f_\alpha(\mathbf{x}, t) + O(\varepsilon^4). \quad (\text{A3})$$

Under the assumption of a small Knudsen number, the Chapman–Enskog expansion is performed on f_α ,

$$f_\alpha = f_\alpha^{(0)} + \sum_{n=1}^{\infty} \varepsilon^n f_\alpha^{(n)}, \quad (\text{A4})$$

where $f_\alpha^{(0)} \equiv f_\alpha^{eq}$. Introducing t_0, t_1, t_2, t_3 as different scale times, defined as

$$t_i = \varepsilon^i t, i = 0, 1, 2, 3. \quad (\text{A5})$$

and

$$\frac{\partial}{\partial t} = \sum_{n=0}^3 \varepsilon^n \frac{\partial}{\partial t_n} + O(\varepsilon^4). \quad (\text{A6})$$

From Equations (A3) to (A6), the equations for each order of ε are given as follows:

$$C_1 \Delta f_\alpha^{(0)} = -\frac{1}{\tau} f_\alpha^{(1)}, \quad (\text{A7})$$

$$\frac{\partial}{\partial t_1} f_\alpha^{(0)} + C_2 \Delta^2 f_\alpha^{(0)} = -\frac{1}{\tau} f_\alpha^{(2)} + \Omega_\alpha^{(2)}, \quad (\text{A8})$$

$$C_3 \Delta^3 f_\alpha^{(0)} + 2C_2 \Delta \frac{\partial}{\partial t_1} f_\alpha^{(0)} + \frac{\partial}{\partial t_2} f_\alpha^{(0)} + \tau \Delta \Omega_\alpha^{(2)} = -\frac{1}{\tau} f_\alpha^{(3)}, \quad (\text{A9})$$

$$C_4 \Delta^4 f_\alpha^{(0)} + 3C_3 \Delta^2 \frac{\partial}{\partial t_1} f_\alpha^{(0)} + 2C_2 \Delta \frac{\partial}{\partial t_2} f_\alpha^{(0)} + \frac{\partial}{\partial t_3} f_\alpha^{(0)} + C_2 \frac{\partial^2}{\partial t_1^2} f_\alpha^{(0)} + \frac{\partial}{\partial t_1} \tau \Omega_\alpha^{(2)} + \tau C_2 \Delta^2 \Omega_\alpha^{(2)} = -\frac{1}{\tau} f_\alpha^{(4)}. \quad (\text{A10})$$

where the partial differential operator $\Delta \equiv \frac{\partial}{\partial t_0} + \mathbf{e}_\alpha \frac{\partial}{\partial \mathbf{x}}$.

Equations (A7)–(A10) represent a series of partial differential equations across various time scales [23]. In these equations, C_i is the polynomial of the relaxation time factor τ .

$$C_1 = 1, \quad (\text{A11})$$

$$C_2 = \frac{1}{2} - \tau, \quad (\text{A12})$$

$$C_3 = \tau^2 - \tau + \frac{1}{6}, \quad (\text{A13})$$

$$C_4 = -\tau^3 + \frac{3}{2}\tau^2 - \frac{7}{12}\tau + \frac{1}{24}. \quad (\text{A14})$$

Based on Equations (3) and (A4), it follows that

$$\sum_{\alpha} f_{\alpha}^{(n)}(\mathbf{x}, t) = 0, \text{ for } n \geq 1 \quad (\text{A15})$$

Equation (A15) indicates that the moment at zero vanishes for each order $n \geq 1$ of ε .

The equilibrium distribution function is characterized by certain moments, which are denoted in the following manner:

$$\sum_{\alpha} f_{\alpha}^{(0)}(\mathbf{x}, t) e_{\alpha} \equiv m^0(\mathbf{x}, t), \quad (\text{A16})$$

$$\sum_{\alpha} f_{\alpha}^{(0)}(\mathbf{x}, t) e_{\alpha}^2 \equiv \pi^0(\mathbf{x}, t) \quad (\text{A17})$$

$$\sum_{\alpha} f_{\alpha}^{(0)}(\mathbf{x}, t) e_{\alpha}^3 \equiv P^0(\mathbf{x}, t) \quad (\text{A18})$$

$$\sum_{\alpha} f_{\alpha}^{(0)}(\mathbf{x}, t) e_{\alpha}^4 \equiv Q^0(\mathbf{x}, t). \quad (\text{A19})$$

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