Notes on the Overconvergence of Fourier Series and Hadamard–Ostrowski Gaps

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Abstract: This paper examines the relationship between the overconvergence of Fourier series and the existence of Hadamard–Ostrowski gaps. Ostrowski’s result on the overconvergence of power series serves as a motivating factor for obtaining a natural generalization: the overconvergence of Fourier series. The connection between Hadamard–Ostrowski gaps and the overconvergence of Fourier series is clarified by applying the Hadamard three-circle theorem and the theory of orthogonal polynomials. Our main result is obtained by applying the Hadamard three-circle theorem.

Keywords: Fourier series; overconvergence; orthogonal polynomials; Hadamard–Ostrowski gaps

MSC: 30B10; 30B30; 30B40

1. Introduction

The classical work of Ostrowski [1] established close connections between the gap structure of a complex power series and the convergence properties of subsequences of its partial sums beyond the disk of convergence. More general results for the overconvergence of power series are obtained in [2]. Kovacheva presents new results on the overconvergence of Fourier series in [3,4]. For the used methodology one should check [5–8].

2. Analytic Continuation and Domain of Analyticity

Definition 1. Let \( G_i, i = 1, 2, \) be domains in \( \mathbb{C} \), such that \( G_1 \cap G_2 \neq \emptyset \). If \( f_i : G_i \rightarrow \mathbb{C}, i = 1, 2, \) are analytic functions and \( f_1 \equiv f_2 \) in \( G_1 \cap G_2 \), then \( f_2 \) is an analytic continuation of \( f_1 \) in \( G_2 \) and \( f_1 \) is an analytic continuation of \( f_2 \) in \( G_1 \).

Theorem 1. If \( f : G_1 \rightarrow \mathbb{C} \) has an analytic continuation in \( G_2 \), then the continuation is unique.

Let \( G_1, G_2 \) and \( G_1 \cup G_2 \) be domains, and \( f_i : G_i \rightarrow \mathbb{C}, i = 1, 2, \) be analytic functions, where \( f_2 \) is an analytic continuation of \( f_1 \) in \( G_2 \). The function, \( f : G_1 \cup G_2 \rightarrow \mathbb{C} \), defined by \( f(z) = f_1(z) \) for \( z \in G_1 \) is an analytic continuation of \( f_1 \) in \( G = G_1 \cup G_2 \).

Let the series, \( f(z) = \sum_{n=0}^{\infty} a_n z^n \), be convergent in the disk, \( D(0, r) : |z| < r \). Any point on the boundary circle \( K : |z| = r \) that has a neighborhood in which \( f \) has an analytic continuation is called regular, otherwise, the point is deemed singular. If \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) has a radius of convergence, \( r \), then there is at least one singular point on the circle, \( K(0, r) \). If all points on \( K(0, r) \) are singular, then the disk, \( D(0, r) \), is termed the domain of analyticity for \( f \).
3. Overconvergence of Power Series

Let \( \sum_{n=0}^{\infty} a_n z^n \) be the radius of convergence, \( r \). The sequence, \( \{S_k(z)\} \), of partial sums of the series diverges at any point outside \( |z| \leq r \). However, it is possible for a suitably chosen subsequence, \( \{S_{n_k}(z)\} \), of the sequence of partial sums to turn out to be uniformly convergent for some \( z : |z| > r \). In such a case, we say that the series \( \sum_{n=0}^{\infty} a_n z^n \) exhibits the property of overconvergence. Consider Porter’s example of a series demonstrating overconvergence [9]:

Example 1.

\[
f(z) = \sum_{n=0}^{\infty} \frac{(z(1-z))^n}{r_n},
\]

where \( r_n \) is the coefficient with the maximal absolute value among the coefficients in the binomial \((1-z)^n\).

We write down the considered functional series in terms of powers of \( z \): \( f(z) = \sum_{n=0}^{\infty} a_n z^n \). Considering that the lowest degree of \( z \) in \( (z(1-z))^{n+1} \) is \( 4^n + 1 > 2.4^n = \deg(z(1-z))^n \), the power series is obtained without rearranging the terms of the original series. Therefore, the original series is convergent in the disk of convergence of the power series derived from it. We have \( |a_n| \leq 1 \), and this equality is fulfilled for infinitely many \( n \). Using Hadamard’s formula, we determine the radius of convergence of the considered power series to be \( R = 1 \). The sequence of partial sums of the grouped series is the subsequence \( \{S_{2.4^n}(z)\} \) of the sequence \( \{S_n(z)\} \) of partial sums of the resulting power series. Therefore, \( \{S_{2.4^n}(z)\} \) is convergent in the unit disk, and using \( S_{2.4^n}(z) = S_{2.4^n}(1-z) \), we find that the considered sequence is also convergent in the disk, \( D(1,1) \), which means the overconvergence of \( \sum_{n=0}^{\infty} a_n z^n \). Another possible justification is the following: we have the inequality \( \frac{2^{n+1}}{4^n + 1} < r_n < 2^{n+1} \), then \( \limsup |r_n|^{\frac{1}{n+1}} = 2 \). Thus, the series \( \sum_{n=0}^{\infty} \frac{a_n z^n}{r_n} \) is convergent in the circle, \( |w| < 2 \), which with respect to \( z \), is the interior of the lemniscate \( |z(1-z)| = 2 \). Therefore, the sequence \( \{S_{2.4^n}(z)\} \) is convergent inside the lemniscate and divergent outside of it. On the other hand, since \( \sum_{n=0}^{\infty} a_n z^n \) is obtained from a grouped series without rearrangement, its circle of convergence is the largest circle centered at the origin contained in the lemniscate \( |z(1-z)| = 2 \). Obviously, this is the unit circle, and we again show that the overconvergence \( \sum_{n=0}^{\infty} a_n z^n \) possesses overconvergence.

Let us determine those \( n \in \mathbb{N} : a_n = 0 \). We calculate

\[
f(z) = \sum_{n=0}^{\infty} \frac{(z(1-z))^n}{r_n} = \sum_{n=0}^{\infty} a_n z^n = \frac{1}{r_0} + \frac{z^4 + \cdots + z}{r_1} + \frac{z^{16} - 16z^{17} + \cdots + 16z^{31} + z^{32}}{r_2} + \cdots + \frac{z^{4^k+1} - \cdots + z^{2^k} + z^{2^k+1}}{r_{k+1}} + \cdots.
\]

We define the sequences \( \{p_i\}_{i=1}^{\infty} \) and \( \{q_i\}_{i=1}^{\infty} \) by \( p_i = 2.4^i + 1, q_i = 4^{i+1} - 1 \). Then, \( a_n = 0 \) for all \( n \in \bigcup_{i=1}^{\infty} [p_i, q_i] \) and, therefore, \( S_n = S_{2.4^i} \) for each \( k \) and every integer \( n_k \in [2.4^k, 4^{k+1} - 1] \) \( \equiv |p_k - 1, q_k| \). Each sequence

\[
\{S_{p_k - 1}\}_{k=1}^{\infty}, \{S_{p_k}\}_{k=1}^{\infty}, \ldots, \{S_{q_k}\}_{k=1}^{\infty}
\]

trivially ensures the overconvergence of the power series for \( f(z) \). On the other hand,

\[
\liminf_k \frac{q_k}{p_k} = \liminf_k \frac{4^{k+1} - 1}{2.4^k + 1} = 2.
\]

The above calculations are motivated by the following definition:

**Definition 2.** We will say that the series \( \sum_{n=0}^{\infty} a_n z^n \) admits Ostrowski gaps with respect to its coefficients if there exist sequences \( \{p_k\}_{k=1}^{\infty} \) and \( \{q_k\}_{k=1}^{\infty} \), satisfying the following conditions:

(a) \( q_k - 1 \leq p_k < q_k \);  
(b) \( \liminf_{k \to \infty} \frac{q_k}{p_k} > 1 \);
(c) if \( p \in \bigcup_{k=1}^{\infty} [p_k, q_k] \), then \( a_p = 0 \).

4. Relationship between Overconvergence, Hadamard–Ostrowski Gaps, and Analytic Continuation

Consider again the series \( \sum_{n=0}^{\infty} a_n z^n \), for which there exist infinitely many indices \( s: n_{s+1} > (1 + \theta) n_s \), with fixed \( \theta > 0 \). In such a case, we will say that the considered series has Ostrowski gaps (sequences of zero coefficients). He was the first to establish that with these types of gaps in the coefficients and the presence of analytic continuation, a sufficient condition for overconvergence is obtained, which is the content of the theorem proved below. Moreover, these two conditions turn out to be necessary, given that Ostrowski proves the following theorem: if a series possesses the property of overconvergence, then it can be represented as a sum of two series, one having Ostrowski gaps with respect to its coefficients, and the second with a radius of convergence greater than that of the original series. Assume that the radius of convergence of the power series is 1. A generalization of Ostrowski gaps is introduced, where the overconvergence property becomes equivalent to analytic continuity plus Hadamard–Ostrowski gaps:

Definition 3. The series \( \sum_{n=0}^{\infty} a_n z^n \) admits Hadamard–Ostrowski gaps if there exist sequences \( \{p_k\}_{k=1}^{\infty} \) and \( \{q_k\}_{k=1}^{\infty} \) satisfying the following conditions:

(a) \( q_{k-1} \leq p_k < q_k \).
(b) \( \lim_{k \to \infty} \frac{q_k}{p_k} = 1 \).
(c) \( \limsup_{l \to \infty} |a_l|^{1/l} < 1 \) for \( l \in \bigcup_{k=1}^{\infty} [p_k, q_k] \).

Relation between \( p_k \), \( q_k \) and \( n_k \)

With respect to the overconvergence property, the Ostrowski and Hadamard–Ostrowski gaps are equivalent. Indeed, let \( \sum_{n=0}^{\infty} a_n z^n \) admit Hadamard–Ostrowski gaps. Through equality

\[
 f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{j \in \cup_{k=1}^{\infty} [p_k, q_k]} a_j z^j + \sum_{j \in \cup_{k=1}^{\infty} (p_k, q_k]} a_j z^j = f_1(z) + f_2(z),
\]

the question of the overconvergence of \( f \) reduces to the same question about \( f_2 \), as \( f_1 \) has a radius of convergence greater than 1. The opposite direction is also trivial. This observation shows that in the case of overconvergence, the subsequences of the series of partial sums of \( f \) that ensure its overconvergence have the same indices as those of \( f_2 \). As seen in Example 1, we can choose \( n_k \in [p_k - 1, q_k] \).

5. Ostrowski’s Theorem and Its Consequences

Theorem 2. Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \), \( |z| < r \), and as there are infinitely many indices, \( s \), we denote them by \( s_k \), \( k = 0, 1, 2 \ldots \) with the following: \( n_{s_k+1} \geq (1 + \theta) n_s \) with \( \theta > 0 \). Then, in a neighborhood of every regular point of the boundary of the disk of convergence, the series possesses the property of overconvergence, i.e., the subsequence \( \{S_n(z)\}_{k=1}^{\infty} \) is uniformly convergent in a sufficiently small neighborhood of every regular point of \( |z| = r \).

Proof. ([9]). □

An interesting consequence of Ostrowski’s result is the following Hadamard theorem:

Theorem 3. Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) have a radius of convergence of 1, such that \( n_{s_k+1} > (1 + \theta) n_s \) for all sufficiently large \( s \), and \( \theta > 0 \) is fixed. Then the series \( f \sum_{n=0}^{\infty} a_n z^n \) has no analytical continuation.

Proof. Assume that the series, \( \sum_{n=0}^{\infty} a_n z^n \), has an analytic continuation in a neighborhood, \( U \), of a point, \( z = e^{i\phi} \). Then, the assumptions of Ostrowski’s theorem are fulfilled; therefore, there exists a neighborhood \( V \subset U \) of \( z = e^{i\phi} \) in which \( S_n(z) \to \phi(z) \). But \( s_k = k \) for all sufficiently large \( k \) since \( n_{s_k+1} > (1 + \theta) n_s \). We have that \( \{S_n(z)\}_{k=1}^{\infty} \equiv \{S_s(z)\}_{s=1}^{\infty} \) resulting in \( S_n(z) \to \phi(z) \) in \( V \). The latter leads to a contradiction since the partial sums of the considered series cannot be convergent outside the circle of convergence. Thus, the assumption that the power series has analytic continuation is wrong, and the unit disk is a domain of analyticity for the series. □
Example 2. For example, the power series, $\sum_{n=0}^{\infty} \frac{z^n}{n!}$, has a radius of convergence of 1 and Ostrowski gaps, but no overconvergence, because it is analytically noncontinuable, according to Hadamard’s theorem.

Every power series, $\sum_{n=0}^{\infty} a_n z^n$, with a nonzero radius of convergence, $R > 0$, is uniformly convergent in every circle, $|z| \leq r < R$. Therefore, the overconvergence property could be considered as an extension of uniform convergence in a region outside the convergence circle, $|z| < R$. Now, we will prove another Ostrowski theorem, which is the opposite of the one proved above:

**Theorem 5.** If the power series $\sum_{n=0}^{\infty} a_n z^n$ has the overconvergence property, then it has Hadamard–Ostrowski gaps, i.e., $\sum_{n=0}^{\infty} a_n z^n$ can be represented as the sum of two power series, the first with a radius of convergence greater than the radius of convergence of the initial power series, and the second series allows Ostrowski gaps in terms of their coefficients.

Again, without loss of generality, we will assume that the initial power series has a radius of convergence of $r = 1$. We will prove that there exist sequences $\{p_k\}_{k=0}^{\infty}$ and $\{q_k\}_{k=0}^{\infty}$ that satisfy the conditions of the definition for Hadamard–Ostrowski gaps. We will now write down the power series in the following form:

$$\sum_{n=0}^{\infty} b_n z^n = \sum_{j \in \{1, \ldots, p\}} b_{j1}z^j + \sum_{j \in \{1, \ldots, q\}} b_{j2}z^j.$$

In such a case, the first power series on the right-hand side of the equality has a radius of convergence $R = \limsup_{|z| \to r_n} |z| > 1$; this is because $l \in \bigcup_{k=0}^{\infty} [p_k, q_k]$, and the second power series, according to the choice of the sequence, $\{p_k\}_{k=0}^{\infty}$ and $\{q_k\}_{k=0}^{\infty}$ exhibits Ostrowski gaps. We will prove the existence of $\{p_k\}_{k=1}^{\infty}$ and $\{q_k\}_{k=1}^{\infty}$ with the requested properties by the following:

**Theorem 4.** If the power series $\sum_{n=0}^{\infty} b_n z^n$ has the overconvergence property, then it has Hadamard–Ostrowski gaps.
\[
\left| b_{n+p} \right|^\frac{1}{\tau} \leq \frac{2^\frac{p}{r}}{r} \left( \frac{n + p}{p} \right)^\frac{1}{\tau} \leq \frac{2^\frac{p}{r}}{r} \left( \frac{(n + p)^\frac{n}{p} + p}{n^\frac{n}{p} p} \right)^\frac{1}{\tau} = \frac{2^\frac{p}{r} 1 + \tau}{r} \frac{1}{\tau} = g(\tau) \frac{r}{p}.
\]

For \( \tau \to 0 \implies g(\tau) \to 1 \), we can redefine \( g \) by continuity at point \( \tau = 0 \): \( g(0) = 1 \). Since \( r > 1 \), there exists \( \theta > 0 \) and \( 0 < \rho < 1 \), such that \( \frac{\theta r^k}{\rho} < \rho < 1 \) for all \( \tau < \theta \). Hence, \( \left| b_{n+p} \right|^\frac{1}{\tau} \leq \frac{\theta r^k}{\rho} < \rho \).

Let \( f(z) \) be the analytic continuation of \( f \) in \( V \). Let \( F(z) = \left\{ \begin{array}{ll} f(z) & \text{for } z \in D(0,1) \\
 f_1(z) & \text{for } z \in V \end{array} \right. \) and we can replace \( \phi(z) \) with

\[
\phi_n(z) = F(z) - S_n(z) = b_{n+1}z^{n+1} + b_{n+2}z^{n+2} + \cdots,
\]

for sufficiently large \( k \), according to the overconvergence condition, we have \( |\phi_n(z)| = |F(z) - S_n(z)| < 1 \). Then the inequality \( |b_{n+p}|^\frac{1}{\tau} \leq \frac{\theta r^k}{\rho} < \rho \) is written in the following form:

\[
\left| b_{n+1+p} \right|^\frac{1}{\tau} \leq \frac{\theta r^k}{\rho} \leq \frac{\theta r^k}{\rho} < \rho.
\]

Taking into account that \( \theta = \tau = \frac{p}{n+1} \), we obtain \( p \leq \theta(n+1) \).

Thus, the inequality \( |b_{n+1+p}|^\frac{1}{\tau} \leq \frac{\theta r^k}{\rho} < \rho \) holds for all \( p : 0 \leq p \leq \theta(n+1) \) and \( n_k + 1 \leq n_k + 1 + p < (1 + \theta)(n_k + 1) \).

We obtain \( |b_k|^{\frac{1}{\tau}} < \rho \) for all \( n_k + 1 \leq s \leq (1 + \theta)(n_k + 1) \).

We can define the sequences \( \{p_k\}_{k=1}^\infty \) and \( \{q_k\}_{k=1}^\infty \) as follows: \( p_k = n_k + 1 \), \( q_k = (1 + \theta)n_k \). The sequences defined in this way satisfy the conditions of definition 3, which we check immediately:

\[
q_{k-1} = (1 + \theta)n_{k-1} < n_k < n_k + 1 = p_k < (1 + \theta)n_k = q_k \implies q_{k-1} \leq p_k < q_k.
\]

\[
\limsup_{s \to \infty} \left| p_k \right|^{\frac{1}{\tau}} < 1, \quad s \in \bigcup_{k=1}^\infty \left[ p_k, q_k \right).
\]

\[\square\]

Second shorter proof (R. Kovacheva). The main idea is to employ a harmonic majorant for a sequence of analytic functions, each having a unique modulus. \[\square\]

Definition 4. Let \( v(z) \) be a harmonic function in a region, \( R \), of the complex plane, and let \( \{v_n(z)\}_{n \geq 1} \) be a sequence of analytic functions in \( R \) having a unique modulus \( |v_n(z)| \) in \( R \). If for every connected compact \( K \subset R \), the condition

\[
\limsup_{n \to \infty} \max_{z \in K} |v_n(z)| \leq \max_{z \in K} e^{\theta(z)},
\]

is satisfied, then we will say that \( v(z) \) is the harmonic majorant of the sequence \( \{v_n(z)\}_{n \geq 1} \) in \( R \).

We will use the following theorem [10]:

Theorem 6. Let \( v(z) \) be a harmonic majorant of \( \{v_n(z)\}_{n \geq 1} \) in \( R \). Let there exist a connected compact \( K \subset R \) consisting of more than one point for which

\[
\limsup_{n \to \infty} \max_{z \in K} |v_n(z)| < \max_{z \in K} e^{\theta(z)},
\]

then the strict inequality holds for every connected compact in \( R \).

Let \( \sum_{n=0}^\infty b_n z^n \) be a power series with a radius of convergence of 1, and \( V \) be a neighborhood of a regular point of \( |z| = 1 \). Let \( U \subset V \), \( U \cap D(0,1) = \emptyset \) be a compact subset of \( V \) such that \( d = dist(D(0,1),U) < 1 \). Let \( \{S_n(z)\}_{n \geq 1} \) be uniformly convergent in \( D(0,1) \cup V \). We will prove that
for all sufficiently large \( k \), there exists \( l_k \in (n_k - 1, n_k) \), and \( \limsup_{s \to \infty} |b_n|^{1/2} < 1 \), \( s \in \mathbb{R}^n \) where \( \theta > 0 \) is a fixed number that we will determine. Putting \( p_k = l_k \), \( q_k = (1 + \theta) l_k \), we obtain the sequences, defining the Hadamard–Ostrowski gaps. We will first prove \( \limsup_{k \to \infty} |b_n|^{1/2} < 1 \) and, subsequently, determine \( l_k \).

We fix \( 0 < \epsilon < \frac{\pi}{2} \), and applying the Corollary 1, we have the following:

\[
\max_{|z|=1+\epsilon} |S_n(z)| \leq \max_{|z|=1-\epsilon} |S_n(z)| \left( 1 + \frac{\nu}{1-\epsilon} \right) = C_1 \left( 1 + \frac{\nu}{1-\epsilon} \right) \quad \Rightarrow \\
\max_{|z|=1+\epsilon} \left| \frac{S_n(z)}{z^{n_k}} \right| \leq \left( \frac{C_1}{1-\epsilon} \right)^{\nu} \quad \Rightarrow \\
\left( \max_{|z|=1} \left| \frac{S_n(z)}{z^{n_k}} \right| \right)^{\frac{1}{\nu}} \leq \lim_{\epsilon \to 0} \left( \max_{|z|=1+\epsilon} \left| \frac{S_n(z)}{z^{n_k}} \right| \right)^{\frac{1}{\nu}} \leq \lim_{\epsilon \to 0} \frac{1}{1-\epsilon} = 1 \\
\limsup_{k \to \infty} \left( \max_{|z|=1} \left| \frac{S_n(z)}{z^{n_k}} \right| \right)^{\frac{1}{\nu}} \leq 1. \quad (*)
\]

Let \( F \) be the analytic continuation of \( f \) in \( D = D(0,1) \cup U \). Due to \( S_n(z) \Rightarrow F(z) \) in the compact subsets of \( D \), then for all sufficiently large \( k \), we will have the following:

\[
\max_{|z|=U} |S_n(z)| \leq \max_{z \in U} |F| + \epsilon_1 = C_2 \quad \Rightarrow \\
\max_{|z|=U} \left| \frac{S_n(z)}{z^{n_k}} \right| \leq \frac{C_2}{(1+d)^{\nu_k}} \quad \Rightarrow \\
\Rightarrow \exists q \in (\frac{1}{1+d}, 1): \left( \max_{U} \left| \frac{S_n(z)}{z^{n_k}} \right| \right)^{\frac{1}{\nu}} \leq q \quad (**) \text{ and let consider} \\
\{v_{\nu_k}(z)\} \text{ defined by } v_{\nu_k}(z) = \left( \frac{S_n(z)}{z^{n_k}} \right)^{\frac{1}{\nu}} \text{ for } z \in D_1 = \mathbb{C} \setminus D(0,1).
\]

According to (**), zero is the harmonic majorant of the sequence \( \{v_{\nu_k}(z)\}_{k=1}^{\infty} \) in \( D_1 \), and from (**), it is established that on the compact \( U \), the strict inequality \( |v_{\nu_k}(z)| \leq q < 1 \) holds. According to Theorem 6, the strict inequality holds for every compact \( K \subset D_1 \), i.e., \( \limsup_k (\max_{z \in K} |v_{\nu_k}(z)|) = q_k < 1 \). We fix \( \delta : 0 < \delta < d \), consider the compact \( K = \{ z : |z| = 1 + \delta \} \subset D_1 \). Then, there exists \( q = q_\delta \), depending on \( \delta \), such that \( \limsup_k \left( \max_{|z|=1+\delta} |v_{\nu_k}(z)| \right) = q < 1 \) and, hence, we obtain the following:

\[
\max_{|z|=1+\delta} \left| \frac{S_n(z)}{z^{n_k}} \right| \leq C_\delta q^{n_k} s \cdot q < 1 \text{ and } C_\delta \frac{1}{\nu_k} \to 1. \text{ Evaluate} \\
|b_n| = \frac{|S_n(0)|}{n_k!} = \frac{1}{2\pi} \int_{|z|=1+\delta} \frac{S_n(z) \ dz}{z^{n_k}} \leq \frac{1}{2\pi} \int_{|z|=1+\delta} \left| \frac{S_n(z)}{z^{n_k}} \right| \cdot |dz| \leq \\
\leq C_\delta q^{n_k} \int_{|z|=1+\delta} \frac{|dz|}{z^{n_k}} = C_\delta q^{n_k} \Rightarrow b_n \leq C_\delta q^{n_k} \limsup_{k \to \infty} |b_n|^{\frac{1}{\nu_k}} < 1.
\]

Now, let us define \( l_k \) in the following form: \( l_k = n_k - s_k \):

\[
|b_{n_k-s_k}| = \frac{|S_{n_k-s_k}(0)|}{(n_k-s_k)!} = \frac{1}{2\pi} \int_{|z|=1+\delta} \frac{S_{n_k-s_k}(z) \ dz}{z^{n_k-s_k+1}} \leq \frac{1}{2\pi} \int_{|z|=1+\delta} \left| \frac{S_{n_k-s_k}(z)}{z^{n_k-s_k}} \right| \cdot |z^{n_k-s_k-1}| \cdot |dz| \leq \\
\leq C_\delta q^{n_k} (1+\delta) \Rightarrow |b_{n_k-s_k}| \leq C_\delta q^{n_k} (1+\delta) \Rightarrow |b_{n_k-s_k}|^{\frac{1}{n_k-s_k}} \leq q^{n_k} (1+\delta)^{\frac{n_k-s_k}{n_k}} \text{,} \\
\text{where } q < 1, \text{ and we can choose it in such a way to satisfy } \frac{1}{\frac{1}{1+\delta}} < q \text{ and define } s_k, \text{ such that we have the following:}
\]

\[
\limsup_{k \to \infty} |b_{n_k-s_k}|^{\frac{1}{n_k}} \leq 1 \quad \Rightarrow \quad q^{\frac{n_k}{n_k-s_k}} (1+\delta)^{\frac{n_k-s_k}{n_k}} \leq 1 \quad \iff \\
\frac{n_k}{n_k-s_k} \ln q + \frac{s_k}{n_k-s_k} \ln(1+\delta) \leq \ln \tau, \text{ where } q < \tau \quad \iff \\
\iff s_k \leq n_k \frac{\ln \tau}{\ln (1+\delta)} \text{ and put } s_k = n_k \frac{\ln \frac{1}{\tau}}{\ln (1+\delta)}.
\]
therefore for all \( m : 0 \leq m \leq s_k \) is valid \( \lim_{k \to \infty} \frac{|b_{n_k - m}|^{1/\ell_k}}{\ell_k} \leq \tau \implies \forall l \in [l_k, n_k] \) we obtain \( \lim_{k \to \infty} |b_l|^1 / \tau^1 < 1 \) and put \( \theta = \frac{\ln \frac{z}{q}}{\ln \frac{q(1 + \delta)}{z}} \implies \)

\[
\implies p_k = l_k = n_k - s_k = \frac{n_k \ln \frac{2(1 + \delta)}{z}}{\ln \frac{q(1 + \delta)}{z}}; q_k = (1 + \theta)l_k = \left(1 + \frac{\ln \frac{z}{q}}{\ln \frac{q(1 + \delta)}{z}}\right) \frac{n_k \ln \frac{q(1 + \delta)}{z}}{\ln \frac{q(1 + \delta)}{z}} = n_k.
\]

The sequences \( \{p_k\} \) and \( \{q_k\} \) are correctly defined because the constants \( q \) and \( \tau \) are chosen such that \( 1/\tau^2 < q < \tau < 1 \), and the proof is complete.

### 6. Decomposition of Analytic Functions in Series by Orthogonal Polynomials:

#### Overconvergence and Examples

**6.1. Regular Compacts and Green’s Function**

Let \( E \) be a singly connected compact in \( \mathbb{C} \), and \( G = \mathbb{C} \setminus E \) be a domain in the extended complex plane \( \mathbb{C} \), and suppose there exists a function \( g_\infty(z) : G \to \mathbb{R} \) with the following properties:

1. \( g_\infty(z) \) is a harmonic function in \( G \) except at point \( z = \infty \);
2. The function \( g_\infty(z) - \ln(z) \) is harmonic in a neighborhood of \( z = \infty \);
3. \( \lim_{z \to \infty} g_\infty(z) = 0 \).

We will denote by \( g_\infty(z) \) the Green’s function for the domain, \( G \), with a pole at the infinity point. It is unique. If we assume that \( h_\infty(z) \) satisfies the conditions (1)–(3), then \( g_\infty(z) - h_\infty(z) \) is harmonic in \( G \) and vanishes on \( \partial G \). According to the maximum principle, \( g_\infty(z) - h_\infty(z) \equiv 0 \). Due to (2), we have \( \lim_{z \to \infty} g_\infty(z) = +\infty \) and, therefore, it is nonnegative on the boundary of \( G \setminus \{\infty\} \) and is a harmonic function inside. From the maximum principle, we obtain \( g_\infty(z) > 0 \) in \( G \).

**Definition 5.** We will call a simply connected compact \( E \) regular if its complement, \( \mathbb{C} \setminus E \), in the extended complex plane possesses the Green’s function with a pole at the infinite point.

Consider a regular and convex compact, \( E \), with its complement, \( G \). Let \( g(z) \) represent the Green’s function with a pole at the infinity point for \( G \) and let \( h(z) \) represent the harmonic conjugate of \( g(z) \) in \( G \). Then, the function \( g(z) + i \cdot h(z) \) is analytic in \( G \), and we consider the function \( w = \varphi(z) = g(z) + i \cdot h(z) \). We will show that \( \varphi \) conformally maps \( G \) to the exterior of the unit circle. Indeed, if \( z_0 \in \partial G \), then

\[
\varphi(z_0) = e^{g(z_0) + i \cdot h(z_0)} \implies |
\varphi(z_0)| = e^{g(z_0)} = e^0 = 1 \implies \varphi(\partial G) \subset \partial D(0, 1)
\]

and taking into account \( \lim_{z \to \infty} g_\infty(z) = +\infty \), we have that \( \varphi \) maps \( G \) to the exterior of the unit circle.

It suffices to show that \( \frac{\partial \varphi}{\partial z} \neq 0 \) in \( G \), which is equivalent to \( 0 \neq q' \equiv q'(z) = (g'(z) + i \cdot h'(z))e^{g(z) + i \cdot h(z)} = \frac{1}{2} \left( \frac{\partial (g + i \cdot h)}{\partial x} - i \cdot \frac{\partial (g + i \cdot h)}{\partial y} \right) e^{g(z) + i \cdot h(z)} = \frac{1}{2} \left( g'_x + h'_y + i \cdot (h'_x - g'_y) \right) e^{g(z) + i \cdot h(z)}.
\]

If we assume that \( q'(z_0) = 0 \) for some \( x_0 + iy_0 = z_0 \in G \), then we have \( g'_x(z_0) + h'_y(z_0) = h'_x(z_0) - g'_y(z_0) = 0 \), which combined with the Cauchy-Riemann equations \( g'_x = h'_y \) and \( g'_y = -h'_x \), this gives \( g'_x(x_0, y_0) = g'_y(x_0, y_0) = h'_x(z_0) = h'_y(z_0) = 0 \). Thus, we establish that \( q'(z_0) = 0 \implies g'_x(x_0, y_0) = g'_y(x_0, y_0) = 0 \). We will show that this is impossible for points in \( G \), referencing the following:

**Theorem 7.** All points, \( z = x + iy \in \mathbb{C} \), for which the partial derivatives \( g'_x \) and \( g'_y \) simultaneously vanish, belong to the convex hull of \( E \).

**Proof.** ([11]). □

Since \( E \) is a convex compact, it coincides with its convex hull and, therefore, it contains all (if any) zeros of \( q'(z) \), meaning that there are no points in \( G \) annihilating \( q'(z) \). Thus, we establish that the map \( G \to \{ w \in \mathbb{C}_\infty : |w| > 1 \} \) is conformal; moreover, according to the inverse mapping theorem, \( \varphi \) is locally invertible on \( G \).
Let $R > 1$ be a real number and consider the image of the circle $|w| = R$ by $q^{-1}$, i.e., $\{ z \in G : |w| = |q(z)| = R \}$. We put $\ln(R) = \rho$ and have $\sigma(f) = |q(z)| = R \implies g(z) = \rho$,

$$q^{-1} : \{ w \in C : |w| = R > 1 \} \rightarrow \{ z = x + iy \in G : g(x, y) = \rho > 0 \},$$

which means that the image is an analytical Jordan curve, whose support we will denote by $\Gamma_{\rho}$. Thus, every level curve $\Gamma_{\rho}, \rho > 0$ of the Green’s function of $G$ is an analytic Jordan curve, which is the image of a circle with a radius greater than 1. Using this result, we immediately obtain the following properties:

1. If $0 < \rho_1 < \rho_2$, then the curves $\Gamma_{\rho_1}$ and $\Gamma_{\rho_2}$ do not intersect; moreover, the first one is contained inside the bounded region enclosed by $\Gamma_{\rho_2}$.

2. Through each point, $x_0 + iy_0 = z_0 \in G$ passes a unique level curve $\Gamma_{\rho_0}, \rho_0 = g(x_0, y_0)$, namely $g(x, y) = g(x_0, y_0)$.

Based on these properties, an analyticity parameter is defined for a function analytic in a regular and convex compact. Denote by $E_{\rho}$, the bounded region in $C$ is enclosed by $\Gamma_{\rho}$, i.e., $E_{\rho} = \{ z = x + iy \in G : g(x, y) < \rho \}$ and $\Gamma_{\rho} = \{ z = x + iy \in G : g(x, y) = \rho \}$, $\Gamma_{\rho} = \partial E_{\rho}$. Let $f$ be analytic in $E$ and suppose that it is not an entire function, which means that there are singular points in the finite plane, $C$. If any singular points lie on $\partial E$, then we set the analyticity parameter, $\rho(f) = 1$. Otherwise, the singularities of $f$ lie in $G$. Since they are isolated, by using properties (1) and (2), we prove the existence of $\rho > 0$, such that on $\Gamma_{\rho}$, there is at least one singularity, while there are no singularities of $f$ in $E_{\rho}$, and we have $\rho(f) = e^\rho$. Indeed, we may set $\rho := \sup_r \{ r : f \text{ is analytic in } E_r \}$.

6.2. Capacity of Compact Set, Regular Measure, and Orthogonal Polynomials

Let $E$ be a regular convex compact with complement $G = C_{\infty} \setminus E$. There are three fundamentally different approaches to defining the capacity of a compact set [12]:

1. Fekete’s method:

   Given an arbitrary set of $n \geq 2$ distinct points $z_1, z_2, \ldots, z_n$, belonging to $E$, we define the following:

   $$h_n(z_1, z_2, \ldots, z_n) = \prod_{i,j} (z_i - z_j); \quad h_n = \max_{z_1, z_2, \ldots, z_n \in E} \{ h_n(z_1, z_2, \ldots, z_n) \}; \quad d_n = \frac{1}{h_n^{\frac{1}{n}}}. $$

   The sequence $(d_n)_{n=2}^{\infty}$ is monotonically decreasing and bounded below; hence, it is convergent. The number $d = \lim_{n \to \infty} d_n$ is called the transfinite Fekete diameter of $E$.

2. Chebyshev’s constant:

   Let $n$ be a natural number. We consider all polynomials $p \in \mathbb{C}[z]$ of degree $\deg p = n$, with leading coefficients equal to 1. The minimax problem

   $$\min_{p : \deg p = n} \max_{z \in E} |p(z)|$$

   has a solution and we can put $m_n = \min_{p : \deg p = n} \max_{z \in E} |p(z)|$.

   We define the sequence $(\tau_n)_{n=1}^{\infty}$ by $\tau_n = m_n^{-\frac{1}{n}}$. It has been proved that this series is convergent and the limit $\tau = \lim_{n \to \infty} \tau_n$ is called the Chebyshev’s constant for $E$.

   For each natural, $n$, we let $T_n(z, E)$ the polynomial with leading coefficient 1 that has the smallest maximum of the modulus over $E$, and by $\tilde{T}_n(z, E)$, the polynomial with zeros in $E$ that has the same min–max property on $E$. According to the introduced notations, $m_n = \max_{z \in E} |T_n(z, E)|$ and $\tilde{m}_n = \max_{z \in E} |\tilde{T}_n(z, E)|$. Similar to $\tau_n$, $\tilde{\tau}_n$ is defined, with the equality, $\tau_n = \tilde{\tau}_n$, being valid.

3. Robben’s constant:

   Since the function, $g_{\infty}(z) - \ln |z|$, is harmonic in a neighborhood of $z = \infty$, there exists the limit, $\lim_{z \to \infty} (g_{\infty}(z) - \ln |z|)$; let us denote it by $\gamma$. A capacity of $E$ is defined as $\text{Cap}(E) = e^{-\gamma}$. Hence,

   $$\lim_{z \to \infty} \frac{g(z)}{z} = \lim_{z \to \infty} \frac{|g(z)|}{|z|} = \lim_{z \to \infty} e^{g_{\infty}(z) - \ln |z|} = e^\gamma = \frac{1}{\text{Cap}(E)}.$$
We will use the following standard notations:

\[ \| \cdot \| \]

we can separate an orthonormal basis, for example, by the Gram–Schmidt orthogonalization of the orthogonal system \( \{ p_n(z) \}_{n=0}^{\infty} \). If the measure, \( \mu \), is completely regular, it can be chosen to have compact support matching \( \mu \). This shows the existence of a series of polynomials, \( \{ p_n(z) \}_{n=0}^{\infty} \), deg \( p_n = n \), with \( < p_n, p_m > = \int_E p_n(z) p_m(z) d\mu(z) = \delta_{n,m} \).

We consider all polynomials, \( q \in \mathbb{C}[z] \), of degree deg \( q = n \), and with leading coefficients equal to 1. The minimization problem,

\[ \min_{q_n} < q_n, q_n > = \min_{q_n} \int_E q_n(z) \overline{q}_n(z) d\mu(z) \]

has a unique solution about \( q_n \), namely \( P_n(z) = \frac{1}{\mathcal{C}} p_n(z) \), where \( p_n(z) = c_n z^n + \cdots + c_0 \) belongs to the orthogonal system \( \{ p_n(z) \}_{n=0}^{\infty} \).

We will use the following standard notations: \( ||f||_{L_2(\mu)} = \sqrt{<f,f> \text{ and } ||f||_E = \max_{z \in E} |f|} \).

**Definition 6.** We will call the measure, \( \mu \), regular if its support is a regular compact, and completely regular if it is regular with

\[ \lim_{n \to \infty} \max_{z \in E} |p_n(z)|^{\frac{1}{2}} \leq 1. \]

Since \( < p_n, p_n > = 1 \), the inequality in the definition becomes an equality:

\[ \lim_{n \to \infty} \max_{z \in E} |p_n(z)|^{\frac{1}{2}} = 1. \]

**Proof.** Theorem 9. \( \Box \)

**Example 3.** Fully regular Borel measures: ([13]) If \( R \) is a bounded region, whose boundary is a Jordan curve, the measure, \( \mu \), defined by \( d\mu(z) = w(x, y) dxdy \) is completely regular at \( w \mid R \geq 0 \) and \( \int_R \frac{dxdy}{w^{\alpha}(x,y)} < \infty \), for the fixed \( \alpha > 0 \).

Let \( p_n(z) = c_n z^n + \cdots + c_1 z + c_0 \).

**Theorem 9.** If the measure, \( \mu \), is completely regular, then

\[ \lim_{n \to \infty} \max_{z \in E} |p_n(z)|^{\frac{1}{2}} = 1 \text{ and } \lim_{n \to \infty} |c_n|^{\frac{1}{2}} = \frac{1}{\mathcal{C} \text{ap}(E)}. \]

**Proof.** ([14]). Let \( P_n(z) = \frac{1}{\mathcal{C}} p_n(z) \) and \( T_n(z, E) \) be the polynomials solving the corresponding minimization problems on the \( L_2(\mu) \) norm and Chebyshev max norm. Therefore,

\[ \frac{1}{|c_n|} \left( \int_E P_n(z) \overline{P}_n(z) d\mu(z) \right)^{\frac{1}{2}} \leq \left( \int_E T_n(z, E) \overline{T}_n(z, E) d\mu(z) \right)^{\frac{1}{2}} \leq \mu(E)^{\frac{1}{2}} ||T_n(z, E)||_E \]

\[ \Rightarrow |c_n| \geq \frac{1}{\mu(E)^{\frac{1}{2}} ||T_n(z, E)||_E} \Rightarrow \lim_{n \to \infty} |c_n|^{\frac{1}{2}} \geq \lim_{n \to \infty} \frac{1}{\mu(E)^{\frac{1}{2}} m_n^2} = \frac{1}{\mathcal{C} \text{ap}(E)}, \]

Since \( \mu \) is completely regular, we have \( ||p_n(z)||_E^{\frac{1}{2}} \leq 1 \); therefore,

\[ \frac{1}{|c_n|} ||p_n(z)||_E = ||p_n(z)||_E \geq ||T_n(z, E)||_E \Rightarrow |c_n| \leq \frac{||p_n(z)||_E}{||T_n(z, E)||_E} \]

\[ \Rightarrow \lim_{n \to \infty} |c_n|^{\frac{1}{2}} \leq \lim_{n \to \infty} \frac{||p_n(z)||_E^{\frac{1}{2}}}{m_n^2} \leq \frac{1}{\mathcal{C} \text{ap}(E)} \Rightarrow \]
\[
\limsup_n |c_n|^{\frac{1}{4}} \leq \frac{\liminf_n \|p_n(z)\|^2_F}{\text{Cap}(E)} \leq \frac{1}{\text{Cap}(E)} \leq \limsup_n |c_n|^{\frac{1}{2}} \Rightarrow \limsup n \|p_n(z)\|^2_F = 1 \geq \limsup n \|p_n(z)\|^2_F \Rightarrow \limsup n \|p_n(z)\|_E^2 = 1.
\]

The following important theorem establishes a relationship between the asymptotics of the orthogonal system, \(\{p_n(z)\}_{n=0}^{\infty}\) and the conformal map, \(\varphi(z) = e^{g(z)}\).

**Lemma 1.** The following equality holds: \(\lim_{n \to \infty} |p_n(z)|^{\frac{1}{2}} = |\varphi(z)|, \forall z \in G = \mathbb{C}_\infty \setminus E.\)

**Proof.** ([14]). \(\square\)

**Theorem 10.** For an arbitrary polynomial \(P(z)\) of degree \(n\), the following inequality holds:

\[|P(z)| \leq ||P||_E \cdot |\varphi(z)|^{n}, \quad z \in G.\]

**Proof.** Theorem 16 in Section 7. As a consequence of the orthogonal system, \(\{p_n(z)\}_{n=0}^{\infty}\) at \(z \in G\), we have the following:

\[|p_n(z)| \leq ||p_n||_E \cdot |\varphi(z)|^{n} \Rightarrow \frac{p_n(z)}{|\varphi(z)|^{n}} \leq \limsup_n \frac{|p_n(z)|^{\frac{1}{2}}}{|\varphi(z)|^{\frac{1}{2}}} \leq \limsup n \frac{|p_n(z)|^{\frac{1}{2}}}{|\varphi(z)|^{\frac{1}{2}}} = 1.\]

\(\square\)

### 6.3. Fourier Series for Analytic Functions Overconvergence

Let \(f\) be an analytic function within a regular convex compact, \(E\), and let \(\{p_n(z)\}_{n=0}^{\infty}\) be the orthogonal system of polynomials defined above. Regarding \(f\), we can write the formal orthogonal series as follows: \(\sum_{n=0}^{\infty} a_n p_n(z), z \in E,\) with partial sums \(S_n(z) = \sum_{n=0}^{m} a_n p_n(z),\) where

\[a_n = a_n(f) = \int_E f(z) \overline{p_n(z)} d\mu(z) = \int_E S_m(z) \overline{p_n(z)} d\mu(z); m \geq n.\]

We will assume that \(f\) is not an entire function and possesses an analyticity parameter, \(\rho(f) = e^\rho > 1.\)

**Lemma 2.** Under the assumptions above, the following holds: \(\limsup_{n \to \infty} |a_n|^{\frac{1}{2}} = \frac{1}{P(f)}\), and the sequence of partial sums, \(\{S_n(z)\}\), is uniformly convergent in the compact subsets of \(E\) to \(f\).

**Proof.** ([3]). \(\square\)

The series \(\sum_{n=0}^{\infty} a_n p_n(z)\) is called the Fourier series of \(f\) on the orthogonal system, \(\{p_n(z)\}_{n=0}^{\infty}\), with a radius of convergence of

\[\rho(f) = \frac{1}{\limsup_{n \to \infty} |a_n|^{\frac{1}{2}}} = e^\rho.\]

**Definition 7.** Let \(f(z) = \sum_{n=0}^{\infty} a_n p_n(z)\) be a Fourier series with a finite radius of convergence \(\rho(f) > 1\). We say that a Fourier series has the overconvergence property if there exists a subsequence, \(\{S_{n_k}(z)\}_{k=0}^{\infty}\), of the sequence of partial sums, \(\{S_n(z)\}_{n=0}^{\infty}\), which is uniformly convergent in a domain located outside \(E_f\).

**Definition 8.** We state that \(f = \sum_{n=0}^{\infty} a_n p_n(z)\) admits Hadamard–Ostrowski gaps with respect to the coefficients, \(a_n\), if there exist sequences \(\{p_k\}_{k=1}^{\infty}\) and \(\{q_k\}_{k=1}^{\infty}\), which satisfy the following conditions:

(a) \(q_{k-1} \leq p_k < q_k;\)
(b) \(\liminf_{k \to \infty} \frac{q_k}{p_k} = \lambda > 1;\) and
(c) \(\limsup_{l \to \infty} \frac{|a_l|^{\frac{1}{2}}}{P(f)} < 1\) for \(l \in \cup_{k=1}^{\infty} \{p_k, q_k\}\).

The following sections contain the information needed to state the main result of the paper: the refined and complete proof of Theorem 11.
Theorem 11. Let \( f \) be analytic in \( E \) with Fourier series, \( f(z) = \sum_{n=0}^{\infty} a_n p_n(z) \), and a radius of convergence, \( \rho(f) > 1 \), such that the sequence of coefficients, \( \{a_n\} \), has Hadamard–Ostrowski gaps. Then, in a sufficiently small neighborhood of every regular point on \( \Gamma_\rho \), the considered series possesses overconvergence.

Proof. The main idea is contained in [3]; here, we clarify the details and remove some inaccuracies. Let \( \rho(f) = \rho_0 \) and \( z_0 \in \Gamma_\rho \) be a regular point, with \( \phi(z) \) being the analytic extension of \( f \) in the neighborhood, \( U \), of \( z_0 \). We define \( F(z) = \begin{cases} f(z) & \text{for } z \in \Gamma_\rho, \\ \phi(z) & \text{for } z \in U \end{cases} \). In \( \Gamma_\rho \), the following representation holds:

\[
f(z) = \sum_{n=0}^{\infty} a_n p_n(z) = \sum_{j \in \Upsilon_{n_k} [p_n, a_k]} a_j p_j(z) + \sum_{j \in \Upsilon_{n_\rho} [p_n, a_k]} a_j p_j(z).
\]

The first series on the right-hand side of the equality has a radius of convergence \( \rho_1 > \rho_0 \), which gives a reason to consider only the series

\[
\sum_{j \in \Upsilon_{n_k} [p_n, a_k]} a_j p_j(z) = \sum_{j=0}^{p_{1_k}-1} a_j p_j(z) + \sum_{j=p_{1_k}}^{\infty} a_j p_j(z),
\]

since \( \Gamma_\rho \) is located outside \( E_\rho \) and, therefore, intersects \( U \) and the part, \( V \subset U \), outside \( E_\rho \). Thus, the considerations are reduced to the series, \( \tilde{f} = \sum_{j \in \Upsilon_{n_k} [p_n, a_k]} a_j p_j(z) \), for which we have \( S_{\rho_k}(z) = S_{\rho_k}(z) \). We keep the old notation, but by \( \tilde{f} \), we will mean \( f \).

Let us again \( \phi(z) = \mathrm{e}^{(z+i) \frac{\beta}{2}} \) fix \( \alpha = \frac{1}{2}(1 + \frac{1}{\rho_0}) < \alpha < 1 \), and we choose \( z_1 \in \Gamma_\rho \) such that \( \phi(z_1) = \alpha \phi(z_0) \) and put

\[
R_1 = |\phi(z_0) - \phi(z_1)| e^{-\frac{\pi}{\alpha}}, \quad R_2 = \sqrt{R_1 R_3}, \quad R_3 = |\phi(z_0) - \phi(z_1)| e^\alpha, \quad t > 0.
\]

We define the curves \( \gamma_i(t) = \{z : |\phi(z) - \phi(z_1)| = R_i, \ i = 1, 2, 3\} \), which are images of circles through \( \phi^{-1} \). Since \( \max_{z \in \gamma_i} |\phi(z)| = R_1 + |\phi(z_1)| = |\phi(z_0)|(\alpha + (1 - \alpha)e^{-\frac{\pi}{\alpha}}) < |\phi(z_0)| \), then \( \gamma_1(t) \) is contained in \( \Gamma_\rho \). Analogously, \( \max_{z \in \gamma_2} |\phi(z)| = |\phi(z_0)|(\alpha + (1 - \alpha)e^\alpha) > |\phi(z_0)| \), then \( \gamma_3(t) \) intersects \( U \) in the part outside \( E_\rho \). The same is true for the curve \( \gamma_2(t) \), because

\[
|\phi(z_0) - \phi(z_1)| e^\alpha > |\phi(z_0) - \phi(z_1)| \quad \text{and} \quad \max_{z \in \gamma_2} |\phi(z)| > |\phi(z_0)|.
\]

The curves \( \gamma_i(t) \) do not intersect since their images through \( \phi \) are non-intersecting circles. We can choose a small enough \( t > 0 \) so that the outermost curve \( \gamma_3(t) \) lies in \( \Gamma_\rho \cup V \).

We will use Theorem 10, and for brevity, put \( L_{\mathfrak{a}} = ||p_n(z)|| \frac{1}{\rho_0} \) and \( |d_z|^{\frac{1}{2}} \leq \frac{1}{\rho_0} \). Then, \( \lim_{n \to \infty} L_{\mathfrak{a}} = 1 \), and we can choose \( \{\eta_n\} \) such that \( \lim_{n \to \infty} \eta_n = 1 \). Therefore, \( \lim_{n \to \infty} \eta_n = 1 \). We find the following estimates:

\[
\max_{z \in \gamma_1} |\tilde{F}(z) - S_{\rho_k}(z)| = \max_{z \in \gamma_1} |\tilde{F}(z) - S_{\rho_k}(z)| = \max_{z \in \gamma_1} \sum_{n=q_k}^{\infty} a_n p_n(z) \leq \sum_{n=q_k}^{\infty} a_n p_n(z) \leq \sum_{n=q_k}^{\infty} a_n |L_{\mathfrak{a}}^n| |\phi(z)|^n \leq \max_{z \in \gamma_1} \sum_{n=q_k}^{\infty} \left( \frac{\eta_n L_{\mathfrak{a}}^n}{\rho_0} \right)^n |\phi(z)|^n = \max_{z \in \gamma_1} \sum_{n=q_k}^{\infty} \left( \frac{\eta_n \eta_n}{\rho_0} \right)^n |\phi(z)|^n \leq \sum_{n=q_k}^{\infty} \left( \eta_n |\phi(z)|(\alpha + (1 - \alpha)e^{-\frac{\pi}{\alpha}}) \right)^n \leq \sum_{n=q_k}^{\infty} \left( \eta_n |\phi(z)|(\alpha + (1 - \alpha)e^\alpha) \right)^n \leq \sum_{n=q_k}^{\infty} (\eta_n \cdot s(t))^n = \left( \eta_n (\alpha + (1 - \alpha)e^\alpha) \right)^{q_k} \left( \sum_{n=q_k}^{\infty} (\eta_n \cdot s(t))^n \right).
The function $s(t) = \alpha + (1 - \alpha)e^{-\frac{t}{t_0}}$ is decreasing at $t \in (0, \infty)$; therefore, $\forall \epsilon > 0 \exists N = N(\epsilon)$ such that for $n > N$ and $t \geq \epsilon$, $\eta_n - s(t) < 1$ holds. With $\epsilon$ fixed, we can choose $\eta_1: \eta_n < \eta_\epsilon < \frac{1}{\eta_1} < \frac{1}{s(t)}$ at all $n > N(\epsilon)$ and $t \geq \epsilon$. At $t \geq \epsilon$, we have the following:

$$\sum_{n=0}^{N(\epsilon)} (\eta_n - s(t))^n = \sum_{n=0}^{N(\epsilon)} (\eta_n - s(t))^n + \sum_{n>N(\epsilon)} (\eta_n - s(t))^n \leq \sum_{n=0}^{N(\epsilon)} (\eta_n - s(t))^n + \sum_{n>N(\epsilon)} (\eta_n - s(t))^n \leq C(\epsilon) + \sum_{n>N(\epsilon)} (\eta_n - s(t))^n = C(\epsilon) + \left(\frac{\eta_\epsilon - s(t)}{1 - \eta_\epsilon - s(t)}\right)^{N(\epsilon)} = C_1(\epsilon) \implies$$

$$\limsup_{z \to \infty} |\tilde{F}(z) - S_{p_\epsilon}(z)| \leq C_1(\epsilon) \lim_{z \to \infty} |\tilde{F}(z) - S_{p_\epsilon}(z)| = C_1(\epsilon) \left(\frac{\eta_\epsilon - s(t)}{1 - \eta_\epsilon - s(t)}\right)^{N(\epsilon)} \implies$$

$$\lim_{z \to \infty} \max_{z \in \mathbb{H}_3} |\tilde{F}(z) - S_{p_\epsilon}(z)| \leq C_1(\epsilon) \lim_{z \to \infty} |\tilde{F}(z) - S_{p_\epsilon}(z)| = C_1(\epsilon) \left(\frac{\eta_\epsilon - s(t)}{1 - \eta_\epsilon - s(t)}\right)^{N(\epsilon)} \implies$$

We again use Theorem 10 and keep the notations of $L_n$, $r_n$, and $\eta_n$. On $\gamma_3$, we have the following:

$$\max_{z \in \mathbb{H}_3} |\tilde{F}(z) - S_{p_\epsilon}(z)| \leq \max_{z \in \mathbb{H}_3} |\tilde{F}| + \max_{z \in \mathbb{H}_3} |S_{p_\epsilon}(z)| = M + \max_{z \in \mathbb{H}_3} \left| \sum_{n=0}^{p_\epsilon} a_n p_n(z) \right| \leq$$

$$\leq M + \max_{z \in \mathbb{H}_3} \left| \sum_{n=0}^{p_\epsilon} a_n |p_n(z)| \right| \leq M + \max_{z \in \mathbb{H}_3} \left| \sum_{n=0}^{p_\epsilon} \left( \frac{\eta_n}{\rho_0^2} \right)^n \cdot L_n \cdot |p_n(z)| \right| \leq M + \left( \frac{1}{\rho_0^2} \right)^n \cdot \left( \sum_{n=0}^{p_\epsilon} \frac{1}{\eta_n} \right) \cdot \left( \sum_{n=0}^{p_\epsilon} \frac{1}{\eta_n} \right) \leq$$

$$= M + \left( \frac{1}{\rho_0^2} \right)^n \cdot \left( \sum_{n=0}^{p_\epsilon} \frac{1}{\eta_n} \right) \cdot \left( \sum_{n=0}^{p_\epsilon} \frac{1}{\eta_n} \right) \leq M + \left( \frac{1}{\rho_0^2} \right)^n \cdot \left( \sum_{n=0}^{p_\epsilon} \frac{1}{\eta_n} \right) \cdot \left( \sum_{n=0}^{p_\epsilon} \frac{1}{\eta_n} \right) \leq$$

The function $k(t) = \alpha + (1 - \alpha)e^{\frac{-t}{t_0}}$ is increasing; therefore, $\forall \epsilon > 0 \exists N_1 = N_1(\epsilon)$ such that for $n > N_1$ and $t \geq \epsilon$, it holds that $\eta_n - k(t) > 1$. With $\epsilon$ fixed, we can choose $\eta_1: \eta_n > \eta_\epsilon > \frac{1}{\eta_1} > \frac{1}{k(t)}$ at all $n > N_1(\epsilon)$ and $t \geq \epsilon$. When $t \geq \epsilon$, we have the following:

$$\sum_{n=0}^{N(\epsilon)} (\eta_n - k(t))^n = \sum_{n=0}^{N(\epsilon)} (\eta_n - k(t))^n + \sum_{n>N(\epsilon)} (\eta_n - k(t))^n \leq$$

$$\leq \sum_{n=0}^{N(\epsilon)} (\eta_n - k(t))^n + \sum_{n>N(\epsilon)} (\eta_n - k(t))^n \leq C(\epsilon) + \left(\frac{\eta_\epsilon - k(t)}{1 - \eta_\epsilon - k(t)}\right)^{N(\epsilon)} = C_2(\epsilon) \implies$$

$$\limsup_{z \to \infty} |\tilde{F}(z) - S_{p_\epsilon}(z)| \leq C_2(\epsilon) \lim_{z \to \infty} |\tilde{F}(z) - S_{p_\epsilon}(z)| = C_2(\epsilon) \left(\frac{\eta_\epsilon - k(t)}{1 - \eta_\epsilon - k(t)}\right)^{N(\epsilon)} \implies$$

Let $K_i = \mathfrak{F}(\gamma_i) = \{w \in \mathbb{C} : |w - \varphi_i(z)| = R_i\}, i = 1, 2, 3$ and put $K_k(z) = \tilde{F}(z) - S_{p_\epsilon}(z)$. The function

$$R_k \circ \varphi^{-1} = R_k(\varphi^{-1}(w)) = \tilde{F} \circ \varphi^{-1} - S_{p_\epsilon} \circ \varphi^{-1} = \tilde{F}(\varphi^{-1}(w)) - S_{p_\epsilon}(\varphi^{-1}(w))$$

is analytic in the ring between circles $K_1$ and $K_3$, as well as on them. By applying Hadamard’s three-circle theorem [15] and using $\varphi^{-1}(K_i) = K_i$, we obtain the following:

$$\ln \frac{R_3}{R_1} \left( \max_{w \in K_2} \left| R_k(\varphi^{-1}(w)) \right| \right) \leq \ln \frac{R_3}{R_2} \left( \max_{w \in K_1} \left| R_k(\varphi^{-1}(w)) \right| \right) + \ln \frac{R_2}{R_3} \left( \max_{w \in K_3} \left| R_k(\varphi^{-1}(w)) \right| \right) \implies$$
Analytical functions that conformally map a region onto another region of the complex plane form a
vanish. Single-sheet analytic functions perform conformal and bijective correspondence. Writing
if
\( f \)
them in the Taylor series provides estimates for the modulus of their coefficients.

7. Auxiliary Results

\[ \lambda \left( \sum_{t=1}^{\infty} T(t) \right) \leq \lambda \left( \sum_{t=1}^{\infty} W(t) \right) + \lambda \left( \sum_{t=1}^{\infty} S(t) \right) \]

For all sufficiently large \( k \) and those \( t \geq \varepsilon \), for which \( D(t) < 0 \), we have the following:

\[ D(t) + \lambda \cdot \ln(\eta_k) + \ln(\eta_k) < 0, \text{ since } \lim_{n \to \infty} \eta_n = 1. \]

We will prove that for all sufficiently small \( t > 0 \), we have \( D(t) < 0 \). We calculate

\[ D'(t) = \frac{-\sqrt{\lambda} \cdot (1-a) e^{-\frac{t}{\lambda}}}{a + (1-a) e^{-\frac{t}{\lambda}}} + \frac{(1-a) e^{-\frac{t}{\lambda}}}{a + (1-a) e^{-\frac{t}{\lambda}}} \rightarrow (1-a)(1-\sqrt{\lambda}), \text{ for } t \to 0. \]

Therefore, \( D'(0) < 0 \), and since it is a continuous function of \( t \), there exists a neighborhood \((0, t_0)\)
in which \( D'(t) \) remains negative. Therefore, \( D(t) \) is decreasing in the interval \((0, t_0)\), but \( D(0) = 0 \Rightarrow D(t) < 0 \) at \( t \in (0, t_0) \). Therefore, for any fixed \( \varepsilon \in (0, \frac{t_0}{2}) \exists N_2(\varepsilon) : \forall k > N_2(\varepsilon) \Rightarrow \lambda \cdot \ln(\eta_k) + \ln(\eta_k) < -D(\varepsilon) \Rightarrow D(t) + \lambda \cdot \ln(\eta_k) + \ln(\eta_k) < 0 \text{ for all } t \in [\varepsilon, \frac{t_0}{2}] \text{ for all } k > N_2.

Let \( t_0' \leq \frac{t_0}{2} \): \( \eta_3(t) \in E_0 \cup V \forall t \in (0, t_0') \). Finally, if we fix \( \varepsilon \in (0, t_0') \), then for every \( t \in [\varepsilon, t_0'] \), we have the following:

\[ C_5(\varepsilon) + \frac{q_k}{\lambda} \left( D(t) + \lambda \cdot \ln(\eta_k) + \ln(\eta_k) \right) \frac{R_3}{R_2} \to -\infty \text{ for } k \to \infty \]

\[ \Rightarrow \frac{R_3}{R_1} \ln \left( \max_{z \in \gamma} |R_k(z)| \right) \to -\infty \Rightarrow \max_{z \in \gamma} |R_k(z)| \to 0. \]

We denote by \( W_{\gamma_2} \) the compact set in \( C \) with boundary \( \gamma_2 \) and let \( W \subset V \cap W_{\gamma_2} \). Since \( \{R_k(z)\}_{k \geq 1} \) is a series of analytic functions in \( W_{\gamma_2} \), then from the maximum principle, we obtain

\[ ||R_k||_W \leq ||R_k||_{W_{\gamma_2}} = ||R_k||_{\gamma_2} \to 0 \Rightarrow ||R_k||_W \to 0 \Rightarrow S_{\gamma_2}(z) \to \tilde{F}(z) \text{ in } W. \]

\[ \square \]

7. Auxiliary Results

Here, we will state the additional results that are used in the proofs of Ostrowski’s theorems. Analytical functions that conformally map a region onto another region of the complex plane form a special class of maps that preserve the angles between curves in magnitude and direction. Single-valued analytic functions realize conformal mapping at all points where their derivatives do not vanish. Single-sheet analytic functions perform conformal and bijective correspondence. Writing them in the Taylor series provides estimates for the modulus of their coefficients.

**Definition 9.** We will state that the power series with positive coefficients, \( g(z) = \sum_{n=0}^{\infty} b_n z^n \), dominates the series \( f(z) = \sum_{n=0}^{\infty} a_n z^n \), and we will write it as follows: \( f \prec g \) if \( |a_n| < b_n, \forall n \).

The following properties follow from the definition of the sum and multiplication of the power series, as follows: if \( f_i \leq g_i \), then \( \sum f_i(z) \leq \sum g_i(z) \), also \( \prod f_i(z) \leq \prod g_i(z) \). As a consequence, if \( f \leq F \) and \( g \leq G \), then \( f(g) \leq F(G) \).
Theorem 12. Let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) be analytic and single-sheeted in \(|z| < 1\), i.e., bijective and conformally mapping the open unit disk onto a region of the complex plane. Then \( |a_2| < 2 \) and \( \frac{r}{r-2} \leq |f(z)| \leq \frac{1}{1-r} \) when \(|z| < r < 1\).

Theorem 13. Let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) be analytic and single-sheeted in \(|z| < 1\), i.e., bijective and conformally mapping the open unit circle in a region of the complex plane. Then, \( |a_n| < \frac{26}{5} n^2 \) and \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \ll \frac{b}{r^{n+1}} \), where \( b \geq 7 \), and \( r > 1 \) is a positive number.

Proof. (\[9]) Applying the previous Theorem 12 and Cauchy's inequality for arbitrary \( r < 1 \), we have \( |a_n| \leq \frac{r}{r^{1/r^n}} \). Let us fix \( n \) and define \( r < 1 \) so that the value of \( \frac{r}{r^{1/r^n}} = \frac{1}{g(r)} \) is minimal. That is, we will determine the maximum of \( g(r) = r^{n-1}(1-r)^2 \), and directly find \( g'(r) = (n+1)r^{n-2}(r-1)(r-\frac{1}{r+1}) \) from where it follows that a desired value is reached at \( r = \frac{n+1}{n+2} \). Therefore, \(|a_n| < \frac{(n+1)^2}{4} \left( \frac{n+1}{n-1} \right) n^{n-1} = \frac{n^2 - 1}{4} \left( 1 + \frac{2}{n-1} \right) r^{n-2} \leq \frac{n^2}{4} n^{n-3} < \frac{n^2}{4} < \frac{26}{5} n^2 \) for \( n \geq 3 \). From Theorem 12, it follows that \( |a_2| < 2 \). For \( b \geq 7 \), we obtain \( \frac{26}{5} n^2 - b^{n-1} \implies \exists r > 1 \) such that \( \frac{26}{5} n^2 < \frac{b^{n-1}}{r^{n+1}} \). Consequently, \(|a_n| < \frac{26}{5} n^2 < \frac{b^{n-1}}{r^{n+1}} \implies f(z) \ll z + \sum_{n=2}^{\infty} \frac{b^{n-1}}{r^{n+1}} z^n \). \( \square \)

Theorem 14. If \( n \) and \( k \) are positive integers, then \( \frac{(n+k)!}{n^k} < \frac{(n+k)^{n-k}}{n^k} \).

Proof. We carry out the proof by induction on \( k \). For \( k = 1 \), we have \( \frac{(n+1)!}{n^1} > \frac{(n+1)!}{n^n} \). Suppose that for \( k = s \) it is valid that \( \frac{(n+s)!}{n^s} > \frac{(n+s)!}{n^s} \), and consider \( k = s + 1 \):

\[
\frac{(n+s+1)!}{n^{(s+1)}} = \frac{n+s+1}{s+1} \frac{(n+s)!}{n^s} > \frac{n+s+1}{s+1} \frac{(n+s)!}{n^s} > \frac{n+s+1}{s+1} \frac{(n+s)!}{n^s} > \frac{n+s+1}{s+1} \frac{(n+s)!}{n!} \frac{(n+s)!}{(n+s+1)!} > \frac{s+1}{n!} \frac{(n+1)!}{n!} > \frac{(n+1)!}{n^s} \]

with the latter being true because the sequence \( \frac{1}{n!} \frac{(n+1)!}{n^s} \) is strictly increasing. \( \square \)

Theorem 15. The coefficient of \( t^p \) in the development of \((1 + t + t^2 + \cdots)^n \) is equal to \( \left( \frac{n + p - 1}{p} \right) \).

Proof. Assume that \( |t| < 1 \) and set \( f(t) = (1 + t + t^2 + \cdots)^n \). We obtain

\[ f(t) = (1 + t + t^2 + \cdots)^n = (1 - t)^{-n} \] by calculating the coefficients of \( f(t) = \sum_{k=0}^{\infty} b_k t^k \):

\[
b_k = \frac{f^{(k)}(0)}{k!} = \frac{1}{k!} \frac{d^k}{dt^k} (1 - t)^{-n} \bigg|_{t=0} = \frac{n(n+1)\cdots(n+k-1)(1-t)^{-(n+k)}}{k!} \bigg|_{t=0} = \frac{n(n+1)\cdots(n+k-1)}{k!} \frac{(n+k-1)!}{(n-1)!} = \left( \frac{n+k-1}{n-1} \right) \left( \frac{n+k-1}{n} \right).
\] \( \square \)
Theorem 16. For an arbitrary polynomial, \( P(z) \), of degree \( n \), the following inequality holds:

\[
\max_{z \in E} |P(z)| \leq \max_{z \in E} |P(z)| \cdot e^{\rho n}.
\]

Proof. The function, \( \frac{P(z)}{e^{\rho n(z)}} \), is analytic and single-valued in \( G \), and therefore reaches its maximum on \( \partial G \). But, \( \partial G = \partial \mathbb{D} \) and \( \frac{P(z)}{e^{\rho n(z)}} \big|_{z \in \partial \mathbb{D}} = 1 \); therefore,

\[
\max_{z \in \mathbb{D}} \frac{|P(z)|}{e^{\rho n(z)}} = \max_{z \in G} \frac{|P(z)|}{e^{\rho n(z)}} = \max_{z \in \partial \mathbb{D}} \frac{|P(z)|}{e^{\rho n(z)}} = \max_{z \in \partial \mathbb{D}} |P(z)| = \max_{z \in E} |P(z)| = L \implies
\]

\[
L = \max_{z \in \mathbb{D}} \frac{|P(z)|}{e^{\rho n(z)}} = \max_{z \in \partial \mathbb{D}} \frac{|P(z)|}{e^{\rho n(z)}} \max_{z \in \partial G} \frac{|P(z)|}{e^{\rho n(z)}} = \max_{z \in \partial G} |P(z)|, \max_{z \in \partial G} \frac{|P(z)|}{e^{\rho n(z)}} \implies
\]

\[
\implies L e^{\rho n} = \max_{z \in \partial \mathbb{D}} |P(z)| \cdot e^{\rho n}, \max_{z \in \partial \mathbb{D}} |P(z)| = \max_{z \in \partial \mathbb{D}} |P(z)| \implies
\]

\[
\implies \max_{z \in \mathbb{D}} |P(z)| \leq L e^{\rho n} = \max_{z \in E} |P(z)| \cdot e^{\rho n}.
\]

Theorem 10 immediately follows from here; for example:

\[
\max_{z \in \mathbb{D}} |P(z)| \leq \max_{z \in \mathbb{D}} |P(z)| \cdot e^{\rho n} = ||P||_{E} \cdot |\phi(z)|_{E}^{n}.
\]

If \( z_0 \in G \) is an arbitrary point, distinct from \( \infty \), then there is a single level line; we denote it by \( \Gamma_{\rho r} \), which passes through \( z_0 \). Then, \( |P(z_0)| \leq ||P||_{\Gamma_{\rho r}} \leq ||P||_{E} \cdot |\phi(z)|_{E}^{n} = ||P||_{E} \cdot |\phi(z_0)|^{n}. \)

Corollary 1. If \( E \) is the disk \( |z| \leq r \), then the Green’s function for \( G \) has the form \( \ln \frac{r}{|z|} \). Therefore, \( |\phi(z)| = e^{\ln \frac{r}{|z|}} = \frac{r}{|z|} \) and the inequality from Theorem 10 takes the following form:

\[
\max_{|z|=R} |P(z)| = \max_{|z|=r} |P(z)| \leq \max_{|z|=r} |P(z)| \cdot |\phi(z)|_{|z|=R}^{n} = \max_{|z|=r} |P(z)| \left( \frac{r}{R} \right)^{n} \implies
\]

\[
\max_{|z|=R} |P(z)| \leq \max_{|z|=r} |P(z)| \left( \frac{r}{R} \right)^{n}.
\]

In particular, if \( E \) is the unit disk, then \( \phi(z) = z \), and putting \( \rho = \ln R > 0 \), the inequality takes the following form:

\[
\max_{|z| \leq R} |P(z)| = \max_{1 \leq |z| \leq R} |P(z)| \leq \max_{|z| \leq 1} |P(z)| e^{\rho n} = \max_{|z| \leq 1} |P(z)| R^{n} = LR^{n}
\]

for an arbitrary polynomial \( P(z) \) of degree \( n \).

8. Conclusions

The paper provides a brief overview of the necessary theoretical material used to prove the main results concerning the overconvergence of functional series. Ostrowski’s result on the overconvergence of power series serves as a motivating factor for obtaining the natural generalization: the overconvergence of Fourier series. The connection between Hadamard–Ostrowski gaps and the overconvergence of Fourier series is clarified by applying the Hadamard three-circle theorem and the theory of orthogonal polynomials.

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**References**


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