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Geometric Approximation of Point Interactions in Three-Dimensional Domains

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Abstract: In this paper, we study a three-dimensional second-order elliptic operator with a point interaction in an arbitrary domain. The operator is supposed to be self-adjoint. We cut out a small cavity around the center of the interaction and consider an operator in such perforated domain with the Robin condition on the boundary of the cavity. Our main result states that once the coefficient in this Robin condition is appropriately chosen, the operator in the perforated domain converges to that with the point interaction in the norm resolvent sense. We also succeed in establishing order-sharp estimates for the convergence rate.

Keywords: point interaction; small cavity; Robin condition; norm resolvent convergence; convergence rate

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1. Introduction

Operators with singular point interactions are a popular model in modern mathematical physics, which have attracted a lot of attention. They have been used to model physical systems, in which an interaction is supported in a small area [1]. While for one-dimensional operators such operators look rather simple, the two- and three-dimensional cases are more delicate. In the pioneering work [2], Berezin and Faddeev provided a method of dealing with such cases. After that, there appeared many works devoted to operators with point interactions. Here, we mention only a famous monograph [3] and refer to many references provided therein.

One of the directions of studying operators with point interactions is a corresponding perturbation theory. Namely, there is a natural question of how to approximate such operators by the ones with regular coefficients in the norm resolvent sense. A usual method is to use operators with regular coefficients and to suppose that some of these coefficients are located in a small area and are large in the area. The results of such kind are discussed in much detail in [3]; see also [4,5].

In our recent works [6,7], we suggested a completely new alternative approach to approximating two-dimensional operators with point interactions via an appropriate geometric perturbation. In [6], we considered differential operators with a fixed differential expression and the perturbation was a small cavity about the center of the interaction, which was cut out from the domain. On the boundary of the cavity, a special Robin boundary condition was imposed. The coefficient in this condition was large and depended on a small parameter, which governed the size of the cavity. Once the cavity shrank to the center of the interaction, we showed that, in the sense of the norm resolvent convergence, the perturbed operator converges to an operator with a point interaction and the latter is determined by the shape of the cavity and the coefficient in the Robin condition. However, it turned out that in this way, we could approximate not all values of the coupling constant, and the admissible values of such coupling constant should satisfy a certain upper bound. At the same time, an important feature of our result is that it was established for an operator with a general differential expression and not just for the Laplacian, which has been treated in many previous works. To the best of our knowledge, a general definition of operators with point interaction on manifolds with arbitrary differential expressions was given for the first time in a very recent work [8].

In [7], we succeeded in dealing with non-self-adjoint operators, but the boundary condition on the boundary of the cavity was non-local. Such non-locality as well as non-self-adjointness allowed us to omit the aforementioned upper bound from [6] for the admissible values of the coupling constant.

It should be noted that small cavities are a very classical example in singular perturbation theory. The case of classical boundary conditions has been studied many times. Here, we mention only some books [9–12] as well as many references therein. Typical results show a convergence of the solutions for given right hand sides and the convergence is either weak or strong in appropriate Sobolev spaces. Once the right-hand sides in a problem are smooth enough, it is also possible to construct asymptotic expansions for the solutions, and this has been performed in many situations in a series of works. We also mention some recent results on norm resolvent convergence for problems in perforated domains [13–19]. However, in all these works, the boundary conditions were not too singular and could not produce point interactions in the limit.

In this present paper, we extend the approach of [6,7] to the three-dimensional case. Namely, we consider an arbitrary second-order differential operator in an arbitrary three-dimensional domain with varying coefficients. As in [6], we suppose that this operator is self-adjoint. Then, we add a point interaction to this operator and show how to approximate it by cutting out a small cavity. On the boundary of this cavity, we, again, impose a Robin condition with an appropriately scaled coefficient. Then, we show that once the coupling constant satisfies an appropriate upper bound, the operator on the domain with the cavity approximates the operator with the point interaction in the resolvent sense. Moreover, we succeed in providing estimates for the convergence rate and show that they are order-sharp. The established norm resolvent convergence implies the convergence of the spectrum and of the associated spectral projections.

Our technique generally follows the lines of [6,7]. However, the three-dimensional case turns out to be much more difficult. The main difficulty is due to the completely different behavior of the fundamental solution of the Laplace operator in comparison with the two-dimensional case. Such difference destroys certain crucial local estimates from [6,7], and this is why, instead, we have to analyze a special Steklov problem corresponding to the considered cavity. Such analysis turns out to be an independent problem, which we solve in Section 4.1, and nothing like this is needed in the two-dimensional case.

2. Problem and Results

In the three-dimensional space $\mathbb{R}^3$, we choose an arbitrary non-empty domain, which is either bounded or unbounded, and we denote this domain by Ω. The situation in which Ω coincides with the entire space is possible. Once the boundary of the domain Ω is non-empty, we suppose that its smoothness is C^2. We use x_0 to denote an arbitrary fixed point in Ω, while ω is a bounded simply connected domain in $\mathbb{R}^3$ containing the origin; the boundary of ω is C^3-smooth. We introduce a small cavity around the point x_0 as $\omega_\varepsilon := \{x : (x - x_0)\varepsilon^{-1} \in \omega\}$, where $x = (x_1, x_2, x_3)$ are the Cartesian coordinates in $\mathbb{R}^3$ and ε is a small positive parameter.

Let $A_{ij} = A_{ij}(x)$, $A_j = A_j(x)$, and $A_0 = A_0(x)$ be real functions defined on the closure $\overline{\Omega}$ possessing the following smoothness: $A_{ij} \in C^4(\overline{\Omega})$, $A_j \in C^3(\overline{\Omega})$, $A_0 \in C^2(\overline{\Omega})$. The functions A_{ij} obey the standard ellipticity condition

$$A_{ij} = A_{ji}, \qquad \sum_{i,j=1}^{3} A_{ij}(x)\xi_i\xi_j \geqslant c_0(\xi_1^2 + \xi_2^2 + \xi_3^2)$$

for all $\xi_i \in \mathbb{R}$ and $x \in \overline{\Omega}$ with a fixed positive constant c_0 independent of x and ξ.

We consider a self-adjoint, scalar second-order differential operator $\mathcal{H}_\varepsilon$ with the differential expression

$$\hat{\mathcal{H}} := -\sum_{i,j=1}^{3} \frac{\partial}{\partial x_i} A_{ij} \frac{\partial}{\partial x_j} + \mathrm{i}\sum_{j=1}^{3}\left(A_j \frac{\partial}{\partial x_j} + \frac{\partial}{\partial x_j} A_j\right) + A_0$$

in $\Omega_\varepsilon := \Omega \setminus \overline{\omega_\varepsilon}$ subject to boundary conditions

$$\mathcal{B}u = 0 \quad \text{on} \quad \partial\Omega, \tag{1}$$

$$\frac{\partial u}{\partial \mathrm{n}} + \alpha(x,\varepsilon)u = 0 \quad \text{on} \quad \partial\omega_\varepsilon, \tag{2}$$

where

$$\alpha(x,\varepsilon) := \alpha_0(x - x_0) + \alpha_1\big((x - x_0)\varepsilon^{-1}\big), \tag{3}$$

$$\frac{\partial}{\partial \mathrm{n}} := \sum_{i,j=1}^{3} A_{ij}\nu_i \frac{\partial}{\partial x_i} - \mathrm{i}\sum_{j=1}^{3} \nu_j A_j,$$

and $\nu = (\nu_1, \nu_2, \nu_3)$ stands for the unit normal on $\partial\omega_\varepsilon$ directed inside ω_ε. $\mathcal{B}$ denotes an arbitrary boundary operator. The only restriction for this operator is that it should obey implicit assumptions, which we impose in what follows. Particular examples for the operator $\mathcal{B}$ are the ones corresponding to the Dirichlet, Neumann, Robin, or quasi-periodic boundary conditions. If $\partial\Omega$ is empty, then boundary condition (1) is not needed. The function α_0 is introduced as

$$\alpha_0(x) := -|\mathrm{A}_0^{-\frac{1}{2}}x|\nu \cdot \mathrm{A}_0 \nabla_x |\mathrm{A}_0^{-\frac{1}{2}}x|^{-1} = \frac{\nu \cdot x}{|\mathrm{A}_0^{-\frac{1}{2}}x|^2}, \tag{4}$$

where ν is the unit normal on $\partial\omega$ directed inside ω and $\mathrm{A}_0 := \mathrm{A}(0)$,

$$\mathrm{A}(x) := \begin{pmatrix} A_{11}(x) & A_{12}(x) & A_{13}(x) \\ A_{21}(x) & A_{22}(x) & A_{23}(x) \\ A_{31}(x) & A_{32}(x) & A_{33}(x) \end{pmatrix}.$$

The function $\alpha_1 = \alpha_1(s)$ is supposed to be real and continuous on $\partial\omega$, and it will be fixed later.

This paper aims to study the behavior of the resolvent of the operator $\mathcal{H}_\varepsilon$ for a small ε. Before formulating our main result, we need to introduce additional notation. $B_r(a)$ denotes the open ball of radius r centered at a point a. The definition of the cavity ω_ε implies the chain of inclusions

$$\omega_\varepsilon \subset B_{R_1\varepsilon}(x_0) \subset B_{2R_1\varepsilon}(x_0) \subset B_{R_2}(x_0) \subset B_{2R_2}(x_0) \subset \Omega_0 \subset \Omega$$

with some fixed positive constants R_1, R_2 independent of ε.

Let $\mathcal{H}_\Omega$ be the operator in $L_2(\Omega)$ with the differential expression $\hat{\mathcal{H}}$ subject to boundary condition (1); the associated sesquilinear form is denoted by $\mathfrak{h}_\Omega$. We make the following assumptions on the operator $\mathcal{H}_\Omega$ and its form $\mathfrak{h}_\Omega$, which are, in fact, implicit assumptions for the coefficients A_{ij}, A_j, and A_0 and for the boundary operator $\mathcal{B}$. The operator $\mathcal{H}_\Omega$ is self-adjoint and is lower semi-bounded, while the form $\mathfrak{h}_\Omega$ is closed and symmetric, and its domain $\mathfrak{D}(\mathfrak{h}_\Omega)$ is a subspace of $W_2^1(\Omega)$. The domain Ω contains a subdomain Ω_0 such that $x_0 \in \Omega_0$ and the restriction of each function from the domain $\mathfrak{D}(\mathcal{H}_\Omega)$ to Ω_0 belongs to $W_2^2(\Omega_0)$. The estimate

$$\mathfrak{h}_\Omega(u,u) - \mathfrak{h}_{\Omega_0}(u,u) + c_1\|u\|^2_{L_2(\Omega\setminus\Omega_0)} \geqslant c_2\|u\|^2_{W_2^1(\Omega\setminus\Omega_0)} \tag{5}$$

holds for all $u \in \mathfrak{D}(\mathfrak{h}_\Omega)$ with constants c_1, c_2 independent of u, and the constant c_2 is strictly positive. Given an arbitrary subdomain $\tilde{\Omega} \subset \Omega$ on $W_2^1(\tilde{\Omega})$, we introduce an auxiliary form:

$$\begin{aligned}\mathfrak{h}_{\tilde{\Omega}}(u,v) := \sum_{i,j=1}^{3}\left(A_{ij}\frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_i}\right)_{L_2(\tilde{\Omega})} + \mathrm{i}\sum_{j=1}^{3}\left(\frac{\partial u}{\partial x_j}, A_j v\right)_{L_2(\tilde{\Omega})} \\ - \mathrm{i}\sum_{j=1}^{3}\left(A_j u, \frac{\partial v}{\partial x_j}\right)_{L_2(\tilde{\Omega})} + (A_0 u, v)_{L_2(\tilde{\Omega})}.\end{aligned}$$

We suppose that for bounded subdomains $\tilde{\Omega}$ such that $\partial\tilde{\Omega} \cap \partial\Omega = \emptyset$, this auxiliary form satisfies the lower bound

$$\mathfrak{h}_{\tilde{\Omega}}(u,u) + c_1\|u\|^2_{L_2(\tilde{\Omega})} \geqslant c_2\|u\|^2_{W_2^1(\tilde{\Omega})} \tag{6}$$

with the constants c_1, c_2 from (5).

Rigorously, we introduce the operator $\mathcal{H}_\varepsilon$ in terms of the operator $\mathcal{H}_\Omega$ in the same way as in the two-dimensional case in [6]. Namely, we first introduce an auxiliary infinitely differentiable cut-off function χ with values in $[0,1]$ equal to the ones in $B_{2R_2}(x_0)$ and vanishing outside Ω_0. Then, $\mathcal{H}_\varepsilon$ is the operator in $L_2(\Omega_\varepsilon)$ with the differential expression $\hat{\mathcal{H}}$ on the domain $\mathfrak{D}(\mathcal{H}_\varepsilon)$, which consists of the functions u satisfying condition (2) and

$$(1-\chi)u \in \mathfrak{D}(\mathcal{H}_\Omega), \qquad \chi u \in W_2^2(\Omega_0 \setminus \omega_\varepsilon).$$

On this domain, the operator $\mathcal{H}_\varepsilon$ acts as follows:

$$\mathcal{H}_\varepsilon u := \mathcal{H}_\Omega(1-\chi)u + \hat{\mathcal{H}}\chi u.$$

It is proven in Section 3 in Lemma 3 that the boundary value problem

$$(\hat{\mathcal{H}} + c_1)G = 0 \quad \text{in} \quad \Omega\setminus\{x_0\}, \qquad \mathcal{B}G = 0 \quad \text{on} \quad \partial\Omega, \tag{7}$$

where c_1 is the constant from (5) and (6), possesses a unique solution in the space $W_2^2(\Omega \setminus B_\delta(x_0)) \cap C^2(\overline{\Omega_0} \setminus \{x_0\})$ for some $\delta > 0$ with the differentiable asymptotic at x_0:

$$\begin{aligned}&G(x) = G_{-1}(x-x_0) + G_0(x-x_0) + a_0 + O(|x-x_0|), \qquad x \to x_0, \\ &G_{-1}(x) := |\mathrm{A}_0^{-\frac{1}{2}}x|^{-1}, \\ &G_0(x) := \sum_{i,j=1}^{3} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}|\mathrm{A}_0^{-\frac{1}{2}}x| + \sum_{i,j,k=1}^{3} a_{ijk}\frac{\partial^3}{\partial x_i\partial x_j\partial x_k}|\mathrm{A}_0^{-\frac{1}{2}}x|^3 + \sum_{j=1}^{3} a_j\frac{\partial}{\partial x_j}|\mathrm{A}_0^{-\frac{1}{2}}x|,\end{aligned} \tag{8}$$

where a_{ij} are homogeneous polynomials of order 1 with real coefficients; a_{ijk}, a_0 are real constants; and a_j are complex constants. We denote

$$\begin{aligned}\beta_0 :=& \sum_{j=1}^{3} \int\limits_{\partial\omega} x_j G_{-1}(x)\nu \cdot \frac{\partial \mathrm{A}}{\partial x_j}(x_0)\nabla G_{-1}(x)\,ds + \sum_{j=1}^{3} \int\limits_{\partial\omega} G_{-1}(x)\nu \cdot \mathrm{A}_0 \nabla \operatorname{Re} G_0(x)\,ds \\ &- \sum_{j=1}^{3} \int\limits_{\partial\omega} \operatorname{Re} G_0(x)\nu \cdot \mathrm{A}_0 \nabla G_{-1}(x)\,ds.\end{aligned} \tag{9}$$

It is shown in Lemma 3 that this constant is real.

We consider an auxiliary eigenvalue problem

$$\begin{aligned}&\operatorname{div}_\xi \mathrm{A}_0 \nabla_\xi \psi = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \omega, \qquad \lambda\nu \cdot \mathrm{A}_0 \nabla_\xi \psi + \alpha_0 \psi = 0 \quad \text{on} \quad \partial\omega, \\ &\psi(\xi) = C|\mathrm{A}_0^{-\frac{1}{2}}\xi|^{-1} + O(|\mathrm{A}_0^{-\frac{1}{2}}\xi|^{-2}), \qquad \xi \to \infty,\end{aligned} \tag{10}$$

where C is some constant depending on the choice of the function ψ. We show in Section 4.1 that this problem has at most countably many eigenvalues, each of these eigenvalues is real, and the greatest eigenvalue is equal to 1 and is simple. $\varkappa$ denotes the distance from 1 to the next closest eigenvalue of problem (10).

We let

$$\beta := a_0 - \frac{1}{4\pi(\det \mathrm{A}_0)^{\frac{1}{4}}}\left(\beta_0 + \int\limits_{\partial\omega} \alpha_1(s) G_{-1}^2(x)\,ds\right) \tag{11}$$

and assume that $\beta \neq a_0$. We also impose the condition

$$\beta_0 + \int\limits_{\partial\omega} \alpha_1(s) G_{-1}^2(x)\,ds < \varkappa \|G\|^2_{L_2(\Omega)}. \tag{12}$$

$\mathcal{H}_{0,\beta}$ denotes the operator in $L_2(\Omega)$ with the differential expression $\hat{\mathcal{H}}$ and a point interaction at the point x_0. The domain of this operator and its action read as follows:

$$\mathfrak{D}(\mathcal{H}_{0,\beta}) := \Big\{u = u(x) : u(x) = v(x) + (\beta - a)^{-1} v(x_0) G(x),\ v \in \mathfrak{D}(\mathcal{H}_\Omega)\Big\} \tag{13}$$

$$\mathcal{H}_{0,\beta} u = \mathcal{H}_\Omega v s. - c_1 (\beta - a)^{-1} v(x_0) G. \tag{14}$$

Here, the constant c_1 comes from (5) and (6), $\|\cdot\|_{X\to Y}$ denotes the norm of a bounded operator acting from a Hilbert space X into a Hilbert space Y, while $\sigma(\,\cdot\,)$ stands for a spectrum of an operator.

Our main result is as follows.

Theorem 1. *The operators $\mathcal{H}_\varepsilon$ and $\mathcal{H}_{0,\beta}$ are self-adjoint and satisfy the estimates*

$$\|(\mathcal{H}_\varepsilon - \lambda)^{-1} - (\mathcal{H}_{0,\beta} - \lambda)^{-1}\|_{L_2(\Omega)\to L_2(\Omega_\varepsilon)} \leqslant C\varepsilon^{\frac{1}{2}}, \tag{15}$$

$$\|\chi_{\tilde\Omega}\big((\mathcal{H}_\varepsilon - \lambda)^{-1} - (\mathcal{H}_{0,\beta} - \lambda)^{-1}\big)\|_{L_2(\Omega)\to\mathfrak{D}(\mathfrak{h}_\Omega)} \leqslant C\varepsilon^{\frac{1}{2}}. \tag{16}$$

Here, $\tilde\Omega$ is an arbitrary fixed subdomain of Ω, the closure of which does not contain the point x_0, while $\chi_{\tilde\Omega}$ is an infinitely differentiable cut-off function equal to one on $\tilde\Omega$ and vanishing outside some larger fixed domain, the closure of which also does not contain the point x_0. The symbol C denotes positive constants independent of ε but depending on λ and additionally on the choice of $\tilde\Omega$ in (16). These estimates are order-sharp.

The convergence of the resolvents established in the above theorem implies the convergence of the spectrum and spectral projections. Such convergence can be established by a literal reproduction of the proof of Theorem 2.2 in [6]. This gives our second main result; in the following theorem, $\sigma(\,\cdot\,)$ denotes the spectrum of an operator.

Theorem 2. *The spectrum of the operator $\mathcal{H}_\varepsilon$ converges to that of $\mathcal{H}_{0,\beta}$ as $\varepsilon \to +0$. Namely, if $\lambda \notin \sigma(\mathcal{H}_{0,\beta})$, then $\lambda \notin \sigma(\mathcal{H}_\varepsilon)$ provided ε is small enough. If $\lambda \in \sigma(\mathcal{H}_{0,\beta})$; then, there exists a point $\lambda_\varepsilon \in \sigma(\mathcal{H}_\varepsilon)$ such that $\lambda_\varepsilon \to \lambda$ as $\varepsilon \to +0$. For any $\varrho_1, \varrho_2 \notin \text{œ}(\mathcal{H}_{0,\beta})$, $\varrho_1 < \varrho_2$, the spectral projection of $\mathcal{H}_\varepsilon$ corresponding to the segment $[\varrho_1, \varrho_2]$ converges to the spectral projection of $\mathcal{H}_{0,\beta}$ corresponding to the same segment in the sense of the norm $\|\cdot\|_{L_2(\Omega)\to L_2(\Omega_\varepsilon)}$.*

For each fixed segment $J := [\varrho_1, \varrho_2]$ of the real line, the inclusion

$$\sigma(\mathcal{H}_\varepsilon) \cap J \subset \left\{\lambda \in J : \operatorname{dist}(\lambda, \sigma(\mathcal{H}_{0,\beta}) \cap J) \leqslant C\varepsilon^{\frac{1}{2}}\right\}$$

holds, where C is a fixed constant independent of ε but depending on Q. If λ_0 is an isolated eigenvalue of $\mathcal{H}_{0,\beta}$ of multiplicity n, there exist exactly n eigenvalues of the operator $\mathcal{H}_\varepsilon$, counting multiplicities, which converge to λ_0 as $\varepsilon \to +0$. The total projection $\mathcal{P}_\varepsilon$ associated with these perturbed eigenvalues and the projection $\mathcal{P}_{0,\beta}$ onto the eigenspace associated with λ_0 satisfy estimates similar to (15) and (16).

Let us briefly discuss our problem and the main results. First of all, we stress that the operators we consider are rather general, namely, they have general differential expressions with variable coefficients and these coefficients can have a rather arbitrary behavior outside the domain Ω_0. Namely, once it is possible to define properly the operator $\mathcal{H}_\Omega$, our scheme works, and we can introduce the operators $\mathcal{H}_\varepsilon$ and $\mathcal{H}_{0,\beta}$. Such approach worked perfectly for two-dimensional operators in [6,7], and, here, we extend it to three-dimensional operators.

Our first main result, Theorem 1, states that a general three-dimensional operator with a point interaction can be approximated by cutting out a small hole around the center of the point interaction and by imposing a special Robin condition on its boundary. This condition is given by (1), and in view of the definition of the function α_0 in (4), we immediately see that

$$\alpha(x,\varepsilon) = \varepsilon^{-1}\alpha_0\left(\frac{x-x_0}{\varepsilon}\right) + \alpha_1\left(\frac{x-x_0}{\varepsilon}\right),$$

which means that the coefficient in this Robin condition grows as ε tends to zero. Under an appropriate choice of the function α_1, Theorem 1 states the convergence of the resolvent of $\mathcal{H}_\varepsilon$ to that of $\mathcal{H}_{0,\beta}$ in the operator norm $\|\cdot\|_{L_2(\Omega)\to L_2(\Omega_\varepsilon)}$ (see (15)). The convergence rate is $O(\varepsilon^{\frac{1}{2}})$, which is shown to be order-sharp. The second convergence expressed in estimate (16) means that once we consider the restriction of the resolvent of the operator $\mathcal{H}_\varepsilon$ to a subdomain of Ω separated from the point x_0, then the convergence also holds a stronger $\|\cdot\|_{L_2(\Omega)\to\mathfrak{D}(\mathfrak{h}_{\tilde{\Omega}})}$-norm. The mentioned subdomain is controlled by the cut-off function. We also stress that both estimates (15) and (16) are order-sharp, and in Section 6, we adduce examples proving this statement. We also note that the norm $\|(\mathcal{H}_\varepsilon-\lambda)^{-1} - (\mathcal{H}_{0,\beta}-\lambda)^{-1}\|_{L_2(\Omega)\to W_2^1(\Omega_\varepsilon)}$ does not go to zero as $\varepsilon \to 0$ since an example from Section 6 shows that we only have

$$\|(\mathcal{H}_\varepsilon-\lambda)^{-1} - (\mathcal{H}_{0,\beta}-\lambda)^{-1}\|_{L_2(\Omega)\to W_2^1(\Omega_\varepsilon)} = O(1), \qquad \varepsilon \to 0. \tag{17}$$

The constant β describing the point interaction in the operator $\mathcal{H}_{0,\beta}$ cannot take all values on the real line because of assumption (12). This condition is, in fact, an upper bound for β, and it involves the constant $\varkappa$, which is an implicit characteristic of the cavity ω. At the same time, we a priori know that $\varkappa > 0$, and to obey (12), it is sufficient to suppose that

$$\beta_0 + \int\limits_{\partial\omega} \alpha_1(s) G_{-1}^2(x)\, ds < 0.$$

A more gentle sufficient condition for (12) could be given once we have a lower bound for $\varkappa$ expressed in some geometric characteristics of the boundary $\partial\omega$. Unfortunately, we fail in trying to find such lower bound. A possible way of getting it could be based on using a nice formula for the eigenvalues of (10), which we establish in this work (see (53)), and trying to obtain an appropriate minimax principle on its base. However, we fail in trying to find an appropriate set of functions over which we can take such minimax. We also mention that in the two-dimensional case, for self-adjoint operators, we have an upper bound for admissible values of β (see [6]).

Comparing our results with the ones established in [6,7] for the two-dimensional case, we mention the following important differences. The first of them is that the convergence rates in estimates (15) and (16) are now powers of ε, while in [6,7], similar rates are powers of $|\ln\varepsilon|^{-1}$. This means that for the three-dimensional operators, our approximation is better. A deep reason explaining this situation is a difference between the fundamental solutions of the Laplace operator in two and three dimensions. Due to the same reason, we to modify quite essentially a part of our proof for the three-dimensional operator, and this is the second important difference. Namely, one of the key ingredients is a lower-semiboundedness of the form associated with the perturbed operator, and we do need an explicit lower bound for this form. In the two-dimensional case, such lower bound is based on certain local estimates similar to the ones in Lemma 5 below. In the three-dimensional case, these local estimates are not enough, and we have to analyze an auxiliary Steklov problem; see Section 4.1 below.

Once we have the resolvent convergence stated in Theorem 1, it is possible to prove the convergence of the spectrum and the associated spectral projections. This can be performed by a literal reproduction of the proof of a similar theorem from [6], and it leads us to Theorem 2. This is why we do not provide the proof of Theorem 2 in this paper.

3. Auxiliary Statements

Here, we establish several lemmas, which are important ingredients in the proof of Theorem 1.

Lemma 1. *The identities*

$$\int\limits_{\partial\omega} \frac{\alpha_0(s)}{|\mathrm{A}_0^{-\frac{1}{2}}x|}\,\mathrm{d}s = \int\limits_{\partial\omega} \frac{\nu\cdot x}{|\mathrm{A}_0^{-\frac{1}{2}}x|^3}\,ds = 4\pi\big(\det \mathrm{A}_0\big)^{\frac{1}{4}}, \tag{18}$$

$$\int\limits_{\partial\omega} \frac{\alpha_0(s)}{|\mathrm{A}_0^{-\frac{1}{2}}x|^2}\,\mathrm{d}s = -\int\limits_{\mathbb{R}^3\setminus\omega} \frac{dx}{|\mathrm{A}_0^{-\frac{1}{2}}x|^4} \tag{19}$$

hold true.

Proof. We begin with an obvious equation:

$$\operatorname{div} \mathrm{A}_0\nabla_x|\mathrm{A}_0^{-\frac{1}{2}}x|^{-1} = 0, \qquad x\in\mathbb{R}^3\setminus\{0\}.$$

We integrate by parts this equation over $\omega\setminus\{x : |x| < \delta\}$ with a sufficiently small δ:

$$\begin{aligned} 0 &= -\int\limits_{\partial\omega} \nu\cdot \mathrm{A}_0\nabla_x|\mathrm{A}_0^{-\frac{1}{2}}x|^{-1}\,ds + \int\limits_{\{x:\,|x|=\delta\}} \frac{x}{|x|}\cdot \mathrm{A}_0\nabla_x|\mathrm{A}_0^{-\frac{1}{2}}x|^{-1}\,ds \\ &= \int\limits_{\partial\omega} \frac{\nu\cdot x}{|\mathrm{A}_0^{-\frac{1}{2}}x|^3}\,ds - \int\limits_{\{x:\,|x|=\delta\}} \frac{|x|}{|\mathrm{A}_0^{-\frac{1}{2}}x|^3}\,ds = \int\limits_{\partial\omega} \frac{\nu\cdot x}{|\mathrm{A}_0^{-\frac{1}{2}}x|^3}\,ds - \int\limits_{\{x:\,|x|=1\}} \frac{ds}{|\mathrm{A}_0^{-\frac{1}{2}}x|^3}. \end{aligned} \tag{20}$$

Since the matrix A_0 is positive definite and Hermitian, there exists an orthogonal matrix reducing A_0 to its diagonal form. Performing the change of the variables with this matrix in the latter integral, then passing to the spherical coordinates, and denoting with Λ_j the eigenvalues of the matrix $\mathrm{A}_0^{-\frac{1}{2}}$, we obtain the following:

$$
\begin{aligned}
\int\limits_{\{x:\,|x|=1\}} \frac{1}{|A_0^{-\frac{1}{2}}x|^3}\,ds &= \int\limits_{\{x:\,|x|=1\}} \left(\sum_{j=1}^{3}\Lambda_j x_j^2\right)^{-\frac{3}{2}} ds \\
&= \int\limits_0^{2\pi} d\phi \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos\vartheta\, d\vartheta}{\big((\Lambda_1\cos^2\phi+\Lambda_2\sin^2\phi)\cos^2\vartheta+\Lambda_3\sin^2\vartheta\big)^{\frac{3}{2}}} \\
&= \int\limits_0^{2\pi} \frac{\sin\vartheta}{(\Lambda_1\cos^2\phi+\Lambda_2\sin^2\phi)\big((\Lambda_1\cos^2\phi+\Lambda_2\sin^2\phi)\cos^2\vartheta+\Lambda_3\sin^2\vartheta\big)^{\frac{1}{2}}}\Bigg|_{\vartheta=-\frac{\pi}{2}}^{\vartheta=\frac{\pi}{2}} d\phi \\
&= \frac{2}{\sqrt{\Lambda_3}} \int\limits_0^{2\pi} \frac{d\phi}{\Lambda_1\cos^2\phi+\Lambda_2\sin^2\phi} = \frac{8}{\sqrt{\Lambda_3}} \int\limits_0^{\frac{\pi}{2}} \frac{d\phi}{\Lambda_1\cos^2\phi+\Lambda_2\sin^2\phi} \\
&= \frac{8}{\sqrt{\Lambda_1\Lambda_2\Lambda_3}} \arctan\frac{\sqrt{\Lambda_2}}{\sqrt{\Lambda_1}}\tan\phi\Bigg|_{\phi=0}^{\phi=\frac{\pi}{2}} = 4\pi\big(\det A_0\big)^{\frac{1}{4}}.
\end{aligned}
$$

This formula and (20) imply an identity (18).

Similar to the above calculations, we integrate by parts as follows:

$$
\begin{aligned}
0 &= \lim_{R\to+\infty} \int\limits_{B_R(0)\setminus\omega} |A_0^{-\frac{1}{2}}x|^{-1}\,\mathrm{div}_x\, A_0|A_0^{-\frac{1}{2}}x|^{-1}\,dx \\
&= \int\limits_{\partial\omega} |A_0^{-\frac{1}{2}}x|^{-1}\nu\cdot A_0\nabla_x|A_0^{-\frac{1}{2}}x|^{-1}\,ds - \int\limits_{\mathbb{R}^3\setminus\omega} A_0\nabla_x|A_0^{-\frac{1}{2}}x|^{-1}\cdot\nabla_x|A_0^{-\frac{1}{2}}x|^{-1}\,dx \\
&= -\int\limits_{\partial\omega} \frac{\alpha_0(s)}{|A_0^{-\frac{1}{2}}x|^2}\,\mathrm{ds} - \int\limits_{\mathbb{R}^3\setminus\omega} \frac{dx}{|A_0^{-\frac{1}{2}}x|^4},
\end{aligned}
$$

and this proves (19). The proof is complete. □

With $\mathbb{Z}_+$ we denote the set of non-negative integral numbers, that is, $\mathbb{Z}_+ := \mathbb{N}\cup\{0\}$.

Lemma 2. *For each $m\in\mathbb{Z}_+$, each polynomial $P=P(x)$, and each multi-index $\gamma\in\mathbb{Z}_+^3$, the equation*

$$
\sum_{i,j=1}^{3} A_{ij}(x_0)\frac{\partial^2 u(x)}{\partial x_i\partial x_j} = P(x-x_0)\frac{\partial^\gamma}{\partial x^\gamma}|A_0^{-\frac{1}{2}}(x-x_0)|^{2m-1}, \qquad x\in\mathbb{R}^3\setminus\{x_0\},
$$

possesses a solution of the form

$$
u(x) = \sum_{\substack{\theta\in\mathbb{Z}_+^3\\ |\theta|\leqslant\deg P}} Q_\theta(x-x_0)\frac{\partial^{\gamma+\theta}}{\partial x^{\gamma+\theta}}|A_0^{-\frac{1}{2}}(x-x_0)|^{2m+1+2|\theta|}, \tag{21}
$$

where Q_θ are some polynomials with degrees obeying the inequality

$$
\deg Q_\theta \leqslant \deg P - |\theta|.
$$

Proof. It is sufficient to study only the case when A_0 coincides with the unit matrix E and $x_0=0$, since the general case is reduced to the one mentioned by the linear change $y=A_0^{-\frac{1}{2}}(x-x_0)$. This is why we provide only the proof of the particular mentioned case.

We prove the lemma by induction in the degree of the polynomial P. We first consider the case $\deg P = 0$, that is, P is a constant. Then, it is straightforward to confirm that the equation

$$\Delta u = P\frac{\partial^\gamma}{\partial x^\gamma}|x|^{2m-1}, \qquad x \in \mathbb{R}^3 \setminus \{0\}, \tag{22}$$

possesses a solution

$$u(x) = \frac{P}{(2m+1)(2m+2)}\frac{\partial^\gamma}{\partial x^\gamma}|x|^{2m+1}.$$

Suppose that Equation (22) possesses a solution of the form in (21), with $\mathrm{A}_0 = \mathrm{E}$ and $x_0 = 0$ for all γ and all polynomials P with $\deg P \leqslant k$ for some $k \in \mathbb{Z}_+$. We then consider Equation (22) with a polynomial P such that $\deg P = k+1$ and seek its solution as

$$u(x) = \frac{P(x)}{(2m+1)(2m+2)}\frac{\partial^\gamma}{\partial x^\gamma}|x|^{2m+1} - \tilde{u}(x) \tag{23}$$

and for $\tilde{u}$, we then obtain the equation

$$\Delta u = \frac{2}{(2m+1)(2m+2)}\sum_{i=1}^{3}\frac{\partial P}{\partial x_i}\frac{\partial}{\partial x_i}\frac{\partial^\gamma}{\partial x^\gamma}|x|^{2m+1} + \Delta P\frac{\partial^\gamma}{\partial x^\gamma}|x|^{2m+1}.$$

The degrees of the polynomials $\frac{\partial P}{\partial x_i}$ and ΔP are at most $\deg P - 1$ and $\deg P - 2$, respectively, and by the induction assumption, the above equation possesses a solution of the form in (21), namely,

$$\tilde{u} = \sum_{\substack{\theta\in\mathbb{Z}_+^3 \\ |\theta|\leqslant k}} Q_\theta(x)\frac{\partial^{\gamma+\theta}}{\partial x^{\gamma+\theta}}|x|^{2m+3+2|\theta|}$$

with some polynomials Q_θ of degrees $\deg Q_\theta \leqslant k - |\theta|$. Substituting this formula into (23), we arrive at (21) for $\deg P = k+1$. The proof is complete. □

Estimates (5) and (6) show that the spectrum of the self-adjoint operator $\mathcal{H}_\Omega$ is a subset of the half-line $[c_2 - c_1, \infty)$. Then, the positivity of the constant c_2 implies that the resolvent $(\mathcal{H}_\Omega + c_1)^{-1}$ is well-defined.

We introduce an auxiliary sesquilinear form

$$\begin{aligned}\mathfrak{g}_\varepsilon(u,v) :=& \mathfrak{h}_\Omega\big((1-\chi)u,(1-\chi)v\big) + \mathfrak{h}_{\Omega_0\setminus\omega_\varepsilon}\big(\chi u,(1-\chi)v\big) \\ &+ \mathfrak{h}_{\Omega_0\setminus\omega_\varepsilon}\big((1-\chi)u,\chi v\big) + \mathfrak{h}_{\Omega_0\setminus\omega_\varepsilon}(\chi u,\chi v)\end{aligned} \tag{24}$$

on the domain

$$\mathfrak{D}(\mathfrak{g}_\varepsilon) := \Big\{u:\ (1-\chi)u \in \mathfrak{D}(\mathfrak{h}_\Omega),\ \chi u \in W_2^1(\Omega_0 \setminus \omega_\varepsilon)\Big\}. \tag{25}$$

It is clear that this form is symmetric.

Lemma 3. *The boundary value problem (7) possesses a unique solution in $W_2^2(\Omega \setminus B_{2R_2}(x_0)) \cap C^2(\overline{\Omega_0} \setminus \{x_0\})$ with a differentiable asymptotic (8). The identity*

$$\left(\frac{\partial G}{\partial \mathrm{n}} + \alpha_0 G, G\right)_{L_2(\partial\omega_\varepsilon)} = \beta_0 + O(\varepsilon) \tag{26}$$

holds true, where β_0 is introduced in (9), and this constant is real.

Proof. We expand the coefficients A_{ij}, A_j, and A_0 of the differential expression $\hat{\mathcal{H}}$ by the Taylor formula about the point x_0, and using Lemma 2, we see that there exists a function $G_1(x)$ of the form

$$G_1(x) = G_0(x) + \sum_{j,\theta} P_{j,\theta}(x) \frac{\partial^\theta}{\partial x^\theta} |\mathrm{A}_0^{-\frac{1}{2}}(x)|^j, \tag{27}$$

where the sum is finite and is taken over $j \in \mathbb{Z}_+$ and $\theta \in \mathbb{Z}_+^3$, and $P_{j,\theta}$ are some polynomials, where

$$\sum_{j,\theta} P_{j,\theta}(x) \frac{\partial^\theta}{\partial x^\theta} |\mathrm{A}_0^{-\frac{1}{2}}(x)|^j = O(1), \qquad x \to 0, \tag{28}$$

such that

$$(\hat{\mathcal{H}} + c_1)(G_{-1}(x - x_0) + G_1(x - x_0)) = F_0(x),$$

where F_0 is continuous; the Lipschitz in $\overline{B_{2R_2}(x_0)}$ is infinitely differentiable in $B_{2R_2}(x_0) \setminus \{x_0\}$, and

$$F_0(x) = O(|x - x_0|), \qquad x \to x_0.$$

We seek the solution to the boundary value problem (7), (24) as

$$G(x) = G_2(x) + G_3(x), \qquad G_2(x) := \big(G_{-1}(x - x_0) + G_1(x - x_0)\big)\chi(x), \tag{29}$$

and the unknown function G_2 should solve the equation

$$(\mathcal{H}_\Omega + c_1)G_3 = F_2, \qquad F_2 := -\chi F_0 + F_1, \tag{30}$$

and F_1 is a certain polynomial expression of the derivatives of G_0 and χ up to the second order. This yields $F_2 \in L_2(\Omega) \cap C^\gamma(\overline{\Omega_0})$ for each $\gamma \in (0, 1)$.

Since the point $-c_1$ is outside the resolvent set of the operator $\mathcal{H}_\Omega$, Equation (30) is uniquely solvable in $\mathfrak{D}(\mathcal{H}_\Omega)$. The standard Schauder estimates [20] imply that this solution belongs to $C^{2+\gamma}(\overline{\Omega_0})$. Therefore, the function G_3 satisfies the Taylor formula

$$G_3(x) = a_0 + O(|x - x_0|), \qquad x \to x_0,$$

with some constant a_0. Now, recovering the function G by Formula (29), we conclude that problem (7), (24) is uniquely solvable in $W_2^2(\Omega \setminus B_{2R_2}(x_0)) \cap C^2(\overline{\Omega_0} \setminus \{x_0\})$, and the solution satisfies asymptotics (8).

We confirm that the constant a_0 is real. We proceed as in the proof of Lemma 3.2 in [6], namely, as in Equations (3.9)–(3.12) in [6], and we obtain the following:

$$\mathfrak{h}_\Omega(G_3, G_3) + c_1\|G_3\|^2_{L_2(\Omega)} - \big((\hat{\mathcal{H}} + c_1)G_2, G_2\big)_{L_2(\Omega)} = -\big((\hat{\mathcal{H}} + c_1)G_2, G\big)_{L_2(\Omega)}$$

and, denoting $\Omega^\delta := \Omega \setminus B_\delta(x_0)$,

$$\begin{aligned} &((\mathcal{H}_\Omega + c_1)G_2, G_2)_{L_2(\Omega)} - \big((\hat{\mathcal{H}} + c_1)G_2, G\big)_{L_2(\Omega)} \\ &= \lim_{\delta \to +0} \left(\sum_{i,j=1}^{3} \left(A_{ij} \frac{\partial G_2}{\partial x_j}, \frac{\partial G_2}{\partial x_i} \right)_{L_2(\Omega^\delta)} - 2\,\mathrm{Im} \sum_{j=1}^{2} \left(A_j \frac{\partial G_2}{\partial x_j}, G_2 \right)_{L_2(\Omega^\delta)} \right. \\ &\quad \left. + ((A_0 + c_1)G_2, G_2)_{L_2(\Omega^\delta)} - \int\limits_{\partial B_\delta(x_0)} G \overline{\frac{\partial G_2}{\partial \mathrm{n}}}\, ds \right). \end{aligned} \tag{31}$$

We let

$$b_\delta(x) := \sum_{j=1}^{3} \nu_j A_j(x_0), \qquad x \in \partial B_\delta(x_0),$$
$$G_4(x) := G_{-1}(x-x_0) + \operatorname{Re} G_0(x-x_0), \qquad G_5(x) := \operatorname{Im} G_0(x-x_0),$$

where $x_{0,i}$ are the coordinates of the point x_0, and we observe that the functions b_δ and G_5 are odd with respect to each of the variables $x_i - x_{0,i}$, $i = 1,2,3$. Then, it follows from asymptotics (8) and Formulas (27)–(29) that, as $\delta \to +0$,

$$\begin{aligned}
\int\limits_{\partial B_\delta(x_0)} G \overline{\frac{\partial G_2}{\partial \mathrm{n}}}\, ds = & \int\limits_{\partial B_\delta(x_0)} (G_2 + a_0) \overline{\frac{\partial G_2}{\partial \mathrm{n}}}\, ds \\
= & -\frac{1}{\delta} \int\limits_{\partial B_\delta(x_0)} (G_2 + a_0) \sum_{i,j=1}^{3} A_{ij}(x_i - x_{i,0}) \frac{\partial \overline{G_2}}{\partial x_i}\, ds \\
& - \mathrm{i} \int\limits_{\partial B_\delta(x_0)} b_\delta |\mathrm{A}_0^{-\frac{1}{2}}(x-x_0)|^{-2}\, ds + o(1) \\
= & -\frac{a_0}{\delta} \int\limits_{\partial B_\delta(x_0)} \sum_{i,j=1}^{3} A_{ij}(x_0)(x_i - x_{i,0}) \frac{\partial}{\partial x_i} G_{-1}(x-x_0)\, ds \\
& - \frac{1}{\delta} \int\limits_{\partial B_\delta(x_0)} G_4 \sum_{i,j=1}^{3} A_{ij}(x)(x_i - x_{i,0}) \frac{\partial}{\partial x_i} G_{-1}(x-x_0)\, ds \\
& - \frac{\mathrm{i}}{\delta} \int\limits_{\partial B_\delta(x_0)} G_5 \sum_{i,j=1}^{3} A_{ij}(x_0)(x_i - x_{i,0}) \frac{\partial}{\partial x_i} G_{-1}(x-x_0)\, ds + o(1) \\
= & 4\pi a_0 \big(\det \mathrm{A}_0\big)^{\frac{1}{4}} - \frac{1}{\delta} \int\limits_{\partial B_\delta(x_0)} G_4 \sum_{i,j=1}^{3} A_{ij}(x)(x_i - x_{i,0}) \frac{\partial}{\partial x_i} G_{-1}(x-x_0)\, ds + o(1).
\end{aligned}$$

We substitute this identity into (31), and this leads us to a formula for a_0, which implies that this constant is real.

We proceed to proving (26). We first represent the function G as $G = \chi G + (1-\chi)G$, and we see that the function $(1-\chi)G$ is the solution of the equation

$$(\mathcal{H}_\Omega + c_1)(1-\chi)G = -(\hat{\mathcal{H}} + c_1)\chi G. \tag{32}$$

Hence,

$$\begin{aligned}
\mathfrak{h}_\Omega((1-\chi)G, (1-\chi)G) + c_1 \|(1-\chi)G\|^2_{L_2(\Omega)} = & -\big((\hat{\mathcal{H}} + c_1)\chi G, (1-\chi)G\big)_{L_2(\Omega_0)} \\
= & -\mathfrak{h}_{\Omega_0 \setminus \omega_\varepsilon}\big(\chi G, (1-\chi)G\big)_{L_2(\Omega_0)} \\
& - c_1\big(\chi G, (1-\chi)G\big)_{L_2(\Omega_0)}.
\end{aligned} \tag{33}$$

We then consider Equation (32) pointwise in Ω_ε, multiply it by χG in $L_2(\Omega_0)$, and integrate it once by parts. This gives the following:

$$\begin{aligned}
&\big((\mathcal{H}_\Omega + c_1)(1-\chi)G, \chi G\big)_{L_2(\Omega_0 \setminus \omega_\varepsilon)} = -\big((\hat{\mathcal{H}} + c_1)\chi G, \chi G\big)_{L_2(\Omega_0 \setminus \omega_\varepsilon)}, \\
&\mathfrak{h}_{\Omega_0 \setminus \omega_\varepsilon}((1-\chi)G, \chi G) = -\mathfrak{h}_{\Omega_0 \setminus \omega_\varepsilon}(\chi G, \chi G)_{L_2(\Omega_0 \setminus \omega_\varepsilon)} - \left(\frac{\partial G}{\partial \mathrm{n}}, G\right)_{L_2(\partial \omega_\varepsilon)}.
\end{aligned} \tag{34}$$

Summing this identity with (33) and taking into consideration definition (24) of the form $\mathfrak{g}_\varepsilon$, we find that

$$\mathfrak{g}_\varepsilon(G,G)+\|G\|^2_{L_2(\Omega_\varepsilon)}=\left(\frac{\partial G}{\partial \mathrm{n}},G\right)_{L_2(\partial\omega_\varepsilon)}. \tag{35}$$

In view of asymptotics (8) and definition (4) of the function α_0, we then see that

$$\mathfrak{g}_\varepsilon(G,G)+\|G\|^2_{L_2(\Omega_\varepsilon)}+(\alpha_0 G,G)_{L_2(\partial\omega_\varepsilon)}=\left(\frac{\partial G}{\partial \mathrm{n}}+\alpha_0 G,G\right)_{L_2(\partial\omega_\varepsilon)}=\tilde\beta_0+O(\varepsilon),$$

where

$$\begin{aligned}\tilde\beta_0:=&\sum_{j=1}^{3}\int\limits_{\partial\omega} x_j G_{-1}(x)\nu\cdot\frac{\partial \mathrm{A}}{\partial x_j}(x_0)\nabla G_{-1}(x)\,ds+\sum_{j=1}^{3}\int\limits_{\partial\omega} G_{-1}(x)\nu\cdot \mathrm{A}_0\nabla G_0(x)\,ds\\&-\sum_{j=1}^{3}\int\limits_{\partial\omega}\overline{G_0(x)}\nu\cdot\mathrm{A}_0\nabla G_{-1}(x)\,ds-\mathrm{i}\sum_{j=1}^{3}\nu_j A_j(x_0)\int\limits_{\partial\omega}G_{-1}^2(x)\,ds.\end{aligned}$$

Since the initialexpression in the above formulas is real due to Formula (35), the same is true for the constant $\tilde\beta_0$, and identity (26) holds true. Moreover, since the constant $\tilde\beta_0$ is real, we immediately see that $\tilde\beta_0=\operatorname{Re}\tilde\beta_0=\beta_0$, and this completes the proof. □

We let $\Pi_\varepsilon:=B_{2R_2}(x_0)\setminus\omega_\varepsilon$.

Lemma 4. *These estimates hold:*

$$\|v\|^2_{L_2(\partial\omega_\varepsilon)}\leqslant C\varepsilon\|v\|^2_{W_2^1(\Pi_\varepsilon)},\qquad v\in W_2^1(\Pi_\varepsilon), \tag{36}$$

$$\|v\|^2_{L_2(\omega_\varepsilon)}\leqslant C\varepsilon^2\|v\|^2_{W_2^1(B_{2R_2}(x_0))},\qquad v\in W_2^1(B_{2R_2}(x_0)), \tag{37}$$

where C is a fixed constant independent of ε and v.

Estimates (36) and (37) are proven in Lemmas 2.1 and 2.2 in [14]. Using these estimates and reproducing the proof of Lemmas 3.4 and 3.5 in [6] with obvious minor changes, we arrive at the following statement.

Lemma 5. *For all $v\in W_2^1(\Pi_\varepsilon)$ satisfying the condition*

$$\int\limits_{\partial\omega_\varepsilon} v\,ds=0 \tag{38}$$

the inequality

$$\|v\|^2_{L_2(\partial\omega_\varepsilon)}\leqslant C\varepsilon\|\nabla v\|^2_{L_2(\Pi_\varepsilon)} \tag{39}$$

holds, where C is a constant independent of ε and v. If, in addition, the function v is defined on the entire ball $B_{2R_2}(x_0)$ and $v\in W_2^2(B_{2R_2}(x_0))$, then

$$\|v\|^2_{L_2(\partial\omega_\varepsilon)}\leqslant C\varepsilon^3\|v\|^2_{W_2^2(B_{2R_2}(x_0))}, \tag{40}$$

where C is a constant independent of ε and v.

For all $\varphi\in C^1(\partial\omega)$ and all $v\in W_2^2(B_{2R_2}(x_0))$, the inequality

$$\left|\varepsilon^{-2}\int\limits_{\partial\omega_\varepsilon}\varphi\left(\frac{x-x_0}{\varepsilon}\right)v(x)\,\mathrm{d}s-c(\varphi)v(x_0)\right|\leqslant C\varepsilon^{\frac12}\|v\|_{W_2^2(B_{2R_2}(x_0))},\quad c(\varphi):=\int\limits_{\partial\omega}\varphi(x)\,\mathrm{d}s, \tag{41}$$

holds true, where C is a constant independent of ε and v.

Proof. For each function $v \in W_2^1(\Pi_\varepsilon)$, we let $\tilde{v}(\xi) := v(x_0 + \varepsilon\xi)$. The latter function is an element of $W_2^1(B_{R_1}(0) \setminus \omega)$, and

$$\int\limits_{\partial\omega} \tilde{v}\, ds = 0.$$

Hence,

$$\|\tilde{v}\|^2_{L_2(\partial\omega)} \leqslant C\|\nabla_\xi \tilde{v}\|^2_{L_2(B_{R_1}(0)\setminus\omega)},$$

where C is a fixed constant independent of $\tilde{v}$. Rewriting the obtained inequality in terms of the function v, we obtain

$$\|v\|^2_{L_2(\partial\omega_\varepsilon)} \leqslant C\varepsilon\|\nabla v\|^2_{L_2(B_{R_1\varepsilon}(x_0)\setminus\omega_\varepsilon)}, \tag{42}$$

where C is a constant independent of ε and v. This proves (39). If, in addition, $v \in W_2^2(B_{2R_2}(x_0))$, then we apply estimate (37) with v replaced by its derivatives and ω_ε replaced by $B_{R_1\varepsilon}(x_0)$ to the right hand side of (42), and this leads us to (40).

We proceed to proving (41). The boundary value problem

$$\Delta_\xi Y = 0 \quad \text{in} \quad \omega \setminus \{0\}, \qquad \frac{\partial Y}{\partial \nu} = \varphi \quad \text{on} \quad \partial\omega, \qquad Y(\xi) = -\frac{c(\varphi)}{4\pi|\xi|} + O(1)$$

is solvable and possesses a unique solution, such that

$$\int\limits_{\omega} Y(\xi)\, d\xi = 0.$$

By the standardSchauder estimates, the function $Y + \frac{c(\varphi)}{4\pi|\cdot|}$ belongs to $C^{2+\gamma}(\overline{\omega})$ for each $\gamma \in (0,1)$.

Let $v \in C^2(B_{2R_2}(x_0))$, then the function $\tilde{v}(\xi) := v(x_0 + \varepsilon\xi)$ is an element of $C^2(\overline{\omega})$. Using the above definition of the function and integrating by parts, we easily find that

$$0 = \lim_{r\to+0} \int\limits_{\omega\setminus B_r(0)} \tilde{v}\Delta_\xi Y\, ds = -\int\limits_{\partial\omega} \tilde{v}\varphi\, ds + c(\varphi)\tilde{v}(0) + \int\limits_{\partial\omega} Y\frac{\partial\tilde{v}}{\partial\nu}\, ds + \int\limits_{\omega} Y\Delta_\xi\tilde{v}\, ds.$$

Returning back tothe function v, we obtain

$$\varepsilon^{-2}\int\limits_{\partial\omega_\varepsilon} v\varphi\left(\frac{\cdot - x_0}{\varepsilon}\right) ds - c(\varphi)v(x_0) = \varepsilon^{-1}\int\limits_{\partial\omega_\varepsilon} Y\left(\frac{\cdot - x_0}{\varepsilon}\right)\frac{\partial v}{\partial\nu}\, ds + \varepsilon^{-1}\int\limits_{\omega} Y\left(\frac{\cdot - x_0}{\varepsilon}\right)\Delta v\, ds.$$

Using the aforementionedsmoothness of the function Y and estimating the right-hand side of the obtained identity, in view of (36), we obtain

$$\begin{aligned}\left|\varepsilon^{-2}\int\limits_{\partial\omega_\varepsilon} v\varphi\left(\frac{\cdot - x_0}{\varepsilon}\right) ds - c(\varphi)v(x_0)\right| \leqslant & C\varepsilon^{-1}\left\|\frac{\partial v}{\partial\nu}\right\|_{L_2(\partial\omega_\varepsilon)}\left\|Y\left(\frac{\cdot - x_0}{\varepsilon}\right)\right\|_{L_2(\partial\omega_\varepsilon)} \\ & + C\varepsilon^{-1}\|v\|_{W_2^2(\omega_\varepsilon)}\left\|Y\left(\frac{\cdot - x_0}{\varepsilon}\right)\right\|_{L_2(\omega_\varepsilon)} \\ \leqslant & C\varepsilon^{\frac{1}{2}}\|v\|_{W_2^2(B_{2R_2}(x_0))},\end{aligned}$$

where the Cs are some constants independent of ε and v. Since the space $C^2(\overline{B_{2R_2}(x_0)})$ is dense in $W_2^2(B_{2R_2}(x_0))$, the above estimate also holds for all $v \in W_2^2(B_{2R_2}(x_0))$, and we arrive at (41). The proof is complete. □

4. Lower Semi-Boundedness and Self-Adjointness

In this section, we establish the self-adjointness of the operators $\mathcal{H}_\varepsilon$ and $\mathcal{H}_{0,\beta}$. In addition, we show that the operator $\mathcal{H}_\varepsilon$ is lower semi-bounded, and this is a key ingredient in proving estimates (15) and (16).

We introduce a sesquilinear form

$$\mathfrak{h}_\varepsilon(u,v) := \mathfrak{g}_\varepsilon(u,v) + (\alpha u, v)_{L_2(\partial\omega_\varepsilon)} \tag{43}$$

on the domain $\mathfrak{D}(\mathfrak{h}_\varepsilon) := \mathfrak{D}(\mathfrak{g}_\varepsilon)$, and we recall that the form $\mathfrak{g}_\varepsilon$ and its domain are introduced in (24) and (25). The form $\mathfrak{h}_\varepsilon$ is symmetric. Literally reproducing Equations (4.4)–(4.7) from [6], we see that the form $\mathfrak{h}_\varepsilon$ is associated with the operator $\mathcal{H}_\varepsilon$. Proceeding, then, as in inequalities (4.16)–(4.18) from [6], we also obtain

$$\mathfrak{g}_\varepsilon(u,u) + c_1\|u\|^2_{L_2(\Omega_\varepsilon)} \geqslant c_2\|u\|^2_{W_2^1(\Omega_\varepsilon)} \tag{44}$$

for all $u \in \mathfrak{D}(\mathfrak{g}_\varepsilon)$.

The proof of the self-adjointness of the operator $\mathcal{H}_\varepsilon$ is based on the lower semi-boundedness of its form $\mathfrak{h}_\varepsilon$. In order to prove the latter, we need to study an auxiliary operator similar to a Neumann-to-Dirichlet map and an associated Steklov problem.

4.1. Auxiliary Operator

We first establish the closedness of the form $\mathfrak{g}_\varepsilon$.

Lemma 6. *The form $\mathfrak{g}_\varepsilon$ is closed.*

Proof. We recall that, by our assumptions, the form $\mathfrak{h}_\Omega$ is closed. Let $u_n \in \mathfrak{D}(\mathfrak{g}_\varepsilon)$ be a sequence such that $\mathfrak{g}_\varepsilon(u_n - u_m) \to 0$ as $n, m \to +\infty$ and $u_n \to u$ in $L_2(\Omega_\varepsilon)$. Then, by inequality (44), we immediately conclude that u is an element of $W_2^1(\Omega_\varepsilon)$ and $u_n \to u$ in the norm of this space. Hence,

$$\begin{aligned}\mathfrak{h}_{\Omega_0\setminus\omega_\varepsilon}\big(\chi(u_n-u),(1-\chi)(u_n-u)\big) + \mathfrak{h}_{\Omega_0\setminus\omega_\varepsilon}\big((1-\chi)(u_n-u),\chi(u_n-u)\big)&\\ + \mathfrak{h}_{\Omega_0\setminus\omega_\varepsilon}\big(\chi(u_n-u),\chi(u_n-u)\big) \to 0,& \qquad n\to+\infty,\end{aligned}$$

and, therefore, by definition (43) of the form $\mathfrak{g}_\varepsilon$, we see that

$$\mathfrak{h}_\Omega\big((1-\chi)(u_n-u),(1-\chi)(u_n-u)\big) \to 0 \quad \text{as} \quad n\to+\infty.$$

The closedness of the form $\mathfrak{h}_\Omega$ then implies that $(1-\chi)u \in \mathfrak{D}(\mathfrak{g}_\varepsilon)$, and by the definition of the cut-off function, we conclude that $u = (1-\chi)u + \chi u \in \mathfrak{D}(\mathfrak{g}_\varepsilon)$ and $\mathfrak{h}_\Omega(u_n - u, u_n - u) \to 0$ as $n \to +\infty$. The proof is complete. □

We equip the linear space $\mathfrak{D}(\mathfrak{g}_\varepsilon)$ with the scalar product

$$(\,\cdot\,,\,\cdot\,)_{\mathfrak{g}_\varepsilon} := \mathfrak{g}_\varepsilon(\,\cdot\,,\,\cdot\,) + c_1(\,\cdot\,,\,\cdot\,)_{L_2(\mathbb{R}^3\setminus\omega)}$$

and owing to the symmetricity and closedness of the form, as well as to inequality (44), this makes the space $\mathfrak{D}(\mathfrak{g}_\varepsilon)$ a Hilbert one. Since by (44) we have $W_2^1(\Omega_\varepsilon) \subseteq \mathfrak{D}(\mathfrak{g}_\varepsilon)$, each $u \in \mathfrak{D}(\mathfrak{g}_\varepsilon)$ possesses a trace on $\partial\omega_\varepsilon$. The operator, which maps $u \in \mathfrak{D}(\mathfrak{g}_\varepsilon)$ into its trace on $\partial\omega_\varepsilon$, is well-defined as a bounded one from $\mathfrak{D}(\mathfrak{g}_\varepsilon)$ into $L_2(\partial\omega_\varepsilon)$; we denote this operator by $\mathcal{T}_\varepsilon$. In view of inequalities (36) and (44) and the compactness of the trace operator from $W_2^1(\Pi_\varepsilon)$ into $L_2(\partial\omega_\varepsilon)$, the operator $\mathcal{T}_\varepsilon : \mathfrak{D}(\mathfrak{g}_\varepsilon) \to L_2(\partial\omega_\varepsilon)$ is compact and satisfies the estimate

$$\|\mathcal{T}_\varepsilon\|_{\mathfrak{D}(\mathfrak{g}_\varepsilon)\to L_2(\partial\omega_\varepsilon)} \leqslant C\varepsilon^{\frac{1}{2}}, \tag{45}$$

where C is a constant independent of ε.

For each $\phi \in \mathfrak{D}(\mathfrak{g}_\varepsilon)$, we consider the boundary value problem

$$(\hat{\mathcal{H}} + c_1)u = 0 \quad \text{in} \quad \Omega_\varepsilon, \qquad \mathcal{B}u = 0 \quad \text{on} \quad \partial\Omega, \qquad \frac{\partial u}{\partial \mathrm{n}} = -\alpha\phi \quad \text{on} \quad \partial\omega_\varepsilon. \tag{46}$$

The solution isunderstood in the generalized sense, namely, a solution is a function $u \in \mathfrak{D}(\mathfrak{g}_\varepsilon)$ such that

$$(u, v)_{\mathfrak{g}_\varepsilon} + (\alpha\phi, v)_{L_2(\partial\omega_\varepsilon)} = 0 \quad \text{for all} \quad v \in \mathfrak{D}(\mathfrak{g}_\varepsilon). \tag{47}$$

Since $\mathfrak{g}_\varepsilon$ is the scalar product on the Hilbert space $\mathfrak{D}(\mathfrak{g}_\varepsilon)$, boundary value problem (46) is uniquely solvable for each $\phi \in L_2(\partial\omega_\varepsilon)$. By $\mathcal{A}_\varepsilon^0$, we denote the operator mapping ϕ into the solution of problem (46). This operator is bounded as acting from $L_2(\partial\omega_\varepsilon)$ into $\mathfrak{D}(\mathfrak{g}_\varepsilon)$. Moreover, by estimates (36), (44) we easily find that

$$\|u\|_{\mathfrak{g}_\varepsilon}^2 = -(\alpha\phi, u)_{L_2(\partial\omega_\varepsilon)} \leqslant C\varepsilon^{-\frac{1}{2}} \|\phi\|_{L_2(\partial\omega\varepsilon)} \|u\|_{W_2^1(\Pi_\varepsilon)} \leqslant C\varepsilon^{-\frac{1}{2}} \|\phi\|_{L_2(\partial\omega\varepsilon)} \|u\|_{\mathfrak{g}_\varepsilon}, \tag{48}$$

where the Cs are constants independent of ε, u, and ϕ. Hence,

$$\|\mathcal{A}_\varepsilon^0\|_{L_2(\partial\omega_\varepsilon)\to\mathfrak{D}(\mathfrak{g}_\varepsilon)} \leqslant C\varepsilon^{-\frac{1}{2}}, \tag{49}$$

where C is a constant independent of ε. It also follows from the symmetricity of the form $\mathfrak{g}_\varepsilon$ and the identity (47) that the operator $\mathcal{A}_\varepsilon := \mathcal{A}_\varepsilon^0 \mathcal{T}_\varepsilon$ acting on $\mathfrak{D}(\mathfrak{g}_\varepsilon)$ is self-adjoint. Since the operator $\mathcal{T}_\varepsilon$ is compact, the same is true for $\mathcal{A}_\varepsilon$. Estimates (45) and (49) imply that

$$\|\mathcal{A}_\varepsilon\|_{\mathfrak{D}(\mathfrak{g}_\varepsilon)\to\mathfrak{D}(\mathfrak{g}_\varepsilon)} \leqslant C,$$

where C is a constant independent of ε. The spectrum of the operator $\mathcal{A}_\varepsilon$ consists of discrete eigenvalues, which can accumulate only at zero, and the latter is the only possible point of the essential spectrum.

It is possible to construct an asymptotic expansion for the operator $\mathcal{A}_\varepsilon$ as $\varepsilon \to 0$ on the base of the classical method of matching asymptotic expansions similarly to Chapter III in [10] and Chapter II, Section 2.3.4 in [12]. The application of this technique shows that

$$\begin{aligned} &\|\mathcal{A}_\varepsilon - \mathcal{L}_\varepsilon\|_{\mathfrak{D}(\mathfrak{g}_\varepsilon)\to\mathfrak{D}(\mathfrak{g}_\varepsilon)} \to 0, \qquad \varepsilon \to 0, \qquad \mathcal{L}_\varepsilon := \zeta_\varepsilon \mathcal{S}_\varepsilon^{-1} \mathcal{L} \mathcal{S}_\varepsilon \mathcal{T}_\varepsilon + \varepsilon(1-\zeta_\varepsilon) G \mathcal{C} \mathcal{S}_\varepsilon \mathcal{T}_\varepsilon, \\ &(\mathcal{S}_\varepsilon u)(\xi) := u(\varepsilon\xi), \qquad \xi \in \mathbb{R}^3 \setminus \omega, \qquad \mathcal{C}u := \frac{1}{4\pi(\det \mathrm{A}_0)^{\frac{1}{4}}} \int\limits_{\partial\omega} \alpha_0(\xi) u(\xi)\, ds. \end{aligned} \tag{50}$$

Here, $\zeta_\varepsilon(x) = \zeta(|x|\varepsilon^{-\frac{1}{2}})$, and $\zeta = \zeta(t)$ is an infinitely differentiable cut-off function equal to one as $t < 1$ and vanishing as $t > 2$. By $\mathcal{L}$, we denote an operator from $L_2(\partial\omega)$ into $W_2^1(B_{R_1}(0) \setminus \omega) \cap C^\infty(\mathbb{R}^3 \setminus \omega)$ mapping each function $\phi \in L_2(\partial\omega)$ into the unique solution $U = U(\xi)$ of the boundary value problem

$$\begin{aligned} &\operatorname{div}_\xi \mathrm{A}_0 \nabla_\xi U = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \omega, \qquad \nu \cdot \mathrm{A}_0 \nabla_\xi U + \alpha_0 \phi = 0 \quad \text{on} \quad \partial\omega, \\ &U(\xi) = \mathcal{C}(\phi) |\mathrm{A}_0^{-\frac{1}{2}} \xi|^{-1} + O(|\mathrm{A}_0^{-\frac{1}{2}} \xi|^{-2}), \qquad \xi \to \infty, \end{aligned}$$

and the above asymptotic for U is differentiable. Since the operator $\mathcal{T}_\varepsilon$ is compact and $\mathcal{C}$ is a linear functional, it follows from the definition of the operator $\mathcal{L}_\varepsilon$ in (50) that this operator is compact. Hence, its spectrum consists of eigenvalues of finite multiplicities, which can accumulate only at zero, and the latter is the only possible point of the continuous spectrum.

Let $\mathcal{T} : W_2^1(B_{R_1}(0) \setminus \omega) \to L_2(\partial\omega)$ be the operator of taking the trace on $\partial\omega$; this operator is obviously compact. Then, the operator $\mathcal{T}\mathcal{L} : L_2(\partial\omega) \to L_2(\partial\omega)$ is compact as well. The eigenvalues of this operator coincide with those of the operator $\mathcal{L}_\varepsilon$, counting the multiplicities. Indeed, let $\lambda \neq 0$ be an eigenvalue of the operator $\mathcal{T}\mathcal{L}$. This means that there exists a non-trivial solution of boundary value problem (10). Hence, λ is an eigenvalue of the operator $\mathcal{L}_\varepsilon$, and the associated eigenfunction is $\zeta_\varepsilon \mathcal{S}_\varepsilon^{-1}\psi + \varepsilon\lambda^{-1}(1 - \zeta_\varepsilon)G\mathcal{C}(\psi)$. Furthermore, vice versa, let $\lambda \neq 0$ be an eigenvalue of the operator $\mathcal{L}_\varepsilon$ and ψ be an associated eigenfunction. Then, we consider the eigenvalue equation $\mathcal{L}_\varepsilon\psi = \lambda\psi$ as the identity for two functions defined for $x \in B_{\varepsilon^{\frac{1}{2}}} \setminus \omega_\varepsilon$, and we see immediately that ψ solves problem (10) and, hence, λ is an eigenvalue of the operator $\mathcal{TL}$.

Since the operator $\mathcal{TL}$ is compact, its spectrum consists of discrete eigenvalues, which can accumulate only at zero, and the latter is the only possible point of the essential spectrum. Since the eigenvalues of the operator $\mathcal{TL}$ coincide with those of the operator $\mathcal{L}_\varepsilon$, in view of the convergence in (50), we conclude that the eigenvalues of $\mathcal{TL}$ are the limits of the eigenvalues of $\mathcal{L}_\varepsilon$ as $\varepsilon \to 0$, counting the multiplicities, and, hence, the eigenvalues of $\mathcal{TL}$ are real.

Let $\lambda \neq 0$ be an eigenvalue of the operator $\mathcal{TL}$, then problem (10) possesses a non-trivial solution. We multiply the equation in (10) by ψ in $L_2(\mathbb{R}^3 \setminus \omega)$ and integrate once by parts using the boundary condition in (10). This gives

$$\lambda = -\frac{(\alpha_0\psi, \psi)_{L_2(\partial\omega)}}{(\mathrm{A}_0\nabla_\xi\psi, \nabla_\xi\psi)_{L_2(\mathbb{R}^3\setminus\omega)}}. \tag{51}$$

We represent ψ as

$$\psi(\xi) = E(\xi)\Psi(x), \qquad E(\xi) := |\mathrm{A}_0^{-\frac{1}{2}}\xi|^{-1}. \tag{52}$$

Since the function E is non-zero on $\mathbb{R}^3 \setminus \omega$, the above representation for ψ is well-defined, and in view of the asymptotic at infinity in problem (10), the function $\Psi(\xi)$ possesses the following differentiable asymptotic at infinity:

$$\Psi(\xi) = \lambda^{-1}\mathcal{C}(\psi) + O(|\mathrm{A}_0^{-\frac{1}{2}}\xi|^{-1}), \qquad \xi \to \infty.$$

We substitute representation (52) into the denominator of (51) and integrate by parts using the definition of E and α_0:

$$\begin{aligned}
(\mathrm{A}_0\nabla_\xi\psi, \nabla_\xi\psi)_{L_2(\mathbb{R}^3\setminus\omega)} =& (\mathrm{A}_0 E\nabla_\xi\Psi, E\nabla_\xi\Psi)_{L_2(\mathbb{R}^3\setminus\omega)} + (\mathrm{A}_0\Psi\nabla_\xi E, \Psi\nabla_\xi E)_{L_2(\mathbb{R}^3\setminus\omega)} \\
&+ (\mathrm{A}_0\Psi\nabla_\xi E, E\nabla_\xi\Psi)_{L_2(\mathbb{R}^3\setminus\omega)} + (\mathrm{A}_0 E\nabla_\xi\Psi, \Psi\nabla_\xi E)_{L_2(\mathbb{R}^3\setminus\omega)} \\
=& (\mathrm{A}_0 E\nabla_\xi\Psi, E\nabla_\xi\Psi)_{L_2(\mathbb{R}^3\setminus\omega)} + \int\limits_{\partial\omega} \overline{\Psi} E\nu \cdot \mathrm{A}_0\Psi\nabla_\xi E\, ds \\
&- \int\limits_{\mathbb{R}^3\setminus\omega} E \operatorname{div} \mathrm{A}_0|\Psi|^2\nabla_\xi E\, d\xi + (\mathrm{A}_0\Psi\nabla_\xi E, E\nabla_\xi\Psi)_{L_2(\mathbb{R}^3\setminus\omega)} \\
&+ (\mathrm{A}_0 E\nabla_\xi\Psi, \Psi\nabla_\xi E)_{L_2(\mathbb{R}^3\setminus\omega)} \\
=& (\mathrm{A}_0 E\nabla_\xi\Psi, E\nabla_\xi\Psi)_{L_2(\mathbb{R}^3\setminus\omega)} - (\alpha_0 E\Psi, E\Psi)_{L_2(\partial\omega)},
\end{aligned}$$

and the final expression is positive since the same is true for the initial scalar product $(\mathrm{A}_0\nabla_\xi\psi, \nabla_\xi\psi)_{L_2(\mathbb{R}^3\setminus\omega)}$. Substituting these identities and representation (52) into (51), we obtain

$$\lambda = \frac{-(\alpha_0 E\Psi, E\Psi)_{L_2(\partial\omega)}}{(\mathrm{A}_0 E\nabla_\xi\Psi, E\nabla_\xi\Psi)_{L_2(\mathbb{R}^3\setminus\omega)} - (\alpha_0 E\Psi, E\Psi)_{L_2(\partial\omega)}}. \tag{53}$$

Since the denominatorof the obtained quotient is positive, we immediately conclude that $\lambda < 1$ once Ψ is not identically one. It is straightforward to confirm that $\lambda = 1$ is an eigenvalue of the operator $\mathcal{TL}$, and the corresponding non-trivial solution of problem (10) is $\psi = E$. Identity (53), then, implies that $\lambda = 1$ is a simple eigenvalue of the operator $\mathcal{TL}$.

In view of the established facts on the eigenvalues of the operator $\mathcal{TL}$ and the convergence in (50), the greatest eigenvalue of the operator $\mathcal{A}_\varepsilon$ is simple and converges to 1 as $\varepsilon \to 0$. We denote the next eigenvalue of the operator $\mathcal{A}_\varepsilon$ by $\tilde{\lambda}_\varepsilon$. This eigenvalue converges to the next eigenvalue of the operator $\mathcal{TL}$, which is strictly less than one. Let

ψ_ε be a normalized eigenfunction, in $\mathfrak{D}(\mathfrak{g}_\varepsilon)$, of the operator $\mathcal{A}_\varepsilon$ associated with its greatest eigenvalue. Then, by the minimax principle applied to the operator $\mathcal{A}_\varepsilon$, we find

$$\frac{(\mathcal{A}_\varepsilon u, u)_{\mathfrak{g}_\varepsilon}}{\|u\|^2_{\mathfrak{g}_\varepsilon}} \leqslant \tilde{\lambda}_\varepsilon \quad \text{for all} \quad u \in \mathfrak{D}(\mathfrak{g}_\varepsilon) \quad \text{such that} \quad (u, \psi_\varepsilon)_{\mathfrak{g}_\varepsilon} = 0. \tag{54}$$

In view of identity (47), we can rewrite this inequality as

$$-(\mathcal{A}_\varepsilon u, u)_{\mathfrak{g}_\varepsilon} = (\alpha u, u)_{L_2(\partial\omega_\varepsilon)} \geqslant -\tilde{\lambda}_\varepsilon \|u\|^2_{\mathfrak{g}_\varepsilon} \tag{55}$$

for all $u \in \mathfrak{D}(\mathfrak{g}_\varepsilon)$ obeying the orthogonality condition from (54).

We also need asymptotics for the eigenvalue λ_ε and the associated eigenfunction ψ_ε; let us find them. It follows from problem (7), (24); the definition (4) of the function α_0; and Lemma 3 that the function G solves the following boundary value problem

$$\begin{aligned} &(\hat{\mathcal{H}} + c_1)G = 0 \quad \text{in} \quad \Omega_\varepsilon, \qquad \mathcal{B}G = 0 \quad \text{on} \quad \partial\Omega, \\ &\frac{\partial G}{\partial \mathrm{n}} = -\alpha G + \varepsilon^{-1} h_\varepsilon \quad \text{on} \quad \partial\omega_\varepsilon, \end{aligned} \tag{56}$$

where h_ε is a continuous function in $\partial\omega_\varepsilon$ bounded uniformly in the spatial variables in $\partial\omega_\varepsilon$ and the small parameter ε. U_ε denotes the solution of the problem

$$(\hat{\mathcal{H}} + c_1)U_\varepsilon = 0 \quad \text{in} \quad \Omega_\varepsilon, \qquad \mathcal{B}U_\varepsilon = 0 \quad \text{on} \quad \partial\Omega, \qquad \frac{\partial U_\varepsilon}{\partial \mathrm{n}} = h_\varepsilon \quad \text{on} \quad \partial\omega_\varepsilon, \tag{57}$$

and in view of the uniform boundedness of h, similarly to (48), we immediately obtain the following:

$$\|U_\varepsilon\|_{\mathfrak{g}_\varepsilon} = O(\varepsilon^{\frac{3}{2}}). \tag{58}$$

Lemma 3 also implies that

$$\|G\|_{L_2(\Omega_\varepsilon)} = \|G\|_{L_2(\Omega)} + O(\varepsilon^{\frac{1}{2}}), \qquad \|G\|_{\mathfrak{g}_\varepsilon} = \|G^2_{-1}\|_{L_2(\mathbb{R}^3\setminus\omega)} \varepsilon^{-\frac{1}{2}} + O\big(\varepsilon^{\frac{1}{2}}|\ln\varepsilon|\big), \tag{59}$$

where C is a positive constant independent of ε.

Comparing problems (46) and (56), we see that $G = \mathcal{A}_\varepsilon G + \varepsilon^{-1} U_\varepsilon$ and, hence,

$$G_\varepsilon = \mathcal{A}_\varepsilon G_\varepsilon + U_\varepsilon, \qquad G_\varepsilon := \frac{G}{\|G\|_{\mathfrak{g}_\varepsilon}}, \qquad V_\varepsilon := \frac{\varepsilon^{-1} U_\varepsilon}{\|G\|_{\mathfrak{g}_\varepsilon}}, \qquad \|V_\varepsilon\|_{\mathfrak{g}_\varepsilon} = O(\varepsilon). \tag{60}$$

We apply the resolvent $(\mathcal{A}_\varepsilon - 1)^{-1}$ to the obtained equation and employ standard results on the behavior of the resolvents of the self-adjoint operators near the isolated eigenvalues (see Chapter V, Section 3.5 in [21]). This gives the identity

$$G_\varepsilon = \frac{(V_\varepsilon, \psi_\varepsilon)_{\mathfrak{g}_\varepsilon}}{1 - \lambda_\varepsilon} \psi_\varepsilon + \mathcal{R}_\varepsilon V_\varepsilon, \tag{61}$$

where $\mathcal{R}_\varepsilon$ is the reduced resolvent at the point 1, and this is an operator in $\mathfrak{D}(\mathfrak{g}_\varepsilon)$ bounded uniformly in ε and acting into the orthogonal complement to ψ_ε in $\mathfrak{D}(\mathfrak{g}_\varepsilon)$. Hence,

$$\|\mathcal{R}_\varepsilon V_\varepsilon\|_{\mathfrak{g}_\varepsilon} = O(\varepsilon). \tag{62}$$

This estimate and identity (61) imply that

$$\|G_\varepsilon - c_\varepsilon \psi_\varepsilon\|_{\mathfrak{g}_\varepsilon} = O(\varepsilon), \qquad c_\varepsilon := \frac{(V_\varepsilon, \psi_\varepsilon)_{\mathfrak{g}_\varepsilon}}{1 - \lambda_\varepsilon} = (G_\varepsilon, \psi_\varepsilon)_{\mathfrak{g}_\varepsilon}. \tag{63}$$

Calculating the scalarproduct in $\mathfrak{D}(\mathfrak{g}_\varepsilon)$ of both sides of identity (61) with G_ε, in view of identity (62), we immediately see that

$$1 = |c_\varepsilon|^2 + O(\varepsilon). \tag{64}$$

Calculating the scalarproduct in $\mathfrak{D}(\mathfrak{g}_\varepsilon)$ of both sides of identity (61) with $c_\varepsilon\psi_\varepsilon$, by (58), (62), and (63), the definition of V_ε in (60), and the normalization of G_ε and ψ_ε in $\mathfrak{D}(\mathfrak{g}_\varepsilon)$, we see that

$$\lambda_\varepsilon - 1 = -\frac{1}{\varepsilon\|G\|_{\mathfrak{g}_\varepsilon}}\frac{(U_\varepsilon, c_\varepsilon\psi_\varepsilon)_{\mathfrak{g}_\varepsilon}}{(G_\varepsilon, c_\varepsilon\psi_\varepsilon)_{\mathfrak{g}_\varepsilon}} = -\frac{(U_\varepsilon, G)_{\mathfrak{g}_\varepsilon}}{\varepsilon\|G\|^2_{\mathfrak{g}_\varepsilon}}\big(1 + O(\varepsilon)\big). \tag{65}$$

Let us find the scalar product $(U_\varepsilon, G)_{\mathfrak{g}_\varepsilon}$. In order to do this, we write the definition of the generalized solution of problems (56), (57) with G as the test function:

$$(G, G)_{\mathfrak{g}_\varepsilon} + (\alpha G, G)_{L_2(\partial\omega_\varepsilon)} - \varepsilon^{-1}(h_\varepsilon, G)_{L_2(\partial\omega_\varepsilon)} = 0, \qquad (U_\varepsilon, G)_{\mathfrak{g}_\varepsilon} - (h_\varepsilon, G)_{\partial\omega_\varepsilon} = 0.$$

Hence,

$$\varepsilon^{-1}(U_\varepsilon, G)_{\mathfrak{g}_\varepsilon} = \varepsilon^{-1}(h_\varepsilon, G)_{L_2(\partial\omega_\varepsilon)} = \|G\|^2_{\mathfrak{g}_\varepsilon} + (\alpha G, G)_{L_2(\partial\omega_\varepsilon)}.$$

It also follows from asymptotics (8) and the definition of the function α that

$$\begin{aligned}(\alpha G, G)_{L_2(\partial\omega_\varepsilon)} =& \varepsilon^{-1}(\alpha_0 G_{-1}, G_{-1})_{L_2(\partial\omega)} + 2\operatorname{Re}(\alpha_0 G_0, G_{-1})_{L_2(\partial\omega)} \\ &+ (\alpha_1 G_{-1}, G_{-1})_{L_2(\partial\omega)} + O(\varepsilon).\end{aligned}$$

This identity and (11), (26) yield

$$\varepsilon^{-1}(U_\varepsilon, G)_{\mathfrak{g}_\varepsilon} = \beta_0 + (\alpha_1 G_{-1}, G_{-1})_{L_2(\partial\omega)} + O(\varepsilon) = -4\pi\big(\det \mathrm{A}_0\big)^{\frac{1}{4}}(\beta - a_0) + O(\varepsilon), \quad \varepsilon \to 0.$$

The obtained formula and (59), (19) allow us to rewrite (65) as

$$\lambda_\varepsilon - 1 = \varepsilon\frac{4\pi(\beta - a_0)\big(\det \mathrm{A}_0\big)^{\frac{1}{4}}}{\|G^2_{-1}\|^2_{L_2(\mathbb{R}^3\setminus\omega)}} + O(\varepsilon^2|\ln\varepsilon|).$$

4.2. Lower Semi-Boundedness

In this subsection, we prove the lower-semiboundedness of the form $\mathfrak{h}_\varepsilon$. We represent each function $u \in \mathfrak{D}(\mathfrak{g}_\varepsilon)$ as

$$u = u^\perp + (u, \psi_\varepsilon)_{\mathfrak{D}(\mathfrak{g}_\varepsilon)}\psi_\varepsilon, \qquad (u^\perp, \psi_\varepsilon)_{\mathfrak{D}(\mathfrak{g}_\varepsilon)} = 0.$$

Then, by (55), for all $u \in \mathfrak{D}(\mathfrak{g}_\varepsilon)$, we have the following:

$$\begin{aligned}\mathfrak{h}_\varepsilon(u, u) + c_1\|u\|^2_{L_2(\Omega_\varepsilon)} =& (u - \mathcal{A}_\varepsilon u, u)_{\mathfrak{g}_\varepsilon} \\ =& (u^\perp - \mathcal{A}_\varepsilon u^\perp, u^\perp)_{\mathfrak{g}_\varepsilon} + (1 - \lambda_\varepsilon)|(u, \psi_\varepsilon)_{\mathfrak{D}(\mathfrak{g}_\varepsilon)}|^2 \\ \geqslant& (1 - \tilde{\lambda}_\varepsilon)\|u\|^2_{\mathfrak{g}_\varepsilon} + (1 - \lambda_\varepsilon)|(u, \psi_\varepsilon)_{\mathfrak{D}(\mathfrak{g}_\varepsilon)}|^2 \\ =& (\kappa + c_3(\varepsilon))\|u\|^2_{\mathfrak{g}_\varepsilon} + (1 - \lambda_\varepsilon)|(u, \psi_\varepsilon)_{\mathfrak{D}(\mathfrak{g}_\varepsilon)}|^2.\end{aligned}$$

As it is established in the previous section, the eigenvalue $\tilde{\lambda}_\varepsilon$ converges to the second eigenvalue of the operator $\mathcal{TL}$, and this is why $1 - \tilde{\lambda}_\varepsilon = \kappa + c_3(\varepsilon)$, where $c_3(\varepsilon) \to 0$ as $\varepsilon \to +0$. This allows us to the rewrite the above estimate as

$$\mathfrak{h}_\varepsilon(u, u) + c_1\|u\|^2_{L_2(\Omega_\varepsilon)} \geqslant (\kappa + c_3(\varepsilon))\|u\|^2_{\mathfrak{g}_\varepsilon} + (1 - \lambda_\varepsilon)|(u, \psi_\varepsilon)_{\mathfrak{D}(\mathfrak{g}_\varepsilon)}|^2. \tag{66}$$

For each $u \in \mathfrak{D}(\mathfrak{g}_\varepsilon)$, the function

$$u^\perp := u - (u, \psi_\varepsilon)_{\mathfrak{g}_\varepsilon}\psi_\varepsilon \tag{67}$$

satisfies the orthogonality condition in (54), and in view of (66), we have

$$\begin{aligned}\mathfrak{h}_\varepsilon(u,u)+c_1\|u\|^2_{L_2(\Omega_\varepsilon)} =&(u-\mathcal{A}_\varepsilon u,u)_{\mathfrak{g}_\varepsilon}=(u^\perp-\mathcal{A}_\varepsilon u^\perp,u^\perp)_{\mathfrak{g}_\varepsilon}+(1-\lambda_\varepsilon)|(u,\psi_\varepsilon)_{\mathfrak{g}_\varepsilon}|^2\\ \geqslant&(\kappa+c_3(\varepsilon))\|u^\perp\|^2_{\mathfrak{g}_\varepsilon}+(1-\lambda_\varepsilon)|(u,\psi_\varepsilon)_{\mathfrak{g}_\varepsilon}|^2\\ \geqslant&(\kappa+c_3(\varepsilon))\|u^\perp\|^2_{\mathfrak{g}_\varepsilon}\\ &-\varepsilon\left(\frac{4\pi(\underline{-}a_0)\big(\det \mathrm{A}_0\big)^{\frac{1}{4}}}{\|G^2_{-1}\|^2_{L_2(\mathbb{R}^3\setminus\omega)}}+C\varepsilon|\ln\varepsilon|\right)|(u,\psi_\varepsilon)_{\mathfrak{g}_\varepsilon}|^2\end{aligned} \tag{68}$$

where C is some constant independent of ε and u.

For further purposes, it is more convenient to introduce another representation similar to (67): we let

$$\begin{aligned}&u_\perp:=u-(u,G_\varepsilon)_{\mathfrak{g}_\varepsilon}G_\varepsilon, \qquad \psi_{\varepsilon,\perp}:=\psi_\varepsilon-(\psi_\varepsilon,G_\varepsilon)_{\mathfrak{g}_\varepsilon}G_\varepsilon,\\ &(u_\perp,G_\varepsilon)_{\mathfrak{g}_\varepsilon}=(\psi_{\varepsilon,\perp},G_\varepsilon)_{\mathfrak{g}_\varepsilon}=0.\end{aligned} \tag{69}$$

Comparing the above definition of $\psi_{\varepsilon,\perp}$ with (63), (64), we immediately see that

$$\|\psi_{\varepsilon,\perp}\|_{\mathfrak{g}_\varepsilon}=O(\varepsilon), \qquad \psi_{\varepsilon,\perp}=\psi_\varepsilon-\overline{c_\varepsilon}G_\varepsilon. \tag{70}$$

It follows from (67), (69) that

$$u^\perp=u_\perp+(u,G_\varepsilon-c_\varepsilon\psi_\varepsilon)_{\mathfrak{g}_\varepsilon}G_\varepsilon-(u,\psi_\varepsilon)_{\mathfrak{g}_\varepsilon}\psi_{\varepsilon,\perp}$$

and in view of the orthogonality conditions in (69), we find

$$\|u^\perp\|^2_{\mathfrak{g}_\varepsilon}=\|u_\perp-(u,\psi_\varepsilon)_{\mathfrak{g}_\varepsilon}\psi_{\varepsilon,\perp}\|^2_{\mathfrak{g}_\varepsilon}+|(u,G_\varepsilon-c_\varepsilon\psi_\varepsilon)_{\mathfrak{g}_\varepsilon}|^2. \tag{71}$$

By the Cauchy–Schwarz inequality and (70), we obtain

$$\begin{aligned}\|u_\perp-(u,\psi_\varepsilon)_{\mathfrak{g}_\varepsilon}\psi_{\varepsilon,\perp}\|^2_{\mathfrak{g}_\varepsilon}\geqslant&\|u_\perp\|^2_{\mathfrak{g}_\varepsilon}(1-|\ln\varepsilon|^{-1})-(|\ln\varepsilon|-1)|(u,\psi_\varepsilon)_{\mathfrak{g}_\varepsilon}|^2\|\psi_{\varepsilon,\perp}\|^2_{\mathfrak{g}_\varepsilon}\\ \geqslant&\|u_\perp\|^2_{\mathfrak{g}_\varepsilon}(1-|\ln\varepsilon|^{-1})-C\varepsilon^2|\ln\varepsilon||(u,\psi_\varepsilon)_{\mathfrak{g}_\varepsilon}|^2,\end{aligned} \tag{72}$$

where the Cs are some positive constants independent of ε and u. It also follows from the Cauchy–Schwarz inequality and (63), (64) that

$$\begin{aligned}|(u,\psi_\varepsilon)_{\mathfrak{g}_\varepsilon}|^2=&|c_\varepsilon|^{-2}|(u,c_\varepsilon\psi_\varepsilon)_{\mathfrak{g}_\varepsilon}|^2=|c_\varepsilon|^{-2}|(u,G_\varepsilon)_{\mathfrak{g}_\varepsilon}-(u,G_\varepsilon-c_\varepsilon\psi_\varepsilon)_{\mathfrak{g}_\varepsilon}|^2\\ \geqslant&(1-|\ln\varepsilon|^{-1})|(u,G_\varepsilon)_{\mathfrak{g}_\varepsilon}|^2-(|\ln\varepsilon|-1)|(u,G_\varepsilon-c_\varepsilon\psi_\varepsilon)_{\mathfrak{g}_\varepsilon}|^2,\\ |(u,\psi_\varepsilon)_{\mathfrak{g}_\varepsilon}|^2\geqslant& 2|(u,G_\varepsilon)_{\mathfrak{g}_\varepsilon}|^2+2|(u,G_\varepsilon-c_\varepsilon\psi_\varepsilon)_{\mathfrak{g}_\varepsilon}|^2.\end{aligned} \tag{73}$$

Substituting the latter inequality and (72) into (71), we obtain the following:

$$\|u^\perp\|^2_{\mathfrak{g}_\varepsilon}\geqslant\|u_\perp\|^2_{\mathfrak{g}_\varepsilon}(1-|\ln\varepsilon|^{-1})-C\varepsilon^2|\ln\varepsilon||(u,G_\varepsilon)_{\mathfrak{g}_\varepsilon}|^2+\frac{1}{2}|(u,G_\varepsilon-c_\varepsilon\psi_\varepsilon)_{\mathfrak{g}_\varepsilon}|^2$$

with some constant C independent of ε and u. This estimate and (73) allow us to rewrite (68) as

$$\begin{aligned}\mathfrak{h}_\varepsilon(u,u)+c_1\|u\|^2_{L_2(\Omega_\varepsilon)}\geqslant&(\kappa+c_3(\varepsilon))\|u_\perp\|^2_{\mathfrak{g}_\varepsilon}(1-|\ln\varepsilon|^{-1})\\ &-\varepsilon\left(\frac{4\pi(\beta-a_0)\big(\det \mathrm{A}_0\big)^{\frac{1}{4}}}{\|G^2_{-1}\|^2_{L_2(\mathbb{R}^3\setminus\omega)}}+C|\ln\varepsilon|^{-1}\right)|(u,G_\varepsilon)_{\mathfrak{g}_\varepsilon}|^2\end{aligned} \tag{74}$$

where C is a constant independent of ε and u.

By the Cauchy–Schwarz inequality and (59), (67), we find that

$$\begin{aligned}\|u\|^2_{L_2(\Omega_\varepsilon)} =&\|u_\perp\|^2_{L_2(\Omega_\varepsilon)} + 2\operatorname{Re}(u,G_\varepsilon)_{\mathfrak{g}_\varepsilon}(G_\varepsilon,u_\perp)_{L_2(\Omega_\varepsilon)} + |(u,G_\varepsilon)_{\mathfrak{g}_\varepsilon}|^2\|G_\varepsilon\|^2_{L_2(\Omega_\varepsilon)}\\ \geqslant& -\eta\|u_\perp\|^2_{L_2(\Omega_\varepsilon)} + \frac{\eta\|G_\varepsilon\|^2_{L_2(\Omega_\varepsilon)}}{1+\eta}|(u,G_\varepsilon)_{\mathfrak{g}_\varepsilon}|^2\\ \geqslant& -\eta\|u_\perp\|^2_{L_2(\Omega_\varepsilon)} + \frac{\varepsilon\eta\|G\|^2_{L_2(\Omega)}(1-C\varepsilon^{\frac12})}{(1+\eta)\|G^2_{-1}\|^2_{L_2(\mathbb{R}^3\setminus\omega)}}|(u,G_\varepsilon)_{\mathfrak{g}_\varepsilon}|^2\end{aligned}$$

for an arbitrary $\eta\in(0,1)$ with some constant C independent of ε, u, and η. By (74), for an arbitrary $c_4>0$, we then obtain

$$\begin{aligned}\mathfrak{h}_\varepsilon(u,u) + (c_1+c_4)\|u\|^2_{L_2(\Omega_\varepsilon)} \geqslant& \big(\kappa + c_3(\varepsilon) - c_4\eta(1-|\ln\varepsilon|^{-1})\big)\|u_\perp\|^2_{\mathfrak{g}_\varepsilon}\\ &+ \frac{\varepsilon\|G\|^2_{L_2(\Omega)}}{\|G^2_{-1}\|^2_{L_2(\mathbb{R}^3\setminus\omega)}}\left(\frac{c_4\eta}{(1+\eta)} - \frac{4\pi(\beta-a_0)\big(\det \mathrm{A}_0\big)^{\frac14}}{\|G\|^2_{L_2(\Omega)}} - C|\ln\varepsilon|^{-1}\right)|(u,G_\varepsilon)_{\mathfrak{g}_\varepsilon}|^2.\end{aligned}$$

Hence, choosing η small enough and c_4 large enough, in view of condition (12), we conclude on the existence of the constants η and c_4 such that

$$\mathfrak{h}_\varepsilon(u,u) + (c_1+c_4)\|u\|^2_{L_2(\Omega_\varepsilon)} \geqslant c_5\|u_\perp\|^2_{\mathfrak{g}_\varepsilon} + c_5\varepsilon|(u,G_\varepsilon)_{\mathfrak{g}_\varepsilon}|^2 \tag{75}$$

for all $u\in\mathfrak{D}(\mathfrak{g}_\varepsilon)$ with a fixed positive constant c_5 independent of ε and u.

4.3. Self-Adjointness

We proceed to proving the self-adjointness of the operators $\mathcal{H}_\varepsilon$ and $\mathcal{H}_{0,\beta}$. We begin with the operator $\mathcal{H}_\varepsilon$. Since the form $\mathfrak{h}_\varepsilon$ is symmetric and lower-semi-bounded and is associated with the operator $\mathcal{H}_\varepsilon$, it is sufficient to show that it is closed, and then this will imply the self-adjointness of the operator $\mathcal{H}_\varepsilon$.

We choose an arbitrary sequence $u_n\in\mathfrak{D}(\mathfrak{h}_\varepsilon)$ such that

$$\|u_n-u\|_{L_2(\Omega_\varepsilon)}\to 0,\qquad \mathfrak{h}_\varepsilon(u_n-u_m,u_n-u_m)\to 0\quad\text{as}\quad n,m\to\infty \tag{76}$$

for some $u\in L_2(\Omega_\varepsilon)$. We also observe that since

$$\|v\|^2_{\mathfrak{g}_\varepsilon} = \|v_\perp\|^2_{\mathfrak{g}_\varepsilon} + |(v,G_\varepsilon)_{\mathfrak{g}_\varepsilon}|^2$$

for each $v\in\mathfrak{D}(\mathfrak{g}_\varepsilon)$, then it follows from (75) that

$$\mathfrak{h}_\varepsilon(v,v) + (c_1+c_4)\|v\|^2_{L_2(\Omega_\varepsilon)} \geqslant c_5\varepsilon\|v\|^2_{\mathfrak{g}_\varepsilon}.$$

This estimate and (76) yield

$$\|u_n-u_m\|_{\mathfrak{g}_\varepsilon}\to 0\quad\text{as}\quad n,m\to\infty.$$

Since the space $\mathfrak{D}(\mathfrak{g}_\varepsilon)$ is Hilbert and is a subspace of $L_2(\Omega_\varepsilon)$, the sequence u_n converges in $\mathfrak{D}(\mathfrak{g}_\varepsilon)$, and the limit is necessarily u. Hence, $u\in\mathfrak{D}(\mathfrak{g}_\varepsilon)$ and $\|u_n-u\|_{\mathfrak{g}_\varepsilon}\to 0$ as $n\to\infty$. By estimates (36) and (44), we also see that $\big(\alpha(u_n-u),(u_n-u)\big)_{L_2(\partial\omega_\varepsilon)}\to 0$ as $n\to+\infty$. Therefore, $\mathfrak{h}_\varepsilon(u_n-u)\to 0$ as $n\to+\infty$, and the form $\mathfrak{h}_\varepsilon$ is closed. This yields the self-adjointness of the operator $\mathcal{H}_\varepsilon$.

We proceed to the operator $\mathcal{H}_{0,\beta}$. We consider the adjoint operator $\mathcal{H}^*_{0,\beta}$, and by the definition of an adjoint operator, the domain of $\mathcal{H}^*_{0,\beta}$ consists of all functions $v\in L_2(\Omega)$, for which there exists a function $g\in L_2(\Omega)$ obeying the identity

$$(\mathcal{H}_{0,\beta}u,v)_{L_2(\Omega)} = (u,g)_{L_2(\Omega)}\quad\text{for all}\quad u\in\mathfrak{D}(\mathcal{H}_{0,\beta}),\qquad \mathcal{H}^*_{0,\beta}v=g.$$

Substituting the representation in (13) for the functions from the domain of the operator $\mathcal{H}_{0,\beta}$ into the above identity, we obtain

$$(\mathcal{H}_\Omega u_0, v)_{L_2(\Omega)} - (\beta - a)^{-1} u_0(x_0)(G, c_1 v + g)_{L_2(\Omega)} = (u_0, g)_{L_2(\Omega)}, \qquad u_0 \in \mathfrak{D}(\mathcal{H}_\Omega). \quad (77)$$

Similarly to (32)–(34), we confirm that

$$\big((\mathcal{H}_\Omega + c_1)u_0, G\big)_{L_2(\Omega\setminus B_\delta(x_0))} = -\left(\frac{\partial u_0}{\partial \mathrm{n}}, G\right)_{L_2(\partial B_\delta(x_0))} + \left(u_0, \frac{\partial G}{\partial \mathrm{n}}\right)_{L_2(\partial B_\delta(x_0))} \quad (78)$$

Since $u_0 \in W_2^2(\Omega)$, by (18), (36), and (41), we find that

$$\left(\frac{\partial u_0}{\partial \mathrm{n}}, G\right)_{L_2(\partial B_\delta(x_0))} \to 0, \qquad \left(u_0, \frac{\partial G}{\partial \mathrm{n}}\right)_{L_2(\partial B_\delta(x_0))} \to -4\pi\big(\det \mathrm{A}_0\big)^{\frac{1}{4}} u_0(x_0)$$

Passing, then, to the limit in (78), we obtain

$$\big((\mathcal{H}_\Omega + c_1)u_0, G\big)_{L_2(\Omega)} = -4\pi\big(\det \mathrm{A}_0\big)^{\frac{1}{4}} u_0(x_0). \quad (79)$$

This allows us to rewrite (77) as

$$(\mathcal{H}_\Omega u_0, v)_{L_2(\Omega)} - (\beta - a)^{-1}\kappa\big((\mathcal{H}_\Omega + c_1)u_0, G\big)_{L_2(\Omega)} = (u_0, g)_{L_2(\Omega)},$$
$$\rho := -\frac{\pi\sqrt{\det \mathrm{A}_0}(G, c_1 v + g)_{L_2(\Omega)}}{4},$$

which yields

$$(\mathcal{H}_\Omega u_0, v - (\beta - a)^{-1}\overline{\rho} G)_{L_2(\Omega)} = (u_0, g + (\beta - a)^{-1} c_1 \overline{\rho} G)_{L_2(\Omega)}.$$

Due to the self-adjointness of the operator $\mathcal{H}_\Omega$, we then obtain the identities

$$w := v - (\beta - a)^{-1}\overline{\rho} G \in \mathfrak{D}(\mathcal{H}_\Omega), \qquad \mathcal{H}_\Omega w = g + (\beta - a)^{-1} c_1 \overline{\rho} G. \quad (80)$$

Applying identity (79) with u_0 replaced by w, we find that

$$-4\pi\big(\det \mathrm{A}_0\big)^{\frac{1}{4}} w(x_0) = \big((\mathcal{H}_\Omega + c_1)w, G\big)_{L_2(\Omega)} = (g + c_1 v, G)_{L_2(\Omega)} = -4\pi\big(\det \mathrm{A}_0\big)^{\frac{1}{4}} \overline{\rho}.$$

This identity and (80) imply that

$$v = w + (\beta - a)^{-1} w(x_0) G, \qquad \mathcal{H}_{0,\beta}^* w = g = \mathcal{H}_\Omega w - (\beta - a)^{-1} c_1 w(x_0)$$

and, hence, $\mathcal{H}_{0,\beta}^* = \mathcal{H}_{0,\beta}$.

In the next section, we also need the following auxiliary lemma, the proof of which literally reproduces that of Lemma 4.3 in [6].

Lemma 7. *Let $f \in L_2(\Omega)$, $\operatorname{Im}\lambda \neq 0$, $u := (\mathcal{H}_{0,\beta} - \lambda)^{-1} f$. Then, the function u satisfies the representation*

$$u(x) = v(x) + (\beta - a_0)^{-1} v(x_0) G(x), \qquad v \in \mathfrak{D}(\mathcal{H}_\Omega), \quad (81)$$

and the estimate

$$\mathfrak{h}_\Omega(v, v) + c_1\|v\|^2_{L_2(\Omega)} + \|v_0\|^2_{W_2^2(B_{2R_2}(x_0))} + |v_0(x_0)|^2 \leqslant C(\lambda)\|f\|^2_{L_2(\Omega)} \quad (82)$$

holds, where $C(\lambda)$ is a constant independent of f.

5. Resolvent Convergence

In this section, we prove estimates (15) and (16). The operators $\mathcal{H}_\varepsilon$ and $\mathcal{H}_{0,\beta}$ are self-adjoint, and this is why their resolvents are well-defined for λ with a non-zero imaginary part. We arbitrarily fix such λ and a function $f \in L_2(\Omega)$, and we let

$$u_0 := (\mathcal{H}_{0,\beta} - \lambda)^{-1} f, \qquad u_\varepsilon := (\mathcal{H}_\varepsilon - \lambda)^{-1} f, \qquad v_\varepsilon := u_\varepsilon - u_0. \tag{83}$$

The function v_ε is an element of $W_2^2(\Omega_\varepsilon)$ and solves the boundary value problem

$$(\hat{\mathcal{H}} - \lambda) v_\varepsilon = 0 \quad \text{in} \quad \Omega_\varepsilon, \qquad \mathcal{B} v_\varepsilon = 0 \quad \text{on} \quad \partial\Omega, \qquad \frac{\partial v_\varepsilon}{\partial \mathrm{n}} = -\alpha v_\varepsilon + p_\varepsilon \quad \text{on} \quad \partial\omega_\varepsilon,$$

where

$$p_\varepsilon := \left(\frac{\partial}{\partial \mathrm{n}} + \alpha\right) u_0.$$

The associated integral identity with v_ε as the test function reads as

$$\mathfrak{h}_\varepsilon(v_\varepsilon, v_\varepsilon) - \lambda \|v_\varepsilon\|^2_{L_2(\Omega_\varepsilon)} = (p_\varepsilon, v_\varepsilon)_{L_2(\partial\omega_\varepsilon)}. \tag{84}$$

Our next step is to estimate the right hand of this identity.

Since $u_0 \in \mathfrak{D}(\mathcal{H}_{0,\beta})$, it satisfies representation (81) with $v = v_0$ and estimate (82), while by (14), for the function f, we have

$$f = (\mathcal{H}_{0,\beta} - \lambda) u_0 = (\mathcal{H}_\Omega - \lambda) v_0 - (\beta - a_0)^{-1} (\lambda + c_1) v_0(x_0) G.$$

Following (69), we let

$$v_{\varepsilon,\perp} := v_\varepsilon - (v_\varepsilon, G_\varepsilon)_{\mathfrak{g}_\varepsilon} G_\varepsilon, \qquad (v_{\varepsilon,\perp}, G_\varepsilon)_{\mathfrak{g}_\varepsilon} = 0, \qquad v_\varepsilon = v_{\varepsilon,\perp} + (v_\varepsilon, G_\varepsilon)_{\mathfrak{g}_\varepsilon} G_\varepsilon. \tag{85}$$

Then, we represent the function p_ε as

$$\begin{aligned} &p_\varepsilon = p_{\varepsilon,1} + p_{\varepsilon,2} + p_{\varepsilon,3} + p_{\varepsilon,4}, \qquad p_{\varepsilon,1} := \frac{\partial v_0}{\partial \mathrm{n}}, \\ &p_{\varepsilon,2} := (v_0 - \langle v_0 \rangle_{\partial\omega_\varepsilon}) \alpha, \qquad p_{\varepsilon,3} := (\langle v_0 \rangle_{\partial\omega_\varepsilon} - v_0(x_0)) \alpha, \\ &p_{\varepsilon,4} := v_0(x_0)(\beta - a_0)^{-1} \left(\frac{\partial G}{\partial \mathrm{n}} + \alpha G\right) + \alpha v_0(x_0), \end{aligned} \tag{86}$$

where

$$\langle v_0 \rangle_{\partial\omega_\varepsilon} := \frac{1}{\varepsilon^2 \operatorname{mes} \partial\omega} \int\limits_{\partial\omega_\varepsilon} v_0(x)\, ds$$

and $\operatorname{mes} \partial\omega$ is the area of $\partial\omega$.

By estimates (36) and (82), we immediately obtain

$$\big|(p_{\varepsilon,1}, v_\varepsilon)_{L_2(\partial\omega_\varepsilon)}\big| \leqslant C\varepsilon \|v_0\|_{W_2^2(\Omega_0)} \|v_\varepsilon\|_{W_2^1(\Omega_0)} \leqslant C\varepsilon \|f\|_{L_2(\Omega)} \big(\|v_{\varepsilon,\perp}\|_{\mathfrak{g}_\varepsilon} + |(v_\varepsilon, G_\varepsilon)_{\mathfrak{g}_\varepsilon}|\big); \tag{87}$$

here and till the end of this section, C denotes various constants independent of f, u_0, u_ε, v_ε, ε, and spatial variables.

The function $v_0 - \langle v_0 \rangle_{\partial\omega_\varepsilon}$ satisfies condition (38) and belongs to $W_2^2(B_{2R_2}(x_0))$. This is why, by (36), (40), (82), and the definition of α, we obtain the following:

$$\begin{aligned} \big|(p_{\varepsilon,2}, v_\varepsilon)_{L_2(\partial\omega_\varepsilon)}\big| \leqslant& C\varepsilon^2 \|v_0\|_{W_2^2(B_{2R_2}(x_0))} \|v_\varepsilon\|_{W_2^1(\Omega_0)} \\ \leqslant& C\varepsilon^2 \|f\|_{L_2(\Omega)} \big(\|v_{\varepsilon,\perp}\|_{\mathfrak{g}_\varepsilon} + |(v_\varepsilon, G_\varepsilon)_{\mathfrak{g}_\varepsilon}|\big). \end{aligned} \tag{88}$$

Applying estimate (41) with $\phi = 1$ to the function v_0 and estimates (36) and (82), we obtain the following:

$$\begin{aligned}\big|(p_{\varepsilon,3}, v_\varepsilon)_{L_2(\partial\omega_\varepsilon)}\big| \leqslant & C|(v_0 - \langle v_0\rangle_{\partial\omega_\varepsilon})|\|v_\varepsilon\|_{L_2(\partial\omega_\varepsilon)} \leqslant C\varepsilon\|f\|_{L_2(\Omega)}\|v_\varepsilon\|_{\mathfrak{g}_\varepsilon} \\ \leqslant & C\varepsilon\|f\|_{L_2(\Omega)}\|v_\varepsilon\|_{\mathfrak{g}_\varepsilon}\big(\|v_{\varepsilon,\perp}\|_{\mathfrak{g}_\varepsilon} + |(v_\varepsilon, G_\varepsilon)_{\mathfrak{g}_\varepsilon}|\big).\end{aligned} \tag{89}$$

In view of the definition of $v_{\varepsilon,\perp}$ in (85), we have

$$(p_{\varepsilon,4}, v_\varepsilon)_{L_2(\partial\omega_\varepsilon)} = (p_{\varepsilon,4}, v_{\varepsilon,\perp})_{L_2(\partial\omega_\varepsilon)} + (v_\varepsilon, G_\varepsilon)_{\mathfrak{g}_\varepsilon}\frac{(p_{\varepsilon,4}, G)_{L_2(\partial\omega_\varepsilon)}}{\|G\|_{\mathfrak{g}_\varepsilon}}. \tag{90}$$

Using, then, the definition of the function α in (3), asymptotics for $\|G\|_{\mathfrak{g}_\varepsilon}$ in (59), estimate (36) applied for $v_{\varepsilon,\perp}$, inequality (82) for v_0, the boundary condition on $\partial\omega_\varepsilon$ in (56), and the uniform boundedness of the function, we find that

$$\big|(p_{\varepsilon,4}, v_{\varepsilon,\perp})_{L_2(\partial\omega_\varepsilon)}\big| \leqslant C\varepsilon^{\frac{1}{2}}\|p_{\varepsilon,4}\|_{L_2(\partial\omega_\varepsilon)}\|v_{\varepsilon,\perp}\|_{\mathfrak{g}_\varepsilon} \leqslant C\varepsilon^{\frac{1}{2}}\|f\|_{L_2(\Omega)}\|v_{\varepsilon,\perp}\|_{\mathfrak{g}_\varepsilon}. \tag{91}$$

Employing asymptotics(8) and (26), condition (11), and estimate (82), we find that

$$\big|(p_{\varepsilon,4}, G)_{L_2(\partial\omega_\varepsilon)}\big| \leqslant C\varepsilon\|f\|_{L_2(\Omega)}$$

and, hence, in view of (59), (90), and (91),

$$\big|(p_{\varepsilon,4}, v_\varepsilon)_{L_2(\partial\omega_\varepsilon)}\big| \leqslant C\varepsilon^{\frac{1}{2}}\|f\|_{L_2(\Omega)}\|v_{\varepsilon,\perp}\|_{\mathfrak{g}_\varepsilon} + C\varepsilon^{\frac{3}{2}}\|f\|_{L_2(\Omega)}|(v_\varepsilon, G_\varepsilon)_{\mathfrak{g}_\varepsilon}|.$$

Summing up this estimate and (87)–(89), in view of (86), we obtain

$$\big|(p_\varepsilon, v_\varepsilon)_{L_2(\partial\omega_\varepsilon)}\big| \leqslant C\varepsilon^{\frac{1}{2}}\|f\|_{L_2(\Omega)}\|v_{\varepsilon,\perp}\|_{\mathfrak{g}_\varepsilon} + C\varepsilon\|f\|_{L_2(\Omega)}|(v_\varepsilon, G_\varepsilon)_{\mathfrak{g}_\varepsilon}|.$$

We take the imaginary part of identity (84) and use the above estimate:

$$\|v_\varepsilon\|^2_{L_2(\Omega_\varepsilon)} \leqslant C\varepsilon^{\frac{1}{2}}\|f\|_{L_2(\Omega)}\|v_{\varepsilon,\perp}\|_{\mathfrak{g}_\varepsilon} + C\varepsilon\|f\|_{L_2(\Omega)}|(v_\varepsilon, G_\varepsilon)_{\mathfrak{g}_\varepsilon}|.$$

Then, we take the imaginary part of identity (84) and employ the above inequality and (75):

$$\|v_{\varepsilon,\perp}\|_{\mathfrak{g}_\varepsilon} + \varepsilon^{\frac{1}{2}}|(v_\varepsilon, G_\varepsilon)_{\mathfrak{g}_\varepsilon}| \leqslant C\varepsilon^{\frac{1}{2}}\|f\|_{L_2(\Omega)}.$$

This implies that

$$\|v_{\varepsilon,\perp}\|_{\mathfrak{g}_\varepsilon} \leqslant C\varepsilon^{\frac{1}{2}}\|f\|_{L_2(\Omega)}, \qquad |(v_\varepsilon, G_\varepsilon)_{\mathfrak{g}_\varepsilon}| \leqslant C\|f\|_{L_2(\Omega)}. \tag{92}$$

Inequality (15) follows from the above estimates, (83), (85), and (59). It is also easy to see that for an arbitrary domain $\tilde{\Omega}$ described in the formulation of the theorem, we have

$$\|\chi_{\tilde{\Omega}}G\|_{\mathfrak{D}(\mathfrak{h}_\Omega)} \leqslant C.$$

Using this estimate and (92), we arrive at (16).

6. Order Sharpness

In this section, we show that estimates (15) and (16) are order-sharp by providing an appropriate example. We let

$$\begin{aligned}&\Omega := B_1(0), \qquad x_0 := 0, \qquad \Omega_0 := B_{\frac{1}{2}}(0), \\ &\hat{\mathcal{H}} := -\Delta, \qquad c_1 := 0, \qquad \mathcal{B}u = u,\end{aligned}$$

Then, A_0 is the unit matrix and, assuming that α_1 is a constant function,

$$G(x) = |x|^{-1} - 1, \qquad \alpha_0(x) = -|x|^{-1},$$
$$a_0 = -1, \qquad \beta_0 = 0, \qquad \alpha_1 = -\beta - 1.$$

We choose u_0 as

$$u_0(x) := v_0(x) + (\beta+1)^{-1}G(x), \qquad v_0(x) := w(|x|\varepsilon^{-1}),$$

where $w = w(\xi)$ is an infinitely differentiable even function on $\mathbb{R}$, vanishing outside $[-2,2]$ and obeying the conditions

$$w(\xi) \equiv 1 \quad \text{on} \quad [-1,1], \qquad f_0(\xi) := w''(\xi) + 2\xi^{-1}w'(\xi) \not\equiv 0 \quad \text{on} \quad [-2,2]\setminus[-1,1]. \tag{93}$$

The function v_0 obviously belongs to $W_2^2(\Omega)$ and vanishes on $\partial\Omega$. The function u_0 solves the equation

$$(\mathcal{H}_{0,\beta} + \mathrm{i})u_0 = f,$$
$$f(x) := -\varepsilon^{-2}f_0(|x|\varepsilon^{-1}) + \mathrm{i}w(|x|\varepsilon^{-1}) + \mathrm{i}(\beta+1)^{-1}G(x).$$

In view of the assumption of f_0 in (93), we immediately see that

$$\|f\|^2_{L_2(\Omega)} \geqslant C\varepsilon^{-4}\|f_0(|\cdot|\varepsilon^{-1})\|^2_{L_2(B_{2\varepsilon}(0))} - C \geqslant C\varepsilon^{-1},$$

where the Cs are some positive constants independent of ε. Using the first assumption in (93), it is also straightforward to confirm that

$$\begin{aligned}\left(\frac{\partial}{\partial\nu} + \alpha_0 + \alpha_1\right)u_0 =& \varepsilon^{-1}(\alpha_1 - \beta)(\beta+1)^{-1} + \alpha_1\beta(\beta+1)^{-1} \\ =& -\varepsilon^{-1}\big(2 - (\beta+1)^{-1}\big) - \beta.\end{aligned} \tag{94}$$

The function

$$Q(x) := \frac{1}{|x|}\sinh\frac{1+\mathrm{i}}{\sqrt{2}}(|x|-1)$$

solves the problem

$$\begin{aligned}&(-\Delta + \mathrm{i})Q = 0 \quad \text{in} \quad x \in \Omega\setminus\{0\}, \qquad Q = 0 \quad \text{on} \quad \partial\Omega,\\ &\left(\frac{\partial}{\partial\nu} + \alpha_0 + \alpha_1\right) = -\varepsilon^{-1}\frac{1+\mathrm{i}}{\sqrt{2}}\left(\cosh\frac{1+\mathrm{i}}{\sqrt{2}}(1-\varepsilon) + (2+\beta)\sinh\frac{1+\mathrm{i}}{\sqrt{2}}(1-\varepsilon)\right).\end{aligned} \tag{95}$$

We also observe that

$$\|Q\|_{L_2(\Omega_\varepsilon)} \geqslant C, \qquad \|Q\|_{W_2^1(\Omega_\varepsilon)} \geqslant C\varepsilon^{-\frac{1}{2}}, \qquad \|Q\|_{W_2^1(\Omega\setminus B_r(0))} \geqslant C(r), \tag{96}$$

where C and $C(r)$ are some fixed positive constants independent of ε.

Using problem (95) and identity (94), we easily see that the corresponding function $u_\varepsilon = (\mathcal{H}_\varepsilon + \mathrm{i})^{-1}f$ reads as $u_\varepsilon = u_0 - c_\varepsilon Q$, where

$$\begin{aligned}c_\varepsilon :=& \frac{\sqrt{2}}{1+\mathrm{i}}\frac{2 - (\beta+1)^{-1} + \varepsilon\beta}{\left(\cosh\frac{1+\mathrm{i}}{\sqrt{2}}(1-\varepsilon) + (2+\beta)\sinh\frac{1+\mathrm{i}}{\sqrt{2}}(1-\varepsilon)\right)}\\ =& \frac{\sqrt{2}}{1+\mathrm{i}}\frac{2 - (\beta+1)^{-1}}{\left(\cosh\frac{1+\mathrm{i}}{\sqrt{2}} + (2+\beta)\sinh\frac{1+\mathrm{i}}{\sqrt{2}}\right)} + O(\varepsilon).\end{aligned}$$

Hence, in view of (96),

$$\begin{aligned}
&\frac{\|u_\varepsilon - u_0\|_{L_2(\Omega\setminus B_r(0))}}{\|f\|_{L_2(\Omega)}} \geqslant C\varepsilon^{\frac{1}{2}},\\
&\frac{\|u_\varepsilon - u_0\|_{L_2(\Omega_\varepsilon)}}{\|f\|_{L_2(\Omega)}} \geqslant C(r)\varepsilon^{\frac{1}{2}},\\
&\frac{\|u_\varepsilon - u_0\|_{W_2^1(\Omega_\varepsilon)}}{\|f\|_{L_2(\Omega)}} \geqslant C,
\end{aligned} \tag{97}$$

where C and $C(r)$ are some fixed constants independent of ε. The first estimate shows that estimate (15) is order-sharp, while the second estimate does the same for (16). Estimate (97) ensures that estimate (17) is order-sharp. The proof of Theorem 1 is complete.

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