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Existence, Uniqueness, and Averaging Principle of Fractional Neutral Stochastic Differential Equations in the $\mathbb{L}^p$ Space with the Framework of the Ψ-Caputo Derivative

Abdelhamid Mohammed Djaouti [1,*], Zareen A. Khan [2,*,†], Muhammad Imran Liaqat [3,†] and Ashraf Al-Quran [1,†]

[1] Department of Mathematics and Statistics, Faculty of Sciences, King Faisal University, Hofuf 31982, Saudi Arabia; aalquran@kfu.edu.sa
[2] Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia
[3] Abdus Salam School of Mathematical Sciences, Government College University, 68-B, New Muslim Town, Lahore 54600, Pakistan; imran_liaqat_22@sms.edu.pk
* Correspondence: adjaout@kfu.edu.sa (A.M.D.); zakhan@pnu.edu.sa (Z.A.K.)
† These authors contributed equally to this work.

Abstract: In this research work, we use the concepts of contraction mapping to establish the existence and uniqueness results and also study the averaging principle in $\mathbb{L}^p$ space by using Jensen's, Grönwall–Bellman's, Hölder's, and Burkholder–Davis–Gundy's inequalities, and the interval translation technique for a class of fractional neutral stochastic differential equations. We establish the results within the framework of the Ψ-Caputo derivative. We generalize the two situations of $p = 2$ and the Caputo derivative with the findings that we obtain. To help with the understanding of the theoretical results, we provide two applied examples at the end.

Keywords: fractional calculus; Ψ-Caputo derivative; neutral stochastic differential equations; existence and uniqueness; averaging principle

MSC: 26A33; 69H07; 37C25

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1. Introduction

Fractional calculus (Fr-C) provides the basis for improving our understanding as well as our modeling skills in many different fields. Memory effects can be incorporated into mathematical models employing Fr-C. Many real-world systems have inherited or memory characteristics, meaning that their current actions are dependent on their previous states. A foundation for precisely capturing these memory effects is provided by Fr-C, allowing for more realistic modeling of complicated systems. Fr-C can represent non-local behaviors, in contrast to integer-order calculus, which focuses mostly on local interactions. This is especially helpful in systems where spatial correlations or long-range interactions are important. These non-local processes can be theoretically described using fractional-order derivatives (FrODs) and integrals. Integer-order calculus is treated as a specific instance within the more comprehensive mathematical framework provided by Fr-C.

Fr-C has many uses, of which the following are a few: when evaluating signals with fractal-like features or long-range correlations, Fr-C techniques prove to be beneficial in signal-processing work. Fractional differentiation and integration operators facilitate the meaningful information extraction process from these signals, resulting in improved processing and analysis capacities. Materials that are viscoelastic display characteristics of both liquids and solids. To better understand and forecast how they will respond to stress and strain, Fr-C is used to simulate their behavior. This is crucial in fields such as material science and structural engineering. Fr-C is used to explain electrochemical processes, such as charge transport in batteries and supercapacitors, and to analyze data from impedance spectroscopy. For more details about the uses of Fr-C, see [1–7].

FrODs are developed as various types, each with its own definition based on unique mathematical techniques and characteristics. Some of them are Grünwald–Letnikov, Riemann–Liouville (R-L), Caputo–Fabrizio, Atangana–Baleanu, Caputo, conformable, and Ψ-Caputo [8–14]. These are a few of the FrODs that are most frequently utilized. The choice of which type to utilize depends on the particular problem being addressed and the desired qualities of the FrODs.

In [15], Almeida proposed a novel FrOD concerning a kernel function known as the Ψ-Caputo derivative (Ψ-Cap-D) and extended the results of other academics [16–19]. They provide a flexible framework for modeling and analyzing complex dynamical systems with memory effects and non-local behaviors. The Ψ-Cap-D has some advantages over the classical Cap-D, such as being more flexible and having a higher order of accuracy. The Ψ-Cap-D has been widely used in various fields of science and engineering, including viscoelasticity, fractional control theory, and more. Some research publications in this field have expressed interest in the Ψ-Cap-D; for example, ref. [20] proposes a numerical investigation of the non-integer-order relaxation–oscillation equations in terms of the Ψ-Cap-D. Research on Ulam stability in the sense of the Ψ-Cap-D for Langevin non-integer order differential equations has been presented in [21]. An iterative technique was used by the authors in [22] to investigate an initial value problem for differential equations in the context of the Ψ-Cap-D.

Stochastic processes are used to describe systems that evolve over time in a random manner. Common types of stochastic processes include Brownian motion, Poisson processes, and Markov processes. Stochastic differential equations (SDEs) are mathematical equations that incorporate both deterministic dynamics and random fluctuations. They are widely used to model systems that evolve over time under the influence of both deterministic forces and stochastic influences. Fractional stochastic differential equations (FrSDEs) are differential equations that incorporate both stochastic processes and fractional derivatives. They are used to model systems where randomness and memory effects play significant roles. FrSDEs find applications in various fields, including physics, biology, anomalous diffusion processes, engineering, finance, population dynamics, and economics [23–26].

Fractional neutral stochastic differential equations (FrNSDEs) are a special kind of equation that depends on past and present values but also involves derivatives with delays as well as the function itself, Fr-C, and stochasticity. These equations are used to model systems where the evolution of a quantity depends not only on its current state but also on its past states, and where Fr-C is used to describe non-local behaviors or memory effects while incorporating randomness due to stochastic processes.

FrNSDEs find applications across various fields where systems exhibit memory effects or hereditary properties influenced by randomness. Some notable applications include the following: FrNSDEs can model population dynamics with memory effects, where the growth rate depends not only on the current population size but also on past populations. This is particularly relevant in ecological studies, where species interactions and environmental factors influence population growth. In economic modeling, FrNSDEs can capture the memory effects and long-term dependencies observed in economic time series data. They are used in modeling financial markets to account for the impact of past market behavior on future dynamics, such as in modeling stock price movements or interest rate fluctuations. FrNSDEs are utilized in control theory to model systems with memory effects, such as systems with delays or systems influenced by past inputs. They provide a more accurate representation of dynamical systems with feedback mechanisms, allowing for better control strategies in various engineering applications. For more details about applications of FrNSDEs, see [27–30]. These applications demonstrate the versatility of FrNSDEs in modeling diverse systems with memory effects and stochastic dynamics, offering insights into complex phenomena across various disciplines.

To use FrNSDEs effectively, we need to ensure that they have the properties of existence and uniqueness (Ex-Un). Existence means that there is at least one function that satisfies the FrNSDEs for a given set of initial conditions and random inputs. Uniqueness means that there is only one such function, and no other function can satisfy the same FrNSDEs. Ex-Un are important because they guarantee that the FrNSDEs has consistent and predictable behavior and that we can find and analyze its solution using various methods. The Ex-Un of FrNSDEs depends on the type of FrOD, the initial or boundary conditions, and the properties of the nonlinearities involved. Establishing the Ex-Un of solutions to FrNSDEs is crucial for several reasons, especially when dealing with complex systems exhibiting both randomness and memory effects.

The following are the main points that show the importance of Ex-Un [31,32]:

1. Just as with deterministic fractional differential equations (FrDEs), ensuring the Ex-Un of solutions to FrNSDEs is essential to validate the mathematical models used to describe real-world phenomena. If solutions exist and are unique, it suggests that the model accurately represents the underlying stochastic process and its dynamics.
2. Knowing that solutions to FrNSDEs exist and are unique provides confidence in the predictability and stability of stochastic systems over time. Uniqueness ensures that the solution is well defined and not affected by multiple possible outcomes, while existence guarantees that solutions can be found for given initial conditions.
3. In fields such as finance, engineering, and control systems, where FrNSDEs are commonly used to model stochastic processes, the Ex-Un of solutions are crucial for risk assessment, optimal control design, and decision-making. Unambiguous solutions enable accurate predictions and effective risk management strategies.
4. The Ex-Un of solutions to FrNSDEs is important for statistical analysis and inference, such as estimating parameters, predicting future behavior, and assessing confidence intervals. Having well-defined solutions allows researchers to make meaningful interpretations and draw reliable conclusions from observed data.
5. Ex-Un results guide the development and validation of numerical methods and algorithms for solving FrNSDEs. Knowing that solutions exist and are unique helps ensure the accuracy, stability, and convergence of numerical simulations, providing reliable tools for analyzing stochastic systems in practice.
6. Establishing the Ex-Un of solutions to FrNSDEs contributes to a better understanding of the underlying stochastic processes and their behavior. It allows researchers to interpret solutions in a physically meaningful way and gain insights into the complex interplay between randomness, memory effects, and system dynamics.

FrNSDEs research is highly popular these days. The Ex-Un, stability, artificial intelligence, electrical and electronic engineering, robust control, and most likely asymptotic estimations of the solution as well as random periodic solutions for FrNSDEs were the subject of study [33–41]. Approximate controllability and optimal control of FrNSDEs with a time lag in control were found in [38,42].

A useful technique for streamlining both stochastic and deterministic systems is the averaging principle. The averaging principle is a crucial and essential approximation theory that serves as a foundation for averaging procedures in mathematics, engineering mechanics, control, and other complex problems. It is an approximation principle that can balance complicated and simple systems to some extent. The fundamental idea behind the averaging principle is to prove an approximation theorem for SDEs, which in a way replaces the original system, and to provide the optimal order convergence theorem that goes along with it.

The averaging concept of FrSDEs has drawn the interest of certain academics in recent years. For instance, under certain innovative assumptions, Luo et al. derived an averaging principle for the solution of a class of FrSDEs with time delays [43]. Xu et al. [44] established the averaging principle in the mean square sense for FrSDEs with Cap-D driven by Brownian motion. The authors in [45] studied the averaging principle for SDEs with Poisson noises. For SDEs driven by Lévy noise, Xu [46] studied averaging principle.

The approximation theorem was studied by the authors [47] as an averaging concept for the solutions of Itô-Doob-type FrSDEs with non-Lipschitz coefficients in the sense of probability and mean square. For FrSDEs in the framework of Caputo–Hadamard, Liu et al. [48] obtained the averaging principle findings established in $\mathbb{L}^2$ space. Similarly, Yang et al. [49] developed the averaging principle results for Hilfer FrSDEs driven by Lévy noise, also in the sense of $\mathbb{L}^2$ space. Additionally, Liu et al. [50] established the results of the averaging principle for impulsive FrSDEs driven by fractional Brownian motion, with the findings also established in $\mathbb{L}^2$ space. Likewise, Duan et al. [51] investigated the averaging principle of a class of Caputo FrSDEs driven by fractional Brownian motion with delays, with the results similarly established in $\mathbb{L}^2$ space. Furthermore, Xu et al. [52] established the averaging principle for FrSDEs with Lévy motion in the sense of the Cap-D, with the results established in $\mathbb{L}^2$ space.

As can be seen from the foregoing discussion, the averaging principle of FrNSDEs does not receive much attention in the literature. Furthermore, the majority of earlier research works address the averaging principle of FrSDEs in terms of $\mathbb{L}^2$ convergence. Inspired by the aforementioned results, we first demonstrate the Ex-Un of solutions to a class of FrNSDEs using the Banach fixed-point theorem. The purpose of the second section is to study a class of the FrNSDEs averaging principle in the sense of the pth moment by utilizing the Grönwall–Bellman's inequality (Grön-Bell-Ineq), Jensen's inequality (Jen-Ineq), Burkholder–Davis–Gundy's (BHDG-Ineq) inequality, Hölder's inequality (Höld-Ineq) and the interval translation approach. Furthermore, we are establishing our results within the framework of the Ψ-Cap-D. Additionally, we provide examples and conduct numerical simulations to scrutinize the theoretical outcomes established in our research.

Compared with the research results of [34–54], the major contributions of this paper include at least the following three aspects:

1. In contrast to [43–54], the system we study is more generalized because in this manuscript we establish the results of FrNSDEs, which are more generalized than FrSDEs.
2. In this manuscript, we successfully establish the results in $\mathbb{L}^p$ space. In past studies, there have been many articles on the case of $p = 2$, such as [35–53].
3. In contrast to [34–54], in our research work, we establish results regarding the existence, uniqueness, and averaging principle in the sense of the Ψ-Cap-D.

We investigate a family of FrNSDEs that accurately describe real-world phenomena. This type of model might more accurately capture the impact of the system's historical status. We examine the following FrNSDEs:

$$\begin{cases} \mathscr{T}_\tau^{\vartheta,\Psi}\left[\mathbb{X}(\tau) - \mathbb{C}\big(\tau,\mathbb{X}(\tau),\mathbb{X}(\tau-\varpi)\big)\right] \\ = \mathbb{Y}\big(\tau,\mathbb{X}(\tau),\mathbb{X}(\tau-\varpi)\big) + \mathbb{Z}\big(\tau,\mathbb{X}(\tau),\mathbb{X}(\tau-\varpi)\big)\frac{\mathrm{d}\mathfrak{B}_\tau}{\mathrm{d}\tau},\ \tau \in [0,\mathbb{T}] \\ \mathbb{X}(\tau) = \boldsymbol{\mathfrak{X}}(\tau), \tau \in [-\varpi, 0], \end{cases} \tag{1}$$

where ϑ represents the order of the Ψ-Cap-D with $\vartheta \in (\frac{1}{2}, 1]$, $\mathbb{Y} : [0,\mathbb{T}] \times \mathcal{R}^{\mathfrak{f}} \times \mathcal{R}^{\mathfrak{f}} \to \mathcal{R}^{\mathfrak{f}}$, $\mathbb{Z} : [0,\mathbb{T}] \times \mathcal{R}^{\mathfrak{f}} \times \mathcal{R}^{\mathfrak{f}} \to \mathcal{R}^{\mathfrak{f}\times\mathfrak{b}}$, each of them being a measurable, continuous function. A complete probability space $(\Omega, \mathcal{F}, \mathfrak{P})$ defines the $\mathfrak{b}$-dimensional Brownian motion. The function $\boldsymbol{\mathfrak{X}} : [-\varpi, 0] \to \mathcal{R}^{\mathfrak{f}}$ is continuous. Suppose $\|.\|$ with $\mathscr{E}[\|\boldsymbol{\mathfrak{X}}(\tau)\|^2] < \infty$ is the norm for $\mathcal{R}^{\mathfrak{f}}$.

Our research work is structured as follows: We provide some definitions and fundamental assumptions in the section that follows, which act as a basis for deriving results about FrNSDEs. We establish the Ex-Un of the solution of FrNSDE in the first subsection of Section 3, establish the averaging principle theorem in the second section, and include examples to support our findings in Section 4. Section 5 then presents the conclusion.

2. Preliminaries

In this section, we present some basic definitions and assumptions that serve as the foundation for the results established in this paper.

Definition 1 ([15]). *The* Ψ*-Cap-D of order* ϑ *is defined as follows: assume* $\rho - 1 < \vartheta < \rho$, $\mathbb{G}(\tau) \in \mathfrak{C}^{\rho}[\mathfrak{r}, \mathfrak{q}]$, *and* $\Psi \in \mathfrak{C}^{\rho}[\mathfrak{r}, \mathfrak{q}]$ *is an increasing function with* $\Psi'(\tau) \neq 0\ \forall \tau \in [\mathfrak{r}, \mathfrak{q}]$, *then we have*

$$\begin{cases} \mathscr{T}_{\tau}^{\vartheta,\Psi}\mathbb{G}(\tau) = \big(\mathbb{I}_{\tau}^{\rho-\vartheta,\Psi}\mathbb{G}^{[\rho]}\big)(\tau) \\ = \frac{1}{\Gamma(\rho-\vartheta)}\int_{\mathfrak{r}}^{\tau}\Psi'(\lambda)\big(\Psi(\tau)-\Psi(\lambda)\big)^{\rho-\vartheta-1}\mathbb{G}^{[\rho]}(\lambda)\mathrm{d}\lambda, \end{cases}$$

where $\rho = [\vartheta] + 1$ *and* $\mathbb{G}^{[\rho]}(\tau) = \left(\frac{1}{\Psi'(\tau)}\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^{\rho}\mathbb{G}(\tau)$ *on* $[\mathfrak{r}, \mathfrak{q}]$.

Definition 2 ([15]). *The* Ψ*-fractional integral of order* ϑ *for* $\mathbb{G}(\tau)$ *is defined as follows: suppose that* $\vartheta > 0$, $\mathbb{G}(\tau)$ *is an integrable function defined on* $[\mathfrak{r}, \mathfrak{q}]$ *and* $\Psi \in \mathfrak{C}^1[\mathfrak{r}, \mathfrak{q}]$ *is an increasing function with* $\Psi'(\tau) \neq 0\ \forall \tau \in [\mathfrak{r}, \mathfrak{q}]$, *then we have*

$$\mathbb{I}_{\mathfrak{r}^+}^{\vartheta,\Psi}\mathbb{G}(\tau) = \frac{1}{\Gamma(\vartheta)}\int_{\mathfrak{r}}^{\tau}\Psi'(\lambda)\big(\Psi(\tau)-\Psi(\lambda)\big)^{\vartheta-1}\mathbb{G}(\lambda)\mathrm{d}\lambda.$$

Definition 3. *If* $\mathbb{X}(\tau)$ *is* $\mathcal{F}(\tau)$*-adapted and* $\mathscr{E}\left[\int_{-\varpi}^{\mathbb{T}}\|\mathbb{X}(\tau)\|\mathrm{d}\tau\right] < \infty$, $\boldsymbol{\mathfrak{X}}(0) = \boldsymbol{\mathfrak{X}}_0$, *and it satisfies the following conditions, then an* $\mathcal{R}^{\mathfrak{f}}$*-value stochastic process* $\{\mathbb{X}(\tau)\}_{-\varpi \le \tau \le \mathbb{T}}$ *is a unique solution to Equation (1):*

$$\begin{cases} \mathbb{X}(\tau) = & \boldsymbol{\mathfrak{X}}_0 - \mathbb{C}(0,\boldsymbol{\mathfrak{X}}(0),\boldsymbol{\mathfrak{X}}(-\varpi)) + \mathbb{C}(\tau,\mathbb{X}(\tau),\mathbb{X}(\tau-\varpi)) \\ & +\frac{1}{\Gamma(\vartheta)}\int_0^{\tau}\big(\Psi(\tau)-\Psi(\lambda)\big)^{\vartheta-1}\Psi'(\lambda)\mathbb{Y}(\lambda,\mathbb{X}(\lambda),\mathbb{X}(\lambda-\varpi))\mathrm{d}\lambda \\ & +\frac{1}{\Gamma(\vartheta)}\int_0^{\tau}\big(\Psi(\tau)-\Psi(\lambda)\big)^{\vartheta-1}\Psi'(\lambda)\mathbb{Z}(\lambda,\mathbb{X}(\lambda),\mathbb{X}(\lambda-\varpi))\mathrm{d}\mathfrak{B}_{\lambda},\ \tau \in [0,\mathbb{T}], \\ \mathbb{X}(\tau) = & \boldsymbol{\mathfrak{X}}(\tau), \tau \in [-\varpi, 0]. \end{cases} \tag{2}$$

Now we make the assumption that coefficient $\mathbb{C}$ with $\|\mathbb{C}\big(0,\boldsymbol{\mathfrak{X}}(0),\boldsymbol{\mathfrak{X}}(-\varpi)\big)\| < \infty$, and the uniformly continuous functions $\mathbb{Y}$ and $\mathbb{Z}$ in Equation (1) when $\forall \mathscr{G}_1, \mathscr{G}_2, \mathcal{Y}_1, \mathcal{Y}_2, \mathscr{G}, \mathcal{Y} \in \mathcal{R}^{\mathfrak{f}}$, $\tau \in [-\varpi, \mathbb{T}]$, there are $\xi_1, \xi_2, \xi_3 > 0$ that meet the following requirements:
$(\mathscr{A}_1)$:

$$\|\mathbb{C}(\tau,\mathscr{G}_1,\mathscr{G}_2) - \mathbb{C}(\tau,\mathcal{Y}_1,\mathcal{Y}_2)\| \le \xi_1\big(\|\mathscr{G}_1 - \mathcal{Y}_1\| + \|\mathscr{G}_2 - \mathcal{Y}_2\|\big).$$

$(\mathscr{A}_2)$:

$$\begin{aligned} \|\mathbb{Y}(\tau,\mathscr{G}_1,\mathscr{G}_2) - \mathbb{Y}(\tau,\mathcal{Y}_1,\mathcal{Y}_2)\| \vee \|\mathbb{Z}(\tau,\mathscr{G}_1,\mathscr{G}_2) - \mathbb{Z}(\tau,\mathcal{Y}_1,\mathcal{Y}_2)\| \\ \le \xi_2\big(\|\mathscr{G}_1 - \mathcal{Y}_1\| + \|\mathscr{G}_2 - \mathcal{Y}_2\|\big). \end{aligned}$$

where $\mathbb{Y} \vee \mathbb{Z} = \max(\mathbb{Y}, \mathbb{Z})$.
$(\mathscr{A}_3)$:

$$\|\mathbb{Y}(\tau,\mathscr{G},\mathcal{Y})\| \vee \|\mathbb{Z}(\tau,\mathscr{G},\mathcal{Y})\| \le \xi_3\big(1 + \|\mathscr{G}\| + \|\mathcal{Y}\|\big).$$

$(\mathscr{A}_4)$: Functions $\widetilde{\mathbb{Y}}_1$ and $\widetilde{\mathbb{Z}}_2$ exist and for $\mathbb{T}_1 \in [0,\mathbb{T}]$, $\tau \in [0,\mathbb{T}]$, and $p \ge 2$, we are able to identify positively bound functions $\mathscr{Y}_1(\mathbb{T}_1)$ and $\mathscr{Y}_2(\mathbb{T}_1)$ that fulfill

$$\frac{1}{\mathbb{T}_1}\int_0^{\mathbb{T}_1}\|\mathbb{Y}(\tau,\mathscr{G},\mathcal{Y}) - \widetilde{\mathbb{Y}}_1(\tau,\mathscr{G},\mathcal{Y})\|^p\mathrm{d}\tau \le \mathscr{Y}_1(\mathbb{T}_1)\big(1 + \|\mathscr{G}\|^p + \|\mathcal{Y}\|^p\big),$$
$$\frac{1}{\mathbb{T}_1}\int_0^{\mathbb{T}_1}\|\mathbb{Z}(\tau,\mathscr{G},\mathcal{Y}) - \widetilde{\mathbb{Z}}_2(\tau,\mathscr{G},\mathcal{Y})\|^p d\tau \le \mathscr{Y}_2(\mathbb{T}_1)\big(1 + \|\mathscr{G}\|^p + \|\mathcal{Y}\|^p\big).$$

Lemma 1 ([43]). *Assume that there are real numbers* $\sigma_1, \sigma_2, \ldots, \sigma_{\upsilon}(\upsilon \in \mathbb{N})$ *and meet* $\sigma_{\jmath} \ge 0, (\jmath = 1, 2, \ldots, \upsilon)$. *Then,*

$$\left(\sum_{\jmath=1}^{\upsilon}\sigma_{\jmath}\right)^p \le \upsilon^{p-1}\sum_{\jmath=1}^{\upsilon}\sigma_{\jmath}^p,\ \forall p > 1.$$

3. The Main Results

In this part, first of all, we determine a useful lemma, and then, by utilizing the Banach fixed-point theorem (BFPT), we establish the results of Ex-Un of Equation (1). The BFPT is an effective method widely used by researchers to study the Ex-Un of solutions to FrSDEs [52,54–56]. The BFPT guarantees the Ex-Un of solutions to certain types of mappings in complete metric spaces. This is particularly valuable in the context of FrSDEs, where ensuring the Ex-Un of solutions is crucial for understanding the behavior of the underlying stochastic processes. BFPT offers a powerful tool for generalizing classical results from deterministic differential equations to stochastic and fractional settings. By establishing Ex-Un results using Banach's theorem, researchers extend classical theories to more complex systems governed by FrSDEs.

The importance of BFPT relative to other fixed-point theorems lies in its generality, simplicity of formulation, and wide applicability. BFPT does not impose strong assumptions on the nature of the mapping or the structure of the space, making it applicable to a wide range of mathematical problems. Its statement is concise and easy to understand, involving a simple contraction condition on the mapping, which is straightforward to verify in many cases. This simplicity facilitates its use in various mathematical contexts and makes it accessible to a wide range of researchers.

Lemma 2. *For every $\mathbb{T}_1 \in [0, \mathbb{T}]$, we can derive the following growth requirements for $\widetilde{\mathbb{Z}}_2$, utilizing assumptions $(\mathscr{A}_3)$, $(\mathscr{A}_4)$:*

$$\|\widetilde{\mathbb{Z}}_2(\mathscr{G}, \mathcal{Y})\|^p \le \xi_4(1 + \|\mathscr{G}\|^p + \|\mathcal{Y}\|^p),$$

where $\xi_4 = \left(2^{p-1}\mathscr{Y}_2(\mathbb{T}_1) + 6^{p-1}\xi_3^p\right)$.

Proof. Considering Jen-Ineq and assumptions $(\mathscr{A}_3)$, $(\mathscr{A}_4)$, we derive the following result:

$$\begin{aligned}
\|\widetilde{\mathbb{Z}}_2(\mathscr{G}, \mathcal{Y})\|^p &\le 2^{p-1}\|\mathbb{Z}(\tau, \mathscr{G}, \mathcal{Y}) - \widetilde{\mathbb{Z}}_2(\mathscr{G}, \mathcal{Y})\|^p + 2^{p-1}\|\mathbb{Z}(\tau, \mathscr{G}, \mathcal{Y})\|^p \\
&\le 2^{p-1}\mathscr{Y}_2(\mathbb{T}_1)(1 + \|\mathscr{G}\|^p + \|\mathcal{Y}\|^p) + 2^{p-1}\xi_3^p(1 + \|\mathscr{G}\| + \|\mathcal{Y}\|)^p \\
&\le \left(2^{p-1}\mathscr{Y}_2(\mathbb{T}_1) + 6^{p-1}\xi_3^p\right)(1 + \|\mathscr{G}\|^p + \|\mathcal{Y}\|^p).
\end{aligned}$$

□

Theorem 1. *If $(\mathscr{A}_1)$ and $(\mathscr{A}_2)$ are valid, then the problem Equation (1) has a unique solution with the following condition:*

$$\begin{aligned}
\mathbb{S} =& 2 \cdot 6^{p-1}\xi_1^p\mathbb{D} + 6^{p-1}\xi_2^p(\Psi(\tau) - \Psi(0))^{p\vartheta}\left(\frac{p-1}{\vartheta p - 1}\right)^{p-1}\frac{1}{\Gamma^p(\vartheta)} \\
&+ \mathbb{D}^{\frac{p}{2}}\frac{2^p 3^{p-1}\xi_2^p\big(\Psi(\tau) - \Psi(0)\big)^{\frac{(2\vartheta-1)p}{2}}}{(2\vartheta - 1)^{\frac{p}{2}}}\left(\frac{p^{p+1}}{2(p-1)^{p-1}}\right)^{\frac{p}{2}}\frac{1}{\Gamma^p(\vartheta)},
\end{aligned} \tag{3}$$

where $\mathbb{S}$ is a non-negative number, and $\mathbb{S} < 1$

Proof. We construct an operator $\mathbb{A} : \beta \to \beta$ by $\mathbb{X}(\tau) = \boldsymbol{\mathfrak{X}}(\tau),\ \tau \in [-\varpi, 0]$ and the subsequent equality is valid:

$$\begin{aligned}
\mathbb{A}\big(\mathbb{X}(\tau)\big) =& \boldsymbol{\mathfrak{X}}_0 - \mathbb{C}(0, \boldsymbol{\mathfrak{X}}(0), \boldsymbol{\mathfrak{X}}(-\varpi)) + \mathbb{C}\big(\tau, \mathbb{X}(\tau), \mathbb{X}(\tau - \varpi)\big) \\
&+ \frac{1}{\Gamma(\vartheta)}\int_0^\tau \big(\Psi(\tau) - \Psi(\lambda)\big)^{\vartheta-1}\Psi'(\lambda)\mathbb{Y}\big(\lambda, \mathbb{X}(\lambda), \mathbb{X}(\lambda - \varpi)\big)\mathrm{d}\lambda \\
&+ \frac{1}{\Gamma(\vartheta)}\int_0^\tau \big(\Psi(\tau) - \Psi(\lambda)\big)^{\vartheta-1}\Psi'(\lambda)\mathbb{Z}\big(\lambda, \mathbb{X}(\lambda), \mathbb{X}(\lambda - \varpi)\big)\mathrm{d}\boldsymbol{\mathfrak{B}}_\lambda.
\end{aligned} \tag{4}$$

Step 1: We are going to first demonstrate that $\mathbb{A}$ maps β into β. Suppose $\mathbb{X}(\tau) \in \beta$, and here $\mathbb{X}(\tau)$ is arbitrary. We have the following $\forall \tau \in [0, \mathbb{T}]$ from the description of $\mathbb{A}(\mathbb{X}(\tau))$ as in Equation (4) and the Jen-Ineq:

$$\begin{aligned}
&\mathscr{E}\left[\|\mathbb{A}(\mathbb{X}(\tau))\|^p\right] \leq 4^{p-1}\mathscr{E}\left[\|\boldsymbol{\varkappa}_0\|^p\right] \\
&+ 4^{p-1}\mathscr{E}\left[\|\mathbb{C}(\tau, \mathbb{X}(\tau), \mathbb{X}(\tau-\varpi)) - \mathbb{C}(0, \boldsymbol{\varkappa}_0, \boldsymbol{\varkappa}(-\varpi))\|^p\right] \\
&+ 4^{p-1}\frac{1}{\Gamma^p(\vartheta)}\mathscr{E}\left[\left\|\int_0^\tau \left(\Psi(\tau)-\Psi(\lambda)\right)^{\vartheta-1}\Psi'(\lambda)\mathbb{Y}(\lambda, \mathbb{X}(\lambda), \mathbb{X}(\lambda-\varpi))\mathrm{d}\lambda\right\|^p\right] \\
&+ 4^{p-1}\frac{1}{\Gamma^p(\vartheta)}\mathscr{E}\left[\left\|\int_0^\tau \left(\Psi(\tau)-\Psi(\lambda)\right)^{\vartheta-1}\Psi'(\lambda)\mathbb{Z}(\lambda, \mathbb{X}(\lambda), \mathbb{X}(\lambda-\varpi))\mathrm{d}\mathfrak{B}_\lambda\right\|^p\right] \\
&= \mathfrak{I}_1 + \mathfrak{I}_2 + \mathfrak{I}_3 + \mathfrak{I}_4.
\end{aligned} \tag{5}$$

By employing $(\mathscr{A}_1)$ and Jen-Ineq, we achieve the following:

$$\begin{aligned}
\mathfrak{I}_2 &= 4^{p-1}\mathscr{E}\left[\|\mathbb{C}(\tau, \mathbb{X}(\tau), \mathbb{X}(\tau-\varpi)) - \mathbb{C}(0, \boldsymbol{\varkappa}_0, \boldsymbol{\varkappa}(-\varpi))\|^p\right] \\
&\leq 4^{p-1}\xi_1^p\mathscr{E}\left[\|\mathbb{X}(\tau)-\boldsymbol{\varkappa}_0\| + \|\mathbb{X}(\tau-\varpi)-\boldsymbol{\varkappa}(-\varpi)\|\right]^p \\
&\leq 8^{p-1}\xi_1^p\left(\mathscr{E}\left[\|\mathbb{X}(\tau)-\boldsymbol{\varkappa}_0\|^p\right] + \mathscr{E}\left[\|\mathbb{X}(\tau-\varpi)-\boldsymbol{\varkappa}(-\varpi)\|^p\right]\right) \\
&\leq 4\cdot 8^{p-1}\xi_1^p\mathscr{E}\|\mathbb{X}\|^p.
\end{aligned} \tag{6}$$

Applying Höld-Ineq, Jen-Ineq, and $(\mathscr{A}_3)$, we extract the subsequent outcomes:

$$\begin{aligned}
\mathfrak{I}_3 &= 4^{p-1}\frac{1}{\Gamma^p(\vartheta)}\mathscr{E}\left[\left\|\int_0^\tau \left(\Psi(\tau)-\Psi(\lambda)\right)^{\vartheta-1}\Psi'(\lambda)\mathbb{Y}(\lambda, \mathbb{X}(\lambda), \mathbb{X}(\lambda-\varpi))\mathrm{d}\lambda\right\|^p\right] \\
&\leq 4^{p-1}\frac{1}{\Gamma^p(\vartheta)}\left(\int_0^\tau \left(\Psi(\tau)-\Psi(\lambda)\right)^{\frac{p(\vartheta-1)}{p-1}}\left(\Psi'(\lambda)\right)^{\frac{p}{p-1}}\mathrm{d}\lambda\right)^{p-1}\mathscr{E}\left[\int_0^\tau \|\mathbb{Y}(\lambda, \mathbb{X}(\lambda), \mathbb{X}(\lambda-\varpi))\|^p\mathrm{d}\lambda\right] \\
&\leq 4^{p-1}\mathbb{D}\frac{1}{\Gamma^p(\vartheta)}\left(\Psi(\tau)-\Psi(0)\right)^{p\vartheta-1}\left(\frac{p-1}{p\vartheta-1}\right)^{p-1}\mathscr{E}\left[\int_0^\tau \xi_3^p\left(1+\|\mathbb{X}(\lambda)\|+\|\mathbb{X}(\lambda-\varpi)\|\right)^p\mathrm{d}\lambda\right] \\
&\leq 8^{p-1}\mathbb{D}\xi_3^p\left(\Psi(\tau)-\Psi(0)\right)^{p\vartheta}\frac{1}{\Gamma^p(\vartheta)}\left(\frac{p-1}{p\vartheta-1}\right)^{p-1}\left(1+2^p\mathscr{E}\left[\|\mathbb{X}\|^p\right]\right).
\end{aligned} \tag{7}$$

where, $\mathbb{D} = \sup_{\lambda\in[-\varpi,\mathbb{T}]}\Psi'(\lambda)$. Now, applying the BHDG-Ineq, Jen-Ineq, and $(\mathscr{A}_3)$, we get

$$\begin{aligned}
\mathfrak{I}_4 &\leq 4^{p-1}\frac{1}{\Gamma^p(\vartheta)}\mathscr{E}\left[\sup_{\tau\in[-\varpi,\mathbb{T}]}\left\|\int_0^\tau \left(\Psi(\tau)-\Psi(\lambda)\right)^{\vartheta-1}\Psi'(\lambda)\mathbb{Z}(\lambda, \mathbb{X}(\lambda), \mathbb{X}(\lambda-\varpi))\mathrm{d}\mathfrak{B}_\lambda\right\|^p\right] \\
&\leq \left(\frac{(p)^{p+1}}{2(p-1)^{p-1}}\right)^{\frac{p}{2}}\frac{4^{p-1}}{\Gamma^p(\vartheta)}\mathscr{E}\left[\int_0^\tau \left(\Psi(\tau)-\Psi(\lambda)\right)^{2\vartheta-2}\left(\Psi'(\lambda)\right)^2\left\|\mathbb{Z}(\lambda, \mathbb{X}(\lambda), \mathbb{X}(\lambda-\varpi))\right\|^2\mathrm{d}\lambda\right]^{\frac{p}{2}} \\
&\leq \mathbb{D}^{\frac{p}{2}}\frac{2^{\frac{p}{2}}4^{p-1}}{(2\vartheta-1)^{\frac{p}{2}}}\left(\Psi(\tau)-\Psi(0)\right)^{\frac{(2\vartheta-1)p}{2}}\xi_3^p\frac{1}{\Gamma^p(\vartheta)}\left(2(p-1)^{1-p}p^{p+1}\right)^{\frac{p}{2}}\left(1+2\mathscr{E}\left[\|\mathbb{X}\|^2\right]\right)^{\frac{p}{2}} \\
&\leq \mathbb{D}^{\frac{p}{2}}\frac{8^{p-1}\xi_3^p\left(\Psi(\tau)-\Psi(0)\right)^{\frac{(2\vartheta-1)p}{2}}}{(2\vartheta-1)^{\frac{p}{2}}}\left(2(p-1)^{1-p}(p)^{p+1}\right)^{\frac{p}{2}}\frac{1}{\Gamma^p(\vartheta)}\left(1+2^{\frac{p}{2}}\mathscr{E}\left[\|\mathbb{X}\|^p\right]\right).
\end{aligned} \tag{8}$$

By utilizing Equations (6)–(8) in (5), we achieve the following outcomes:

$$\begin{aligned}\mathscr{E}\big[\|\mathbb{A}(\mathbb{X}(\tau))\|^p\big] \leq{}& 4^{p-1}\mathscr{E}\big[\|\boldsymbol{\mathfrak{X}}_0\|^p\big] + 4\cdot 8^{p-1}\xi_1^p\mathscr{E}\big[\|\mathbb{X}\|^p\big] \\ &+ \big(1+2^p\mathscr{E}\big[\|\mathbb{X}\|^p\big]\big)\xi_3^p\big(\Psi(\tau)-\Psi(0)\big)^{p\vartheta}\left(\frac{p-1}{p\vartheta-1}\right)^{p-1}8^{p-1}4^{p-1}\frac{\mathbb{D}}{\Gamma^p(\vartheta)} \\ &+ \frac{8^{p-1}\xi_3^p\big(\Psi(\tau)-\Psi(0)\big)^{\frac{(2\vartheta-1)p}{2}}}{(2\vartheta-1)^{\frac{p}{2}}}\big(2(p-1)^{1-p}p^{p+1}\big)^{\frac{p}{2}}\big(1+2^{\frac{p}{2}}\mathscr{E}\big[\|\mathbb{X}\|^p\big]\big)4^{p-1}\frac{1}{\Gamma^p(\vartheta)}\mathbb{D}^{\frac{p}{2}}.\end{aligned} \tag{9}$$

When put together with the information from the previous discussion, it is simply gained that the following condition is satisfied by a constant $\mathscr{C}$:

$$\mathscr{E}\big[\|\mathbb{A}(\mathbb{X}(\tau))\|^p\big] \leq \mathscr{C}\big(1+\mathscr{E}\big[\|\mathbb{X}\|^p\big]\big).$$

In other words, $\mathbb{A}$ maps β into β.

Step 2: Now, we will prove that the map $\mathbb{A}$ is contractive for this purpose. Let $\mathbb{X}(\tau)$, $\boldsymbol{\mathfrak{E}}(\tau)$ be arbitrary. We obtain the following $\forall\tau\in[0,\mathbb{T}]$ from Equation (4) and Jen-Ineq:

$$\begin{aligned}&\mathscr{E}\big[\|\mathbb{A}(\mathbb{X}(\tau))-\mathbb{A}(\boldsymbol{\mathfrak{E}}(\tau))\|^p\big] \\ \leq{}&3^{p-1}\mathscr{E}\big[\|\mathbb{C}(\tau,\mathbb{X}(\tau),\mathbb{X}(\tau-\varpi))-\mathbb{C}(\tau,\boldsymbol{\mathfrak{E}}(\tau),\boldsymbol{\mathfrak{E}}(\tau-\varpi))\|^p\big] \\ &+3^{p-1}\mathscr{E}\left[\left\|\frac{1}{\Gamma(\vartheta)}\int_0^\tau\big(\Psi(\tau)-\Psi(\lambda)\big)^{\vartheta-1}\Psi'(\lambda)\big(\mathbb{Y}(\lambda,\mathbb{X}(\lambda),\mathbb{X}(\lambda-\varpi))-\mathbb{Y}(\lambda,\boldsymbol{\mathfrak{E}}(\lambda),\boldsymbol{\mathfrak{E}}(\lambda-\varpi))\big)\mathrm{d}\lambda\right\|^p\right] \\ &+3^{p-1}\mathscr{E}\left[\left\|\frac{1}{\Gamma(\vartheta)}\int_0^\tau\big(\Psi(\tau)-\Psi(\lambda)\big)^{\vartheta-1}\Psi'(\lambda)\big(\mathbb{Z}(\lambda,\mathbb{X}(\lambda),\mathbb{X}(\lambda-\varpi))-\mathbb{Z}(\lambda,\boldsymbol{\mathfrak{E}}(\lambda),\boldsymbol{\mathfrak{E}}(\lambda-\varpi))\big)\mathrm{d}\boldsymbol{\mathfrak{B}}_\lambda\right\|^p\right] \\ ={}&\mathfrak{I}_5+\mathfrak{I}_6+\mathfrak{I}_7.\end{aligned} \tag{10}$$

Employing Jen-Ineq and $(\mathscr{A}_1)$, the following is achieved as a result:

$$\begin{aligned}\mathfrak{I}_5 &= 3^{p-1}\mathscr{E}\big[\|\mathbb{C}(\tau,\mathbb{X}(\tau),\mathbb{X}(\tau-\varpi))-\mathbb{C}(\tau,\boldsymbol{\mathfrak{E}}(\tau),\boldsymbol{\mathfrak{E}}(\tau-\varpi))\|^p\big] \\ &\leq 3^{p-1}\xi_1^p\mathscr{E}\big[\|\mathbb{X}(\tau)-\boldsymbol{\mathfrak{E}}(\tau)\|+\|\mathbb{X}(\tau-\varpi)-\boldsymbol{\mathfrak{E}}(\tau-\varpi)\|\big]^p \\ &\leq 6^{p-1}\xi_1^p\Big(\mathscr{E}\big[\|\mathbb{X}(\tau)-\boldsymbol{\mathfrak{E}}(\tau)\|^p\big]+\mathscr{E}\big[\|\mathbb{X}(\tau-\varpi)-\boldsymbol{\mathfrak{E}}(\tau-\varpi)\|^p\big]\Big) \\ &\leq 2\cdot 6^{p-1}\xi_1^p\sup_{\tau\in[-\varpi,\mathbb{T}]}\mathscr{E}\big[\|\mathbb{X}(\tau)-\boldsymbol{\mathfrak{E}}(\tau)\|^p\big].\end{aligned} \tag{11}$$

Using the Höld-Ineq and $(\mathscr{A}_2)$, we obtain

$$\begin{aligned}\mathfrak{I}_6 \leq{}&3^{p-1}\frac{1}{\Gamma^p(\vartheta)}\left(\int_0^\tau\big(\Psi(\tau)-\Psi(\lambda)\big)^{\frac{(\vartheta-1)p}{p-1}}\big(\Psi'(\lambda)\big)^{\frac{p}{p-1}}d\lambda\right)^{p-1} \\ &\mathscr{E}\left[\int_0^\tau\|\mathbb{Y}(\lambda,\mathbb{X}(\lambda),\mathbb{X}(\lambda-\varpi))-\mathbb{Y}(\lambda,\boldsymbol{\mathfrak{E}}(\lambda),\boldsymbol{\mathfrak{E}}(\lambda-\varpi))\|^p\mathrm{d}\lambda\right] \\ \leq{}&3^{p-1}\mathbb{D}\big(\Psi(\tau)-\Psi(0)\big)^{p\vartheta-1}\frac{1}{\Gamma^p(\vartheta)}\left(\frac{p-1}{\vartheta p-1}\right)^{p-1} \\ &\int_0^\tau\xi_2^p\mathscr{E}\big[\|\mathbb{X}(\lambda)-\boldsymbol{\mathfrak{E}}(\lambda)\|+\|\mathbb{X}(\lambda-\varpi)-\boldsymbol{\mathfrak{E}}(\lambda-\varpi)\|^p\mathrm{d}\lambda\big] \\ \leq{}&6^{p-1}\mathbb{D}\frac{1}{\Gamma^p(\vartheta)}\xi_2^p\big(\Psi(\tau)-\Psi(0)\big)^{\vartheta p}\left(\frac{p-1}{\vartheta p-1}\right)^{p-1}\sup_{\tau\in[-\varpi,\mathbb{T}]}\mathscr{E}\big[\|\mathbb{X}(\tau)-\boldsymbol{\mathfrak{E}}(\tau)\|^p\big].\end{aligned} \tag{12}$$

However, using $(\mathscr{A}_2)$ and the BHDK-Ineq, we have

$$
\begin{aligned}
\mathfrak{J}_7 = & 3^{p-1}\mathscr{E}\Bigg[\sup_{\tau\in[-\varpi,\mathbb{T}]}\Big\|\frac{1}{\Gamma(\vartheta)}\int_0^\tau\big(\Psi(\tau)-\Psi(\lambda)\big)^{\vartheta-1}\Psi'(\lambda)\\
&\Big(\mathbb{Z}(\lambda,\mathbb{X}(\lambda),\mathbb{X}(\lambda-\varpi))-\mathbb{Z}(\lambda,\mathfrak{E}(\lambda),\mathfrak{E}(\lambda-\varpi))\Big)\mathrm{d}\mathfrak{B}_\lambda\Big\|^p\Bigg]\\
\leq & 3^{p-1}\left(\frac{(p)^{p+1}}{2(p-1)^{p-1}}\right)^{\frac{p}{2}}\frac{1}{\Gamma^p(\vartheta)}\\
&\mathscr{E}\left[\int_0^\tau\big(\Psi(\tau)-\Psi(\lambda)\big)^{2\vartheta-2}\big(\Psi'(\lambda)\big)^2\|\mathbb{Z}(\lambda,\mathbb{X}(\lambda),\mathbb{X}(\lambda-\varpi))-\mathbb{Z}(\lambda,\mathfrak{E}(\lambda),\mathfrak{E}(\lambda-\varpi))\|^2\mathrm{d}\lambda\right]^{\frac{p}{2}}\\
\leq & \mathbb{D}^{\frac{p}{2}}\frac{2^p3^{p-1}\zeta_2^p\big(\Psi(\tau)-\Psi(0)\big)^{\frac{(2\vartheta-1)p}{2}}}{(2\vartheta-1)^{\frac{p}{2}}}\left(\frac{p^{p+1}}{2(p-1)^{p-1}}\right)^{\frac{p}{2}}\frac{1}{\Gamma^p(\vartheta)}\sup_{\tau\in[-\varpi,\mathbb{T}]}\mathscr{E}\big[\|\mathbb{X}(\tau)-\mathfrak{E}(\tau)\|^p\big]. \qquad (13)
\end{aligned}
$$

By utilizing Equations (11)–(13) in (10), we extract the following outcomes:

$$
\begin{aligned}
&\mathscr{E}\big[\|\mathbb{A}(\mathbb{X}(\tau))-\mathbb{A}(\mathfrak{E}(\tau))\|^p\big]\\
&\leq\Bigg(2\cdot 6^{p-1}\zeta_1^p\mathbb{D}+6^{p-1}\zeta_2^p\big(\Psi(\tau)-\Psi(0)\big)^{p\vartheta}\left(\frac{p-1}{\vartheta p-1}\right)^{p-1}\frac{1}{\Gamma^p(\vartheta)}\\
&+\mathbb{D}^{\frac{p}{2}}\frac{2^p3^{p-1}\zeta_2^p\big(\Psi(\tau)-\Psi(0)\big)^{\frac{(2\vartheta-1)p}{2}}}{(2\vartheta-1)^{\frac{p}{2}}}\left(\frac{p^{p+1}}{2(p-1)^{p-1}}\right)^{\frac{p}{2}}\frac{1}{\Gamma^p(\vartheta)}\Bigg)\sup_{\tau\in[-\varpi,\mathbb{T}]}\mathscr{E}\big[\|\mathbb{X}(\tau)-\mathfrak{E}(\tau)\|^p\big]\\
&\leq\mathbb{S}\|\mathbb{X}(\tau)-\mathfrak{E}(\tau)\|^p. \qquad (14)
\end{aligned}
$$

From Equation (3), we obtain $\mathbb{S}<1$, the operator $\mathbb{A}$ is a contractive map, so there is a single fixed point $\mathbb{X}(\tau)\in\beta$ of this map, with the initial function $\mathbb{X}(\tau)=\mathfrak{X}(\tau),\ \tau\in[-\varpi,0]$, according to the BFPT. □

Averaging Principle Result

The averaging principle of FrNSDEs in the sense of $\mathbb{L}^p$ is first examined in this section. Initially, the standard form of Equation (2) will be examined:

$$
\begin{aligned}
\mathbb{X}_\varepsilon(\tau)=&\mathfrak{X}_0-\mathbb{C}(0,\mathfrak{X}(0),\mathfrak{X}(-\varpi))+\mathbb{C}\big(\tau,\mathbb{X}_\varepsilon(\tau),\mathbb{X}_\varepsilon(\tau-\varpi)\big)\\
&+\varepsilon\frac{1}{\Gamma(\vartheta)}\int_0^\tau\big(\Psi(\tau)-\Psi(\lambda)\big)^{\vartheta-1}\Psi'(\lambda)\mathbb{Y}\big(\lambda,\mathbb{X}_\varepsilon(\lambda),\mathbb{X}_\varepsilon(\lambda-\varpi)\big)\mathrm{d}\lambda\\
&+\sqrt{\varepsilon}\frac{1}{\Gamma(\vartheta)}\int_0^\tau\big(\Psi(\tau)-\Psi(\lambda)\big)^{\vartheta-1}\Psi'(\lambda)\mathbb{Y}\big(\lambda,\mathbb{X}_\varepsilon(\lambda),\mathbb{X}_\varepsilon(\lambda-\varpi)\big)\mathrm{d}\mathfrak{B}_\lambda, \qquad (15)
\end{aligned}
$$

where $\varepsilon\in(0,\varepsilon_0]$ is a positive small parameter, ε_0 is a fixed point, and $\mathbb{C}$, $\mathbb{Y}$, and $\mathbb{Z}$ satisfy the assumptions $(\mathscr{A}_1)$, $(\mathscr{A}_2)$, and $(\mathscr{A}_3)$. The averaged representation of Equation (15) is thus depicted below:

$$
\begin{aligned}
\mathbb{X}^*_\varepsilon(\tau)=&\mathfrak{X}_0-\mathbb{C}(0,\mathfrak{X}(0),\mathfrak{X}(-\varpi))+\mathbb{C}\big(\tau,\mathbb{X}^*_\varepsilon(\tau),\mathbb{X}^*_\varepsilon(\tau-\varpi)\big)\\
&+\varepsilon\frac{1}{\Gamma(\vartheta)}\int_0^\tau\big(\Psi(\tau)-\Psi(\lambda)\big)^{\vartheta-1}\Psi'(\lambda)\widetilde{\mathbb{Y}}_1\big(\lambda,\mathbb{X}^*_\varepsilon(\lambda),\mathbb{X}^*_\varepsilon(\lambda-\varpi)\big)\mathrm{d}\lambda\\
&+\sqrt{\varepsilon}\frac{1}{\Gamma(\vartheta)}\int_0^\tau\big(\Psi(\tau)-\Psi(\lambda)\big)^{\vartheta-1}\Psi'(\lambda)\widetilde{\mathbb{Z}}_2\big(\lambda,\mathbb{X}^*_\varepsilon(\lambda),\mathbb{X}^*_\varepsilon(\lambda-\varpi)\big)\mathrm{d}\mathfrak{B}_\lambda, \qquad (16)
\end{aligned}
$$

where $\widetilde{\mathbb{Y}}_1:\mathcal{R}^{\mathfrak{f}}\times\mathcal{R}^{\mathfrak{f}}\to\mathcal{R}^{\mathfrak{f}},\widetilde{\mathbb{Z}}_2:\mathcal{R}^{\mathfrak{f}}\times\mathcal{R}^{\mathfrak{f}}\to\mathcal{R}^{\mathfrak{f}\times\mathfrak{b}}$

Theorem 2. *Consider that assumptions* $(\mathscr{A}_1)$ *through* $(\mathscr{A}_4)$ *are met. We can determine the corresponding* $\varepsilon_1 \in (0, \varepsilon_0], \varphi > 0, \chi \in (0,1)$ *satisfies for all* ε *in* $(0, \varepsilon_1]$ *when* $p \in [2, (1-\vartheta)^{-1})$ *and for* $\mathbb{V} > 0$*, which is an arbitrarily small number. The formula for this is obtained as follows:*

$$\mathscr{E}\Big[\sup_{\tau \in [-\varpi, \varphi\varepsilon^{-\chi}]} \big\|\mathbb{X}_\varepsilon(\tau) - \mathbb{X}^*_\varepsilon(\tau)\big\|^p\Big] \leq \mathbb{V}. \tag{17}$$

Proof. We achieve the following outcome for any $\tau \in [0, \mathbb{T}]$ via Equations (15) and (16):

$$\begin{aligned}
&\mathbb{X}_\varepsilon(\tau) - \mathbb{X}^*_\varepsilon(\tau) \\
&= \mathbb{C}\big(\tau, \mathbb{X}_\varepsilon(\tau), \mathbb{X}_\varepsilon(\tau-\varpi)\big) - \mathbb{C}\big(\tau, \mathbb{X}^*_\varepsilon(\tau), \mathbb{X}^*_\varepsilon(\tau-\varpi)\big) \\
&+ \varepsilon \frac{1}{\Gamma(\vartheta)} \int_0^\tau \big(\Psi(\tau) - \Psi(\lambda)\big)^{\vartheta-1} \Psi'(\lambda) \big(\mathbb{Y}\big(\lambda, \mathbb{X}_\varepsilon(\lambda), \mathbb{X}_\varepsilon(\lambda-\varpi)\big) - \widetilde{\mathbb{Y}}_1\big(\mathbb{X}^*_\varepsilon(\lambda), \mathbb{X}^*_\varepsilon(\lambda-\varpi)\big)\big) d\lambda \\
&+ \sqrt{\varepsilon} \frac{1}{\Gamma(\vartheta)} \int_0^\tau \big(\Psi(\tau) - \Psi(\lambda)\big)^{\vartheta-1} \Psi'(\lambda) \big(\mathbb{Z}\big(\lambda, \mathbb{X}_\varepsilon(\lambda), \mathbb{X}_\varepsilon(\lambda-\varpi)\big) - \widetilde{\mathbb{Z}}_2\big(\mathbb{X}^*_\varepsilon(\lambda), \mathbb{X}^*_\varepsilon(\lambda-\varpi)\big)\big) d\mathfrak{B}_\lambda.
\end{aligned} \tag{18}$$

When $\mathbb{Q} \in (0,1), \wp_1, \wp_2 \in \mathcal{R}^{\mathfrak{f}}, p \geq 2$, we have

$$\|\wp_1 + \wp_2\|^p \leq \frac{\|\wp_1\|^p}{\mathbb{Q}^{p-1}} + \frac{\|\wp_2\|^p}{(1-\mathbb{Q})^{p-1}}. \tag{19}$$

Take $\mathbb{Q} = \xi_1$; when we use Equation (18) in Equation (19), and then use $(\mathscr{A}_1)$ and Jen-Ineq, we obtain the following as a result:

$$\begin{aligned}
&\big\|\mathbb{X}_\varepsilon(\tau) - \mathbb{X}^*_\varepsilon(\tau)\big\|^p \\
&\leq \xi_1^{1-p} \big\|\mathbb{C}\big(\tau, \mathbb{X}_\varepsilon(\tau), \mathbb{X}_\varepsilon(\tau-\varpi)\big) - \mathbb{C}\big(\tau, \mathbb{X}^*_\varepsilon(\tau), \mathbb{X}^*_\varepsilon(\tau-\varpi)\big)\big\|^p \\
&+ \frac{2^{p-1}}{(1-\xi_1)^{p-1}} \\
&\bigg\|\varepsilon \frac{1}{\Gamma(\vartheta)} \int_0^\tau \big(\Psi(\tau) - \Psi(\lambda)\big)^{\vartheta-1} \Psi'(\lambda) \Big(\mathbb{Y}\big(\lambda, \mathbb{X}_\varepsilon(\lambda), \mathbb{X}_\varepsilon(\lambda-\varpi)\big) - \widetilde{\mathbb{Y}}_1\big(\mathbb{X}^*_\varepsilon(\lambda), \mathbb{X}^*_\varepsilon(\lambda-\varpi)\big)\Big) d\lambda\bigg\|^p \\
&+ \frac{2^{p-1}}{(1-\xi_1)^{p-1}} \\
&\bigg\|\sqrt{\varepsilon} \frac{1}{\Gamma(\vartheta)} \int_0^\tau \big(\Psi(\tau) - \Psi(\lambda)\big)^{\vartheta-1} \Psi'(\lambda) \Big(\mathbb{Z}\big(\lambda, \mathbb{X}_\varepsilon(\lambda), \mathbb{X}_\varepsilon(\lambda-\varpi)\big) - \widetilde{\mathbb{Z}}_2\big(\mathbb{X}^*_\varepsilon(\lambda), \mathbb{X}^*_\varepsilon(\lambda-\varpi)\big)\Big) d\mathfrak{B}_\lambda\bigg\|^p \\
&\leq 2^{p-1}\xi_1 \big(\big\|\mathbb{X}_\varepsilon(\tau) - \mathbb{X}^*_\varepsilon(\tau)\big\|^p + \big\|\mathbb{X}_\varepsilon(\tau-\varpi) - \mathbb{X}^*_\varepsilon(\tau-\varpi)\big\|^p\big) \\
&+ \frac{1}{\Gamma^p(\vartheta)} \frac{2^{p-1}\varepsilon^p}{(1-\xi_1)^{p-1}} \\
&\bigg\|\int_0^\tau \big(\Psi(\tau) - \Psi(\lambda)\big)^{\vartheta-1} \Psi'(\lambda) \Big(\mathbb{Y}\big(\lambda, \mathbb{X}_\varepsilon(\lambda), \mathbb{X}_\varepsilon(\lambda-\varpi)\big) - \widetilde{\mathbb{Y}}_1\big(\mathbb{X}^*_\varepsilon(\lambda), \mathbb{X}^*_\varepsilon(\lambda-\varpi)\big)\Big) d\lambda\bigg\|^p \\
&+ \frac{1}{\Gamma^p(\vartheta)} \frac{2^{p-1}\varepsilon^{\frac{p}{2}}}{(1-\xi_1)^{p-1}} \\
&\bigg\|\int_0^\tau \big(\Psi(\tau) - \Psi(\lambda)\big)^{\vartheta-1} \Psi'(\lambda) \Big(\mathbb{Z}\big(\lambda, \mathbb{X}_\varepsilon(\lambda), \mathbb{X}_\varepsilon(\lambda-\varpi)\big) - \widetilde{\mathbb{Z}}_2\big(\mathbb{X}^*_\varepsilon(\lambda), \mathbb{X}^*_\varepsilon(\lambda-\varpi)\big)\Big) d\mathfrak{B}_\lambda\bigg\|^p.
\end{aligned} \tag{20}$$

Utilizing Equation (20) in Equation (17):

$$\mathscr{E}\Big[\sup_{0 \leq \tau \leq \mathfrak{c}} \big\|\mathbb{X}_\varepsilon(\tau) - \mathbb{X}^*_\varepsilon(\tau)\big\|^p\Big]$$

$$\leq \frac{2^{p-1}\xi_1}{1-2^{p-1}\xi_1}\mathscr{E}\left[\sup_{0\leq\tau\leq\mathfrak{c}}\left\|\mathbb{X}_\varepsilon(\tau-\omega)-\mathbb{X}^*_\varepsilon(\tau-\omega)\right\|^p\right]$$
$$+\frac{1}{\Gamma^p(\vartheta)}\frac{2^{p-1}\varepsilon^p}{(1-\xi_1)^{p-1}(1-2^{p-1}\xi_1)}$$
$$\mathscr{E}\left[\sup_{0\leq\tau\leq\mathfrak{c}}\left\|\int_0^\tau(\Psi(\tau)-\Psi(\lambda))^{\vartheta-1}\Psi'(\lambda)\Big(\mathbb{Y}(\lambda,\mathbb{X}_\varepsilon(\lambda),\mathbb{X}_\varepsilon(\lambda-\omega))-\widetilde{\mathbb{Y}}(\mathbb{X}^*_\varepsilon(\lambda),\mathbb{X}^*_\varepsilon(\lambda-\omega))\Big)\mathrm{d}\lambda\right\|^p\right]$$
$$+\frac{1}{\Gamma^p(\vartheta)}\frac{2^{p-1}\varepsilon^{\frac{p}{2}}}{(1-\xi_1)^{p-1}(1-2^{p-1}\xi_1)}$$
$$\mathscr{E}\left[\sup_{0\leq\tau\leq\mathfrak{c}}\left\|\int_0^\tau(\Psi(\tau)-\Psi(\lambda))^{\vartheta-1}\Psi'(\lambda)\Big(\mathbb{Z}(\lambda,\mathbb{X}_\varepsilon(\lambda),\mathbb{X}_\varepsilon(\lambda-\omega))-\widetilde{\mathbb{Z}}(\xi^*_\varepsilon(\lambda),\mathbb{X}^*_\varepsilon(\lambda-\omega))\Big)\mathrm{d}\mathfrak{B}_\lambda\right\|^p\right]$$
$$=\mathfrak{V}_1+\mathfrak{V}_2+\mathfrak{V}_3. \tag{21}$$

From $\mathfrak{V}_1$

$$\mathfrak{V}_1\leq\frac{2^{2p-2}\xi_1}{1-2^{p-1}\xi_1}\left(\mathscr{E}\left[\sup_{0\leq\tau\leq\mathfrak{c}}\left\|\mathbb{X}_\varepsilon(\tau-\omega)\right\|^p\right]+\mathscr{E}\left[\sup_{0\leq\tau\leq\mathfrak{c}}\left\|\mathbb{X}^*_\varepsilon(\tau-\omega)\right\|^p\right]\right). \tag{22}$$

From $\mathfrak{V}_2$

$$\mathfrak{V}_2\leq\frac{1}{\Gamma^p(\vartheta)}\frac{2^{2p-2}\varepsilon^p}{(1-\xi_1)^{p-1}(1-2^{p-1}\xi_1)}$$
$$\mathscr{E}\left[\sup_{0\leq\tau\leq\mathfrak{c}}\left\|\int_0^\tau(\Psi(\tau)-\Psi(\lambda))^{\vartheta-1}\Psi'(\lambda)\Big(\mathbb{Y}(\lambda,\mathbb{X}_\varepsilon(\lambda),\mathbb{X}_\varepsilon(\lambda-\omega))-\mathbb{Y}(\lambda,\mathbb{X}^*_\varepsilon(\lambda),\mathbb{X}^*_\varepsilon(\lambda-\omega))\Big)\mathrm{d}\lambda\right\|^p\right]$$
$$+\frac{1}{\Gamma^p(\vartheta)}\frac{2^{2p-2}\varepsilon^p}{(1-\xi_1)^{p-1}(1-2^{p-1}\xi_1)}$$
$$\mathscr{E}\left[\sup_{0\leq\tau\leq\mathfrak{c}}\left\|\int_0^\tau(\Psi(\tau)-\Psi(\lambda))^{\vartheta-1}\Psi'(\lambda)\Big(\mathbb{Y}(\lambda,\mathbb{X}^*_\varepsilon(\lambda),\mathbb{X}^*_\varepsilon(\lambda-\omega))-\widetilde{\mathbb{Y}}_1(\mathbb{X}^*_\varepsilon(\lambda),\mathbb{X}^*_\varepsilon(\lambda-\omega))\Big)\mathrm{d}\lambda\right\|^p\right]$$
$$=\mathfrak{V}_{21}+\mathfrak{V}_{22}. \tag{23}$$

Using Höld-Ineq, Jen-Ineq, and $(\mathscr{A}_2)$ on $\mathfrak{V}_{21}$, we obtain the following result:

$$\begin{aligned}\mathfrak{V}_{21}\leq&\frac{1}{\Gamma^p(\vartheta)}\frac{2^{2p-2}\varepsilon^p}{(1-\xi_1)^{p-1}(1-2^{p-1}\xi_1)}\left(\int_0^{\mathfrak{c}}(\Psi(\mathfrak{c})-\Psi(\lambda))^{\frac{(\vartheta-1)p}{p-1}}(\Psi'(\lambda))^{\frac{p}{p-1}}\mathrm{d}\lambda\right)^{p-1}\\&\mathscr{E}\left[\sup_{0\leq\tau\leq\mathfrak{c}}\int_0^\tau\|\mathbb{Y}(\lambda,\mathbb{X}_\varepsilon(\lambda),\mathbb{X}_\varepsilon(\lambda-\omega))-\mathbb{Y}(\lambda,\mathbb{X}^*_\varepsilon(\lambda),\mathbb{X}^*_\varepsilon(\lambda-\omega))\|^p\mathrm{d}\lambda\right]\\\leq&\frac{\mathbb{D}}{\Gamma^p(\vartheta)}\frac{2^{3p-3}\varepsilon^p(\Psi(\mathfrak{c})-\Psi(0))^{p\vartheta-1}\xi_2^p}{(1-\xi_1)^{p-1}(1-2^{p-1}\xi_1)}\left(\frac{p-1}{p\vartheta-1}\right)^{p-1}\\&\mathscr{E}\left[\sup_{0\leq\tau\leq\mathfrak{c}}\int_0^\tau\|\mathbb{X}_\varepsilon(\lambda)-\mathbb{X}^*_\varepsilon(\lambda)\|^p\mathrm{d}\lambda\right]+\mathscr{E}\left[\sup_{0\leq\tau\leq\mathfrak{c}}\int_0^\tau\|\mathbb{X}_\varepsilon(\lambda-\omega)-\mathbb{X}^*_\varepsilon(\lambda-\omega)\|^p\mathrm{d}\lambda\right]\\=&\mathbb{F}_{21}\varepsilon^p(\Psi(\mathfrak{c})-\Psi(0))^{p\vartheta-1}\left(\int_0^{\mathfrak{c}}\mathscr{E}\left[\sup_{0\leq\varrho\leq\lambda}\|\mathbb{X}_\varepsilon(\varrho)-\mathbb{X}^*_\varepsilon(\varrho)\|^p\right]\mathrm{d}\lambda\right.\\&\left.+\int_0^{\mathfrak{c}}\mathscr{E}\left[\sup_{0\leq\varrho\leq\lambda}\|\mathbb{X}_\varepsilon(\varrho-\omega)-\mathbb{X}^*_\varepsilon(\varrho-\omega)\|^p\right]\mathrm{d}\lambda\right),\end{aligned} \tag{24}$$

where $\mathbb{F}_{21} = \mathbb{D}\frac{2^{3p-3}\xi_2^p}{(1-\xi_1)^{p-1}(1-2^{p-1}\xi_1)}\left(\frac{p-1}{p\vartheta-1}\right)^{p-1}\frac{1}{\Gamma^p(\vartheta)}$.

Using Höld-Ineq, Jen-Ineq, and $(\mathscr{A}_4)$ on $\mathfrak{V}_{22}$, we obtain the following result:

$$\begin{aligned}
\mathfrak{V}_{22} \leq & \frac{1}{\Gamma^p(\vartheta)}\frac{2^{2p-2}\varepsilon^p}{(1-\xi_1)^{p-1}(1-2^{p-1}\xi_1)}\left(\int_0^{\mathfrak{c}}\left(\Psi(\mathfrak{c})-\Psi(\lambda)\right)^{\frac{(\vartheta-1)p}{p-1}}\left(\Psi'(\lambda)\right)^{\frac{p}{p-1}}\mathrm{d}\lambda\right)^{p-1} \\
& \mathscr{E}\left[\sup_{0\leq\tau\leq\mathfrak{c}}\int_0^{\tau}\left\|\mathbb{Y}\left(\lambda,\mathbb{X}_\varepsilon^*(\lambda),\mathbb{X}_\varepsilon^*(\lambda-\varpi)\right)-\widetilde{\mathbb{Y}}_1\left(\mathbb{X}_\varepsilon^*(\lambda),\mathbb{X}_\varepsilon^*(\lambda-\varpi)\right)\right\|^p\mathrm{d}\lambda\right] \\
\leq & \frac{\mathbb{D}}{\Gamma^p(\vartheta)}\frac{2^{2p-2}\varepsilon^p}{(1-\xi_1)^{p-1}(1-2^{p-1}\xi_1)}\left(\frac{p-1}{p\vartheta-1}\right)^{p-1}\left(\Psi(\mathfrak{c})-\Psi(0)\right)^{p\vartheta-1}\mathfrak{c}\mathscr{Y}_1(\mathfrak{c}) \\
& \left(1+\mathscr{E}\|\mathbb{X}_\varepsilon^*(\lambda)\|^p+\mathscr{E}\left\|\mathbb{X}_\varepsilon^*(\lambda-\varpi)\right\|^p\right) \\
= & \mathbb{F}_{22}\varepsilon^p\left(\Psi(\mathfrak{c})-\Psi(0)\right)^{p\vartheta-1},
\end{aligned} \tag{25}$$

where $\mathbb{F}_{22} = \frac{2^{2p-2}\mathscr{Y}_1(\mathfrak{c})\left(1+\mathscr{E}\|\mathbb{X}_\varepsilon^*(\lambda)\|^p+\mathscr{E}\|\mathbb{X}_\varepsilon^*(\lambda-\varpi)\|^p\right)}{(1-\xi_1)^{p-1}(1-2^{p-1}\xi_1)}\left(\frac{p-1}{p\vartheta-1}\right)^{p-1}\frac{\mathfrak{c}}{\Gamma^p(\vartheta)}\mathbb{D}$.

Through the use of Jen-Ineq, $\mathfrak{V}_3$ provides the following:

$$\begin{aligned}
\mathfrak{V}_3 \leq & \frac{1}{\Gamma^p(\vartheta)}\frac{2^{2p-2}\varepsilon^{\frac{p}{2}}}{(1-\xi_1)^{p-1}(1-2^{p-1}\xi_1)} \\
& \left(\mathscr{E}\left[\sup_{0\leq\tau\leq\mathfrak{c}}\left\|\int_0^{\tau}\left(\Psi(\tau)-\Psi(\lambda)\right)^{\vartheta-1}\Psi'(\lambda)\right.\right.\right. \\
& \left.\left.\left.\left[\mathbb{Z}\left(\lambda,\mathbb{X}_\varepsilon(\lambda),\mathbb{X}_\varepsilon(\lambda-\varpi)\right)-\mathbb{Z}\left(\lambda,\mathbb{X}_\varepsilon^*(\lambda),\mathbb{X}_\varepsilon^*(\lambda-\varpi)\right)\right]\mathrm{d}\mathfrak{B}_\lambda\right\|^p\right]\right)+ \\
& \frac{1}{\Gamma^p(\vartheta)}\frac{2^{2p-2}\varepsilon^{\frac{p}{2}}}{(1-\xi_1)^{p-1}(1-2^{p-1}\xi_1)} \\
& \left(\mathscr{E}\left[\sup_{0\leq\tau\leq\mathfrak{c}}\left\|\int_0^{\tau}\left(\Psi(\tau)-\Psi(\lambda)\right)^{\vartheta-1}\Psi'(\lambda)\right.\right.\right. \\
& \left.\left.\left.\left[\mathbb{Z}\left(\lambda,\mathbb{X}_\varepsilon^*(\lambda),\mathbb{X}_\varepsilon^*(\lambda-\varpi)\right)-\widetilde{\mathbb{Z}}\left(\lambda,\mathbb{X}_\varepsilon^*(\lambda),\mathbb{X}_\varepsilon^*(\lambda-\varpi)\right)\right]\mathrm{d}\mathfrak{B}_\lambda\right\|^p\right]\right) = \mathfrak{V}_{31}+\mathfrak{V}_{32}.
\end{aligned} \tag{26}$$

Using $(\mathscr{A}_2)$, Höld-Ineq, and BHDG-Ineq on $\mathfrak{V}_{31}$, we achieve the following outcomes:

$$\begin{aligned}
\mathfrak{V}_{31} \leq & \frac{1}{\Gamma^p(\vartheta)}2^{2p-2}\varepsilon^{\frac{p}{2}}\left(2(p-1)^{1-p}(p)^{p+1}\right)^{\frac{p}{2}}\frac{1}{(1-\xi_1)^{p-1}(1-2^{p-1}p_1)} \\
& \mathscr{E}\left[\int_0^{\mathfrak{c}}\left(\Psi(\mathfrak{c})-\Psi(\lambda)\right)^{2\vartheta-2}\left(\Psi'(\lambda)\right)^2\|\mathbb{Z}(\lambda,\mathbb{X}_\varepsilon(\lambda),\mathbb{X}_\varepsilon(\lambda-\varpi))-\mathbb{Z}(\lambda,\mathbb{X}_\varepsilon^*(\lambda),\mathbb{X}_\varepsilon^*(\lambda-\varpi))\|^2\mathrm{d}\lambda\right]^{\frac{p}{2}} \\
\leq & \frac{1}{\Gamma^p(\vartheta)}\frac{2^{2p-2}\varepsilon^{\frac{p}{2}}\mathfrak{c}^{\frac{p}{2}-1}}{(1-\xi_1)^{p-1}(1-2^{p-1}\xi_1)}\left((p)^{p+1}2(p-1)^{1-p}\right)^{\frac{p}{2}} \\
& \mathscr{E}\left[\int_0^{\mathfrak{c}}\left(\Psi(\mathfrak{c})-\Psi(\lambda)\right)^{(\vartheta-1)p}\left(\Psi'(\lambda)\right)^{(\vartheta-1)p}\|\mathbb{Z}(\lambda,\mathbb{X}_\varepsilon(\lambda),\mathbb{X}_\varepsilon(\lambda-\varpi))-\mathbb{Z}(\lambda,\mathbb{X}_\varepsilon^*(\lambda),\mathbb{X}_\varepsilon^*(\lambda-\varpi))\|^p\mathrm{d}\lambda\right] \\
\leq & \frac{1}{\Gamma^p(\vartheta)}2^{3p-3}\varepsilon^{\frac{p}{2}}\mathfrak{c}^{\frac{p}{2}-1}\xi_2^p\left((p)^{p+1}2(1-p)^{p-1}\right)^{\frac{p}{2}}\frac{1}{(1-\xi_1)^{p-1}(1-2^{p-1}\xi_1)} \\
& \int_0^{\mathfrak{c}}\left(\Psi(\mathfrak{c})-\Psi(\lambda)\right)^{(\vartheta-1)p}\left(\Psi'(\lambda)\right)^{(\vartheta-1)p}
\end{aligned}$$

$$\mathscr{E}\left[\sup_{0\le\varrho\le S}[\|\mathbb{X}_\varepsilon(\varrho)-\mathbb{X}^*_\varepsilon(\varrho)\|^p+\|\mathbb{X}_\varepsilon(\varrho-\varpi)-\mathbb{X}^*_\varepsilon(\varrho-\varpi)\|^p]\mathrm{d}\lambda\right]$$
$$=\mathbb{F}_{31}\varepsilon^{\frac{p}{2}}\mathfrak{c}^{\frac{p}{2}-1}\Bigg(\int_0^{\mathfrak{c}}(\Psi(\mathfrak{c})-\Psi(\lambda))^{(\vartheta-1)p}(\Psi'(\lambda))^{(\vartheta-1)p}\mathscr{E}\left[\sup_{0\le\varrho\le\lambda}\|\mathbb{X}_\varepsilon(\varrho)-\mathbb{X}^*_\varepsilon(\varrho)\|^p\mathrm{d}\lambda\right]$$
$$+\int_0^{\mathfrak{c}}(\Psi(\mathfrak{c})-\Psi(\lambda))^{(\vartheta-1)p}(\Psi'(\lambda))^{(\vartheta-1)p}\mathscr{E}\left[\sup_{0\le\varrho\le\lambda}\|\mathbb{X}_\varepsilon(\varrho-\varpi)-\mathbb{X}^*_\varepsilon(\varrho-\varpi)\|^p\mathrm{d}\lambda\right]\Bigg), \tag{27}$$

where $\mathbb{F}_{31}=\frac{2^{3p-3}\xi_2^p}{(1-\xi_1)^{p-1}(1-2^{p-1}\xi_1)}\left(\frac{(p)^{p+1}}{2(p-1)^{p-1}}\right)^{\frac{p}{2}}\frac{1}{\Gamma^p(\vartheta)}$.

Again using $(\mathscr{A}_2)$, Höld-Ineq, and BHDG-Ineq on $\mathfrak{V}_{32}$, we achieve the following outcomes:

$$\mathfrak{V}_{32}\le\frac{1}{\Gamma^p(\vartheta)}2^{2p-2}\left(2(p-1)^{1-p-}(p)^{p+1}\right)^{\frac{p}{2}}\frac{1}{(1-\xi_1)^{p-1}(1-2^{p-1}\xi_1)}\varepsilon^{\frac{p}{2}}$$
$$\mathscr{E}\left[\int_0^{\mathfrak{c}}\left\|\mathbb{Y}(\lambda,\mathbb{X}^*_\varepsilon(\lambda),\mathbb{X}^*_\varepsilon(\lambda-\varpi))-\widetilde{\mathbb{Z}}_2(\lambda,\mathbb{X}^*_\varepsilon(\lambda),\mathbb{X}^*_\varepsilon(\lambda-\varpi))\right\|^2(\Psi(\mathfrak{c})-\Psi(\lambda))^{2\vartheta-2}(\Psi'(\lambda))^2\mathrm{d}\lambda\right]^{\frac{p}{2}}$$
$$\le\frac{1}{\Gamma^p(\vartheta)}2^{2p-2}\varepsilon^{\frac{p}{2}}\mathfrak{c}^{\frac{p}{2}-1}\frac{1}{(1-\xi_1)^{p-1}(1-2^{p-1}\xi_1)}\left(2(p-1)^{p-1}(p)^{p+1}\right)^{\frac{p}{2}}$$
$$\mathscr{E}\Bigg[\int_0^{\mathfrak{c}}(\Psi(\mathfrak{c})-\Psi(\lambda))^{(\vartheta-1)p}(\Psi'(\lambda))^{(\vartheta-1)p}$$
$$\left(\|\mathbb{Z}(\lambda,\mathbb{X}^*_\varepsilon(\lambda),\mathbb{X}^*_\varepsilon(\lambda-\varpi))\|^p+\left\|\widetilde{\mathbb{Z}}_2(\mathbb{X}^*_\varepsilon(\lambda),\mathbb{X}^*_\varepsilon(\lambda-\varpi))\right\|^p\right)\mathrm{d}\lambda\Bigg]$$
$$\le\frac{1}{\Gamma^p(\vartheta)}\mathfrak{c}^{\frac{p}{2}-1}\frac{2^{3p-3}3^{p-1}\varepsilon^{\frac{p}{2}}(\Psi(\mathfrak{c})-\Psi(0))^{p\vartheta-p+1}\xi_3^p\left(\xi_3^p+\xi_4\right)^p}{(1-\xi_1)^{p-1}(1-2^{p-1}\xi_1)[(\vartheta-1)p+1]}$$
$$\left(2(p-1)^{1-p}(p)^{p+1}\right)^{\frac{p}{2}}(1+\mathscr{E}[\|\mathbb{X}^*_\varepsilon(\lambda)\|^p]+\mathscr{E}[\|\mathbb{X}^*_\varepsilon(\lambda-\varpi)\|^p])$$
$$=\mathbb{F}_{32}\varepsilon^{\frac{p}{2}}(\Psi(\mathfrak{c})-\Psi(0))^{p\vartheta-p+1}, \tag{28}$$

where

$$\mathbb{F}_{32}=\frac{1}{\Gamma^p(\vartheta)}\mathfrak{c}^{\frac{p}{2}-1}\frac{1}{(1-\xi_1)^{p-1}}2^{3p-3}3^{p-1}\xi_3^p\left(\xi_3^p+\xi_4\right)^p\frac{1}{(1-2^{p-1}\xi_1)((\vartheta-1)p+1)}$$
$$\left(2(p-1)^{1-p}(p)^{p+1}\right)^{\frac{p}{2}}(1+\mathscr{E}[\|\mathbb{X}^*_\varepsilon(\lambda)\|^p]+\mathscr{E}[\|\mathbb{X}^*_\varepsilon(\lambda-\varpi)\|^p]).$$

By utilizing Equations (22)–(28) in (21), as a result, we obtain the following outcomes:

$$\mathscr{E}\left[\sup_{0\le\tau\le\mathfrak{c}}\|\mathbb{X}_\varepsilon(\tau)-\mathbb{X}^*_\varepsilon(\tau)\|^p\right]$$
$$\le\frac{2^{2p-2}\xi_1}{1-2^{p-1}\xi_1}\left(\mathscr{E}\left[\sup_{0\le\tau\le\mathfrak{c}}\|\mathbb{X}_\varepsilon(\tau-\varpi)\|^p\right]+\mathscr{E}\left[\sup_{0\le\tau\le\mathfrak{c}}\|\mathbb{X}^*_\varepsilon(\tau-\varpi)\|^p\right]\right)$$
$$+\mathbb{F}_{22}\varepsilon^p(\Psi(\mathfrak{c})-\Psi(0))^{p\vartheta-1}+\mathbb{F}_{32}\varepsilon^{\frac{p}{2}}(\Psi(\mathfrak{c})-\Psi(0))^{p\vartheta-p+1}$$
$$+\int_0^{\mathfrak{c}}\left[\mathbb{F}_{21}\varepsilon^p(\Psi(\mathfrak{c})-\Psi(0))^{p\vartheta-1}+\mathbb{F}_{31}\varepsilon^{\frac{p}{2}}(\Psi(\mathfrak{c})-\Psi(0))^{\frac{p}{2}-1}(\Psi(\mathfrak{c})-\Psi(\lambda))^{(\vartheta-1)p}(\Psi'(\lambda))^{(\vartheta-1)p}\right]$$
$$(\Psi(\mathfrak{c})-\Psi(\lambda))^{(\vartheta-1)p}(\Psi'(\lambda))^{(\vartheta-1)p}\mathscr{E}\left[\sup_{0\le\varrho\le\lambda}\|\mathbb{X}_\varepsilon(\varrho)-\mathbb{X}^*_\varepsilon(\varrho)\|^p\mathrm{d}\lambda\right]$$
$$+\int_0^{\mathfrak{c}}\left[\mathbb{F}_{21}\varepsilon^p(\Psi(\mathfrak{c})-\Psi(0))^{p\vartheta-1}+\mathbb{F}_{31}\varepsilon^{\frac{p}{2}}(\Psi(\mathfrak{c})-\Psi(0))^{\frac{p}{2}-1}\lambda^{(\vartheta-1)p}\right]$$

$$\mathscr{E}\left[\sup_{0\le\varrho\le\lambda}\|\mathbb{X}_\varepsilon(\varrho-\varpi)-\mathbb{X}^*_\varepsilon(\varrho-\varpi)\|^p \mathrm{d}\lambda\right]. \tag{29}$$

Taking $\aleph\big((\Psi(\mathfrak{c})-\Psi(0))\big)=\mathscr{E}\left[\sup_{0\le\tau\le\mathfrak{c}}\|\mathbb{X}_\varepsilon(\tau)-\mathbb{X}^*_\varepsilon(\tau)\|^p\right]$ and $\mathscr{E}\left[\sup_{-\varpi\le\tau\le 0}\|\mathbb{X}_\varepsilon(\tau)-\mathbb{X}^*_\varepsilon(\tau)\|^p\right]=0.$

Based on the assumptions mentioned above, it is achievable:

$$\mathscr{E}\left[\sup_{0\le\varrho\le\lambda}\big\|\mathbb{X}_\varepsilon(\varrho-\varpi)-\mathbb{X}^*_\varepsilon(\varrho-\varpi)\big\|^p\right]=\aleph(\lambda-\varpi).$$

Consequently

$$\begin{aligned}\aleph(\mathfrak{c})\le&\frac{2^{2p-2}\zeta_1}{1-2^{p-1}\zeta_1}\left(\mathscr{E}\left[\sup_{0\le\tau\le\mathfrak{c}}\|\mathbb{X}_\varepsilon(\tau-\varpi)\|^p\right]+\mathscr{E}\left[\sup_{0\le\tau\le\mathfrak{c}}\|\mathbb{X}^*_\varepsilon(\tau-\varpi)\|^p\right]\right)\\&+\mathbb{F}_{22}\varepsilon^p\big(\Psi(\mathfrak{c})-\Psi(0)\big)^{p\vartheta-1}+\mathbb{F}_{32}\varepsilon^{\frac{p}{2}}\big(\Psi(\mathfrak{c})-\Psi(0)\big)^{p\vartheta-p+1}\\&+\int_0^{\mathfrak{c}}\left[\mathbb{F}_{21}\varepsilon^p\big(\Psi(\mathfrak{c})-\Psi(0)\big)^{p\vartheta-1}+\mathbb{F}_{31}\varepsilon^{\frac{p}{2}}\big(\Psi(\mathfrak{c})-\Psi(0)\big)^{\frac{p}{2}-1}\big(\Psi(\mathfrak{c})-\Psi(\lambda)\big)^{(\vartheta-1)p}\big(\Psi'(\lambda)\big)^{(\vartheta-1)p}\right]\\&(\aleph(\lambda)+\aleph(\lambda-\varpi))\mathrm{d}\lambda.\end{aligned}\tag{30}$$

Suppose $\gamma(\mathfrak{c})=\sup_{\alpha\in[-\varpi,\mathfrak{c}]}\aleph(\alpha)$. Therefore, when $\forall\mathfrak{c}\in[0,\mathbb{T}]$, we get $\aleph(\lambda)\le\gamma(\lambda)$, and $\aleph(\lambda-\varpi)\le\gamma(\lambda)$.

So, we have the following outcomes from Equation (29):

$$\begin{aligned}\aleph\big((\Psi(\mathfrak{c})-\Psi(0))\big)\le&\frac{2^{2p-2}\zeta_1}{1-2^{p-1}\zeta_1}\left(\mathscr{E}\left[\sup_{0\le\tau\le\mathfrak{c}}\|\mathbb{X}_\varepsilon(\tau-\varpi)\|^p\right]+\mathscr{E}\left[\sup_{0\le\tau\le\mathfrak{c}}\|\mathbb{X}^*_\varepsilon(\tau-\varpi)\|^p\right]\right)\\&+\mathbb{F}_{22}\varepsilon^p\big(\Psi(\mathfrak{c})-\Psi(0)\big)^{p\vartheta-1}+\mathbb{F}_{32}\varepsilon^{\frac{p}{2}}\big(\Psi(\mathfrak{c})-\Psi(0)\big)^{p\vartheta-p+1}+2\\&\int_0^{\mathfrak{c}}\left[\mathbb{F}_{21}\varepsilon^p\big(\Psi(\mathfrak{c})-\Psi(0)\big)^{p\vartheta-1}+\mathbb{F}_{31}\varepsilon^{\frac{p}{2}}\big(\Psi(\mathfrak{c})-\Psi(0)\big)^{\frac{p}{2}-1}\big(\Psi(\mathfrak{c})-\Psi(\lambda)\big)^{(\vartheta-1)p}\big(\Psi'(\lambda)\big)^{(\vartheta-1)p}\right]\gamma(\lambda)\mathrm{d}\lambda.\end{aligned}$$

For $\forall\alpha\in[0,\mathfrak{c}]$, we have

$$\begin{aligned}\aleph(\alpha)\le&\frac{2^{2p-2}\zeta_1}{1-2^{p-1}\zeta_1}\left(\mathscr{E}\left[\sup_{0\le\tau\le\alpha}\|\mathbb{X}_\varepsilon(\tau-\varpi)\|^p\right]+\mathscr{E}\left[\sup_{0\le\tau\le\alpha}\|\mathbb{X}^*_\varepsilon(\tau-\varpi)\|^p\right]\right)\\&+\mathbb{F}_{22}\varepsilon^p\alpha^{p\vartheta-1}+\mathbb{F}_{32}\varepsilon^{\frac{p}{2}}\alpha^{p\vartheta-p+1}\\&+2\int_0^{\alpha}\left[\mathbb{F}_{21}\varepsilon^p\alpha^{p\vartheta-1}+\mathbb{F}_{31}\varepsilon^{\frac{p}{2}}\alpha^{\frac{p}{2}-1}(\alpha-\lambda)^{(\vartheta-1)p}\right]\gamma(\lambda)\mathrm{d}\lambda\\&\frac{2^{2p-2}\zeta_1}{1-2^{p-1}\zeta_1}\left(\mathscr{E}\left[\sup_{0\le\tau\le\mathfrak{c}}\|\mathbb{X}_\varepsilon(\tau-\varpi)\|^p\right]+\mathscr{E}\left[\sup_{0\le\tau\le\mathfrak{c}}\|\mathbb{X}^*_\varepsilon(\tau-\varpi)\|^p\right]\right)\\&+\mathbb{F}_{22}\varepsilon^p\mathfrak{c}^{p\vartheta-1}+\mathbb{F}_{32}\varepsilon^{\frac{p}{2}}\mathfrak{c}^{p\vartheta-p+1}\\&+2\int_0^{\mathfrak{c}}\left[\mathbb{F}_{21}\varepsilon^p\mathfrak{c}^{p\vartheta-1}+\mathbb{F}_{31}\varepsilon^{\frac{p}{2}}\mathfrak{c}^{\frac{p}{2}-1}\big(\Psi(\mathfrak{c})-\Psi(\lambda)\big)^{(\vartheta-1)p}\big(\Psi'(\lambda)\big)^{(\vartheta-1)p}\right]\gamma(\lambda)\mathrm{d}\lambda.\end{aligned}$$

As a result,

$$\begin{aligned}&\gamma\big((\Psi(\mathfrak{c})-\Psi(0))\big)=\sup_{\alpha\in[-\varpi,\mathfrak{c}]}\aleph(\alpha)\\&\le\max\left\{\sup_{\alpha\in[-\varpi,0]}\aleph(\alpha),\ \sup_{\alpha\in[0,\mathfrak{c}]}\aleph(\alpha)\right\}\end{aligned}$$

$$\leq \frac{2^{2p-2}\xi_1}{1-2^{p-1}\xi_1}\left(\mathscr{E}\left[\sup_{0\leq\tau\leq\mathfrak{c}}\|\mathbb{X}_\varepsilon(\tau-\varpi)\|^p\right]+\mathscr{E}\left[\sup_{0\leq\tau\leq\mathfrak{c}}\|\mathbb{X}^*_\varepsilon(\tau-\varpi)\|^p\right]\right)$$
$$+\mathbb{F}_{22}\varepsilon^p\left(\Psi(\mathfrak{c})-\Psi(0)\right)^{p\vartheta-1}+\mathbb{F}_{32}\varepsilon^{\frac{p}{2}}\left(\Psi(\mathfrak{c})-\Psi(0)\right)^{p\vartheta-p+1}+2$$
$$\int_0^{\mathfrak{c}}\left[\mathbb{F}_{21}\varepsilon^p\mathfrak{c}^{p\vartheta-1}+\mathbb{F}_{31}\varepsilon^{\frac{p}{2}}\mathfrak{c}^{\frac{p}{2}-1}\left(\Psi(\mathfrak{c})-\Psi(\lambda)\right)^{(\vartheta-1)p}\left(\Psi'(\lambda)\right)^{(\vartheta-1)p}\right]\gamma(\lambda)\mathrm{d}\lambda.$$

By Grön-Bell-Ineq, we have the following:

$$\gamma(\mathfrak{c})\leq\left(\frac{2^{2p-2}\xi_1}{1-2^{p-1}\xi_1}\left(\mathscr{E}\left[\sup_{0\leq\tau\leq\mathfrak{c}}\|\mathbb{X}_\varepsilon(\tau-\varpi)\|^p\right]+\mathscr{E}\left[\sup_{0\leq\tau\leq\mathfrak{c}}\|\mathbb{X}^*_\varepsilon(\tau-\varpi)\|^p\right]\right)\right.$$
$$\left.+\mathbb{F}_{22}\varepsilon^p\left(\Psi(\mathfrak{c})-\Psi(0)\right)^{p\vartheta-1}+\mathbb{F}_{32}\varepsilon^{\frac{p}{2}}\left(\Psi(\mathfrak{c})-\Psi(0)\right)^{p\vartheta-p+1}\right)$$
$$\exp\left(2\mathbb{F}_{21}\varepsilon^p\left(\Psi(\mathfrak{c})-\Psi(0)\right)^{p\vartheta-1}+\frac{2\mathbb{F}_{31}}{(\vartheta-1)p+1}\varepsilon^{\frac{p}{2}}\left(\Psi(\mathfrak{c})-\Psi(0)\right)^{p\vartheta-p+1}\right).$$

Consequently, we obtain the subsequent outcome from Equation (30):

$$\mathscr{E}\left[\sup_{0\leq\tau\leq\mathfrak{c}}\|\mathbb{X}_\varepsilon(\tau)-\mathbb{X}^*_\varepsilon(\tau)\|^p\right]$$
$$\leq\left(\frac{2^{2p-2}\xi_1}{1-2^{p-1}\xi_1}\left(\mathscr{E}\left[\sup_{0\leq\tau\leq\mathfrak{c}}\|\mathbb{X}_\varepsilon(\tau-\varpi)\|^p\right]+\mathscr{E}\left[\sup_{0\leq\tau\leq\mathfrak{c}}\|\mathbb{X}^*_\varepsilon(\tau-\varpi)\|^p\right]\right)\right.$$
$$\left.+\mathbb{F}_{22}\varepsilon^p\left(\Psi(\mathfrak{c})-\Psi(0)\right)^{p\vartheta-1}+\mathbb{F}_{32}\varepsilon^{\frac{p}{2}}\left(\Psi(\mathfrak{c})-\Psi(0)\right)^{p\vartheta-p}+1\right)$$
$$\exp\left(2\mathbb{F}_{21}\varepsilon^p\left(\Psi(\mathfrak{c})-\Psi(0)\right)^{p\vartheta-1}+\frac{2\mathbb{F}_{31}}{(\vartheta-1)p+1}\varepsilon^{\frac{p}{2}}\left(\Psi(\mathfrak{c})-\Psi(0)\right)^{p\vartheta-p+1}\right).$$

This implies that for any $\forall\tau\in[0,\varphi\varepsilon^{-\chi}]\subseteq[0,\mathbb{T}]$, there are $\varphi>0$ and $\chi\in(0,1)$ as well:

$$\mathscr{E}\left[\sup_{0\leq\tau\leq\varphi\varepsilon^{-\chi}}\|\mathbb{X}_\varepsilon(\tau)-\mathbb{X}^*_\varepsilon(\tau)\|^p\right]\leq\mathscr{U}\varepsilon^{1-\chi},\tag{31}$$

where

$$\mathscr{U}=\left(\frac{2^{2p-2}\xi_1\varepsilon^{1-\chi}}{1-2^{p-1}\xi_1}\left(\mathscr{E}\left[\sup_{0\leq\tau\leq\varphi\varepsilon^{-\chi}}\|\mathbb{X}_\varepsilon(\tau-\varpi)\|^p\right]+\mathscr{E}\left[\sup_{0\leq\tau\leq\varphi\varepsilon^{-\chi}}\|\mathbb{X}^*_\varepsilon(\tau-\varpi)\|^p\right]\right)\right.$$
$$\left.+\mathbb{F}_{22}\varepsilon^p\left(\Psi(\varphi\varepsilon^{-\chi})-\Psi(0)\right)^{p\vartheta-1}+\mathbb{F}_{32}\varepsilon^{\frac{p}{2}}\left(\Psi(\varphi\varepsilon^{-\chi})-\Psi(0)\right)^{p\vartheta-p}+1\right)$$
$$\exp\left(2\mathbb{F}_{21}\varepsilon^p\left(\Psi(\varphi\varepsilon^{-\chi})-\Psi(0)\right)^{p\vartheta-1}+\frac{2\mathbb{F}_{31}}{(\vartheta-1)p+1}\varepsilon^{\frac{p}{2}}\left(\Psi(\varphi\varepsilon^{-\chi})-\Psi(0)\right)^{p\vartheta-p+1}\right)$$

is a constant. As a result, when $\forall\mathbb{V}>0$, finding $\varepsilon_1\in(0,\varepsilon_0]$ that satisfies $\forall\varepsilon\in(0,\varepsilon_1]$ and $\tau\in[-\varpi,\varphi\varepsilon^{-\chi}]$ allows us to deduce

$$\mathscr{E}\left[\sup_{-\varpi\leq\tau\leq\varphi\varepsilon^{-\chi}}\|\mathbb{X}_\varepsilon(\tau)-\mathbb{X}^*_\varepsilon(\tau)\|^p\right]\leq\mathbb{V}.$$

□

Corollary 1. *Assume that the assumptions* $(\mathscr{A}_1)$ *and* $(\mathscr{A}_4)$ *are valid. Considering any arbitrary number* $\mathbb{V}_1 > 0$*, the subsequent criteria are established:* $\chi \in (0,1)$*,* $\varphi > 0$ *and* $\varepsilon_1 \in (0, \varepsilon_0)$ *occur for* $\forall \varepsilon \in (0, \varepsilon_1]$*, and we possess*

$$\lim_{\varepsilon \to 0} \mathfrak{p}\Big(\sup_{\tau \in [-\varpi, \varphi \varepsilon^{-\chi}} \big\| \mathbb{X}_\varepsilon(\tau) - \mathbb{X}^*_\varepsilon(\tau) \big\| > \mathbb{V}_1 \Big) = 0. \tag{32}$$

Proof. Using the Chebyshev-Markov inequality and Theorem 2, one can deduce the following for any number $\mathbb{V}_1 > 0$:

$$\begin{aligned}
\mathfrak{p}\Big[\sup_{\tau \in [-\varpi, \varphi \varepsilon^{-\chi}]} \big\| \mathbb{X}_\varepsilon(\tau) - \mathbb{X}^*_\varepsilon(\tau) \big\| > \mathbb{V}_1 \Big] \leq & \frac{1}{\mathbb{V}_1^2} \mathscr{E}\Big[\sup_{\tau \in [-\varpi, \varphi \varepsilon^{-\chi}]} \big\| \mathbb{X}_\varepsilon(\lambda) - \mathbb{X}^*_\varepsilon(\lambda) \big\|^2 \Big] \\
\leq & \frac{\mathscr{U} \varepsilon^{1-\chi}}{\mathbb{V}_1^2} \\
\leq & \, 0 \text{ as } \varepsilon \to 0,
\end{aligned}$$

It ends the proof. □

In the following section, we provide two examples to demonstrate the usefulness of our established theoretical outcome.

4. Examples

The following two numerical examples demonstrate how the average principle result can be used to obtain the average system of a complex system.

Example 1. *Consider the following FrNSDE:*

$$\begin{cases} \mathscr{T}_\tau^{0.8,\Psi}\Big[\mathbb{X}_\varepsilon(\tau) - \tau^{\frac{1}{8}} - \frac{1}{2}\sin(\mathbb{X}_\varepsilon(\tau))\Big] \\ = \varepsilon \sin^2(\tau)\mathbb{X}_\varepsilon(\tau - \varpi) + \sqrt{\varepsilon}\sin(\mathbb{X}_\varepsilon(\tau)\frac{\mathrm{d}\mathfrak{B}_\tau}{\mathrm{d}\tau}, \ \tau \in [0, \mathbb{T}], \\ \mathbb{X}(\tau) = \boldsymbol{\varkappa}(\tau), \tau \in [-\varpi, 0], \end{cases} \tag{33}$$

From the above system, we have the following: $\vartheta = 0.8$*,* $\Psi(\tau) = \tau^{\frac{1}{2}}$*, and*

$$\begin{aligned}
\mathbb{C}(\tau, \mathbb{X}(\tau), \mathbb{X}(\tau - \varpi)) &= -\tau^{\frac{1}{8}} - \frac{1}{2}\sin(\mathbb{X}_\varepsilon(\tau)), \\
\mathbb{Y}(\tau, \mathbb{X}(\tau), \mathbb{X}(\tau - \varpi)) &= \sin^2(\tau)\mathbb{X}_\varepsilon(\tau - \varpi), \\
\mathbb{Z}(\tau, \mathbb{X}(\tau), \mathbb{X}(\tau - \varpi)) &= \sin(\mathbb{X}_\varepsilon(\tau)).
\end{aligned}$$

The following forms represent the averages of $\mathbb{Y}$ *and* $\mathbb{Z}$*:*

$$\begin{aligned}
\widetilde{\mathbb{Y}}(\tau, \mathbb{X}(\tau), \mathbb{X}(\tau - \varpi)) &= \frac{1}{\pi}\int_0^\pi \sin^2(\lambda)\mathbb{X}_\varepsilon(\lambda - \varpi)\mathrm{d}\lambda = \frac{1}{2}\mathbb{X}^*_\varepsilon(\tau - \varpi), \\
\widetilde{\mathbb{Z}}(\tau, \mathbb{X}(\tau), \mathbb{X}(\tau - \varpi)) &= \frac{1}{\pi}\int_0^\pi \sin(\mathbb{X}_\varepsilon(\tau))\mathrm{d}\lambda = \sin(\mathbb{X}_\varepsilon(\tau)).
\end{aligned}$$

To construct the average form concerning Equation (33), use the simplified solution $\mathbb{X}^*_\varepsilon(\tau)$ *in place of the original solution* $\mathbb{X}_\varepsilon(\tau)$*. As a result, the simplified averaged equation is presented as follows:*

$$\begin{cases} \mathscr{T}_\tau^{0.8,\Psi}\Big[\mathbb{X}^*_\varepsilon(\tau) - \tau^{\frac{1}{8}} - \frac{1}{2}\sin(\mathbb{X}^*_\varepsilon(\tau))\Big] \\ = \frac{1}{2}\varepsilon\mathbb{X}^*_\varepsilon(\tau - \varpi) + \sqrt{\varepsilon}\sin(\mathbb{X}^*_\varepsilon(\tau))\frac{\mathrm{d}\mathfrak{B}_\tau}{\mathrm{d}\tau} \\ \mathbb{X}(\tau) = \boldsymbol{\varkappa}(\tau), \tau \in [-\varpi, 0]. \end{cases} \tag{34}$$

Thus, all the requirements stated in Theorem 2 are fulfilled. As a result, in the context of $\varepsilon \to 0$, the original solution $\mathbb{X}_\varepsilon(\tau)$ and the average solution $\mathbb{X}^*_\varepsilon(\tau)$ are equivalent in the sense of $\mathbb{L}^p$. The solution $\mathbb{X}_\varepsilon(\tau)$ of the original Equation (33) and the solution $\mathbb{X}^*_\varepsilon(\tau)$ of the averaged Equation (34) are then compared numerically in Figure 1. Figure 1 demonstrates a high degree of agreement between $\mathbb{X}_\varepsilon(\tau)$ and $\mathbb{X}^*_\varepsilon(\tau)$, confirming the accuracy of our established theoretical results.

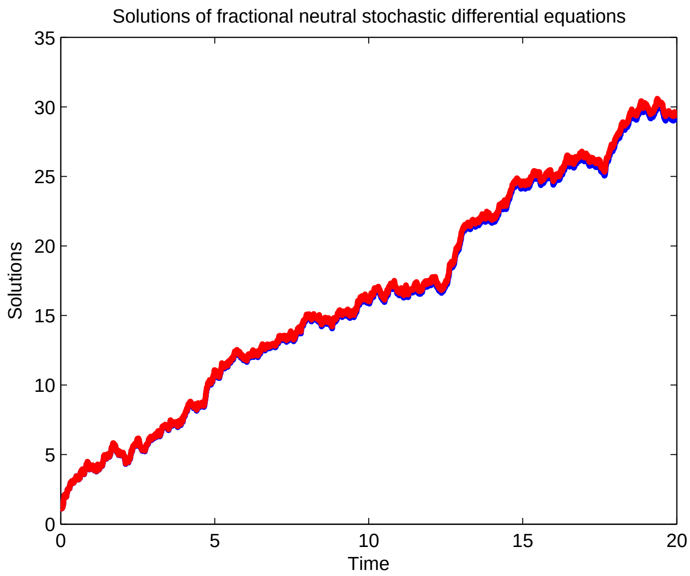

Figure 1. The red color indicates the solution of the original equation, while the blue color represents the solution of the averaged equation, when $\epsilon = 0.001$.

Example 2. *Consider the subsequent FrNSDE:*

$$\begin{cases} \mathscr{T}_\tau^{0.9,\Psi}\left[\mathbb{X}_\varepsilon(\tau) - \frac{1}{8}\tau^{\frac{1}{5}} - \frac{1}{9}\sin(\mathbb{X}_\varepsilon(\tau))\right] \\ = \frac{1}{2}\varepsilon\mathbb{X}_\varepsilon(\tau-\varpi) + \frac{3\pi}{4}\sqrt{\varepsilon}\sin^3\tau.\mathbb{X}_\varepsilon(\tau)\frac{\mathrm{d}\mathfrak{B}_\tau}{\mathrm{d}\tau},\ \tau \in [0,\mathbb{T}], \\ \mathbb{X}(\tau) = \boldsymbol{\mathfrak{X}}(\tau), \tau \in [-\varpi, 0], \end{cases} \tag{35}$$

where $\vartheta = 0.9$, $\Psi(\tau) = \tau^{\frac{1}{2}}$, *and we have the following:*

$$\begin{aligned} \mathbb{C}(\tau,\mathbb{X}(\tau),\mathbb{X}(\tau-\varpi)) &= -\frac{1}{8}\tau^{\frac{1}{5}} - \frac{1}{9}\sin(\mathbb{X}_\varepsilon(\tau)), \\ \mathbb{Y}(\tau,\mathbb{X}(\tau),\mathbb{X}(\tau-\varpi)) &= \frac{1}{2}\mathbb{X}_\varepsilon(\tau-\varpi), \\ \mathbb{Z}(\tau,\mathbb{X}(\tau),\mathbb{X}(\tau-\varpi)) &= \frac{3\pi}{4}\sin^3\tau.\mathbb{X}_\varepsilon(\tau). \end{aligned}$$

The averages of $\mathbb{Y}$ and $\mathbb{Z}$ are presented in the following forms:

$$\begin{aligned} \widetilde{\mathbb{Y}}(\tau,\mathbb{X}(\tau),\mathbb{X}(\tau-\varpi)) &= \frac{1}{\pi}\int_0^\pi \frac{1}{2}\mathbb{X}_\varepsilon(\lambda-\varpi)\mathrm{d}\lambda = \frac{1}{2}\mathbb{X}^*_\varepsilon(\tau-\varpi), \\ \widetilde{\mathbb{Z}}(\tau,\mathbb{X}(\tau),\mathbb{X}(\tau-\varpi)) &= \frac{1}{\pi}\frac{3\pi}{4}\int_0^\pi \sin^3\tau.\mathbb{X}_\varepsilon(\tau)\mathrm{d}\lambda = \mathbb{X}^*_\varepsilon(\tau). \end{aligned}$$

To construct the average form concerning Equation (35), use the simplified solution $\mathbb{X}^*_\varepsilon(\tau)$ in place of the original solution $\mathbb{X}_\varepsilon(\tau)$. Thus, the corresponding averaged FrNSDE of Equation (35) is given below:

$$\begin{cases} \mathscr{T}_{\tau}^{0.9,\Psi}\left[\mathbb{X}_{\varepsilon}^{*}(\tau) - \frac{1}{8}\tau^{\frac{1}{5}} - \frac{1}{9}\sin(\mathbb{X}_{\varepsilon}^{*}(\tau))\right] \\ = \frac{1}{2}\varepsilon\mathbb{X}_{\varepsilon}^{*}(\tau-\varpi) + \sqrt{\varepsilon}\mathbb{X}_{\varepsilon}^{*}(\tau)\frac{d\mathfrak{B}_{\tau}}{d\tau} \\ \mathbb{X}(\tau) = \boldsymbol{\varkappa}(\tau), \tau \in [-\varpi, 0]. \end{cases} \quad (36)$$

Obviously, every condition mentioned in Theorem 2 is satisfied. Therefore, the original solution $\mathbb{X}_{\varepsilon}(\tau)$ *and the average solution* $\mathbb{X}_{\varepsilon}^{*}(\tau)$ *are equivalent in the sense of* $\mathbb{L}^{p}$ *in the context of* $\varepsilon \to 0$*. In Figure 2, the solution* $\mathbb{X}_{\varepsilon}(\tau)$ *of the original Equation (35) and the solution* $\mathbb{X}_{\varepsilon}^{*}(\tau)$ *of the averaged equation Equation (36) are compared numerically. Figure 2 demonstrates the overlapping of solutions* $\mathbb{X}_{\varepsilon}(\tau)$ *and* $\mathbb{X}_{\varepsilon}^{*}(\tau)$*, indicating the reliability of our established theoretical results.*

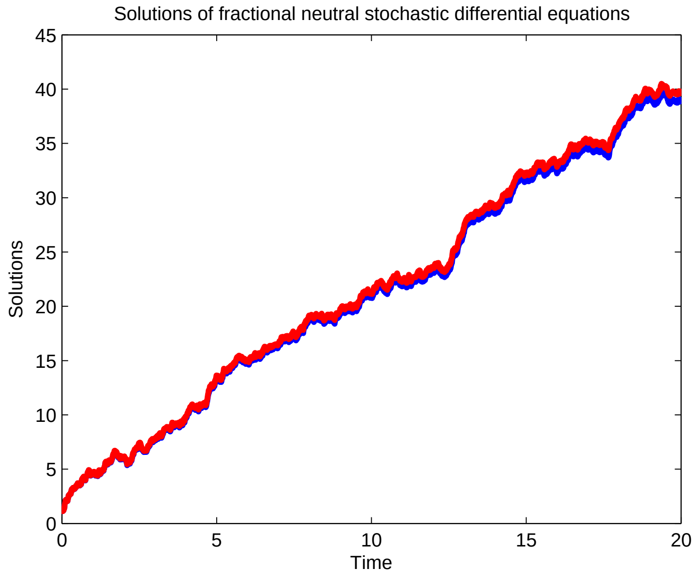

Figure 2. The red color indicates the solution of the original equation, while the blue color represents the solution of the averaged equation, when $\epsilon = 0.001$.

5. Conclusions

In this research work, we generalize the results of the existence and uniqueness of solutions to FrNSDEs and the averaging principle in the framework of Ψ-Caputo derivatives in the $\mathbb{L}^{p}$ space. The concept of contraction mapping is used to investigate the existence and uniqueness of the discussed problem. The averaging principle for FrNSDEs on the $\mathbb{L}^{p}$ space is demonstrated in this study using the following: Grönwall–Bellman's inequality, Gronwall's inequality, Burkholder–Davis–Gundy's inequality, Hölder's inequality, Jensen's inequality, and the interval translation approach. Ultimately, two examples are carried out to comprehend the established outcomes and to demonstrate the correctness of our findings. We will use numerical techniques in our future work to solve various kinds of real-world challenges modeled with FrNSDEs.

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