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Controlled Reflected McKean–Vlasov SDEs and Neumann Problem for Backward SPDEs

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Abstract: This paper is concerned with the stochastic optimal control problem of a 1-dimensional McKean–Vlasov stochastic differential equation (SDE) with reflection, of which the drift coefficient and diffusion coefficient can be both dependent on the state of the solution process along with its law and control. One backward stochastic partial differential equation (BSPDE) with the Neumann boundary condition can represent the value function of this control problem. Existence and uniqueness of the solution to the above equation are obtained. Finally, the optimal feedback control can be constructed by the BSPDE.

Keywords: optimal control; reflected McKean–Vlasov SDE; lifted function; backward stochastic partial differential equation; Neumann problem

MSC: 93E20; 60H15; 91G80



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1. Introduction

Let $p, q \in \mathbb{R}$ with $p < q$, and $T > 0$. Consider the following controlled reflected McKean–Vlasov stochastic differential equation:

$$\begin{cases} dX_t = b_t(X_t, \mathcal{L}(X_t), \alpha_t)dt + \sigma_t(X_t, \mathcal{L}(X_t), \alpha_t)dB_t + dL_t - dU_t, & t \in [0, T]; \\ X_0 = \eta; & L_0 = U_0 = 0; \\ p \leq X_t \leq q, & a.s.; \\ \int_0^T (X_t - p)dL_t = \int_0^T (q - X_s)dU_s = 0 & a.s., \end{cases} \quad (1)$$

where B is a 1-dimensional Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ is the natural filtration generated by B , augmented with an independent σ -algebra $\mathcal{G} \subset \mathcal{F}$. The drift coefficient b and the diffusion coefficient σ are random. Let $\mathcal{P}(\mathbb{R})$ be the set of probability measures on $\mathbb{R}$ and $\mathcal{L}(X_t)$ be the distribution of X_t under $\mathbb{P}$. Let η be a random variable satisfying $\mathbb{E}(|\eta|^2) < \infty$. The control α valued in a Polish space is said to be an admissible control if it is a $\mathbb{F}$ -progressive process such that there is a unique solution to Equation (1). Denote by $\mathcal{A}$ the set of all admissible controls. We denote the Borel σ -algebra of $\mathcal{A}$ as $\mathcal{B}(\mathcal{A})$. L and U are two nondecreasing processes. Through the paper, we aim to minimize a cost function of the control problem over $\mathcal{A}$, whose form is

$$\begin{aligned} J_t(X_t; \alpha) &= \mathbb{E} \left[\int_t^T f_s(X_s, \mathcal{L}(X_s), \alpha_s)ds - \int_t^T g_s(X_s, \mathcal{L}(X_s))dL_s \right. \\ &\quad \left. + \int_t^T g_s(X_s, \mathcal{L}(X_s))dU_s + G(X_T, \mathcal{L}(X_T)) \middle| \mathcal{F}_t \right]. \end{aligned} \quad (2)$$

Define the value function of the reflected McKean–Vlasov control problem as

$$V_t(\eta) := \operatorname{ess\,inf}_{\alpha \in \mathcal{A}} J_t(X_t; \alpha) \Big|_{X_t = \eta}. \quad (3)$$

The optimal stochastic control problems for McKean–Vlasov SDEs have been researched by many authors. Pham and Wei [1,2] studied the stochastic control problem of McKean–Vlasov SDE, whose coefficients might be dependent on the joint distribution of the state of the solution process and its control, then they obtained a dynamic programming Bellman equation for a distribution-dependent optimal control problem. At last, they showed the uniqueness and viscosity property of the value function for the mean-field stochastic control problem by relying on a concept of lifted viscosity solutions to the Bellman equation. Bayraktar and Cosso [3] investigated an optimal stochastic control problem, of which the state process satisfied a McKean–Vlasov stochastic differential equation and the diffusion coefficient might be degenerate. They showed that the value function of the control problem had a nonlinear Feynman–Kac representation, then they strictly proved the dynamic programming principle by using this probabilistic representation. Hafayed and Meherrem [4] proved the sufficient and necessary conditions for optimal stochastic control of the systems driven by controlled McKean–Vlasov SDEs, where the coefficients are dependent on the state of the solution process along with its distribution and control.

The reflected optimal control problems were also investigated by many authors. Ferrari [5] studied the infinite time-horizon singular stochastic optimal control problems reflected at zero for a 1-dimensional diffusion and showed that there are qualitatively different kinds of optimal strategies dependent on the characteristics of the state-dependent instant reward. Baurdoux and Yamazaki [6] investigated the two-sided singular optimal control problems, offered a sufficient condition for an optimal double barrier scheme, and proved that the scheme is optimal when the running cost function is convex. Bayraktar and Qiu [7] studied the stochastic control problem for a 1-dimensional stochastic differential equation with reflection, characterized the value function of the control problem by using a BSPDE, and constructed the optimal control. Elliott et al. [8] investigated a Neumann problem for a BSPDE with the singular terminal condition; they showed that the equation can represent the value function of a constrained stochastic optimal control problem, and then they derived the optimal feedback control.

The stochastic optimal problem of reflected McKean–Vlasov equations has not been widely studied in this field. Some important models in finance, biology, cybernetics, and many other fields involve this problem, so the reflected McKean–Vlasov optimal control problem plays an essential role in practice. In this context, we consider that the drift coefficient and diffusion coefficient of the reflected McKean–Vlasov SDEs can be both dependent on the state of the solution process and its law and control. BSPDEs in the Wasserstein space of probability measures can be written in the Hilbert space by relying on the concept of lifted function. The functions G and g in the cost function depend on the state and its distribution. In addition, the result in [4] for the 1-dimensional case can be covered by our boundary reflected McKean–Vlasov optimal control problem. For future work, we will study the reflected McKean–Vlasov optimal control problem with jump. It is worth noting that [9] obtained the well-posedness of multidimensional reflected McKean–Vlasov SDEs.

The remaining organization of the paper is as follows. In Section 2, we provide some concepts and present the main assumptions and results. In Section 3, we study the existence and uniqueness of the strong solution to general nonlinear BSPDE for Neumann problems, on the basis of which we firstly obtain the priori estimates of strong solution to linear BSPDEs and then prove the existence and uniqueness of the strong solution to nonlinear BSPDEs by using a continuity method. Finally, in Section 4, we prove the main results.

2. Preliminary Knowledge and Main Result

2.1. Notations and Differentiability in Wasserstein Space

The set of probability measures ν on $\mathbb{R}$ is denoted by $\mathcal{P}_2(\mathbb{R})$, where $\|\nu\|_2^2 := \int_{\mathbb{R}} |x|^2 \nu(dx) < \infty$, that is, they are square integrable. For any $\nu \in \mathcal{P}_2(\mathbb{R})$, the set of measurable functions $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is denoted by $L^2_{\nu}(\mathbb{R})$, where functions φ are square integrable with respect to ν . For any $\nu \in \mathcal{P}_2(\mathbb{R})$, the set of measurable functions $\psi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is denoted by $L^2_{\nu \otimes \nu}(\mathbb{R}^2)$, where functions ψ are square integrable with respect to the product measure $\nu \otimes \nu$. Define

$$\nu(\varphi) := \int_{\mathbb{R}} \varphi(x) \nu(dx), \quad \nu \otimes \nu(\psi) := \int_{\mathbb{R} \times \mathbb{R}} \psi(x, x') \nu(dx) \nu(dx').$$

$L^{\infty}_{\nu}(\mathbb{R})$ (respectively, $L^{\infty}_{\nu \otimes \nu}(\mathbb{R} \times \mathbb{R})$) is defined as the subset of elements $\varphi \in L^2_{\nu}(\mathbb{R})$ (respectively, $L^2_{\nu \otimes \nu}(\mathbb{R} \times \mathbb{R})$), where elements φ are bounded ν -a.e. (respectively, $\nu \otimes \nu$ -a.e.). We denote by $\|\cdot\|_{\infty}$ the essential supremum. The set of all square integrable random variables valued in $\mathbb{R}$ on $(\Omega, \mathcal{G}, \mathbb{P})$ is denoted by $L^2(\mathcal{G}; \mathbb{R})$. For any random variable X , its probability law is denoted by $\mathcal{L}(X)$ under $\mathbb{P}$. Without any loss of generality, we assume that $\mathcal{G}$ is sufficiently rich to take $\mathbb{R}$ -valued random variables with square integrable law, that is, $\mathcal{P}_2(\mathbb{R}) = \{\mathcal{L}(\xi) : \xi \in L^2(\mathcal{G}; \mathbb{R})\}$. We equip the measurable space $\mathcal{P}_2(\mathbb{R})$ with the 2-Wasserstein distance

$$\begin{aligned} \mathcal{W}_2(\nu, \nu') &:= \inf \left\{ \left(\int_{\mathbb{R} \times \mathbb{R}} |x - y|^2 \pi(dx, dy) \right)^{\frac{1}{2}} : \pi \in \mathcal{P}_2(\mathbb{R} \times \mathbb{R}) \text{ with marginals } \nu \text{ and } \nu' \right\} \\ &= \inf \left\{ \left(\mathbb{E}|\xi - \xi'|^2 \right)^{\frac{1}{2}} : \xi, \xi' \in L^2(\mathcal{G}; \mathbb{R}) \text{ with } \mathcal{L}(\xi) = \nu, \mathcal{L}(\xi') = \nu' \right\}, \end{aligned}$$

and endow the space $\mathcal{P}_2(\mathbb{R})$ with the corresponding Borel σ -field $\mathcal{B}(\mathcal{P}_2(\mathbb{R}))$.

We shall depend on the concept of derivative about probability measure, which is presented by P.L. Lions in [10]. The concept is established by lifting the functions $u : \mathcal{P}_2(\mathbb{R}) \rightarrow \mathbb{R}$ into function $\tilde{u}$, which is defined on $L^2(\mathcal{G}; \mathbb{R})$ and satisfies $\tilde{u}(\xi) = u(\xi, \mathcal{L}(\xi))$. On the contrary, when function $\tilde{u}$ is defined on $L^2(\mathcal{G}; \mathbb{R})$, we call function u the inverse-lifted function of $\tilde{u}$, which is defined on $\mathcal{P}_2(\mathbb{R})$ and satisfies $u(\xi, \nu) = \tilde{u}(\xi)$ for $\nu = \mathcal{L}(\xi)$. When the lifted function $\tilde{u}$ is Fréchet differentiable (respectively, Fréchet differentiable with respect to continuous derivatives) on $L^2(\mathcal{G}; \mathbb{R})$, then the function u is differentiable (respectively, $\mathcal{C}^1$) on $\mathcal{P}_2(\mathbb{R})$. In terms of Riesz' theorem: $[D\tilde{u}](\xi)(Z) = \mathbb{E}[D\tilde{u}(\xi).Z]$, we use

$$D\tilde{u}(\xi) = \partial_{\nu} u(\mathcal{L}(\xi))(\xi), \quad (4)$$

to represent the Fréchet derivative $[D\tilde{u}](\xi)$ in this case, which is regarded as an element $D\tilde{u}(\xi)$ of $L^2(\mathcal{G}; \mathbb{R})$, where derivative $\partial_{\nu} u(\mathcal{L}(\xi)) : \mathbb{R} \rightarrow \mathbb{R}$ is called as the derivative of function u at $\nu = \mathcal{L}(\xi)$. Moreover, $\partial_{\nu} u(\nu) \in L^2_{\nu}(\mathbb{R})$ for $\nu \in \mathcal{P}_2(\mathbb{R}) = \{\mathcal{L}(\xi) : \xi \in L^2(\mathcal{G}; \mathbb{R})\}$. As mentioned in [11], if the function u is $\mathcal{C}^1$ and for all $\nu \in \mathcal{P}_2(\mathbb{R})$, a version of the mapping $\xi \in \mathbb{R} \mapsto \partial_{\nu} u(\nu)(\xi)$ is continuous such that the mapping $(\nu, \xi) \in \mathcal{P}_2(\mathbb{R}) \times \mathbb{R} \mapsto \partial_{\nu} u(\nu)(\xi)$ is continuous at any point (ν, ξ) and $\xi \in \text{Supp}(\nu)$, and if the mapping $\xi \in \mathbb{R} \mapsto \partial_{\nu} u(\nu)(\xi)$ is differentiable, whose derivative is jointly continuous at any point (ν, ξ) and $\xi \in \text{Supp}(\nu)$, then the function u is partially $\mathcal{C}^2$. We denote by $\partial_x \partial_{\nu} u(\nu)(\xi)$ the gradient of $\partial_{\nu} u(\nu)$. If the function u is partially $\mathcal{C}^2$, $\partial_x \partial_{\nu} u(\nu) \in L^{\infty}_{\nu}(\mathbb{R})$, and for all compact set $\mathcal{K}$ of $\mathcal{P}_2(\mathbb{R})$,

$$\sup_{\nu \in \mathcal{K}} \left[\int_{\mathbb{R}} |\partial_{\nu} u(\nu)(x)|^2 \nu(dx) + \|\partial_x \partial_{\nu} u(\nu)\|_{\infty} \right] < \infty,$$

then we say that the function $u \in \mathcal{C}^2_b(\mathcal{P}_2(\mathbb{R}))$. As mentioned in [11], when the lifted function $\tilde{u}$ is twice continuously Fréchet differentiable on $L^2(\mathcal{G}; \mathbb{R})$ and has Lipschitz continuous Fréchet derivative, then we have $u \in \mathcal{C}^2_b(\mathcal{P}_2(\mathbb{R}))$. In terms of Riesz' theorem, the second Fréchet derivative $D^2\tilde{u}(\xi)$ in this case is viewed as a bilinear form or a symmetric operator (thus it is bounded) on $L^2(\mathcal{G}; \mathbb{R})$, and we also obtain the equality ([12] Appendix A.2) that for all $\xi \in L^2(\mathcal{G}; \mathbb{R})$, $Z \in L^2(\mathcal{G}; \mathbb{R})$,

$$\mathbb{E}[D^2 \bar{u}(\xi)(ZN).ZN] = \mathbb{E}[\partial_x \partial_v u(\mathcal{L}(\xi))(\xi) Z^2], \quad (5)$$

where $N \in L^2(\mathcal{G}; \mathbb{R})$ is independent of (ξ, Z) with zero mean and unit variance.

In order to guarantee the controlled reflected McKean–Vlasov, SDE (1) is well-posed, we state the following assumptions.

(H1)

- (i) For all $(t, x, v, \alpha) \in [0, T] \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \times \mathcal{A}$, there is a constant κ such that $|\sigma_t(x, v, \alpha)|^2 \geq \kappa > 0$.
- (ii) The drift coefficient b and diffusion coefficient σ are measurable functions on $[0, T] \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \times \mathcal{A}$.
- (iii) There is a positive constant C such that for any $t \in [0, T]$, $x, x' \in \mathbb{R}$, $v, v' \in \mathcal{P}_2(\mathbb{R})$ and $\alpha \in \mathcal{A}$,

$$|b_t(x, v, \alpha) - b_t(x', v', \alpha)| + |\sigma_t(x, v, \alpha) - \sigma_t(x', v', \alpha)| \leq C(|x - x'| + \mathcal{W}_2(v, v')),$$

$$|b_t(0, \delta_0, \alpha)| + |\sigma_t(0, \delta_0, \alpha)| \leq C,$$

where

$$\delta_0(A) = \begin{cases} 1, & 0 \in A, \\ 0, & \text{else.} \end{cases}$$

- (iv) For all $(t, x, v) \in [0, T] \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R})$, the functions $\alpha \mapsto b_t(x, v, \alpha), \sigma_t(x, v, \alpha)$ are continuous on $\mathcal{A}$.

Under **(H1)**, we see that b and σ meet the conditions in Theorem 3.2 of [13], and domain $[p, q]$ satisfies the condition on domain $\mathcal{D}$ in Assumption 2.5 of [13], therefore Equation (1) has a unique solution, which is denoted by $\{X_t^\alpha, 0 \leq t \leq T\}$ for $\alpha \in \mathcal{A}$.

The items that appear in cost functional (2) are supposed to fulfill the following conditions.

(H2)

- (i) The functions f, g , and G are measurable, respectively, on $[0, T] \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \times \mathcal{A}$, $[0, T] \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R})$ and $\mathbb{R} \times \mathcal{P}_2(\mathbb{R})$.
- (ii) There is a positive constant C such that for any $(t, x, v, \alpha) \in [0, T] \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \times \mathcal{A}$,

$$|f_t(x, v, \alpha)| + |g_t(x, v)| + |G(x, v)| \leq C(1 + |x|^2 + \|v\|_2^2).$$

- (iii) The coefficients f, g , and G , respectively, on $[0, T] \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \times \mathcal{A}$, $[0, T] \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R})$, and $\mathbb{R} \times \mathcal{P}_2(\mathbb{R})$ are continuous, and are also local Lipschitz continuous, uniformly with respect to $\mathcal{A}$: there is a positive constant C such that for any $t \in [0, T]$, $x, x' \in \mathbb{R}$, $v, v' \in \mathcal{P}_2(\mathbb{R})$ and $\alpha \in \mathcal{A}$,

$$\begin{aligned} & |f_t(x, v, \alpha) - f_t(x', v', \alpha)| + |g_t(x, v) - g_t(x', v')| + |G(x, v) - G(x', v')| \\ & \leq C(1 + |x| + |x'| + \|v\|_2 + \|v'\|_2)(|x - x'| + \mathcal{W}_2(v, v')). \end{aligned}$$

Under **(H2)**, for any $(t, \xi, \alpha) \in [0, T] \times L^2(\mathcal{G}; \mathbb{R}) \times \mathcal{A}$, the cost functional (2) is well-posed and finite.

Considering Peng's pioneering research [14] about non-Markovian stochastic control, the dynamic programming principle related to the reflected McKean–Vlasov stochastic control indicates that the value function V in (3) is the first element of the pair (v, ψ) , which is the solution to the following BSPDE with Neumann boundary condition:

$$\begin{cases} -dv_t(\mathcal{L}(\xi)) = \operatorname{ess\,inf}_{\alpha \in \mathcal{A}} \mathbb{E} \left[\partial_v v_t(\mathcal{L}(\xi))(\xi) b_t(\xi, \mathcal{L}(\xi), \alpha) + \frac{1}{2} \partial_x \partial_v v_t(\mathcal{L}(\xi))(\xi) |\sigma_t(\xi, \mathcal{L}(\xi), \alpha)|^2 \right. \\ \quad \left. + \check{f}_t(\mathcal{L}(\xi), \alpha) \right] dt - \check{\psi}_t(\mathcal{L}(\xi)) dB_t, & (t, \mathcal{L}(\xi)) \in [0, T] \times \mathcal{P}_2([p, q]); \\ \partial_v v_t(\mathcal{L}(\xi))(p) = \check{g}_t(\delta_p), & \partial_v v_t(\mathcal{L}(\xi))(q) = \check{g}_t(\delta_q); \\ v_T(\mathcal{L}(\xi)) = \check{G}(\mathcal{L}(\xi)), & \mathcal{L}(\xi) \in \mathcal{P}_2([p, q]), \end{cases} \quad (6)$$

where dv_t is the derivative of v with respect to t , and $\check{f}, \check{g}, \check{G}, \check{\psi}$ is inverse-lifted function, respectively, of $f(\cdot, \mathcal{L}(\cdot)), g(\cdot, \mathcal{L}(\cdot)), G(\cdot, \mathcal{L}(\cdot)), \psi$.

Therefore, by the lifting identification, we regard the function v as a function on $[0, T] \times L^2(\mathcal{G}; \mathbb{R})$ and reserve the same notation $v_t(\xi) = v_t(\mathcal{L}(\xi))$ (notice that v is dependent on ξ just by its law), then we notice from the relation (4) and (5) between derivatives in the Wasserstein space $\mathcal{P}_2(\mathbb{R})$ and in the Hilbert space $L^2(\mathcal{G}; \mathbb{R})$ that the BSPDE (6) is also written in $[0, T] \times L^2(\mathcal{G}; [p, q])$ as

$$\begin{cases} -dv_t(\xi) = \mathbb{H}_t(\xi, Dv_t(\xi), D^2v_t(\xi))dt - \psi_t(\xi)dB_t, & (t, \xi) \in [0, T] \times L^2(\mathcal{G}; [p, q]); \\ Dv_t(p) = g_t(p, \delta_p), & Dv_t(q) = g_t(q, \delta_q); \\ v_T(\xi) = G(\xi, \mathcal{L}(\xi)), & \xi \in L^2(\mathcal{G}; [p, q]), \end{cases} \quad (7)$$

with Hamiltonian function

$$\begin{aligned} \mathbb{H}_t(\xi, Dv_t(\xi), D^2v_t(\xi)) &= \operatorname{ess\,inf}_{\alpha \in \mathcal{A}} \mathbb{E} \left\{ f_t(\xi, \mathcal{L}(\xi), \alpha) + Dv_t(\xi)b_t(\xi, \mathcal{L}(\xi), \alpha) \right. \\ &\quad \left. + \frac{1}{2}D^2v_t(\xi)|\sigma_t(\xi, \mathcal{L}(\xi), \alpha)N|^2 \right\}, \end{aligned} \quad (8)$$

for $(t, \xi) \in [0, T] \times L^2(\mathcal{G}; [p, q])$; here, $N \in L^2(\mathcal{G}; [p, q])$ is independent of ξ .

2.2. Definition of Solutions to BSPDEs and Main Result

For a Banach space $\mathbb{V}$, the set of $\mathcal{G}$ -measurable and square-integrable $\mathbb{V}$ -valued random variables is denoted by the space $L^2(\mathcal{G}; \mathbb{V})$, and the set of $\mathfrak{P}$ -measurable càdlàg $\mathbb{V}$ -valued process $(X_t)_{t \in [0, T]}$ such that

$$\|X\|_{\mathcal{S}_{\mathcal{G}}^n(0, T; \mathbb{V})} = \mathbb{E} \sup_{t \in [0, T]} \|X_t\|_{\mathbb{V}}^n < \infty$$

is denoted by $\mathcal{S}_{\mathcal{G}}^n(0, T; \mathbb{V})$, where $\mathfrak{P}$ is the predictable σ -algebra corresponding to $\{\mathcal{G}_t\}_{t \geq 0}$ on $\Omega \times [0, \infty)$ and $n \in [1, \infty)$. The set of $\mathfrak{P}$ -measurable $\mathbb{V}$ -valued processes $(v_t)_{t \in [0, T]}$ satisfying

$$\|v\|_{\mathbb{L}_{\mathcal{G}}^n(0, T; \mathbb{V})} = \mathbb{E} \int_0^T \|v_t\|_{\mathbb{V}}^n dt < \infty, \quad n \in [1, \infty);$$

$$\|v\|_{\mathbb{L}_{\mathcal{G}}^\infty(0, T; \mathbb{V})} = \sup_{(\omega, t) \in \Omega \times [0, T]} \|v_t\|_{\mathbb{V}} < \infty, \quad n = \infty$$

is denoted by $\mathbb{L}_{\mathcal{G}}^n(0, T; \mathbb{V})$. For simplicity, we neglect the subscript for the space $\mathcal{S}_{\mathcal{G}}^2(0, T; \mathbb{V})$ and space $\mathbb{L}_{\mathcal{G}}^2(0, T; \mathbb{V})$, particularly if there is no confusion about the adaptedness and filtration.

$H^{m, n}([p, q])$ is denoted as the Sobolev space of real-valued functions φ and its up to m -th order derivatives are in $L^n([p, q])$, which is endowed with the Sobolev norm $\|\varphi\|_{H^{m, n}([p, q])}$, $m \in \mathbb{N}^+$ and $n \in [1, \infty)$. The space of trace-zero functions in $H^{m, n}([p, q])$ is denoted by $H_0^{m, n}([p, q])$. For $m = 0$, $H_0^{0, n}([p, q]) := L^n([p, q])$. For simplicity, $v_1, \dots, v_l \in H^{m, n}([p, q])$ and $\|v\|_{H^{m, n}([p, q])}^n = \sum_{j=1}^l \|v_j\|_{H^{m, n}([p, q])}^n$ can be expressed as $v = (v_1, \dots, v_l) \in H^{m, n}([p, q])$. The norm and the inner product in the usual Hilbert space $L^2([p, q])$ are denoted, respectively, by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$, and the duality between Hilbert space $H^{m, 2}([p, q])$ and their dual spaces is denoted by $\langle \cdot, \cdot \rangle$ when there is no confusion. For $m = 1, 2$, we set

$$\mathcal{H} = \mathcal{S}_{\mathcal{G}}^2(0, T; L^2([p, q])) \cap \mathbb{L}_{\mathcal{G}}^2(0, T; H^{1, 2}([p, q])) \times \mathbb{L}_{\mathcal{G}}^2(0, T; L^2([p, q])),$$

$$\mathcal{H}^m = \mathcal{S}_{\mathcal{G}}^2(0, T; H^{m, 2}([p, q])) \cap \mathbb{L}_{\mathcal{G}}^2(0, T; H^{m+1, 2}([p, q])) \times \mathbb{L}_{\mathcal{G}}^2(0, T; H^{m, 2}([p, q])),$$

and the two spaces are complete and equipped, respectively, with the norms

$$\|(v, \psi)\|_{\mathcal{H}}^2 = \|v\|_{\mathcal{S}_{\mathcal{G}}^2(0, T; L^2([p, q]))}^2 + \|v\|_{\mathbb{L}_{\mathcal{G}}^2(0, T; H^{1, 2}([p, q]))}^2 + \|\psi\|_{\mathbb{L}_{\mathcal{G}}^2(0, T; L^2([p, q]))}^2, \quad \forall (v, \psi) \in \mathcal{H},$$

and

$$\|(v, \psi)\|_{\mathcal{H}^m}^2 = \|v\|_{\mathcal{S}_{\mathcal{G}}^2(0,T;H^{m,2}([p,q]))}^2 + \|v\|_{\mathbb{L}_{\mathcal{G}}^2(0,T;H^{m+1,2}([p,q]))}^2 + \|\psi\|_{\mathbb{L}_{\mathcal{G}}^2(0,T;H^{m,2}([p,q]))}^2, \forall (v, \psi) \in \mathcal{H}^m.$$

Now, the notion of solutions to BSPDEs with general nonlinear coefficients is introduced as follows.

Definition 1. For any $x, x^1, x^2 \in \mathbb{R}$ and any $z, z^1 \in \mathbb{R}$, let R be a random function such that

$$R(\cdot, x, x^1, x^2, z, z^1) : \Omega \times [0, T] \times L^2(\mathcal{G}; [p, q]) \rightarrow \mathbb{R}$$

is $\mathfrak{P} \otimes \mathcal{B}([p, q])$ -measurable, and let $G \in L^2(\mathcal{G}; L^2([p, q]))$. We say that there is a weak solution (v, ψ) to the BSPDE:

$$\begin{cases} -dv_t(\xi) = R_t(\xi, v, Dv, D^2v, \psi, D\psi)dt - \psi_t(\xi)dB_t, & (t, \xi) \in [0, T] \times L^2(\mathcal{G}; [p, q]); \\ Dv_t(p) = g_t(p, \delta_p), & Dv_t(q) = g_t(q, \delta_q); \\ v_T(\xi) = G(\xi, \mathcal{L}(\xi)), & \xi \in L^2(\mathcal{G}; [p, q]), \end{cases} \quad (9)$$

if $(v, \psi) \in \mathcal{H}$ and satisfies Equation (9) in the weak sense, that is, for all $\phi \in C_c^\infty((p, q))$,

$$\langle \phi, R(\cdot, v, Dv, D^2v, \psi, D\psi) \rangle \in \mathbb{L}_{\mathcal{G}}^1(0, T; \mathbb{R})$$

and

$$\langle \phi, v_t \rangle = \langle \phi, G \rangle + \int_t^T \langle \phi, R_s(\cdot, v, Dv, D^2v, \psi, D\psi) \rangle ds - \int_t^T \langle \phi, \psi_s dB_s \rangle \text{ a.s., } t \in [0, T]. \quad (10)$$

If the regularity $(v, \psi) \in \mathcal{H}^1$ has been proved, we call the above (v, ψ) a strong solution.

Particularly, we have a case of nonlinear term R with

$$R_t(\xi, v, Dv, D^2v, \psi, D\psi) = \mathbb{H}_t(\xi, Dv_t(\xi), D^2v_t(\xi)),$$

which is corresponding to BSPDE (7).

In order to obtain that Equation (7) is well-posed, we additionally take the assumptions below.

(H3)

(i) Let

$$G(\cdot, \mathcal{L}(\cdot)) \in L^2(\mathcal{G}; H^{2,2}([p, q])), \quad DG(\cdot, \mathcal{L}(\cdot)) - g_T(\cdot, \mathcal{L}(\cdot)) \in L^2(\mathcal{G}; H_0^{1,2}([p, q])),$$

and along with another function Ψ , the pair (g, Ψ) belongs to $\mathcal{H}^1$ and satisfies BSPDE

$$-dg_t(\xi, \mathcal{L}(\xi)) = Q_t(\xi, \mathcal{L}(\xi))dt - \Psi_t(\xi, \mathcal{L}(\xi))dB_t, \quad \forall \xi \in L^2(\mathcal{G}; [p, q]),$$

in the weak sense with $Q \in \mathbb{L}_{\mathcal{G}}^2(0, T; L^2([p, q]))$.

(ii) For any $u \in \mathcal{S}_{\mathcal{G}}^2(0, T; H^{1,2}([p, q])) \cap \mathbb{L}_{\mathcal{G}}^2(0, T; H^{2,2}([p, q]))$, we assume that $\mathbb{H}(\cdot, u, Du)$, $(D\mathbb{H})(\cdot, u, Du) \in \mathbb{L}_{\mathcal{G}}^2(0, T; L^2([p, q]))$, and there are nonnegative constants K_0 and K_1 such that for any $u_1, u_2 \in \mathcal{S}_{\mathcal{G}}^2(0, T; H^{1,2}([p, q])) \cap \mathbb{L}_{\mathcal{G}}^2(0, T; H^{2,2}([p, q]))$, $(t, \xi) \in [0, T] \times L^2(\mathcal{G}, [p, q])$,

$$|\mathbb{H}_t(\xi, u_1, Du_1) - \mathbb{H}_t(\xi, u_2, Du_2)| \leq K_0|u_1 - u_2| + K_1|D(u_1 - u_2)|.$$

(iii) There is an $\mathcal{A}$ -valued $\mathfrak{P} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$ -measurable function Π satisfying

$$\begin{aligned} \mathbb{H}_t(\xi, y, z) &= \mathbb{E} \left[f_t(\xi, \mathcal{L}(\xi), \Pi_t(\xi, y, z)) + b_t(\xi, \mathcal{L}(\xi), \Pi_t(\xi, y, z))y \right. \\ &\quad \left. + \frac{1}{2} |\sigma_t(\xi, \mathcal{L}(\xi), \Pi_t(\xi, y, z))N|^2 z \right], \end{aligned}$$

that is,

$$\Pi_t(\xi, y, z) \in \arg \operatorname{ess} \inf_{\alpha \in \mathcal{A}} \mathbb{E} \left\{ f_t(\xi, \mathcal{L}(\xi), \alpha) + b_t(\xi, \mathcal{L}(\xi), \alpha)y + \frac{1}{2} |\sigma_t(\xi, \mathcal{L}(\xi), \alpha)N|^2 z \right\},$$

and for any $u \in S_{\mathcal{G}}^2(0, T; H^{1,2}([p, q])) \cap \mathbb{L}_{\mathcal{G}}^2(0, T; H^{2,2}([p, q]))$, there exists a unique solution to the controlled reflected McKean–Vlasov SDE (1) associated with the drift coefficient $b_t(X_t, \mathcal{L}(X_t), \Pi_t(X_t, u_t, Du_t))$.

Finally, we summarize the main theorem. In the next section, we make some preparations to prove it.

Theorem 1. Suppose that (H1), (H2), and (H3) hold. There is a unique strong solution (v, ψ) to BSPDE (7). For the strong solution, we also obtain that $(v, \psi) \in \mathcal{H}^2$. In addition, for any $\eta \in L^2(\mathcal{G}; [p, q])$, $\mathbb{E}[v_t(\eta)] = V_t(\eta)$, which is the value function (3). We obtain the optimal control α^* and the associated state process X^* , respectively, expressed as $\alpha^* = \Pi_t(X_t^*, Dv_t(X_t^*), D^2v_t(X_t^*))$ and

$$\begin{cases} dX_t^* = b_t(X_t^*, \mathcal{L}(X_t^*), \Pi_t(X_t^*, Dv_t(X_t^*), D^2v_t(X_t^*)))dt \\ \quad + \sigma_t(X_t^*, \mathcal{L}(X_t^*), \Pi_t(X_t^*, Dv_t(X_t^*), D^2v_t(X_t^*)))dB_t + dL_t - dU_t; \\ X_0^* = \xi; \quad L_0 = U_0 = 0; \\ p \leq X_t^* \leq q \quad a.s.; \\ \int_0^T (X_t^* - p)dL_t = \int_0^T (q - X_t^*)dU_s = 0 \quad a.s. \end{cases} \quad (11)$$

We take into consideration the conditions (H1) and (H2) as standing assumptions in this context, and they are standard to ensure that the BSPDE (7) is adapted and the controlled McKean–Vlasov SDE with reflection is well-posed.

By assumption (H3) (i), in order to obtain $(v, \psi) \in \mathcal{H}^2$, it is standard for the requests of G (by L^p -theory of BSPDE of [15]); considering the Skorohod conditions of McKean–Vlasov SDE (1) with reflection, one has

$$\int_0^T g_s(X_s, \mathcal{L}(X_s))dL_s = \int_0^T g_s(p, \delta_p)dL_s, \quad \int_0^T g_s(X_s, \mathcal{L}(X_s))dU_s = \int_0^T g_s(q, \delta_q)dU_s,$$

so the reflected control problem only involves $g_s(p, \delta_p)$ and $g_s(q, \delta_q)$.

In assumption (H3) (ii), we suppose that the Hamilton function $\mathbb{H}_t(\xi, u, Du)$ is Lipschitz continuous with respect to u and Du , which implies that $\partial_u \mathbb{H}(\cdot, u, Du), \partial_{Du} \mathbb{H}(\cdot, u, Du) \in L^\infty(\Omega \times [0, T] \times L^2(\mathcal{G}, [p, q]))$ for any $u \in \mathbb{R}$.

3. Existence and Uniqueness of a Strong Solution to General Nonlinear BSPDEs

In this section, the existence and uniqueness of strong solution to general nonlinear BSPDEs with Neumann boundary condition is established. In this context, we only take into account the 1-dimensional case, although it is not essentially difficult with the multidimensional case.

Take into consideration the following Neumann problem for BSPDE:

$$\begin{cases} -dv_t(\xi) = \left[\frac{1}{2} |\sigma_t(\xi, \mathcal{L}(\xi), \alpha)|^2 D^2 v_t(\xi) + \Gamma_t(\xi, v, Dv, D^2 v, \psi, D\psi) \right] dt - \psi_t(\xi) dB_t, \\ Dv_t(p) = 0, \quad Dv_t(q) = 0; \\ v_T(\xi) = G(\xi, \mathcal{L}(\xi)), \quad \xi \in L^2(\mathcal{G}, [p, q]). \end{cases} \quad (12)$$

In this section, we take the following hypothesis.

(H4)

For any $(v, \psi) \in H^{2,2}([p, q]) \times H^{1,2}([p, q])$, $\Gamma_t(\cdot, v, Dv, D^2 v, \psi, D\psi) \in \mathbb{L}^2(0, T; L^2([p, q]))$, and there are nonnegative constants L_1 and L_2 such that for any $(v_i, \psi_i) \in H^{2,2}([p, q]) \times H^{1,2}([p, q])$, $i = 1, 2$, for any $t \in [0, T]$, there holds

$$\begin{aligned} & \|\Gamma_t(\cdot, v_1, Dv_1, D^2 v_1, \psi_1, D\psi_1) - \Gamma_t(\cdot, v_2, Dv_2, D^2 v_2, \psi_2, D\psi_2)\| \\ & \leq L_1 (\|D^2(v_1 - v_2)\| + \|D(\psi_1 - \psi_2)\|) + L_2 (\|v_1 - v_2\|_{H^{1,2}([p, q])} + \|\psi_1 - \psi_2\|_{L^2([p, q])}) a.s.. \end{aligned}$$

Theorem 2. Let $G \in L^2(\mathcal{G}; H^{1,2}([p, q]))$ and assume that **(H2)** and **(H4)** hold. There is a positive constant L_0 depending on L_2 and T , such that when $0 \leq L_1 < L_0$, there is a unique strong solution (v, ψ) to BSPDE (12) which satisfies

$$\|(v, \psi)\|_{\mathcal{H}^1} \leq C' (\|G\|_{L^2(\mathcal{G}; H^{1,2}([p, q]))} + \|\Gamma^0\|_{\mathbb{L}^2(0, T; L^2([p, q]))}), \quad (13)$$

in which $\Gamma^0 := \Gamma(\cdot, 0, 0, 0, 0, 0)$ and C' is dependent on L_1, L_2, κ , and T .

Firstly, we shall establish some priori estimates for linear equations in Section 3.1 and then exploit the method of continuity to finish the proof of Theorem 2 in Section 3.2.

3.1. The Piori Estimates

For every $\gamma \in [0, 1]$, we take into account the following linear BSPDE:

$$\begin{cases} -dv_t(\xi) = \left[\frac{\gamma}{2} |\sigma_t(\xi, \mathcal{L}(\xi), \alpha)|^2 D^2 v_t(\xi) + \frac{1-\gamma}{2} D^2 v_t(\xi) + F_t(\xi) \right] dt - \psi_t(\xi) dB_t, \\ (t, \xi) \in [0, T] \times L^2(\mathcal{G}, [p, q]); \\ Dv_t(p) = 0, \quad Dv_t(q) = 0; \\ v_T(\xi) = G(\xi, \mathcal{L}(\xi)), \quad \xi \in L^2(\mathcal{G}, [p, q]). \end{cases} \quad (14)$$

We place the proof of the following Proposition 1 in Appendix A.

Proposition 1. Assume that **(H2)** holds and

$$F \in \mathbb{L}^2(0, T; L^2([p, q])), \quad G \in L^2(\mathcal{G}; H^{1,2}([p, q])).$$

Assume that (v, ψ) is a strong solution to BSPDE for a Neumann problem (14). Then, this strong solution is unique, satisfying

$$\|(v, \psi)\|_{\mathcal{H}}^2 \leq C_1 \left\{ \|G\|_{L^2(\mathcal{G}; L^2([p, q]))}^2 + \mathbb{E} \left[\int_0^T |\langle F_s, v_s \rangle| ds \right] \right\},$$

and

$$\begin{aligned} \|(v, \psi)\|_{\mathcal{H}^1}^2 & \leq C_2 \left\{ \|G\|_{L^2(\mathcal{G}; H^{1,2}([p, q]))}^2 + \mathbb{E} \left[\int_0^T |\langle F_s, v_s \rangle| + |\langle F_s, D^2 v_s \rangle| ds \right] \right\} \\ & \leq C_3 \left\{ \|G\|_{L^2(\mathcal{G}; H^{1,2}([p, q]))}^2 + \|F\|_{\mathbb{L}^2(0, T; L^2([p, q]))}^2 \right\}, \end{aligned}$$

where the constants C_1, C_2 , and C_3 are only dependent on κ and T , and independent of $\gamma \in [0, 1]$.

3.2. Existence and Uniqueness of Strong Solution to (12) (Proof of Theorem 2)

Firstly, we take into account the BSPDE for Neumann problem with Laplacian operator as follows:

$$\begin{cases} -dv_t(\xi) = [D^2v_t(\xi) + F_t(\xi)]dt - \psi_t(\xi)dB_t, & (t, \xi) \in [0, T] \times L^2(\mathcal{G}, [p, q]); \\ Dv_t(p) = 0, & Dv_t(q) = 0; \\ v_T(\xi) = G(\xi, \mathcal{L}(\xi)), & \xi \in L^2(\mathcal{G}, [p, q]). \end{cases} \quad (15)$$

Proposition 2. Set

$$F \in \mathbb{L}_{\mathcal{G}}^2(0, T; L^2([p, q])), \quad G \in L^2(\mathcal{G}; H^{1,2}([p, q])).$$

Then there is a unique strong solution (v, ψ) to BSPDE (15).

Proof. In view of Proposition 1, we can directly obtain the uniqueness of the strong solution. Hence, it just requires us to prove the existence of the strong solution.

Step 1. Further assume that $F \in \mathbb{L}_{\mathcal{G}}^2(0, T; H^{1,2}([p, q]))$ and $DG \in L^2(\mathcal{G}; H_0^{1,2}([p, q]))$. In view of the theory about the deterministic parabolic PDEs for the Neumann problem of Theorem 7.22 in [16], there is a unique strong solution $\hat{v}$ to PDE:

$$\begin{cases} -\partial_t \hat{v}_t(\xi) = D^2 \hat{v}_t(\xi) + F_t(\xi), & (t, \xi) \in [0, T] \times L^2(\mathcal{G}, [p, q]); \\ D\hat{v}_t(p) = 0, & D\hat{v}_t(q) = 0; \\ \hat{v}_T(\xi) = G(\xi, \mathcal{L}(\xi)), & \xi \in L^2(\mathcal{G}, [p, q]), \end{cases}$$

such that $\hat{v}, D\hat{v}, D^2\hat{v}, \partial_t \hat{v} \in L^2(\mathcal{G}; L^2([0, T] \times [p, q]))$. We first take conditional expectations in Hilbert spaces ([17]) and then set

$$v_t = \mathbb{E}[\hat{v}_t | \mathcal{G}_t] \quad \text{a.s., for each } t \in [0, T],$$

which admits a version in $\mathcal{S}_{\mathcal{G}}^2(0, T; L^2([p, q])) \cap \mathbb{L}_{\mathcal{G}}^2(0, T; H^{2,2}([p, q]))$. Along with $\psi \in \mathbb{L}_{\mathcal{G}}^2(0, T; L^2([p, q]))$, v satisfies $L^2([p, q])$ -valued BSDE:

$$\begin{cases} -dv_t(\xi) = [D^2v_t(\xi) + F_t(\xi)]dt - \psi_t(\xi)dB_t; \\ v_T(\xi) = G(\xi, \mathcal{L}(\xi)), & \xi \in L^2(\mathcal{G}, [p, q]). \end{cases} \quad (16)$$

Considering that the definition of v , v satisfies the zero-Neumann boundary conditions, in view of Definition 1 and Proposition 1, we may easily obtain that (v, ψ) is a weak solution to Equation (15).

Step 2. Now let us show that the weak solution (v, ψ) constructed above is indeed a unique strong solution to Equation (15). Using a similar method for **Step 1**, we can easily check that $D\hat{v}$ is the strong solution of the Dirichlet problem:

$$\begin{cases} -\partial_t \hat{u}_t(\xi) = D^2 \hat{u}_t(\xi) + DF_t(\xi), & (t, \xi) \in [0, T] \times L^2(\mathcal{G}, [p, q]); \\ \hat{u}_t(p) = 0, & \hat{u}_t(q) = 0; \\ \hat{u}_T(\xi) = DG(\xi, \mathcal{L}(\xi)), & \xi \in L^2(\mathcal{G}, [p, q]), \end{cases}$$

and $(Dv, D\psi)$ satisfies $L^2([p, q])$ -valued BSDE (16) corresponding to the coefficients (DF, DG) . Particularly, we obtain

$$(Dv, D\psi) \in \mathcal{S}_{\mathcal{G}}^2(0, T; L^2([p, q])) \times \mathbb{L}_{\mathcal{G}}^2(0, T; L^2([p, q])),$$

and, hence, (v, ψ) is a strong solution to Equation (15). For $F \in \mathbb{L}_{\mathcal{G}}^2(0, T; L^2([p, q]))$ and $G \in L^2(\mathcal{G}; H^{1,2}([p, q]))$, we can choose a sequence

$$\{(F^n, G^n)\}_{n \in \mathbb{N}^+} \subset \mathbb{L}_{\mathcal{G}}^2(0, T; H^{1,2}([p, q])) \times L^2(\mathcal{G}; H^{1,2}([p, q]))$$

with $\{DG^n\}_{n \in \mathbb{N}^+} \subset L^2(\mathcal{G}; H_0^{1,2}([p, q]))$ such that (F^n, G^n) converges to (F, G) in $\mathbb{L}_{\mathcal{G}}^2(0, T; L^2([p, q])) \times L^2(\mathcal{G}; H^{1,2}([p, q]))$. For each (F^n, G^n) , we obtain the strong solution (v^n, ψ^n) to BSPDE (15) associated with the coefficients (F^n, G^n) . Therefore, we can obtain the convergence of (v^n, ψ^n) and the existence of strong solution from the estimates in Proposition 1. $\square$

Finally, we can exploit the method of continuity to complete the proof of Theorem 2.

Proof. Step 1. For each $\gamma \in [0, 1]$ and $F \in \mathbb{L}_{\mathcal{G}}^2(0, T; L^2([p, q]))$, consider the following BSPDE:

$$\begin{cases} -dv_t(\xi) = \left[\frac{\gamma}{2} |\sigma_t(\xi, \mathcal{L}(\xi), \alpha)|^2 D^2 v_t(\xi) + \gamma \Gamma_t(\xi, v, Dv, D^2 v, \psi, D\psi) + (1 - \gamma) D^2 v_t(\xi) \right. \\ \quad \left. + F_t(\xi) \right] dt - \psi_t(\xi) dB_t, & (t, \xi) \in [0, T] \times L^2(\mathcal{G}, [p, q]); \\ Dv_t(p) = 0, & Dv_t(q) = 0; \\ v_T(\xi) = G(\xi, \mathcal{L}(\xi)), & \xi \in L^2(\mathcal{G}, [p, q]). \end{cases} \quad (17)$$

When $\gamma = 1$ and $F = 0$, we observe that the BSPDE (12) is a special case of the above equation. Then we shall extend priori estimates inferred from the linear case to the nonlinear BSPDE (17). Assume that (v, ψ) is a strong solution to Equation (17). For any $t \in [0, T]$, we apply Proposition 1 (see also estimates (A4) and (A7)) and then we obtain, by (H4),

$$\begin{aligned} & \mathbb{E} \left[\sup_{s \in [t, T]} \|v_s\|^2 \right] + \mathbb{E} \int_t^T (\|\psi_s\|^2 + \|Dv_s\|^2) ds \\ & \leq \mathbb{CE} \left[\|G\|^2 + \int_t^T \left| \langle F_s + \gamma \Gamma_s(\cdot, v, Dv, D^2 v, \psi, D\psi), v_s \rangle \right| ds \right] \\ & \leq \mathbb{CE} \left[\|G\|^2 + \varepsilon \int_t^T (\|F_s\|^2 + \|\Gamma_s(\cdot, v, Dv, D^2 v, \psi, D\psi)\|^2) ds + \frac{2}{\varepsilon} \int_t^T \|v_s\|^2 ds \right] \\ & \leq C_1 \mathbb{E} \left[\|G\|^2 + \varepsilon \int_t^T (\|F_s\|^2 + \|\Gamma_s^0\|^2 + \|v_s\|^2 + \|Dv_s\|^2 + \|\psi_s\|^2) ds \right. \\ & \quad \left. + \frac{2}{\varepsilon} \int_t^T \|v_s\|^2 ds + \varepsilon L_1^2 \int_t^T (\|D\psi_s\|^2 + \|D^2 v_s\|^2) ds \right], \end{aligned} \quad (18)$$

and

$$\begin{aligned} & \mathbb{E} \left[\sup_{s \in [t, T]} \|v_s\|_{H^{1,2}([p, q])}^2 \right] + \mathbb{E} \int_t^T (\|v_s\|_{H^{2,2}([p, q])}^2 + \|\psi_s\|_{H^{1,2}([p, q])}^2) ds \\ & \leq \mathbb{CE} \left[\|G\|_{H^{1,2}([p, q])}^2 + \int_t^T \left| \langle F_s + \gamma \Gamma_s(\cdot, v, Dv, D^2 v, \psi, D\psi), v_s \rangle \right| ds \right. \\ & \quad \left. + \int_t^T \left| \langle F_s + \gamma \Gamma_s(\cdot, v, Dv, D^2 v, \psi, D\psi), D^2 v_s \rangle \right| ds \right] \\ & \leq \mathbb{CE} \left[\|G\|_{H^{1,2}([p, q])}^2 + \left(1 + \frac{1}{\varepsilon}\right) \int_t^T (\|F_s\|^2 + \|\Gamma_s^0\|^2 + \|v_s\|^2 + \|Dv_s\|^2 + \|\psi_s\|^2) ds \right. \\ & \quad \left. + \int_t^T \left(\left(\frac{L_1^2}{\varepsilon} + \varepsilon\right) \|D^2 v_s\|^2 + \frac{L_1^2}{\varepsilon} \|D\psi_s\|^2 \right) ds \right]. \end{aligned} \quad (19)$$

Letting $0 < \varepsilon < \frac{1}{C_1}$, by (18) we have

$$\begin{aligned} & \mathbb{E} \left[\sup_{s \in [t, T]} \|v_s\|^2 \right] + \mathbb{E} \int_t^T \left(\|\psi_s\|^2 + \|Dv_s\|^2 \right) ds \\ & \leq C_2 \mathbb{E} \left[\|G\|^2 + \int_t^T \left(\|F_s\|^2 + \|\Gamma_s^0\|^2 + \frac{2}{\varepsilon} \|v_s\|^2 + \varepsilon L_1^2 (\|D\psi_s\|^2 + \|D^2v_s\|^2) \right) ds \right], \end{aligned} \quad (20)$$

with C_2 independent of (ε, L_1) . From (20) and (19), there is a constant L_0 depending on L_2 and T such that as $L_1 < L_0$, setting ε sufficiently small and applying Grownwall inequality, we have

$$\|(v, \psi)\|_{\mathcal{H}^1} \leq C \left(\|G\|_{L^2(\mathcal{G}; H^{1,2}([p, q]))} + \|\Gamma^0\|_{\mathbb{L}(0, T; L^2([p, q]))} + \|F\|_{\mathbb{L}(0, T; L^2([p, q]))} \right), \quad (21)$$

with the constant C depending on κ, L_1, L_2 , and T .

Step 2. Suppose that (v_1, ψ_1) and (v_2, ψ_2) are two strong solutions to Equation (17). Then the pair $(\delta v, \delta \psi) = (v_1 - v_2, \psi_1 - \psi_2)$ satisfies the following BSPDE:

$$\begin{cases} -d\delta v_t(\xi) = \left\{ \gamma \left[\frac{1}{2} |\sigma_t(\xi, \mathcal{L}(\xi), \alpha)|^2 D^2 \delta v_t(\xi) + \Gamma_t(\xi, v_1, Dv_1, D^2v_1, \psi_1, D\psi_1) \right. \right. \\ \quad \left. \left. - \Gamma_t(\xi, v_2, Dv_2, D^2v_2, \psi_2, D\psi_2) \right] + (1 - \gamma) D^2 \delta v_t(\xi) \right\} dt - \delta \psi_t(\xi) dB_t; \\ D\delta v_t(p) = 0, \quad D\delta v_t(q) = 0; \\ \delta v_T(\xi) = 0. \end{cases} \quad (22)$$

Recalling (H4), we obtain

$$\begin{aligned} & \|\Gamma_t(\cdot, v_1, Dv_1, D^2v_1, \psi_1, D\psi_1) - \Gamma_t(\cdot, v_2, Dv_2, D^2v_2, \psi_2, D\psi_2)\| \\ & \leq L_1 (\|D^2(v_1 - v_2)\| + \|D(\psi_1 - \psi_2)\|) + L_2 (\|v_1 - v_2\|_{H^{1,2}([p, q])} + \|\psi_1 - \psi_2\|_{L^2([p, q])}), \end{aligned}$$

a.s. for each $t \in [0, T]$. Using a similar method for **Step 1** and Itô's formula to the square norm of $(\delta v, \delta \psi)$, we can obtain the estimates (18)–(20), in addition to (21) where (G, Γ^0, F) is substituted with zero values. This suggests that the strong solution, respectively, to Equation (17) and (12) is unique.

Step 3. Firstly, we notice that the priori estimate (21) holds in which the constant C is independent of $\gamma \in [0, 1]$. Proposition 2 implies that Equation (17) has a unique strong solution (v, ψ) when $\gamma = 0$. Noticing that BSPDE (12) corresponds to the special case of BSPDE (17) when $\gamma = 1$, then we want to generalize the well-posedness of BSPDE through the interval $[0, 1]$ starting from $\gamma = 0$.

Suppose that for some $\gamma = \gamma_0$, BSPDE (17) which satisfies assumptions (H1), (H2), and (H4) has a unique strong solution (v, ψ) , which holds when $\gamma_0 = 0$ (by Proposition 2 as above). Then, for each $(\check{v}, \check{\psi}) \in \mathcal{H}^1$, the following BSPDE:

$$\begin{cases} -d\check{v}_t(\xi) = \left\{ \gamma_0 \left[\frac{1}{2} |\sigma_t(\xi, \mathcal{L}(\xi), \alpha)|^2 D^2 \check{v}_t(\xi) + \Gamma_t(\xi, v, Dv, D^2v, \psi, D\psi) \right] \right. \\ \quad \left. + (\gamma - \gamma_0) \left[\frac{1}{2} |\sigma_t(\xi, \mathcal{L}(\xi), \alpha)|^2 D^2 \check{v}_t(\xi) + \Gamma_t(\xi, \check{v}, D\check{v}, D^2\check{v}, \check{\psi}, D\check{\psi}) - D^2 \check{v}_t(\xi) \right] \right. \\ \quad \left. + (1 - \gamma_0) D^2 \check{v}_t(\xi) \right\} dt - \check{\psi}_t(\xi) dB_t, \quad (t, \xi) \in [0, T] \times L^2(\mathcal{G}; [p, q]); \\ D\check{v}_t(p) = 0, \quad D\check{v}_t(q) = 0; \\ \check{v}_T(\xi) = G(\xi, \mathcal{L}(\xi)), \quad \xi \in L^2(\mathcal{G}; [p, q]), \end{cases}$$

corresponds to a special case of BSPDE (17) when $\gamma = \gamma_0$ and

$$F_t(\xi) = (\gamma - \gamma_0) \left[\frac{1}{2} |\sigma_t(\xi, \mathcal{L}(\xi), \alpha)|^2 D^2 \check{v}_t(\xi) + \Gamma_t(\xi, \check{v}, D\check{v}, D^2\check{v}, \check{\psi}, D\check{\psi}) - D^2 \check{v}_t(\xi) \right],$$

and it admits a unique strong solution (v, ψ) ; we can further define the following solution map:

$$\mathcal{M}_{\gamma_0} : \mathcal{H}^1 \rightarrow \mathcal{H}^1, \quad (\check{v}, \check{\psi}) \mapsto (v, \psi).$$

For each $(v_i, \psi_i) \in \mathcal{H}^1, i = 1, 2$, using a similar method for **Step 2**, we obtain

$$\begin{aligned} & \| (v_1 - v_2, \psi_1 - \psi_2) \|_{\mathcal{H}^1} \\ & \leq C |\gamma - \gamma_0| \left\| \frac{1}{2} |\sigma_t(\xi, \mathcal{L}(\xi), \alpha)|^2 D^2(\check{v}_1 - \check{v}_2) + \Gamma(\cdot, \check{v}_1, D\check{v}_1, D^2\check{v}_1, \check{\psi}_1, D\check{\psi}_1) \right. \\ & \quad \left. - \Gamma(\cdot, \check{v}_2, D\check{v}_2, D^2\check{v}_2, \check{\psi}_2, D\check{\psi}_2) - D^2(\check{v}_1 - \check{v}_2) \right\|_{\mathbb{L}_{\mathcal{G}}^2(0, T; L^2([p, q]))} \\ & \leq \tilde{C} |\gamma - \gamma_0| \| (\check{v}_1 - \check{v}_2, \check{\psi}_1 - \check{\psi}_2) \|_{\mathcal{H}^1}, \end{aligned}$$

with the constant $\tilde{C}$ being independent of (γ, γ_0) . When $|\gamma - \gamma_0| < \frac{1}{\tilde{C}}$, $\mathcal{M}_{\gamma_0}$ is a contraction mapping admitting a unique fixed point $(v, \psi) \in \mathcal{H}^1$, and it is a strong solution to Equation (17). With this method, when Equation (17) admits a strong solution for γ_0 , then it also satisfies $|\gamma - \gamma_0| < 1/\tilde{C}$ for any γ . We can arrive at $\gamma = 1$ in finite steps starting from $\gamma = 0$. Applying the estimates (21) and the uniqueness obtained in **Step 2**, we can complete the proof. $\square$

4. Proof of Theorem 1

Lemma 1 ([7] Itô-Kunita–Wentzell formula). *Let*

$$X_t = x + \int_0^t \eta_r dr + \int_0^t dA_r + \int_0^t \rho_r dB_r, \quad 0 \leq t \leq T,$$

where $(\eta, \rho) \in \mathbb{L}_{\mathcal{G}}^2(0, T; \mathbb{R}) \times \mathbb{L}_{\mathcal{G}}^2(0, T; \mathbb{R})$ and A is a continuous $\mathcal{G}$ -adapted bounded variation process with $A_0 = 0$. Assume that almost surely $p \leq X_t \leq q$ and

$$v_t(x) = v_0(x) + \int_0^t \mu_r(x) dr + \int_0^t \psi_r(x) dW_r, \quad \text{for all } (t, x) \in [0, T] \times [p, q],$$

holds in the weak sense with (v, μ, ψ) belonging to

$$(\mathcal{S}^2(0, T; H^{2,2}([p, q])) \cap \mathbb{L}^2(0, T; H^{3,2}([p, q]))) \times \mathbb{L}^2(0, T; H^{1,2}([p, q])) \times \mathbb{L}^2(0, T; H^{2,2}([p, q])).$$

Thus, for any $x \in [p, q]$ and $t \in [0, T]$,

$$\begin{aligned} v_t(X_t) - v_0(x) &= \int_0^t \left[\mu_r(X_r) + \eta_r Dv_r(X_r) + \frac{1}{2} |\rho|^2 D^2 v_r(X_r) \right] dr \\ &\quad + \int_0^t Dv_r(X_r) dA_r + \int_0^t (\psi_r(X_r) + Dv_r(X_r) \rho_r) dB_r. \quad \text{a.s.} \end{aligned} \quad (23)$$

Applying Sobolev's embedding theorem, $H^{m,2}(\mathbb{R})$ is embedded into continuous function space C^{m-1} continuously. Therefore, for any $x \in [p, q]$, Equation (23) holds. With this method, Lemma 1 is similar to the first expression of Kunita [18], pp. 118–119, when the whole line $\mathbb{R}$ is replaced by the bounded domain $[p, q]$. For brevity, we just take into account the 1-dimensional case in Lemma 1, and it is not essentially difficult to generalize it to the multidimensional case.

Now, we present a consequence of the Dirichlet problem of BSPDE as below, whose proof is the same as that of [19], Theorem 2, under our assumptions.

Lemma 2. Regard the Dirichlet problem as follows:

$$\begin{cases} -dv_t(\xi) = \left[\frac{1}{2} |\sigma_t(\xi, \mathcal{L}(\xi), \alpha)|^2 D^2 v_t(\xi) + F_t(\xi) \right] dt - \psi_t(\xi) dB_t, \\ (t, \xi) \in [0, T] \times L^2(\mathcal{G}; [p, q]); \\ v_t(p) = 0, \quad v_t(q) = 0; \\ v_T(\xi) = G(\xi, \mathcal{L}(\xi)), \quad \xi \in L^2(\mathcal{G}; [p, q]). \end{cases} \quad (24)$$

with $G \in L^2(\mathcal{G}; H_0^{1,2}([p, q]))$ and $F \in \mathbb{L}^2(0, T; L^2([p, q]))$. Under assumption (H2), there is a unique weak solution (v, ψ) to BSPDE (24) and it is unique with

$$\|(v, \psi)\|_{\mathcal{H}^1} \leq C(\|G\|_{L^2(\mathcal{G}; H^{1,2}([p, q]))} + \|F\|_{\mathbb{L}^2(0, T; L^2([p, q]))}), \quad (25)$$

where the constant C depends on κ and T .

Finally, we begin to accomplish the proof of Theorem 1.

Proof. Firstly, the Neumann problem (7) can be reduced to the instance of zero Neumann boundary condition. By assumption (H3) (i) as well as Definition 1, we set

$$(\hat{g}_t, \hat{Q}_t, \hat{\Psi}_t)(\xi) = \int_0^\xi (g_t, Q_t, \Psi_t)(y, \mathcal{L}(y)) dy, \quad \text{for each } (t, \xi) \in [0, T] \times L^2(\mathcal{G}; [p, q]),$$

and then we obtain $(\hat{g}, \hat{\Psi})$ as the strong solution to the following BSPDE:

$$\begin{cases} -d\hat{g}_t(\xi) = \left[\frac{1}{2} |\sigma_t(\xi, \mathcal{L}(\xi), \alpha)|^2 D^2 \hat{g}_t(\xi) + \hat{f}_t(\xi) \right] dt - \hat{\Psi}_t(\xi) dB_t, \\ (t, \xi) \in [0, T] \times L^2(\mathcal{G}; [p, q]); \\ D\hat{g}_t(p) = g_t(p, \delta_p), \quad D\hat{g}_t(q) = g_t(q, \delta_q); \\ \hat{g}_T(\xi) = \int_0^\xi g_T(y, \mathcal{L}(y)) dy, \quad \xi \in L^2(\mathcal{G}; [p, q]), \end{cases} \quad (26)$$

with $\hat{f} \in \mathbb{L}^2(0, T; H^{1,2}([p, q]))$ being defined as $\hat{f}_t(\xi) = -1/2 |\sigma_t(\xi, \mathcal{L}(\xi), \alpha)|^2 D^2 \hat{g}_t(\xi) + \hat{Q}_t(\xi)$. Hence, the existence and uniqueness of the strong solution (v, ψ) to Equation (7) is equal to that of strong solution $(\tilde{v}, \tilde{\psi}) = (v - \hat{g}, \psi - \hat{\Psi})$ satisfying the following BSPDE:

$$\begin{cases} -d\tilde{v}_t(\xi) = \left[\frac{1}{2} |\sigma_t(\xi, \mathcal{L}(\xi), \alpha)|^2 D^2 \tilde{v}_t(\xi) + \mathbb{H}_t(\xi, D\tilde{v}_t(\xi) + D\hat{g}_t(\xi), D^2 \tilde{v}_t(\xi) + D^2 \hat{g}_t(\xi)) \right. \\ \left. - \hat{f}_t(\xi) \right] dt - \tilde{\psi}_t(\xi) dB_t, \quad (t, \xi) \in [0, T] \times L^2(\mathcal{G}; [p, q]); \\ D\tilde{v}_t(p) = 0, \quad D\tilde{v}_t(q) = 0; \\ \tilde{v}_T(\xi) = G(\xi, \mathcal{L}(\xi)) - \hat{g}_T(\xi), \quad \xi \in L^2(\mathcal{G}; [p, q]). \end{cases} \quad (27)$$

In view of Theorem 2, there is a unique strong solution $(\tilde{v}, \tilde{\psi})$ to BSPDE (27). Taking derivatives of the above equation, we can easily check that

$$(u, \zeta) := (D\tilde{v}, D\tilde{\psi}) = (Dv - D\hat{g}, D\psi - D\hat{\Psi}) = (Dv - g, D\psi - \Psi)$$

is a weak solution to the Dirichlet problem as follows:

$$\begin{cases} -du_t(\xi) = \left[\frac{1}{2} |\sigma_t(\xi, \mathcal{L}(\xi), \alpha)|^2 D^2 u_t(\xi) + R_t(\xi, \alpha) \right] dt - \zeta_t(\xi) dB_t, \\ (t, \xi) \in [0, T] \times L^2(\mathcal{G}; [p, q]); \\ u_t(p) = 0, \quad u_t(q) = 0; \\ u_T(\xi) = DG(\xi, \mathcal{L}(\xi)) - g_T(\xi, \mathcal{L}(\xi)), \quad \xi \in L^2(\mathcal{G}; [p, q]), \end{cases} \quad (28)$$

with

$$\begin{aligned}
& R_t(\xi, \alpha) \\
&= -D\hat{f}_t(\xi) + (\partial_{\xi}\mathbb{H})_t\left(\xi, D\tilde{v}_t(\xi) + g_t(\xi, \mathcal{L}(\xi)), D^2\tilde{v}_t(\xi) + Dg_t(\xi, \mathcal{L}(\xi))\right) \\
&\quad + (\partial_{u_1}\mathbb{H})_t\left(\xi, D\tilde{v}_t(\xi) + g_t(\xi, \mathcal{L}(\xi)), D^2\tilde{v}_t(\xi) + Dg_t(\xi, \mathcal{L}(\xi))\right)\left(D^2\tilde{v}_t(\xi) + Dg_t(\xi, \mathcal{L}(\xi))\right) \\
&\quad + (\partial_{u_2}\mathbb{H})_t\left(\xi, D\tilde{v}_t(\xi) + g_t(\xi, \mathcal{L}(\xi)), D^2\tilde{v}_t(\xi) + Dg_t(\xi, \mathcal{L}(\xi))\right)\left(D^3\tilde{v}_t(\xi) + D^2g_t(\xi, \mathcal{L}(\xi))\right) \\
&= -D\hat{f}_t(\xi) + \operatorname{ess\,inf}_{\alpha \in \mathcal{A}} \mathbb{E}\left\{Df_t(\xi, \mathcal{L}(\xi), \alpha) + [D\tilde{v}_t(\xi) + g_t(\xi, \mathcal{L}(\xi))]Db_t(\xi, \mathcal{L}(\xi), \alpha) \right. \\
&\quad \left. + [D^2\tilde{v}_t(\xi) + Dg_t(\xi, \mathcal{L}(\xi))]b_t(\xi, \mathcal{L}(\xi), \alpha) + \frac{1}{2}[D^3\tilde{v}_t(\xi, \mathcal{L}(\xi)) + D^2g_t(\xi, \mathcal{L}(\xi))] \right. \\
&\quad \left. \cdot |\sigma_t(\xi, \mathcal{L}(\xi), \alpha)N|^2 + [D^2\tilde{v}_t(\xi) + Dg_t(\xi, \mathcal{L}(\xi))] \sigma_t(\xi, \mathcal{L}(\xi), \alpha)N^2 D\sigma_t(\xi, \mathcal{L}(\xi), \alpha) \right\}.
\end{aligned}$$

As $(\tilde{v}, \tilde{\psi})$ is a unique strong solution to BSPDE (27), we arrive at

$$\tilde{v} \in \mathcal{S}_{\mathcal{G}}^2(0, T; H^{1,2}([p, q])) \cap \mathbb{L}_{\mathcal{G}}^2(0, T; H^{2,2}([p, q])) \subset \mathbb{L}^2(0, T; L^2([p, q])).$$

In view of assumptions **(H1)** and **(H2)**, we have $Df, b, Db, \sigma, D\sigma \in \mathbb{L}^2(0, T; L^2([p, q]))$. By assumption (i) of **(H3)**, we obtain $g, Dg, D^2g \in \mathbb{L}^2(0, T; L^2([p, q]))$, and $D\hat{f} \in \mathbb{L}^2(0, T; L^2([p, q]))$; therefore, function $R \in \mathbb{L}^2(0, T; L^2([p, q]))$. Then, we may derive from Lemma 2 that (u, ζ) is a strong solution. Hence $(D\tilde{v}, D\tilde{\psi}) = (u, \zeta) \in \mathcal{H}^1$; furthermore, $(Dv, D\psi) = (D\tilde{v} + g, D\tilde{\psi} + \Psi) \in \mathcal{H}^1$. Therefore, $(v, \psi) \in \mathcal{H}^2$. The regularity and assumption (iii) of **(H3)** imply that $\alpha_t^* = \Pi_t(X_t^*, Dv_t(X_t^*), D^2v_t(X_t^*))$ is admissible. For each admissible control α , exploiting the general Itô–Kunita–Wentzell formula to $v_t(X_t^\alpha)$ implies that for any $\eta \in L^2(\mathcal{G}; [p, q])$, there almost surely holds

$$\begin{aligned}
& \mathbb{E}\left[v_t(X_t^\alpha) - v_T(X_T^\alpha)\right] \\
&= -\mathbb{E} \int_t^T \left[-\mathbb{H}_r(X_r^\alpha, Dv_r(X_r^\alpha), D^2v_r(X_r^\alpha)) + Dv_r(X_r^\alpha)b_r(X_r^\alpha, \mathcal{L}(X_r^\alpha), \alpha_r) \right] dr \\
&\quad - \frac{1}{2} \mathbb{E} \int_t^T D^2v_r(X_r^\alpha) |\sigma_r(X_r^\alpha, \mathcal{L}(X_r^\alpha), \alpha_r)|^2 dr - \mathbb{E} \int_t^T Dv_r(X_r^\alpha) d(L_r - U_r) \\
&= \mathbb{E} \int_t^T \operatorname{ess\,inf}_{\tilde{\alpha} \in \mathcal{A}} \left\{ Dv_r(X_r^\alpha)b_r(X_r^\alpha, \mathcal{L}(X_r^\alpha), \tilde{\alpha}_r) + \frac{1}{2} D^2v_r(X_r^\alpha) |\sigma_r(X_r^\alpha, \mathcal{L}(X_r^\alpha), \tilde{\alpha}_r)N|^2 \right. \\
&\quad \left. + f_r(X_r^\alpha, \mathcal{L}(X_r^\alpha), \tilde{\alpha}_r) \right\} dr - \mathbb{E} \int_t^T \left[Dv_r(X_r^\alpha)b_r(X_r^\alpha, \mathcal{L}(X_r^\alpha), \alpha_r) + \frac{1}{2} D^2v_r(X_r^\alpha) \right. \\
&\quad \left. \cdot |\sigma_r(X_r^\alpha, \mathcal{L}(X_r^\alpha), \alpha_r)|^2 \right] dr - \mathbb{E} \int_t^T Dv_r(X_r^\alpha) d(L_r - U_r) \\
&\leq \mathbb{E} \int_t^T f_r(X_r^\alpha, \mathcal{L}(X_r^\alpha), \alpha_r) dr + \mathbb{E} \int_t^T Dv_r(X_r^\alpha) d(U_r - L_r), \quad \forall t \in [0, T],
\end{aligned}$$

that is, for any $t \in [0, T]$,

$$\begin{aligned}
& \mathbb{E}[v_t(X_t^\alpha)] \\
&\leq \mathbb{E}\left[G(X_T^\alpha, \mathcal{L}(X_T^\alpha)) + \int_t^T f_r(X_r^\alpha, \mathcal{L}(X_r^\alpha), \alpha_r) dr + \int_t^T g(X_r^\alpha, \mathcal{L}(X_r^\alpha)) d(U_r - L_r)\right]. \quad (29)
\end{aligned}$$

Hence, for each admissible control α , it holds almost surely:

$$\mathbb{E}[v_t(X_t^\alpha)] \leq J_t(X_t^\alpha; \alpha), \quad \forall t \in [0, T]. \quad (30)$$

Moreover, using a similar method for inequality (29), for all $\eta \in L^2(\mathcal{G}; [p, q])$ and $t \in [0, T]$,

$$\begin{aligned} & \mathbb{E}[v_t(X_t^*)] \\ &= \mathbb{E}\left[G(X_T^*, \mathcal{L}(X_T^*)) + \int_t^T f_r(X_r^*, \mathcal{L}(X_r^*), \alpha_r^*)dr + \int_t^T g(X_r^*, \mathcal{L}(X_r^*))d(U_r - L_r)\right] \\ &= J_t(X_t^*; \alpha^*) \end{aligned}$$

holds with probability 1. Therefore, by (30), for any $\eta \in L^2(\mathcal{G}; [p, q])$, $\mathbb{E}[v_t(\eta)] = V_t(\eta)$ and optimal stochastic control is given by $\alpha_t^* = \Pi_t(X_t^*, Dv_t(X_t^*), D^2v_t(X_t^*))$ with the associated state process X_t^* , which satisfies BSPDE (11). The proof is completed. $\square$

5. Conclusions

In this paper, we investigated the stochastic optimal control problem of a 1-dimensional McKean–Vlasov stochastic differential equation reflected at $[p, q]$, whose drift and diffusion coefficients are both dependent on the state of the solution process along with its law and control; thus, the results of [4,7] were generalized. We depend on the concept of lifted function presented by P.L. Lions in [10], then the BSPDEs in the Wasserstein space of probability measures can be written in the Hilbert space. Thus, the value function of our control problem can be represented by a BSPDE with Neumann boundary condition in the Hilbert space. The existence and uniqueness of the solution to the above BSPDE were obtained. Finally, we constructed the optimal feedback control of the stochastic control problem by the Itô–Kunita–Wentzell formula. For the future work, we will investigate the multidimensional reflected McKean–Vlasov optimal control problem with jump and singular terminal conditions, and the corresponding applications to population biology, finance, and physics will also be studied.

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Appendix A. Poof of Proposition 1

Proof. **Step 1.** Apply the Itô's formula of Lemma A.1 in [7] to the square norm, which arises

$$\begin{aligned} & \|v_t\|^2 + \int_t^T \|\psi_s\|^2 ds - \|G\|^2 \\ &= \int_t^T \langle v_s, \gamma | \sigma_s(\xi, \mathcal{L}(\xi), \alpha) |^2 D^2 v_s + (1 - \gamma) D^2 v_s + 2F_s \rangle ds - 2 \int_t^T \langle v_s, \psi_s dB_s \rangle \text{ a.s.} \end{aligned}$$

for any $t \in [0, T]$. Considering Neumann boundary conditions, we obtain

$$\int_t^T \langle v_s, (1 - \gamma) D^2 v_s \rangle ds = -(1 - \gamma) \int_t^T \|Dv_s\|^2 ds,$$

and

$$\begin{aligned}
 & \int_t^T \langle v_s, |\sigma_s(\xi, \mathcal{L}(\xi), \alpha)|^2 D^2 v_s \rangle ds \\
 &= - \int_t^T \langle Dv_s, |\sigma_s(\xi, \mathcal{L}(\xi), \alpha)|^2 Dv_s \rangle ds - \int_t^T \langle v_s D|\sigma_s(\xi, \mathcal{L}(\xi), \alpha)|^2, Dv_s \rangle ds \\
 &\leq - \int_t^T \langle Dv_s, |\sigma_s(\xi, \mathcal{L}(\xi), \alpha)|^2 Dv_s \rangle ds + \frac{2C}{\kappa} \int_t^T \|v_s\|^2 ds + \frac{\kappa}{2} \int_t^T \|Dv_s\|^2 ds \\
 &\leq - \frac{\kappa}{2} \int_t^T \|Dv_s\|^2 ds + \frac{2C}{\kappa} \int_t^T \|v_s\|^2 ds.
 \end{aligned} \tag{A1}$$

Hence,

$$\begin{aligned}
 & \|v_t\|^2 + \int_t^T \|\psi_s\|^2 ds - \|G\|^2 \\
 &= \int_t^T \langle v_s, \gamma |\sigma_s(\xi, \mathcal{L}(\xi), \alpha)|^2 D^2 v_s \rangle ds - (1 - \gamma) \int_t^T \|Dv_s\|^2 ds + \int_t^T \langle v_s, 2F_s \rangle ds \\
 &\quad - 2 \int_t^T \langle v_s, \psi_s dB_s \rangle \text{ a.s.}
 \end{aligned} \tag{A2}$$

for all $t \in [0, T]$. Notice that $\gamma \in [0, 1]$ and

$$\frac{\kappa\gamma}{2} + (1 - \gamma) \geq 1 \wedge \frac{\kappa}{2} > 0. \tag{A3}$$

Next, we claim

$$\mathbb{E} \left[\sup_{s \in [t, T]} \|v_s\|^2 \right] + \mathbb{E} \left[\int_t^T (\|\psi_s\|^2 + \|Dv_s\|^2) ds \right] \leq C \mathbb{E} \left[\|G\|^2 + \int_t^T |\langle F_s, v_s \rangle| ds \right]. \tag{A4}$$

In order to prove (A4), we estimate

$$\delta \mathbb{E} \left[\sup_{s \in [t, T]} \|v_s\|^2 \right] + (1 - \delta) \mathbb{E} [\|v_t\|^2] + \mathbb{E} \left[\int_t^T (\|\psi_s\|^2 + \|Dv_s\|^2) ds \right]$$

for $\delta = 0$ and $\delta = 1$ separately.

Firstly, for the case $\delta = 0$, by (A1) and (A2), we have

$$\begin{aligned}
 & \mathbb{E} \left[\|v_t\|^2 + \int_t^T \|\psi_s\|^2 ds \right] \\
 &= \mathbb{E} \left[\|G\|^2 - (1 - \gamma) \int_t^T \|Dv_s\|^2 ds + \int_t^T \langle v_s, \gamma |\sigma_s(\xi, \mathcal{L}(\xi), \alpha)|^2 D^2 v_s + 2F_s \rangle ds \right] \\
 &\leq \mathbb{E} \left[\|G\|^2 - (1 - \gamma) \int_t^T \|Dv_s\|^2 ds - \frac{\gamma\kappa}{2} \int_t^T \|Dv_s\|^2 ds + \frac{2\gamma C}{\kappa} \int_t^T \|v_s\|^2 ds \right. \\
 &\quad \left. + 2 \int_t^T |\langle v_s, F_s \rangle| ds \right],
 \end{aligned}$$

thus, by (A3), we have

$$\begin{aligned}
 & \mathbb{E} [\|v_t\|^2] + \mathbb{E} \left[\int_t^T \|\psi_s\|^2 ds \right] + (1 \wedge \frac{\kappa}{2}) \mathbb{E} \left[\int_t^T \|Dv_s\|^2 ds \right] \\
 &\leq \mathbb{E} [\|v_t\|^2] + \mathbb{E} \left[\int_t^T \|\psi_s\|^2 ds \right] + (\frac{\kappa\gamma}{2} + 1 - \gamma) \mathbb{E} \left[\int_t^T \|Dv_s\|^2 ds \right] \\
 &\leq \mathbb{E} \left[\|G\|^2 + 2 \int_t^T |\langle v_s, F_s \rangle| ds + \frac{2\gamma C}{\kappa} \int_t^T \|v_s\|^2 ds \right].
 \end{aligned}$$

Applying the Gronwall inequality, we obtain

$$\mathbb{E}[\|v_t\|^2] + \mathbb{E}\left[\int_t^T \|\psi_s\|^2 ds\right] + \mathbb{E}\left[\int_t^T \|Dv_s\|^2 ds\right] \leq \mathbb{C}\mathbb{E}\left[\|G\|^2 + \int_t^T |\langle v_s, F_s \rangle| ds\right]. \quad (\text{A5})$$

Secondly, when $\delta = 1$, by (A1) and (A2), we can obtain that

$$\begin{aligned} & \|v_t\|^2 \\ & \leq \|G\|^2 + \int_t^T \langle v_s, \gamma | \sigma_s(\xi, \mathcal{L}(\xi), \alpha) |^2 D^2 v_s \rangle ds + 2 \int_t^T |\langle v_s, F_s \rangle| ds - 2 \int_t^T \langle v_s, \psi_s dB_s \rangle \\ & \leq \|G\|^2 - \frac{\kappa}{2} \int_t^T \|Dv_s\|^2 ds + \frac{2C}{\kappa} \int_t^T \|v_s\|^2 ds + 2 \int_t^T |\langle v_s, F_s \rangle| ds - 2 \int_t^T \langle v_s, \psi_s dB_s \rangle. \end{aligned}$$

Notice, in view of Burkholder–Davis–Gundy inequality ([20]), that

$$\begin{aligned} & 2\mathbb{E}\left[\sup_{\tau \in [t, T]} \left| \int_t^\tau \langle v_s, \psi_s dB_s \rangle \right|\right] \\ & \leq 2\mathbb{E}\left[\left| \int_t^T \langle v_s, \psi_s dB_s \rangle \right|\right] + 2\mathbb{E}\left[\sup_{\tau \in [t, T]} \left| \int_t^\tau \langle v_s, \psi_s dB_s \rangle \right|\right] \\ & \leq 2\mathbb{E}\left[\sup_{\tau \in [t, T]} \left| \int_t^\tau \langle v_s, \psi_s dB_s \rangle \right|\right] + 2\mathbb{E}\left[\sup_{\tau \in [t, T]} \left| \int_t^\tau \langle v_s, \psi_s dB_s \rangle \right|\right] \\ & \leq 4\mathbb{E}\left[\sup_{\tau \in [t, T]} \left| \int_t^\tau \langle v_s, \psi_s dB_s \rangle \right|\right] \leq 16\sqrt{2}\mathbb{E}\left[\left(\int_t^T \|v_s\|^2 \|\psi_s\|^2 ds\right)^{1/2}\right]. \end{aligned}$$

Hence,

$$\begin{aligned} & \mathbb{E}\left[\sup_{s \in [t, T]} \|v_s\|^2\right] \\ & \leq \mathbb{E}\left[\|G\|^2 + 2 \int_t^T |\langle v_s, F_s \rangle| ds + \frac{2C}{\kappa} \int_t^T \|v_s\|^2 ds + 16\sqrt{2}\mathbb{E}\left[\left(\int_t^T \|v_s\|^2 \|\psi_s\|^2 ds\right)^{1/2}\right]\right] \\ & \leq \mathbb{E}\left[\|G\|^2 + 2 \int_t^T |\langle v_s, F_s \rangle| ds + 16^2 \int_t^T \|\psi_s\|^2 ds + \frac{2C}{\kappa} \int_t^T \|v_s\|^2 ds\right] + \frac{1}{2}\mathbb{E}\left[\sup_{s \in [t, T]} \|v_s\|^2\right], \end{aligned}$$

hence,

$$\begin{aligned} & \frac{1}{2}\mathbb{E}\left[\sup_{s \in [t, T]} \|v_s\|^2\right] \leq \mathbb{E}\left[\|G\|^2 + 2 \int_t^T |\langle v_s, F_s \rangle| ds + 16^2 \int_t^T \|\psi_s\|^2 ds + \frac{2C}{\kappa} \int_t^T \|v_s\|^2 ds\right] \\ & \leq \mathbb{E}\left[\|G\|^2 + \left(1 + \frac{2C}{\kappa}\right) \int_t^T \sup_{r \in [s, T]} \|v_r\|^2 ds + \int_t^T \|F_s\|^2 ds + 16^2 \int_t^T \|\psi_s\|^2 ds\right]. \end{aligned}$$

Applying the Gronwall inequality successively, we obtain

$$\mathbb{E}\left[\sup_{s \in [t, T]} \|v_s\|^2\right] \leq 2\mathbb{E}\left[\|G\|^2 + \int_t^T \|F_s\|^2 ds + 16^2 \int_t^T \|\psi_s\|^2 ds\right] e^{(1+2C/\kappa)T}.$$

Thus, by combining (A5), we can obtain that (A4) holds.

Step 2. We first take the partial derivatives with respect to ξ on both sides of BSPDE (14), and then we may obtain without difficulty that $(u, \Phi) := (Dv, D\psi)$ is a weak solution to the BSPDE for the Dirichlet problem as follows:

$$\begin{cases} -du_t(\xi) = \left[\frac{\gamma}{2} |\sigma_t(\xi, \mathcal{L}(\xi), \alpha)|^2 D^2 u_t(\xi) + \frac{1-\gamma}{2} D^2 u_t(\xi) + DF_t(\xi) \right] dt - \Phi_t(\xi) dB_t; \\ u_t(p) = 0, \quad u_t(q) = 0; \\ u_T(\xi) = DG(\xi, \mathcal{L}(\xi)). \end{cases} \quad (\text{A6})$$

We apply the Itô's formula of Lemma A.1 in [7] to the square norm again, which arises as

$$\begin{aligned} & \|u_t\|^2 + \int_t^T \|\Phi_s\|^2 ds - \|DG\|^2 \\ &= \int_t^T \langle u_s, \gamma |\sigma_t(\xi, \mathcal{L}(\xi), \alpha)|^2 D^2 u_t(\xi) + (1-\gamma) D^2 u_s + DF_s \rangle ds \\ & \quad - 2 \int_t^T \langle u_s, \Phi_s dB_s \rangle \quad a.s. \forall t \in [0, T]. \end{aligned}$$

Considering the zero-Dirichlet condition, we have

$$\int_t^T \langle u_s, (1-\gamma) D^2 u_s \rangle ds = -(1-\gamma) \int_t^T \|Du_s\|^2 ds, \quad \int_t^T \langle u_s, DF_s \rangle ds = - \int_t^T \langle Du_s, F_s \rangle ds.$$

Using a method similar for **Step 1**, we have

$$\mathbb{E} \left[\sup_{s \in [t, T]} \|u_s\|^2 \right] + \mathbb{E} \int_t^T (\|Du_s\|^2 + \|\Phi_s\|^2) ds \leq C \mathbb{E} \left[\|DG\|^2 + \int_t^T |\langle F_s, Du_s \rangle| ds \right],$$

that is,

$$\mathbb{E} \left[\sup_{s \in [t, T]} \|Dv_s\|^2 \right] + \mathbb{E} \int_t^T (\|D^2 v_s\|^2 + \|D\psi_s\|^2) ds \leq C \mathbb{E} \left[\|DG\|^2 + \int_t^T |\langle F_s, D^2 v_s \rangle| ds \right],$$

together with (A4), which suggests that

$$\begin{aligned} & \mathbb{E} \left[\sup_{s \in [t, T]} \|v_s\|_{H^{1,2}([p, q])}^2 \right] + \mathbb{E} \int_t^T (\|v_s\|_{H^{2,2}([p, q])}^2 + \|\psi_s\|_{H^{1,2}([p, q])}^2) ds \\ & \leq C \mathbb{E} \left[\|G\|_{H^{1,2}([p, q])}^2 + \int_t^T (|\langle F_s, v_s \rangle| + |\langle F_s, D^2 v_s \rangle|) ds \right] \\ & \leq C \mathbb{E} \left[\|G\|_{H^{1,2}([p, q])}^2 + 8C \int_t^T \|F_s\|^2 ds + \frac{1}{2C} \int_t^T \|v_s\|_{H^{2,2}([p, q])}^2 ds \right], \end{aligned}$$

with C being dependent on κ and T .

$$\begin{aligned} & \mathbb{E} \left[\sup_{s \in [t, T]} \|v_s\|_{H^{1,2}([p, q])}^2 \right] + \frac{1}{2} \mathbb{E} \int_t^T \|v_s\|_{H^{2,2}([p, q])}^2 ds + \mathbb{E} \int_t^T \|\psi_s\|_{H^{1,2}([p, q])}^2 ds \\ & \leq C \mathbb{E} \left[\|G\|_{H^{1,2}([p, q])}^2 \right] + 8C^2 \mathbb{E} \int_t^T \|F_s\|^2 ds, \end{aligned}$$

thus,

$$\begin{aligned} & \mathbb{E} \left[\sup_{s \in [t, T]} \|v_s\|_{H^{1,2}([p, q])}^2 \right] + \mathbb{E} \int_t^T (\|v_s\|_{H^{2,2}([p, q])}^2 + \|\psi_s\|_{H^{1,2}([p, q])}^2) ds \\ & \leq C \mathbb{E} \left[\|G\|_{H^{1,2}([p, q])}^2 + \int_t^T \|F_s\|^2 ds \right]. \end{aligned} \quad (\text{A7})$$

By the estimates and the linearity of the associated BSPDE, we obtain the uniqueness of the strong solution. $\square$

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