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The Law of the Iterated Logarithm for L_p-Norms of Kernel Estimators of Cumulative Distribution Functions

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Abstract: In this paper, we consider the strong convergence of L_p-norms ($p \geq 1$) of a kernel estimator of a cumulative distribution function (CDF). Under some mild conditions, the law of the iterated logarithm (LIL) for the L_p-norms of empirical processes is extended to the kernel estimator of the CDF.

Keywords: L_p-norm; LIL; kernel estimator; empirical CDF

MSC: 60F15; 62G05

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1. Introduction

Consider an independent identically distributed random sample $X_1, X_2, \cdots, X_n$ from a population with an unknown cumulative distribution function (CDF). For the empirical distribution function F_n, defined as follows:

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I(X_i \leq x),\ \forall x \in R^1,$$

with I denoting the indicator function, the classical Glivenko–Cantelli theorem states that $F_n(x)$ converges almost surely (a.s.) to $F(x)$ uniformly in $x \in R^1$, i.e.,

$$\sup_{x\in R^1} |F_n(x) - F(x)| \to 0, \text{ a.s.}$$

The extended Glivenko–Cantelli lemma (in Fabian and Hannan 1985, pp. 80–83 [1]) provides the strong uniform convergence rate as follows:

$$\sup_{x\in R} n^{\alpha}|F_n(x) - F(x)| \to 0 \quad \text{a.s.,} \quad \text{for any } \ 0 < \alpha < 1/2. \tag{1}$$

The law of the iterated logarithm (LIL) for $F_n(t)$, i.e.,

$$\limsup_{n\to\infty} \sqrt{\frac{n}{2\log(\log n)}} \sup_x \left|F_n(x) - F(x)\right| = \frac{1}{2} \quad a.s. \tag{2}$$

was proven by Smirnov (1944) [2] and, independently, Chung (1949) [3].

Finkelstein (1971) [4] obtained the L_2-version of the law of iterated logarithm,

$$\limsup_{n\to\infty} \sqrt{\frac{n}{2\log(\log n)}}\Big[\int_{-\infty}^{\infty}(F_n(x) - F(x))^2 dF(x)\Big]^{1/2} = \frac{1}{\pi} \quad a.s. \tag{3}$$

For any $p \geq 1$, setting

$$C(p) = \frac{1}{2}\Big(\frac{p(p+2)}{\pi}\Big)^{1/2}\Big(\frac{2}{p+2}\Big)^{1/p}\frac{\Gamma(1/p+\frac{1}{2})}{\Gamma(1/p)}, \tag{4}$$

the law of the iterated logarithm for L_p-norm of $F_n(x)$,

$$\limsup_{n\to\infty}\sqrt{\frac{n}{2\log(\log n)}}\Big[\int_{-\infty}^{\infty}|F_n(x)-F(x)|^p dF(x)\Big]^{1/p}=C(p)\ \ a.s. \tag{5}$$

was developped by Gajek, Kahszka, and Lenic (1996) [5]. It is easy to verify that

$$C(1)=\frac{\sqrt{3}}{6}\quad\text{and}\quad C(2)=\frac{1}{\pi}.$$

And (3) is a special case of (5) corresponding to $p=2$.

Notice that there is one serious discontinuity drawback of F_n, regardless of F being continuous or discrete. To treat this deficiency of F_n, Yamato (1973) [6] proposed the following kernel distribution estimator:

$$\hat{F}(x)=\int_{-\infty}^{x}n^{-1}\textstyle\sum_{i=1}^{n}k_h(u-X_i)du,\ x\in\mathbb{R}, \tag{6}$$

in which $h=h_n$ is the usual band width sequence of positive numbers tending to zero, k is a probability density function(PDF) called kernel, and $k_h(u)=k(u/h)/h$.

The aim of this paper is to provide certain conditions to guarantee the LIL of L_p-norm of $\hat{F}$. Some asymptotic properties of the smooth estimator $\hat{F}$ have been established. For example, in Yamato (1973) [6], the asymptotic normality and uniform strong consistency of $\hat{F}$ were obtained. In more general contexts, Winter (1979) [7] considered the convergence rate of perturbed empirical distribution functions. Wang, Cheng, and Yang (2013) [8] developed simultaneous confidence bands for F based on $\hat{F}$. The strong convergence rate of $\hat{F}$ was considered by Cheng (2017) [9], which extended the extended Glivenko–Cantelli Lemma (1) to the kernel estimator $\hat{F}$.

Here, we shall continue to consider the strong convergence of a smooth estimator $\hat{F}$ for F. More specifically, we are interested in extending the LIL of L_p-norm in (5) for $F_n(t)$ to the kernel estimator $\hat{F}$.

The outline of this paper is as follows: Section 2 describes the basic assumptions and main results : the strong uniform closeness between F_n and $\hat{F}$, and the LIL of L_p-norm of $\hat{F}$. Detailed proofs are provided in Section 3.

Note that for the proof of the strong uniform closeness between F_n and $\hat{F}$, we use the Kiefer type approximation for the empirical process (see Csörgő and R*é*v*é*sz (1981) [10]).

Throughout the following all limits are taken as the sample size n tending to ∞.

2. Assumptions and the Main Results

In this section, we start with the assumptions for the kernel function k.

Assumption 1. *k: Functions $k(x)$, $xk(x)$ and $x^2k(x)$ are integrable on the whole real line and satisfy the following properties:*

$$k(x)\geq 0,\ \int_{-\infty}^{+\infty}k(x)dx=1,\int_{-\infty}^{+\infty}xk(x)dx=0\ and\ \int_{-\infty}^{+\infty}x^2k(x)dx<\infty\,.$$

About the band width h, we assume

$$h^{3/2}\log(\log n)\to 0\quad\text{and}\quad nh^4/\log(\log n)\to 0, \tag{7}$$

which are stronger than the assumption $nh^4\to 0$ used in Cheng (2017) [9].

Under the above assumptions, we first state the result for evaluating the uniform closeness between $\hat{F}$ and F_n, which improves Theorem 2.1 in Cheng (2017) [9].

Theorem 1. *Assume that Assumption k and (7) hold. Then, for the continuous CDF F with bounded second order derivative, we have*

$$\sup_{x\in R}\sqrt{\frac{n}{\log(\log n)}}\left|\hat{F}(x)-F_n(x)\right|\to 0, \qquad a.s. \tag{8}$$

Together with LIL in (2), the LIL can be extended to $\hat{F}$, as follows:

Corollary 1. *Under the assumptions of Theorem 1, for the continuous CDF F with bounded second order derivative, we have*

$$\limsup_{n\to\infty}\sqrt{\frac{n}{2\log(\log n)}}\sup_x\left|\hat{F}(x)-F(x)\right|=\frac{1}{2} \quad a.s. \tag{9}$$

Remark 1. *Using a different approach, (9) was verified in Winter (1979) [7].*

Combining (8) with (5), the LIL for L_p-norm of F_n can be extended to $\hat{F}$.

Theorem 2. *Under the assumptions of Theorem 1, for any $p\geq 1$ and the continuous CDF F with bounded second order derivative, we have*

$$\limsup_{n\to\infty}\sqrt{\frac{n}{2\log(\log n)}}\Big[\int_{-\infty}^{\infty}|\hat{F}(x)-F(x)|^p dF(x)\Big]^{1/p}=C(p) \qquad a.s., \tag{10}$$

where $C(p)$ is defined in (4).

Remark 2. *Applying the facts $C(1)=\frac{\sqrt{3}}{6}$ and $C(2)=\frac{1}{\pi}$, Theorem 2 can result in the following corollary:*

Corollary 2. *Under the assumptions of Theorem 1, for the continuous CDF F with bounded second order derivative, we have*

$$\limsup_{n\to\infty}\sqrt{\frac{n}{2\log(\log n)}}\int_{-\infty}^{\infty}|\hat{F}_n(x)-F(x)|dF(x)=\frac{\sqrt{3}}{6} \qquad a.s.$$

and

$$\limsup_{n\to\infty}\sqrt{\frac{n}{2\log(\log n)}}\Big[\int_{-\infty}^{\infty}|\hat{F}_n(x)-F(x)|^2 dF(x)\Big]^{1/2}=\frac{1}{\pi} \qquad a.s.$$

Detailed proofs of the above results are given below.

3. Proof

Set

$$U_n(x) \quad := \quad \frac{1}{n}\sum_{i=1}^{n}\{I(X_i\leq x)-F(x)\}, \quad x\in R.$$

Therefore, (2) guarantees that

$$\limsup_{n\to\infty}\sqrt{\frac{n}{\log(\log n)}}\sup_x\left|U_n(x)\right|=O(1) \quad a.s. \tag{11}$$

For independent uniform $[0,1]$ random variables: $\xi_1,\xi_2,\cdots,\xi_n$, we define

$$V_n(v):=\frac{1}{n}\sum_{i=1}^{n}\Big[I(\xi_i\leq v)-v\Big], \qquad \forall\, v\in[0,1].$$

Then, $V_n(v)$ is a standardized uniform $[0,1]$ empirical process, and $U_n(x)$ has the same distribution as $V_n(F(x))$. Using Theorem 4.4.3 and Theorem 1.15.2 in Csörgő and Révész (1981) [10], applying the Kiefer type approximation of the empirical process, there exists a Kiefer process $\{K(s;t) : 0 \le s \le 1, 0 \le t < \infty\}$ such that

$$\sup_x |nU_n(x) - K(F(x), n)| = O((\log n)^2) \quad \text{a.s.}, \tag{12}$$

with $B_n(v) = K(v,n)/\sqrt{n}, 0 \le v \le 1$ being a Brownian bridge.

The Proof of Theorem 1 involves three parts: (i) applying the the triangular inequality to the distribution functions, (ii) using the the Kiefer approximation of the empirical process, and (iii) applying the the Taylor expansion. See below for details.

Proof of Theorem 1. Rewrite $\hat{F}(x)$,

$$\hat{F}(x) = n^{-1}\sum_{i=1}^{n}\int_{-\infty}^{x} k_h(u-\varepsilon_i)du = n^{-1}\sum_{i=1}^{n} G\left(\frac{x-\varepsilon_i}{h}\right), \tag{13}$$

where $G(x) = \int_{-\infty}^{x} k(u)du$. By the definition of $F_n(x)$, performing integration by parts and a change of variable $u = \frac{x-t}{h}$, we can continue to rewrite $\hat{F}(x)$, as follows:

$$\begin{aligned}\hat{F}(x) &= \int_{-\infty}^{+\infty} G\left(\frac{x-t}{h}\right)dF_n(t) \\ &= G\left(\frac{x-t}{h}\right)F_n(t)\Big|_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} F_n(t)k\left(\frac{x-t}{h}\right)\frac{1}{h}dt \\ &= \int_{-\infty}^{+\infty} F_n(t)\frac{1}{h}k\left(\frac{x-t}{h}\right)dt = \int_{-\infty}^{+\infty} F_n(x-hu)k(u)du. \end{aligned} \tag{14}$$

Combining (14) with the properties $k(u) \ge 0$ and $\int_{-\infty}^{+\infty} k(u)du = 1$, and applying the triangular inequality, we have that

$$\begin{aligned}&\left|\hat{F}(x) - F_n(x)\right| = \left|\int_{-\infty}^{+\infty}[F_n(x-hu) - F_n(x)]k(u)du\right| \\ &\le \left|\int_{-\infty}^{+\infty}\Big\{[F_n(x-hu) - F(x-hu)] - [F_n(x) - F(x)]\Big\}k(u)du\right| \\ &\quad + \left|\int_{-\infty}^{+\infty}[F(x-hu) - F(x)]k(u)du\right| \\ &= \left|\int_{-\infty}^{+\infty}[U_n(x-hu) - U_n(x)]k(u)du\right| + \left|\int_{-\infty}^{+\infty}[F(x-hu) - F(x)]k(u)du\right|.\end{aligned}$$

Moreover, this results in

$$\begin{aligned}&\sup_x \left|\hat{F}(x) - F_n(x)\right| \\ &\le \sup_x \left|\int_{-\infty}^{+\infty}[U_n(x-hu) - U_n(x)]k(u)du\right| + \sup_x\left|\int_{-\infty}^{+\infty}[F(x-hu) - F(x)]k(u)du\right| \\ &=: D_{1n} + D_{2n}, \quad \text{say}.\end{aligned}$$

Thus, to show Theorem 1, it is sufficient to verify that

$$\limsup_{n\to\infty}\sqrt{\frac{n}{\log(\log n)}}D_{1n} = 0 \;\text{ and }\; \lim_{n\to\infty}\sqrt{\frac{n}{\log(\log n)}}D_{2n} = 0 \quad \text{a.s.} \tag{15}$$

By $log(logn) \to \infty$ and the integrability of $k(u)$, it follows that

$$\Big(\int_{log(logn)}^{+\infty} + \int_{-\infty}^{-log(logn)}\Big)k(u)du = o(1). \tag{16}$$

Partitioning the integral in D_{1n} into three parts, and using the triangular inequality, we can obtain

$$\begin{aligned} D_{1n} &= \sup_x \Big|\Big(\int_{-\infty}^{-log(logn)} + \int_{log(logn)}^{+\infty} + \int_{-log(logn)}^{-log(logn)}\Big)[U_n(x-hu) - U_n(x)]k(u)du\Big| \\ \leq\ & \sup_x \Big(\int_{-\infty}^{-log(logn)} + \int_{log(logn)}^{+\infty} + \int_{-log(logn)}^{log(logn)}\Big)\Big|U_n(x-hu) - U_n(t)\Big|k(u)du \\ \leq\ & 2\Big(\sup_x |U_n(x)|\Big)\Big(\int_{-\infty}^{-log(logn)} + \int_{log(logn)}^{+\infty}\Big)k(u)du + \sup_x \int_{-log(logn)}^{log(logn)} \Big|U_n(x-hu) - U_n(x)\Big|k(u)du \\ =:\ & D_{11n} + D_{12n}, \quad \text{say.} \end{aligned} \tag{17}$$

It is easy to see that (11) and (16) imply that

$$\limsup_{n\to\infty} \sqrt{\frac{n}{\log(\log n)}} D_{11n} = 0 \quad \text{a.s.} \tag{18}$$

As for D_{12n}, with the triangular inequality, (12), $\int_{-log(logn)}^{log(logn)} k(u)du \leq 1$ and the continuity of modulus of $B_n(F(x)) = K(F(x), n)/\sqrt{n}$, we have

$$\begin{aligned} & D_{12n} \\ \leq\ & \sup_x \frac{1}{n}\Big|\int_{-log(logn)}^{log(logn)} \{[nU_n(x-hu) - K(F(x-hu), n)] - [nU_n(x) - K(F(x), n)]\}k(u)du\Big| \\ & \quad + \sup_x \frac{1}{n}\Big|\int_{-log(logn)}^{log(logn)} [K(F(x-hu), n) - K(F(x), n)]k(u)du\Big| \\ \leq\ & \frac{2}{n}\Big(\sup_x \Big|nU_n(x) - K(F(x), n)\Big|\Big) + O(h\log(\log n)\sqrt{h\log(\log n)}/\sqrt{n} \\ =\ & O(\frac{(\log n)^2}{n}) + O\Big(h\log(\log n)\sqrt{h\log(\log n)}/\sqrt{n}\Big) \quad \text{a.s.} \end{aligned}$$

Hence, combining the above bound with $\frac{(\log n)^2}{\sqrt{n\log(logn)}} \to 0$ and the assumption $h^{3/2}\log(\log n) \to 0$, it follows that

$$\limsup_{n\to\infty} \sqrt{\frac{n}{\log(\log n)}} D_{12n} = 0. \quad \text{a.s.} \tag{19}$$

Next, we proceed to evaluate D_{2n}. Using the Taylor expansion with integral remainder, the properties $\int_{-\infty}^{+\infty} uk(u)du = 0$, $\int_{-\infty}^{+\infty} u^2k(u)du < \infty$, $\sup_t |f'(t)| < \infty$ and $nh^4/\log(\log n) \to 0$, we obtain

$$
\begin{aligned}
F(x-hu)-F(x) &= -huf(x)+\int_{x}^{x-hu} f^{'}(s)(x-hu-s)ds \\
&= -huf(x)+\int_{0}^{-hu} f^{'}(x-hu-t)tdt, \\
D_{2n} &= \sup_{x}\Big|\int_{-\infty}^{+\infty}[\int_{0}^{-hu} f^{'}(x-hu-t)tdt]k(u)du\Big| \\
&\leq \sup_{x}|f^{'}(x)|\int_{-\infty}^{+\infty}\frac{1}{2}h^2u^2k(u)du = O(h^2), \\
\sqrt{\frac{n}{\log(\log n)}}D_{2n} &= O(\sqrt{\frac{nh^4}{\log(\log n)}}) \to 0. \qquad (20)
\end{aligned}
$$

Therefore, (15) can be produced from (17)–(20). We have completed the proof of Theorem 1. □

Proof of Corollary 1. Decomposing $\hat{F}(x)-F(x)$ into two parts and using the triangular inequality, we have

$$
\begin{aligned}
\big|\hat{F}(x)-F(x)\big| &= \big|\hat{F}(x)-F_n(x)+F_n(x)-F(x)\big| \\
&\leq \big|\hat{F}(x)-F_n(x)\big|+\big|F_n(x)-F(x)\big|, \\
\big|\hat{F}(x)-F(x)\big| &\geq -\big|\hat{F}(x)-F_n(x)\big|+\big|F_n(x)-F(x)\big|.
\end{aligned}
$$

Then, it follows that

$$
\sup_{x\in R}\big|\hat{F}(x)-F(x)\big| \leq \sup_{x\in R}\big|\hat{F}(x)-F_n(x)\big|+\sup_{x\in R}\big|F_n(x)-F(x)\big|
$$

and

$$
\sup_{x\in R}\big|\hat{F}(x)-F(x)\big| \geq -\sup_{x\in R}\big|\hat{F}(x)-F_n(x)\big|+\sup_{x\in R}\big|F_n(x)-F(x)\big|.
$$

Combining the above inequalities with (8) and (2), this guarantees that

$$
\limsup_{n\to\infty}\sqrt{\frac{n}{2\log(\log n)}}\sup_{x}\Big|\hat{F}(x)-F(x)\Big| = \frac{1}{2} \quad a.s.
$$

Thus, we have finished the proof of Theorem 1. □

Proof of Theorem 2. For any $p \geq 1$, using the triangular inequality and the fact that $\int_{-\infty}^{\infty} 1dF(x) = 1$, we have that

$$
\begin{aligned}
&\Big[\int_{-\infty}^{\infty}|\hat{F}(x)-F(x)|^p dF(x)\Big]^{1/p} \\
\leq\ &\Big[\int_{-\infty}^{\infty}|\hat{F}(x)-F_n(x)|^p dF(x)\Big]^{1/p}+\Big[\int_{-\infty}^{\infty}|F_n(x)-F(x)|^p dF(x)\Big]^{1/p} \\
\leq\ &\sup_{x\in R}|\hat{F}(x)-F_n(x)|+\Big[\int_{-\infty}^{\infty}|F_n(x)-F(x)|^p dF(x)\Big]^{1/p} \qquad (21)
\end{aligned}
$$

and

$$\begin{aligned}
&\left[\int_{-\infty}^{\infty}|\hat{F}(x)-F(x)|^p dF(x)\right]^{1/p} \\
\geq\ & -\left[\int_{-\infty}^{\infty}|\hat{F}(x)-F_n(x)|^p dF(x)\right]^{1/p}+\left[\int_{-\infty}^{\infty}|F_n(x)-F(x)|^p dF(x)\right]^{1/p} \\
\geq\ & -\sup_{x\in R}|\hat{F}(x)-F_n(x)|+\left[\int_{-\infty}^{\infty}|F_n(x)-F(x)|^p dF(x)\right]^{1/p}. \qquad (22)
\end{aligned}$$

Note that we have proven

$$\sup_{x\in R}\sqrt{\frac{n}{\log(\log n)}}|\hat{F}(x)-F_n(x)|\to 0, \qquad \text{a.s.}$$

in Theorem 1. Thus, combining it with (21), (22) and the law of the iterated logarithm for L_p-norm of $F_n(x)$ in (5), it follows that

$$\limsup_{n\to\infty}\sqrt{\frac{n}{2\log(\log n)}}\left[\int_{-\infty}^{\infty}|\hat{F}(x)-F(x)|^p dF(x)\right]^{1/p}=C(p)\ \ a.s.$$

Therefore, we have finished the proof of Theorem 2. □

Proof of Corollary 2. Corollary 2 is the special case of results of Theorem 2 with $p=1,2$. □

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