

Article

Asymptotically Newton-Type Methods without Inverses for Solving Equations

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Abstract: The implementation of Newton's method for solving nonlinear equations in abstract domains requires the inversion of a linear operator at each step. Such an inversion may be computationally very expensive or impossible to find. That is why alternative iterative methods are developed in this article that require no inversion or only one inversion of a linear operator at each step. The inverse of the operator is replaced by a frozen sum of linear operators depending on the Fréchet derivative of an operator. The numerical examples illustrate that for all practical purposes, the new methods are as effective as Newton's but much cheaper to implement. The same methodology can be used to create similar alternatives to other methods using inversions of linear operators such as divided differences or other linear operators.

Keywords: Newton-type method; banach space; frozen sum of operators; Fréchet derivative; convergence; inverse of a linear operator

MSC: 65G99; 65H10; 49H17; 49M15



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1. Introduction

A plethora of problems from mathematics, computational science, and other applied disciplines can be reduced using mathematical modeling to an equation of the form

$$F(x) = 0, \quad (1)$$

where $F : D \subset X \rightarrow Y$ is a Fréchet-differentiable operator [1–3]. The symbols X, Y stand for Banach spaces and D denotes an open and convex subset of X . An analytical form of a solution $x^* \in D$ of the Equation (1) can be realized only in some rare cases. Therefore, researchers and practitioners develop iterative methods approximating the solution x^* provided that certain usually sufficient for convergence conditions are imposed on the initial approximation and the operator F [1,2,4]. There are differences among certain types of convergence. In the local convergence the conditions on the solution x^* are imposed and a convergence ball is determined, which provides the degree of difficulty in the selection of the initial approximation x_0 . The main problem in the local convergence is that the solution x^* is unknown (usually). However, in the semi-local convergence, the conditions involve the initial approximation x_0 , and the solution is then found inside a ball at x_0 . There is extensive literature on both types of convergence based on Lipchitz, Holder, or generalized

continuity conditions imposed on the derivative F' in order to control it. The best-known iterative method is without a doubt Newton's one, which is defined as

$$y_0 \in D, y_{n+1} = y_n - F'(y_n)^{-1}F(y_n), n = 0, 1, 2, \dots \quad (2)$$

where $F'(y_n)^{-1}$ denotes the inverse of the Fréchet derivative $F'(y_n)$ of the operator F . The study of Newton's method is carried out via means of local and semi-local convergence. The former uses information about the solution y^* and provides estimates of a possible radius of convergence, error estimates on the norms $\|y^* - y_n\|$, $\|y_{n+1} - y_n\|$ and isolation of the solution findings. The latter type is based on information about the initial approximation and convergence conditions, which guarantee that $\lim_{n \rightarrow +\infty} y_n = y^*$ [3,5–7].

In this article, we address an important issue with the implementation of Newton's method. The order of convergence is two, which is considered fast for a single-step method. **Motivation** There are concerns with the implementation of Newton's method: The inversion of the linear operator $F'(y_n)$ is required at every step in Euclidean, Hilbert or Banach space. This operator may be a large-sized matrix or an operator whose analytical form is computationally expensive or impossible to find or it may be very small (case of small divisors) [6,8–15].

The motivation of this article is to address this issue. This is achieved as follows.

First, let k denote a natural number and $\mathcal{L}(X, Y)$ be the space of all linear operators that are bounded. Suppose that there exists $M \in \mathcal{L}(X, Y)$ such that the operators M and $(I - M^{-1}(M - F'(x)))$ are invertible. Then, Newton's method (2) can be rewritten as

$$y_0 \in D, y_{n+1} = y_n - [I - M^{-1}(M - F'(y_n))]^{-1}M^{-1}F(y_n). \quad (3)$$

Notice that clearly, the iterates of the two preceding versions of Newton's method coincide, since

$$[I - M^{-1}(M - F'(y_n))]^{-1}M^{-1} = [M(I - M^{-1}(M - F'(y_n)))]^{-1} = F'(y_n)^{-1}.$$

But even if the linear operator M^{-1} is fixed, we still need to invert $(I - M^{-1}(M - F'(x_n)))$, which is not a fixed linear operator in general. However, we can avoid this inversion, if we develop the operators $A(x) = A = M^{-1}(M - F'(x))$ and $B_k(x) = B(x) = I + A + \dots + A^k$ for k a fixed natural number, and consider the method for $y_0 = x_0$

$$x_0 \in D, x_{n+1} = x_n - BM^{-1}F(y_n). \quad (4)$$

This method requires the inversion of the fixed linear operator M only once. Therefore, method (4) is a useful alternative to Newton's method using only one inverse.

If $M = F'(x_0)$ and $k = 1$, the method (4) reduces to the modified Newton's method

$$x_0 \in D, x_{n+1} = x_n - F'(x_0)^{-1}F(x_n).$$

We can also set $M = I$. Many other choices of M and k are possible (see the numerical section).

By letting $k \rightarrow +\infty$, we see that $\lim_{k \rightarrow +\infty} B_k M^{-1} = F'(x)^{-1}$, provided that this limit exists. The condition $\|M^{-1}(M - F'(x))\| < 1$ for each $x \in D$ guarantees the existence of this limit. It is worth noting that if the linear operator B is invertible, and the sequence $\{y_n\}$ is convergent to some x^* , then (4) gives

$$B^{-1}(x_n - x_{n+1}) = M^{-1}F(x_n)$$

so

$$0 = \lim_{n \rightarrow +\infty} B^{-1}(x_n - x_{n+1}) = \lim_{n \rightarrow +\infty} M^{-1}F(x_n).$$

Consequently, we deduce that $F(x^*) = 0$, i.e., the limit point x^* , solves the equation $F(x) = 0$. The same argument leads to the development of the method defined for

$$A_1 = (M - F'(x))M^{-1}, B_1 = I + A_1 + \dots + A_1^k$$

and each $n = 0, 1, 2, \dots$ by

$$x_0 \in D, x_{n+1} = x_n - B_1 M^{-1} F(x_n).$$

Clearly, the study of this method is analogous to the method (4). Based on the preceding reasoning it is worth studying the convergence of the method (4).

In particular, for the semi-local analysis, we develop the application of majorizing sequence $\{\alpha_n\}$ defined for $\alpha_0 = 0$, some $\alpha_1 \geq 0$ and each $n = 0, 1, 2, \dots$ by

$$\alpha_{n+2} = \alpha_{n+1} + (\lambda(\alpha_{n+1} - \alpha_n) + \mu)(\alpha_{n+1} - \alpha_n).$$

The scalar sequence $\{\alpha_n\}$ as well as its convergence order are well studied [7,16]. The parameter α_1 depends on the initial approximation x_0 , and the nonnegative parameters λ and μ are related to M and F' (see Sections 2 and 3).

The study of the local convergence reduces to solving the inequality

$$P(t) := \zeta t + \zeta_0 < 1 \text{ for } t \in [0, +\infty),$$

to determine the convergence radius

$$\mu = \frac{1 - \zeta_0}{\zeta}$$

for some $\zeta > 0$ and $\zeta_0 \in [0, 1)$ depending on x^* , M and F' (see Section 4). The uniqueness of the solution is studied in Section 5. In this article, we also study in Section 6 the convergence of the Picard-type or Krasnoselskij method, given as

$$x_{n+1} = P_k(x_n), \quad (5)$$

where

$$P_k(x) = P(x) = (1 - \beta)x - \beta(BM^{-1}F(x) - x)$$

for some $\beta \in [0, 1)$. Notice that if $\beta = 1$, the method (5) reduces to the method (4). The convergence of the method (5) is presented using the celebrated contraction mapping principle [3,8,17]. In Section 7, an error analysis is provided comparing the sequence $\{y_n\}$ to the sequence $\{x_n\}$ generated by (4). Section 8 is developed to further validate the theory of the previous sections and compare the results to Newton's method (2). The concluding remarks of Section 9 contain information about the future direction of our research.

The methodology of this article is very general. That is why it can be extended to other single-step methods using inverses of linear operators. Let us consider the Newton-like method defined by

$$x_0 \in D, x_{n+1} = x_n - \tilde{F}'(x_n)^{-1}F(x_n), \quad (6)$$

where the linear operator $\tilde{F}'(x_n)^{-1} \in \mathcal{L}(Y, X)$ and is considered to be an approximation to the actual derivative F' of the operator F [3,10,16,18]. Then both the local and semi-local results obtained for the method (4) are extended to the method (6) if we simply replace $F'(x_n)$ by $\tilde{F}'(x_n)$ in the definition of the operators A and B , since (6) reduces to (4) if $\tilde{F}' = F'$. The same is true for the Picard method (5) but with operators A and B using $\tilde{F}'$ instead of F . The same idea can be used on two-step, multi-step or multi-point methods using inverses of linear operators in an analogous way [3–7,13–16,19–23].

Novelty of the Article

The article addresses the concerns with the implementation of Newton's or Picard-type method or any other method using inverses of linear operators, as described in the motivation for writing this article. This is achieved by simply replacing the inverse by a finite sum of linear operators. The semi-local and local analysis that is developed in the next sections shows the convergence of these hybrid methods to the solution x^* of the equation $F(x) = 0$. Moreover, the numerical experimentation further demonstrates that the iterations required by the hybrid methods are essentially the same as the original methods but cheaper to find.

2. Convergence of Majorizing Sequence

Let $L > 0, \eta \geq 0$ and $a \in [0, \frac{1}{2})$ be given parameters. It is convenient for the semi-local convergence to be defined: the parameters

$$\begin{aligned}\lambda &= \frac{L(1-a^{k+1})}{2(1-a)}, \quad b = a \frac{1-a^k}{1-a}, \quad \bar{b} = \frac{1}{1-b}, \\ \mu &= \frac{(1-a^{k+1})a^{k+1}\bar{b}}{1-a}, \quad \delta = \frac{1-\sqrt{1-4\lambda\mu}}{2\lambda}, \quad \gamma = \frac{1-2\lambda\mu-\sqrt{1-4\lambda\mu}}{2\lambda^2}\end{aligned}$$

provided that $4\lambda\mu \leq 1$, and the scalar sequence $\{\alpha_n\}$ for $\alpha_0 = 0$, some $\alpha_1 \geq \eta$, and each $n = 0, 1, 2, \dots$ by

$$\alpha_{n+2} = \alpha_{n+1} + (\lambda(\alpha_{n+1} - \alpha_n) + \mu)(\alpha_{n+1} - \alpha_n). \quad (7)$$

Notice that the parameters depend on k , which is fixed. This sequence is important in the study of the semi-local convergence of the method (2). In particular, it is shown to be majorizing for $\{x_n\}$ given by the Formula (7) in the Theorems. However, first, some conditions are developed to that establish the convergence of the sequence $\{\alpha_n\}$. The proving technique of recurrent polynomials has been used in [22–25] to weaken the sufficient semi-local convergence conditions for Newton's method.

Under the preceding notation, the following result can be proven.

Lemma 1. Suppose that the following conditions hold:

$$a \in [0, \frac{1}{2}), \quad 4\lambda\mu \leq 1, \quad \delta < 1, \quad (8)$$

$$\eta \leq \gamma \quad (9)$$

and

$$r_0 = \frac{\eta}{1-\delta} \leq r. \quad (10)$$

Then, the scalar sequence $\{\alpha_n\}$ is non-negative, non-decreasing, and bounded from above by μ . Moreover, the following assertions hold:

$$0 \leq \alpha_{n+1} - \alpha_n \leq \delta(\alpha_n - \alpha_{n-1}) \leq \delta^n \eta \quad (11)$$

and there exists $\alpha \in [0, \frac{\eta}{1-\delta}]$ such that the sequence $\{\alpha_n\}$ converges to α , and

$$0 \leq \alpha - \alpha_n \leq \frac{\delta^n \eta}{1-\delta} \leq r_0 \leq r. \quad (12)$$

Proof. The conditions of the Lemma and the definitions of the parameters imply $\alpha_0 \leq \alpha_1 \leq r_0 \leq r$, $\mu \leq \delta$, $\gamma > 0$, $\lambda > 0$, $\mu > 0$ and $b < 1$. By the definition of the parameters α_0 and α_1 , the assertion (11) holds if $n = 0$. Next, mathematical induction is used to establish the assertions

$$0 \leq \alpha_{m+1} - \alpha_m \leq \delta(\alpha_m - \alpha_{m-1}) \leq \delta^m \eta \quad (13)$$

and

$$0 \leq \alpha_m \leq r_0. \quad (14)$$

These assertions hold for $m = 0$ by the definition of α_0, α_1 and δ . Then it follows by the definition of the sequence $\{\alpha_n\}$ that $0 \leq \alpha_2 - \alpha_1 \leq \delta(\alpha_1 - \alpha_0) \leq \delta\eta$, and

$$0 \leq \alpha_2 \leq \alpha_1 + \delta\eta = \delta + \delta\eta = \frac{1 - \delta^2}{1 - \delta} \eta \leq \frac{\eta}{1 - \delta}$$

Thus, the assertions (13) and (14) hold if $m = 2$. Suppose that these assertions hold for all natural numbers up to m . Then, we have to show

$$\lambda(\alpha_{m+2} - \alpha_{m+1}) + \mu \leq \delta \quad (15)$$

and

$$\alpha_{m+1} \leq r_0. \quad (16)$$

The first assertion holds if instead

$$\lambda\delta^{m+1} + \mu \leq \delta. \quad (17)$$

But this assertion motivates the introduction of consecutive polynomials p_m defined on the interval $[0, 1]$ by

$$f_m(t) = \lambda t^{m+1} - t + \mu. \quad (18)$$

A relation between two consecutive polynomials is useful:

$$f_{m+1}(t) - f_m(t) = \lambda t^{m+2} - \lambda t^{m+1} = \lambda t^{m+1}(t - 1) \leq 0.$$

So, instead of (17), we can show

$$f_1(t) \leq 0 \text{ at } t = \delta, \quad (19)$$

since

$$f_{m+1}(t) \leq f_m(t). \quad (20)$$

But (19) holds by the choice of δ , (9) and (10). Define the function

$$f_\infty(t) = \lim_{m \rightarrow +\infty} f_m(t). \quad (21)$$

This function is given explicitly, since by (21) at $t = \delta$

$$f_\infty(t) = \lim_{m \rightarrow +\infty} (\lambda t^{m+1} - t + \mu) = \mu - \delta \leq 0.$$

Thus, the following holds for all m

$$f_m(t) \leq 0, \quad (22)$$

Hence, (22) holds and by (13)

$$\begin{aligned}\alpha_{m+1} &= \alpha_{m+1} - \alpha_0 = (\alpha_{m+1} - \alpha_m) + (\alpha_m - \alpha_{m-1}) \\ &\quad + \dots + (\alpha_1 - \alpha_0) \leq (\delta^m + \dots + \delta + 1)\eta \\ &= \frac{1 - \delta^{m+1}}{1 - \delta} \eta \leq \frac{\eta}{1 - \delta} = r_0 \leq r.\end{aligned}$$

Therefore, (14) holds. The induction for the assertions (13) and (14) is complete. Hence, the sequence $\{\alpha_n\}$ is non-negative, non-decreasing, and bounded from above by r_0 , and as such it is convergent to some unique $\alpha \in [0, r_0]$.

Then, for j a natural number, we get

$$\begin{aligned}0 &\leq \alpha_{m+j} - \alpha_m = (\alpha_{m+j} - \alpha_{m+j-1}) + \dots + (\alpha_{m+1} - \alpha_m) \\ &\leq (\delta^{m+j-1} + \dots + \delta^m)\eta = \frac{1 - \delta^{m+j}}{1 - \delta} \delta^m \eta.\end{aligned}\quad (23)$$

By letting $j \rightarrow +\infty$, we show (12). $\square$

It is worth noting that the conditions of the Lemma 1 are weaker than those given for the method (7) given in [Lemma 4.2, Page 57] in [7], where, however, λ and μ have a different meaning.

The parameters in Lemma 1 are related to the operators appearing in method (4). These relationships are provided in the next section, where we deal with the semi-local analysis of convergence for method (4).

3. Semi-Local Analysis for Method (4)

Throughout this paper, we use the notation $U(x, \alpha)$ to denote the open ball centered at $x \in X$ with radius $\alpha > 0$, and $U[x, \alpha]$ is the closure of $U(x, \alpha)$. The following Banach Lemma is used to prove our results.

Theorem 1 (Banach Lemma on Invertible Operators [3,4,11]). *If T is a bounded linear operator in X , T^{-1} exists if and only if there is a bounded linear operator T_1 in X such that T_1^{-1} exists and*

$$\|I - T_1 T\| < 1.$$

If T^{-1} exists, then

$$T^{-1} = \sum_{n=0}^{\infty} (I - T_1 T)^n T_1$$

and

$$\|T^{-1}\| \leq \frac{\|T_1\|}{1 - \|I - T_1 T\|}.$$

Further, we use majorizing sequences to prove the semi-local convergence. Recall the definition of a majorizing sequence.

Definition 1 ([3,4,11]). *Let $\{x_n\}$ be a sequence in a normed space X . Then a nonnegative scalar sequence $\{v_n\}$ for which*

$$\|x_{n+1} - x_n\| \leq v_{n+1} - v_n \quad \forall n \geq 0 \quad (24)$$

holds, is a majorizing sequence for $\{x_n\}$. Note that any majorizing sequence is necessarily nondecreasing. Moreover, if the sequence $\{v_n\}$ converges, then $\{x_n\}$ converges too, and for $v^* = \lim_{n \rightarrow \infty} v_n$,

$$\|x^* - x_n\| \leq v^* - v_n.$$

Hence, the study of the convergence of the sequence $\{x_n\}$ reduces to that of $\{v_n\}$.

The following conditions are needed:

Suppose:

(C₁) There exist parameters $\eta \geq 0, a \in [0, \frac{1}{2})$, a point $x_0 \in D$, and an invertible operator M such that

$$\|BM^{-1}F(x_0)\| \leq \eta.$$

(C₂) There exists a parameter $L > 0$ such that for each $v_1, v_2 \in D$

$$\|M^{-1}(F'(v_2) - F'(v_1))\| \leq L\|v_2 - v_1\|.$$

(C₃) The conditions of the Lemma 1 hold and

(C₄) $U[x_0, \alpha] \subset D$.

Theorem 2. Suppose that the conditions (C₁)–(C₄) hold. Then, the following items hold for method (4):

$$\{x_n\} \subset U(x_0, \alpha), \quad (25)$$

$$\|x_{n+1} - x_n\| \leq \alpha_{n+1} - \alpha_n, \quad (26)$$

and there exists a solution $x^* \in U[x_0, \alpha]$ to the equation $F(x) = 0$ such that

$$\|x^* - x_n\| \leq \alpha - \alpha_n. \quad (27)$$

Proof. Notice that by the invertibility of the linear operator M assumed in condition (C₁) and method (4), all iterates $\{x_n\}$ are well defined. The method of induction shall first establish the validity of items (25) and (26). The item (25) trivially holds true for $n = 0$. By the definition of method (4), the scalar sequence (7) and the condition (C₁), we see that

$$\|x_1 - x_0\| = \|BM^{-1}F(x_0)\| \leq \eta = \alpha_1 - \alpha_0 < \alpha.$$

So, the iterate $x_1 \in U(x_0, a)$, and the item (26) holds if $n = 0$.

The definition of the operator B and the condition (C₁) imply that by choosing a ,

$$\begin{aligned} \|I - B\| &= \|A + A^2 + \dots + A^k\| \leq \|A\| + \|A\|^2 + \dots + \|A\|^k \\ &= \|A\| \frac{1 - \|A\|^k}{1 - \|A\|} = b < 1. \end{aligned}$$

Thus, by Theorem 1, the operator B is invertible and

$$\|B^{-1}\| \leq \frac{1}{1 - b} = \bar{b}. \quad (28)$$

By the definition of method (4), we can write in turn that

$$\begin{aligned} F(x_{n+1}) &= F(x_{n+1}) - F(x_n) - MB^{-1}(x_{n+1} - x_n) \\ &= F(x_{n+1}) - F(x_n) - F'(x_n)(x_{n+1} - x_n) \\ &\quad + (F'(x_n) - MB^{-1})(x_{n+1} - x_n) \\ &= F(x_{n+1}) - F(x_n) - F'(x_n)(x_{n+1} - x_n) \\ &\quad + (F'(x_n)B - M)B^{-1}(x_{n+1} - x_n) \\ &= \int_0^1 [F'(x_n + \theta(x_{n+1} - x_n)) - F'(x_n)]d\theta(x_{n+1} - x_n) \\ &\quad + (F'(x_n)B - M)B^{-1}(x_{n+1} - x_n). \end{aligned} \quad (29)$$

But

$$\begin{aligned}
 F'(x_n)B - M &= F'(x_n)(I + A + A^2 + \dots + A^k) - M \\
 &= F'(x_n) - M + (-M + M + F'(x_n))(A + A^2 + \dots + A^k) \\
 &= F'(x_n) - M + M(A + A^2 + \dots + A^k) - (M - F'(x_n)) \\
 &\quad \times (A + A^2 + \dots + A^k) \\
 &= F'(x_n) - M + MA + M(A^2 + \dots + A^k) - (M - F'(x_n)) \\
 &\quad \times (A + A^2 + \dots + A^k) \\
 &= M(A^2 + \dots + A^k) - M(M - F'(x_n))(A + A^2 + \dots + A^k)
 \end{aligned}$$

So,

$$M^{-1}(F'(x_n)B - M) = A^2 + \dots + A^k - A(A + A^2 + \dots + A^k) = -A^{k+1},$$

where we also used the definition of the operator A to get $F'(x_n) - M + MA = 0$. Thus, the preceding Ostrowski representation (29) for $F(x_{n+1})$ can be simplified as

$$\begin{aligned}
 M^{-1}F(x_{n+1}) &= M^{-1} \int_0^1 (F'(x_n + \theta(x_{n+1} - x_n)) - F'(x_n)) d\theta (x_{n+1} - x_n) \\
 &\quad - A^{k+1}B^{-1}(x_{n+1} - x_n).
 \end{aligned} \tag{30}$$

By taking norms in both sides of the identity (30), using the triangle inequality (30), and the conditions (C_1) , (C_2) , we get in turn

$$\begin{aligned}
 \|M^{-1}F(x_{n+1})\| &\leq \frac{L}{2} \|x_{n+1} - x_n\|^2 + a^{k+1}b_k \|x_{n+1} - x_n\| \\
 &\leq \frac{L}{2} (\alpha_{n+1} - \alpha_n)^2 + a^{k+1}\bar{b}(\alpha_{n+1} - \alpha_n).
 \end{aligned} \tag{31}$$

Then, method (4) gives

$$\begin{aligned}
 \|x_{n+2} - x_{n+1}\| &\leq \|B_{k+1}\| \|M^{-1}F(x_{n+1})\| \\
 &\leq (1 + \|A\| + \dots + \|A^k\|) \left(\frac{L}{2} (\alpha_{n+1} - \alpha_n)^2 + a^{k+1}\bar{b}(\alpha_{n+1} - \alpha_n) \right) \\
 &\leq (1 + a + \dots + a^k) \left(\frac{L}{2} (\alpha_{n+1} - \alpha_n)^2 + a^{k+1}\bar{b}(\alpha_{n+1} - \alpha_n) \right) \\
 &= \alpha_{n+2} - \alpha_{n+1}.
 \end{aligned} \tag{32}$$

Consequently, we obtain

$$\begin{aligned}
 \|x_{n+2} - x_0\| &\leq \|x_{n+2} - x_{n+1}\| + \|x_{n+1} - x_0\| \\
 &\leq \alpha_{n+2} - \alpha_{n+1} + \alpha_{n+1} - \alpha_0 = \alpha_{n+2} < \alpha.
 \end{aligned}$$

Hence, the induction for the items (25) and (26) is complete. Then, it follows by (25) and (26) that the sequence $\{x_n\}$ is Cauchy in the Banach space X (since the majorizing sequence $\{\alpha_n\}$ is also Cauchy as convergent to α by the condition (C_3)). Therefore, there exists $x^* \in U[x_0, a]$ such that $\lim_{n \rightarrow +\infty} x_n = x^*$. By letting $n \rightarrow +\infty$ in (31), and the continuity of the operator F , we deduce that $F(x^*) = 0$. That is, the limit point x^* is a solution of the equation $F(x) = 0$. Moreover, for i denoting a natural number, we have by the triangle inequality and (26) that

$$\|x_{n+i} - x_n\| \leq \alpha_{n+i} - \alpha_n.$$

Finally, letting $i \rightarrow +\infty$, we obtain the item (27). $\square$

Remark 1.

- (1) The condition on the parameter a can be replaced as follows. Let $\bar{a} \in [0, \frac{1}{2})$ be fixed. Replace the condition (C_2) with $(C_2)'$. Then there exists a parameter $L_0 > 0$ such that for all $w \in D$,

$$\|M^{-1}(F'(x) - M)\| \leq L_0 \|x - x_0\|.$$

Let $r_1 = \frac{\bar{a}}{L_0}$.

There exists a parameter $L_1 > 0$ (depending on L_0) such that for each $w_1, w_2 \in D_0$,

$$\|M^{-1}(F'(w_2) - F'(w_1))\| \leq L_1 \|w_2 - w_1\|,$$

where $D_0 = U(x_0, \frac{1}{L_0}) \cap D$. Notice that $D_0 \subset D$. So, $L_0 \leq L$ and $L_1 \leq L$.

Moreover, replace $(C4)$ with

$(C4)'$,

$U[x_0, \bar{r}] \subset D$, where $\bar{r} = \min\{r, r_1\}$. Then, the conclusions of Theorem 2 hold with $\bar{a}$, L_1 and $\bar{r}$ replacing a , L and r , respectively.

- (2) A choice for the linear operator can be $M = F'(x_0)$ or $M = I$ or any other operator that satisfies the conditions (C_1) and (C_2) .
- (3) The corresponding results for the method (3) are obtained immediately if $k \rightarrow +\infty$. Notice that if $k = n$ in (4) and $n \rightarrow +\infty$, method (4) becomes (3).

4. Local Analysis of Convergence for the Method (4)

As in the semi-local analysis, it is convenient to define some parameters and functions. Let $d > 0$ and $l_0 > 0$. Define the parameters

$$\zeta_0 = a \frac{1 - a^k}{1 - a},$$

$$\zeta = \frac{l_0}{2} \frac{1 - a^{k+1}}{1 - a},$$

and the polynomial $p_1(t) = \zeta_0 + \zeta t$. These parameters are connected below to the operators on method (4).

Suppose:

- (H₁) There exists a solution x^* of the equation $F(x) = 0$, and an invertible operator $M \in \delta(X, Y)$ such that for all $w \in D$,

$$\|M^{-1}(F'(w) - M)\| \leq l_0 \|w - x^*\|.$$

- (H₂) There exists $a \in [0, \frac{1}{2})$ such that $\|A\| \leq a$. It follows that $\zeta_0 \in [0, 1)$.

Set $\rho = \frac{1 - \zeta_0}{\zeta}$, and

- (H₃) $U[x^*, \rho] \subset D$.

Next, the local analysis of convergence follows using the conditions (H₁)–(H₃) and the developed terminology.

Theorem 3. Suppose that the conditions (H₁)–(H₃) hold and the starting point $x_0 \in S = U(x^*, \rho) - \{x^*\}$. Then, the following items hold for the method (4)

$$\{x_n\} \subset U(x^*, \rho), \quad (33)$$

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq p_1(\|x_n - x^*\|) \|x_n - x^*\| \\ &\leq \|x_n - x^*\| < \rho, \end{aligned} \quad (34)$$

and

$$\lim_{n \rightarrow +\infty} x_n = x^*. \quad (35)$$

Proof. The items (33) and (34) are shown using induction. By the hypothesis $x_0 \in S$, the item (33) holds if $n = 0$. Notice that all iterates $\{x_n\}$ of method (4) are well defined by the condition (H_1) since the linear operator M is invertible. Then, using method (4), we can write in turn

$$\begin{aligned} x_{n+1} - x^* &= x_n - x^* - BM^{-1}F(x_n) \\ &= x_n - x^* - BM^{-1} \int_0^1 F'(x^* + \theta(x_n - x^*))d\theta(x_n - x^*) \\ &= [I - BM^{-1} \left(\int_0^1 F'(x^* + \theta(x_n - x^*))d\theta \right)](x_n - x^*) \\ &= \left[(I - B) - BM^{-1} \int_0^1 (F'(x^* + \theta(x_n - x^*)) - M)d\theta \right](x_n - x^*). \end{aligned} \quad (36)$$

But, by definition B , as in the semi-local analysis,

$$\|B\| \leq \frac{1 - a^{k+1}}{1 - a}$$

and

$$\|I - B\| \leq a \frac{1 - a^k}{1 - a}.$$

By combining (36), using the conditions (H_1) – (H_3) and the triangle inequality, we get in turn that

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq \left[\|A\| \frac{1 - \|A\|^k}{1 - \|A\|} + \frac{l_0}{2} \frac{1 - \|A\|^{k+1}}{1 - \|A\|} \|x_n - x^*\| \right] \|x_n - x^*\| \\ &\leq p_1 (\|x_n - x^*\|) \|x_n - x^*\| \\ &\leq \|x_n - x^*\| < \rho. \end{aligned} \quad (37)$$

Thus, the induction for the items (33) and (34) is completed. Then by the estimation

$$\|x_{n+1} - x^*\| \leq c \|x_n - x^*\| \leq c^{n+1} \|x_n - x^*\| \leq c^{n+1} \rho,$$

where $c = p(\|x_n - x^*\|) \in [0, 1)$, we deduce that $\lim_{n \rightarrow +\infty} x_n = x^*$ and $x_{n+1} \in U(x^*, \rho)$. $\square$

Remark 2.

- (i) A possible choice for the operator M is $M = F'(x^*)$ or $M = I$. However, other choices are possible as long as the conditions (H_1) – (H_3) hold.
- (ii) The parameter ρ is the radius of convergence for method (4).
- (iii) As in the semi-local case, the corresponding local results for method (3) are obtained from Theorem 3 by letting $k \rightarrow +\infty$.

5. Uniqueness of the Solution

In this Section, the uniqueness of the solution x^* is determined in certain regions. In the cases of the semi-local convergence of method (4), we can demonstrate the following:

Proposition 1. Suppose there exists a solution $z \in U(x_0, s)$ for some $s > 0$ of the equation $F(x) = 0$; There exists an invertible operator M and a parameter $l \geq 0$ such that

$$\|M^{-1}(F'(x) - M)\| \leq l \|x - x_0\| \quad (38)$$

for each $x \in D$ and there exists $s_1 \geq s$ such that

$$\frac{1}{2}l(s + s_1) < 1. \quad (39)$$

Define $D_2 = D \cap U[x_0, s_1]$. Then, the only solution of the equation $F(x) = 0$ in the region D_2 is z .

Proof. Suppose that there exists $z_1 \in D_2$ solving the equation $F(x) = 0$. Define the linear operator

$$Q = \int_0^1 F'(z + \theta(z_1 - z))d\theta.$$

Then, it follows by the conditions (38), (39) and the definition of the operator Q in turn that

$$\begin{aligned} \|M^{-1}(Q - M)\| &\leq \int_0^1 l[(1 - \theta)\|z - x_0\| + \|z_1 - x_0\|] \\ &\leq \frac{l}{2}(s + s_1) < 1. \end{aligned}$$

Hence, the linear operator Q is invertible. Then, by using the identity

$$z_1 - z = Q^{-1}(F(z_1) - F(z)) = Q^{-1}(0) = 0,$$

we conclude that $z_1 = z$. $\square$

Remark 3. Notice that the hypotheses of Theorems 2 are used in the Proposition 1. Otherwise, we can set $z = x^*$ and $s = \alpha$.

Next, the corresponding result follows for the local convergence case of method (4).

Proposition 2. There exists parameters $l_1, s_2 > 0$ and an invertible operator M such that for each $x \in U(x^*, s_2)$

$$\|M^{-1}(F'(x) - M)\| \leq l_1\|x - x^*\|, \quad (40)$$

and there exists $s_3 \geq s_2$ such that

$$\frac{l_1 s_3}{2} < 1. \quad (41)$$

Define the region $D_3 = D \cap U[x^*, s_3]$. Then, the only solution of the equation $F(x) = 0$ in the region D_3 is x^* .

Proof. Suppose that there exists $w \in D_3$ solving the equation $F(x) = 0$. Define the linear operator $Q_1 = \int_0^1 F'(x^* + \theta(w - x^*))d\theta$. Then, using the conditions (40), (41), and the definition of the operator Q_1 , we obtain

$$\|M^{-1}(Q_1 - M)\| \leq l_1 \int_0^1 \theta\|w - x^*\|d\theta \leq \frac{1}{2}l_1 s_3 < 1.$$

So, the operator Q_1 is invertible. Then, from the identity

$$w - x^* = Q_1^{-1}(F(w) - F(x^*)) = Q_1^{-1}(0) = 0,$$

we deduce that $w = x^*$. $\square$

Remark 4. We can certainly set $s_2 = \rho$ in Proposition 2 provided that the conditions (H_1) – (H_3) hold.

6. The Picard Iteration

The contraction mapping principle attributed to Banach is a useful tool in establishing the convergence of iterative methods in abstract spaces [3,5–7,19,23].

Theorem 4 ([17,18,26]). Let $G : X \rightarrow X$ be a contraction operator, with Lipschitz constant $q \in [0, 1)$. Then, G has a fixed point $x^* \in X$, i.e., $G(x^*) = x^*$. Moreover, for each starting point $x_0 \in X$, the method of successive approximations or Picard method $x_{n+1} = G(x_n)$, $n = 0, 1, 2, \dots$ converges to $x^* \in X$.

If the operator G has more than one fixed point, Theorem 4 is not applicable. In this case, the fixed points must be separated. Let G_0 be a non-empty closed subset of X_0 . We also have the following results:

Theorem 5 ([17,18]). Suppose that the operator $G : G_0 \rightarrow G_0$ is a contraction with constant $q \in [0, 1)$. Then, the operator G has a unique fixed point $x^* \in G_0$. Moreover, for each $x_0 \in G_0$, the Picard method, $x_{n+1} = G(x_n)$, $n = 0, 1, 2, \dots$ converges to x^* .

Next, we present convergence of method (5) based on Theorem 5.

Theorem 6. Suppose the following: For fixed natural number k , the following conditions hold for $x, y \in D$. Then, there exist an invertible operator M , such that

$$\|M^{-1}(F'(x) - M)\| < 1, \quad (42)$$

$$\|P(x) - P(x_0)\| \leq q_0 \|x - x_0\|, \text{ for some } q_0 \in [0, 1) \quad (43)$$

$$\|P(y) - P(x)\| \leq q \|y - x\|, \text{ for some } q \in [0, 1) \quad (44)$$

$$\|P(x_0) - x_0\| \leq \frac{r}{1 - q_0}, \quad (45)$$

and $U[x_0, r] \subset D$, where $r \in [0, \frac{1}{q_0})$.

Then, the operator $P : D \rightarrow D$ has a unique fixed point $x^* \in D$. Moreover, the Picard iteration $x_{n+1} = P(x_n)$ is convergent to x^* for each $x_0 \in D$.

Proof. The Picard iteration (5) is well defined by the existence of M^{-1} . Moreover, it follows by (44) that the operator P is a contraction on D with Lipschitz constant q . Furthermore, by the definition of r , condition (43), (45) and the estimate

$$\|P(x) - (x_0)\| \leq \|P(x) - P(x_0)\| + \|P(x_0) - (x_0)\| \leq q_0 r + \|P(x_0) - x_0\| \leq r.$$

Hence, $P : U[x_0, r] \rightarrow U[x_0, r]$ is a contraction with constant $q \in [0, 1)$. The result now follows by Theorem 6 for $G_0 = U[x_0, r]$. $\square$

7. Error Analysis for the Method (4)

We need a version of the Newton–Kantorovich Theorem [1,2,27,28] for Newton’s method (2).

Theorem 7. Let $F : D \rightarrow Y$ be a Fréchet-differentiable operator. Suppose the following:

There exist $x_0 \in D$ and $\eta_1 \geq 0$ such that the linear operator $F'(x_0)$ is invertible. Set

$$\|F'(x_0)^{-1}F(x_0)\| \leq \eta_1;$$

There exists $L > 0$ such that for each $x, y \in D$

$$\|M^{-1}(F'(y) - F'(x))\| \leq L\|y - x\|;$$

$$2L\eta_1 \leq 1,$$

and

$$U[x_0, s] \subset D,$$

where $s = \frac{1 - \sqrt{1 - 2L\eta_1}}{L}$. Then, the Newton method (2) converges to a unique solution $x^* \in U[x_0, s]$.

Set $s_1 = \frac{1}{1 - Ls}$.

Next, we relate the sequence $\{y_n\}$ to the sequence $\{x_n\}$.

Lemma 2. Suppose that the conditions (C_1) – (C_4) and those of Theorem 7 hold. Then, the following error estimate holds for each $n = 0, 1, 2, \dots$

$$\|x_{n+1} - y_{n+1}\| \leq \epsilon_n,$$

where

$$\epsilon_n = s_1 \left[\frac{L}{2} \|x_n - y_n\|^2 + \bar{b}(a^{k+1} + L\|x_n - y_n\| \frac{1 - a^{k+1}}{1 - a}) \|x_{n+1} - x_n\| \right].$$

Proof. The iterates $\{y_n\}$ and $\{x_n\}$ are well defined by Theorems 2 and 7. By subtracting (2) from (4) and taking out $F'(y_n)^{-1}$, we have in turn that

$$\begin{aligned} x_{n+1} - y_{n+1} &= x_n - y_n + F'(y_n)^{-1}F(y_n) - BM^{-1}F(x_n) \\ &= F'(y_n)^{-1}[F'(y_n)(x_n - y_n) + F(y_n) - F'(y_n)BM^{-1}F(x_n)] \\ &= F'(y_n)^{-1}[-(F(x_n) - F(y_n) - F'(y_n)(x_n - y_n)) \\ &\quad + (I - F'(y_n)BM^{-1})F(x_n)] \\ &= F'(y_n)^{-1}[-(F(x_n) - F(y_n) - F'(y_n)(x_n - y_n)) \\ &\quad + (M - F'(y_n)B)M^{-1}F(x_n)] \\ &= -F'(y_n)^{-1}[F(x_n) - F(y_n) - F'(y_n)(x_n - y_n)) \\ &\quad + (M - F'(y_n)B)B^{-1}(x_{n+1} - x_n)] \\ &= -F'(y_n)^{-1}(F(x_n) - F(y_n) - F'(y_n)(x_n - y_n)) \\ &\quad - F'(y_n)^{-1} \left((M - F'(x_n)B) \right. \\ &\quad \left. + (F'(x_n) - F'(y_n))B)B^{-1}(x_{n+1} - x_n) \right). \end{aligned} \quad (46)$$

Note that we have the estimates

$$\|F'(y_n)^{-1}M\| \leq \frac{1}{1 - L\|y_n - x_0\|} \leq \frac{1}{1 - Ls} = s_1,$$

$$\|B^{-1}\| \leq \bar{b},$$

$$\|M^{-1}(M - F'(x_n)B)\| = \|A^{k+1}\| \leq a^{k+1},$$

$$\begin{aligned} & \left\| \int_0^1 M^{-1}(F'(y_n + \theta(x_n - y_n)) - F'(y_n)) d\theta \right\| \\ & \leq \frac{L}{2} \|y_n - x_n\|. \end{aligned}$$

Using these estimates, and summing up, we obtain from (46) in turn that

$$\begin{aligned} \|x_{n+1} - y_{n+1}\| & \leq s_1 \left[\frac{L}{2} \|x_n - y_n\|^2 \right. \\ & \quad \left. + \bar{b}(a^{k+1} + L\|x_n - y_n\| + \frac{1 - a^{k+1}}{1 - a}) \|x_{n+1} - x_n\| \right] \\ & = \epsilon_n. \end{aligned}$$

□

Proposition 3. Let all the conditions of the Lemma 2 hold. Define the polynomial ψ for $h_0 = \frac{Ls_1}{2}$, $h_1 = s_1 \bar{b} L \frac{(1-a^{k+1})\eta}{1-a}$, and $h_2 = s_1 \bar{b} a^{k+1} \eta$ by

$$\psi(t) = h_0 t^2 - (1 - h_1)t + h_2.$$

Suppose in addition that

$$(1 - h_1)^2 - 4h_0h_2 > 0 \text{ and } h_1 < 1. \quad (47)$$

Then, the polynomial ψ has two positive roots. Denote the smallest by h^* . Moreover, for each $n = 0, 1, 2, \dots$

$$\|x_n - y_n\| \leq h^*. \quad (48)$$

Proof. It follows by (47) that the polynomial ψ has two positive roots and $h_2 < h^*$. The estimate (46) for $n = 0$ gives $\|x_1 - y_1\| \leq h_2 \leq h^*$, which is true by the choice of h^* . Suppose that (48) holds for all integers m , which are smaller or equal to n . Then, the application of (46) gives

$$\|x_{m+1} - y_{m+1}\| \leq \epsilon_m \leq h_0(h^*)^2 + h_1h^* + h_2 = h^*.$$

□

8. Numerical Examples

In the following example, consider method (4) for cases $M = I$, which remains independent of both x_0 . Additionally, they are compared with method (4) where $M = F'(x_0)$.

Example 1. The solution sought for the nonlinear system

$$\begin{aligned} f1 &= x - 0.1 \sin x - 0.3 \cos y + 0.4 \\ f2 &= y - 0.2 \cos x + 0.1 \sin y + 0.3 \end{aligned}$$

Let $F = (f1, f2)$. Then, the system becomes

$$F(s) = 0 \text{ for } s = (\theta_1, \theta_2)^T.$$

Then

$$F'((x, y)) = \begin{bmatrix} 1 - 0.1 \cos(x) & 0.3 \sin(y) \\ 0.2 \sin(x) & 0.1 \cos(y) + 1 \end{bmatrix}.$$

Method (2)

$$x_{k+1} = x_k - F'(x_k)^{-1} F(x_k).$$

Method (4), $k = 1, M = I$

$$\begin{aligned} B_1(x) &= I + (I - F'(x)), \\ p_1(x) &= x - (I + (I - F'(x)))F(x), \\ x_{n+1} &= p_1(x_n). \end{aligned} \quad (49)$$

Method (4), $k = 2, M = I$

$$\begin{aligned} B_2(x) &= I + (I - F'(x)) + (I - F'(x))^2, \\ p_2(x) &= x - B_2(x)F(x), \\ x_{n+1} &= p_2(x_n). \end{aligned} \quad (50)$$

Method (4), $k = 3, M = I$

$$\begin{aligned} B_3(x) &= I + (I - F'(x)) + (I - F'(x))^2 + (I - F'(x))^3, \\ p_3(x) &= x - B_3(x)F(x), \\ x_{n+1} &= p_3(x_n). \end{aligned} \quad (51)$$

Method (4), $k = 4, M = I$

$$\begin{aligned} B_4(x) &= I + (I - F'(x)) + (I - F'(x))^2 + (I - F'(x))^3 + (I - F'(x))^4, \\ p_4(x) &= x - B_4(x)F(x), \\ x_{n+1} &= p_4(x_n). \end{aligned} \quad (52)$$

Method (4), $k = 5, M = I$

$$\begin{aligned} B_5(x) &= I + (I - F'(x)) + (I - F'(x))^2 + (I - F'(x))^3 + (I - F'(x))^4 + (I - F'(x))^5, \\ p_5(x) &= x - B_5(x)F(x), \\ x_{n+1} &= p_5(x_n). \end{aligned} \quad (53)$$

Method (4), $k = \overline{1,5}, M = F'(x_0)$

$$\begin{aligned} x_{n+1} &= x_n - BM^{-1}F(x_n), \\ A &= M^{-1}(M - F'(x)), \\ B &= I + \sum_{i=1}^k A^i. \end{aligned} \quad (54)$$

Thus, the comparison shows that the behavior of method (4) is essentially the same as Newton's method (2). However, the iterates of method (4) are cheaper to obtain than Newton's. As observed in Tables 1–4, the number of iterations required for the proposed methods with k ranging from 3 to 5 closely aligns with those of Newton's method.

Table 5 shows the results of calculations to determine the Computational Order of Convergence (COC) and the Approximated Computational Order of Convergence (ACOC) aiming to compare the convergence order of method (4) with the convergence order of Newton's method (2).

Table 1. Number of iterations to achieve tolerance $\varepsilon = 10^{-9}$ with initial guess $x_0 = (1, 1)$ and $\|I - F'(x_0)\| = 0.3129 < 1$.

Method	Iterations	Method	Iterations
(2) Newton	4	(2) Newton	4
(49), $k = 1$	6	(54), $k = 1$	8
(50), $k = 2$	5	(54), $k = 2$	6
(51), $k = 3$	4	(54), $k = 3$	5
(52), $k = 4$	4	(54), $k = 4$	5
(53), $k = 5$	4	(54), $k = 5$	4

Table 2. Number of iterations to achieve tolerance $\varepsilon = 10^{-9}$ with initial guess $x_0 = (0, 0)$ and $\|I - F'(x_0)\| = 0.1414 < 1$.

Method	Iterations	Method	Iterations
(2) Newton	3	(2) Newton	3
(49), $k = 1$	5	(54), $k = 1$	3
(50), $k = 2$	4	(54), $k = 2$	3
(51), $k = 3$	3	(54), $k = 3$	3
(52), $k = 4$	3	(54), $k = 4$	3
(53), $k = 5$	3	(54), $k = 5$	3

Table 3. Number of iterations to achieve tolerance $\varepsilon = 10^{-9}$, where $x_0 = (-15, -15)$ and $\|I - F'(x_0)\| = 0.257 < 1$.

Method	Iterations	Method	Iterations
(2) Newton	5	(2) Newton	5
(49), $k = 1$	7	(54), $k = 1$	9
(50), $k = 2$	5	(54), $k = 2$	7
(51), $k = 3$	5	(54), $k = 3$	6
(52), $k = 4$	5	(54), $k = 4$	6
(53), $k = 5$	5	(54), $k = 5$	5

Table 4. Number of iterations to achieve tolerance $\varepsilon = 10^{-12}$, where $x_0 = (-15, -15)$ and $\|I - F'(x_0)\| = 0.257 < 1$.

Method	Iterations	Method	Iterations
(2) Newton	7	(2) Newton	7
(49), $k = 1$	8	(54), $k = 1$	12
(50), $k = 2$	7	(54), $k = 2$	8
(51), $k = 3$	7	(54), $k = 3$	8
(52), $k = 4$	7	(54), $k = 4$	7
(53), $k = 5$	7	(54), $k = 5$	7

Table 5. Computational Order of Convergence and the Approximated Computational Order of Convergence, where $x_0 = (-15, -15)$, $\varepsilon = 10^{-12}$.

Method	COC	ACOC
(2) Newton	1.8624	1.9697
(49), $k = 1$	0.863	1
(50), $k = 2$	0.2695	1.0438
(51), $k = 3$	1.9714	2.3569
(52), $k = 4$	1.8354	1.9453
(53), $k = 5$	1.8642	1.9661

Table 5. Cont.

Method	COC	ACOC
(54), $k = 1$	0.9065	1.0118
(54), $k = 2$	0.5912	0.999
(54), $k = 3$	0.7321	0.9926
(54), $k = 4$	1.933	2.0151
(54), $k = 5$	1.8679	1.9578

Definition 2. Computational order of convergence of a sequence $\{x_n\}_{n \geq 0}$ is defined by

$$\bar{\rho}_n = \frac{\ln |e_{n+1}/e_n|}{\ln |e_n/e_{n-1}|},$$

where x_{n-1}, x_n, x_{n+1} are three consecutive iterations near the root α and $e_n = x_n - \alpha$ [29].

Definition 3. The approximated computational order of convergence of a sequence $\{x_n\}_{n \geq 0}$ is defined by

$$\hat{\rho}_n = \frac{\ln |\hat{e}_{n+1}/\hat{e}_n|}{\ln |\hat{e}_n/\hat{e}_{n-1}|},$$

where $\hat{e}_n = x_n - x_{n-1}$. x_n, x_{n-1}, x_{n-2} are three consecutive iterates [29].

Table 5 demonstrates that the convergence of the proposed methods closely corresponds with the convergence of Newton's method, particularly for values of k ranging from 4 to 5 with the convergence order closely approximating 2.

Example 2. Let $X = Y = \mathbb{R}^3$ and $D = U[x^*, 1]$. The mapping F is defined on D for $a = (a_1, a_2, a_3)^{tr} \in \mathbb{R}^3$ as

$$F(a) = (a_1, e^{a_2} - 1, \frac{e-1}{2}a_3^2 + a_3)^{tr}.$$

Then, the definition of the derivative according to Fréchet [1,8,11,27,30] is given for the mapping F

$$F'(a) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{a_2} & 0 \\ 0 & 0 & (e-1)a_3 + 1 \end{bmatrix}.$$

The point $x^* = (0, 0, 0)^{tr}$ solves the equation $F(a) = 0$. Moreover, $F'(x^*) = I$. The conditions of the Theorem 3 hold provided that $\ell_0 = e - 1$, $k = 1$, $a = a(t) = \ell_0 t$. Then, we can have $\rho \in (0, 0.2909883534)$.

Example 3. Let $K[0, 1]$ stand for the space of continuous functions mapping the interval $[0, 1]$ into the real number system. Let $B_1 = B_2 = K[0, 1]$ and $D = P[x^*, 1]$ with $x^*(\xi) = 0$. The operator E is defined on $K[0, 1]$ as

$$E(z)(\xi) = z(\xi) - 6 \int_0^1 \xi z(\tau)^3 d\tau.$$

Then, the definition of the derivative according to Fréchet [1,8,11,27,30] is given below for the operator E

$$E'(z(w))(\xi) = w(\xi) - 18 \int_0^1 \xi \tau z(\tau)^2 w(\tau) d\tau$$

for each $w \in K[0, 1]$. Therefore, the conditions are validated, since for $x^* = 0$, $E'(x^*(\xi)) = I$ provided that $\ell_0 = 9$, $a = a(t) = \ell_0 t$, and $k = 1$. Then, we can get $\rho \in (0, 0.05)$.

9. Concluding Remarks

The difficulty of implementing Newton's method is addressed in this article. In particular, the computation of $F'(x_n)^{-1}$ required at each step of Newton's method is avoided with the introduction of method (4) (or method (5)), where the inversion only once of a fixed linear operator is required to implement it. The inverse of the linear operator is exchanged with a finite sum of linear operators depending on F' . Both the semi-local and the local convergence analysis of these methods are asymptotically comparable to Newton's. This is guaranteed by Theorem 2, and Remark 1 for the semi-local and Theorem 3, and Remark 2, for the local analysis, respectively. Moreover, the numerical examples are used to demonstrate that method (4) or method (5) are reliable replacements of Newton's method for all practical purposes. In our future research, we shall consider further extensions of the form

$$x_{n+1} = x_n - Q(x_n)F(x_n).$$

Here, Q is a conscious approximation to the inverse of a linear operator (like B), which may be a divided difference or some other operator, and it is also given by $Q(x) = I + L(x) + L^2(x) + \dots + L^k(x)$, where L is a linear operator [6,9,12–14,20,21,24,25,27,28,30–32].

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