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# Rigidity of Holomorphically Projective Mappings of Kähler Spaces with Finite Complete Geodesics

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**Abstract:** In this work, we consider holomorphically projective mappings of (pseudo-) Kähler spaces. We determine the conditions for finite complete geodesics that must be satisfied for the mappings to be trivial; i.e., these spaces are rigid.

**Keywords:** geodesic; holomorphically projective mappings; Kähler space; rigidity; Riemann tensor; symmetric space

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## 1. Introduction

This work develops some new ideas in the theory of holomorphically projective mappings of Kähler spaces. These questions are connected with the compact and complete geodesics, Kähler spaces, and their holomorphically projective mappings and transformations.

In 1954, Westlake [1] and Yano [2] studied the geodesic mappings of Kähler spaces. They proved that if the structure of the Kähler space is preserved, then the mapping is trivial. This result was generalized by Muto [3], for the case where these structures commute. Mikeš proved that geodesic mappings of Kähler spaces can exist [4–7], eventually onto Kähler spaces (see [8–14]). These Kähler spaces are equidistant (Sinyukov [15,16]); i.e., they admit convergent vector fields (Shirokov [17–19]), which are special concircular vector fields (Yano [20]).

Analytically planar curves and holomorphically projective mappings of Kähler spaces introduced by Otsuki and Tashiro [21] are a natural generalization of geodesics and geodesic mappings. In these mappings, analytically planar curves are mapped onto analytically planar curves. They showed that spaces with a constant holomorphic curvature of holomorphically projective mapping have properties similar to those of spaces with a constant curvature with respect to geodesic mappings. An overview of the results up to 1963 on holomorphically projective mappings is available in Beklemishev [22], Yano, and Bochner [23,24], for example.

Mikeš generalized these results for holomorphically projective mappings in different directions [25,26]; some of these results are included in the fifth (last) chapter of Sinyukov's monograph [16]. These results can be found in [10] and in [4,8,11–13]. More results can be found in Mikeš dissertation [4], in particular, concerning $K_n[B]$, see Section 4. These general results were published in [10].

Other problems and ideas in the theory of holomorphically projective mappings were developed by Aminova [27–30] and others. The complex projective space $(\mathbb{C}P(n), g_{\text{Fubini-Study}})$ admits global non-trivial holomorphically projective mappings and transformations with maximal parameters (see [31,32]). Previously, locally, for spaces with constant holomorphic curvature, the same was proved in [25,26]. These calculations were mostly performed in a complex form. The same is true in other works, e.g., [32]. In this work, Equation (3) was not attributed to Mikeš (see [4,10]).

Holomorphically projective mappings of hyperbolic and parabolic Kähler spaces have been dealt with in Prvanović [33], Kurbatova [14,34], and Shiha [35].

Holomorphically projective mappings views have been generalized in many ways. In 1962, A.Z. Petrov [36] studied quasi-geodesic mappings, where he showed that it is possible to simulate physical processes and electromagnetic fields. Similar results were presented in the paper of C.-L. Bejan and O. Kowalski [37]. The abovementioned mapping generalized the $F$-planar mappings of Mikeš and Sinyukov [38]. The almost geodesic mappings $\pi_2$ are also a direct generalization of holomorphically projective mappings (see [16] and [10–13,39]). In a 2019 paper [40] by A. Kozak and A. Borowiec, the authors studied a new physical interpretation of almost geodesic mappings that are special transformations, which genuinely preserve geodesics in space and time.

The problems connected with these topics have been considered in many monographs and reviews, such as [41–48].

Many authors have dealt with **rigidity problems**, i.e., when the holomorphically projective mappings will be affine (trivial). We follow these works on similar problems of rigidity, which were studied for motions (Killing vector fields) and their generalization in compact or complete Riemann and Kähler spaces (see the monographs by Yano and Bochner [23,24]).

Using Bochner's methods (see Stepanov [49]), Tachibana and Ishihara [50,51], Hasegawa and Yamauchi [52], and Akbar-Zadeh and Couty [53,54] also discovered new results. Later, Sinyukov [55] and Mikeš [10,56] also continued this research. Due to the method of Švec [57], even more general results were found [58].

In 1961, Tachibana and Ishihara [59] proved that Ricci symmetric (non-Einstein) spaces do not admit nontrivial analytical holomorphically projective transformations. Then, in 1979, Mikeš proved that these spaces also do not admit nontrivial mappings, while global requirements are not assumed, [4,10]. See also Bácsó and Ilosvay [60].

Sakaguchi [61] used Sinyukov's methods (see [15]) and proved that symmetric and recurrent Kähler spaces of non-constant holomorphically projective curvature do not admit non-trivial holomorphically projective mappings. Domashev and Mikeš [25] generalized Sakaguchi's results for (pseudo-) Kähler spaces.

The main results of our study are Theorems 2 and 3. They clearly state that in order for the mapping to be rigid the space does not have to be complete. It suffices that there exist a finite number of geodesics and their images that are complete. In other words, the space is uniquely defined by the given geodetics, which are the supporting skeleton (reinforcement) of the space.

## 2. Kähler Spaces

*Kähler space* $K_n$ is an $n$ dimensional (pseudo-) Riemannian space in which, along with the metric tensor $g$, an affine structure $F$ is defined that satisfies the relations $F^2 = -Id$, $g(X, FX) = 0$, and $\nabla F = 0$, where $\nabla$ is the Levi–Civita connection, and $X$ is any tangent vector on $K_n$. Necessarily, the spaces $K_n$ are of an even dimension, i.e., $n = 2m$, and $n \geq 4$.

In local coordinates $x \equiv (x^1, x^2, \dots, x^n)$, components $g_{ij}(x)$ and $F_i^h(x)$ of $g$ and $F$ satisfy the relations

$$F_\alpha^h F_i^\alpha = -\delta_i^h; \qquad F_{(i}^\alpha g_{j)\alpha} = 0; \qquad F_{i,j}^h = 0.$$

Here and in what follows, "," denotes a covariant derivative on $K_n$ and the round brackets denote the symmetrization of indices. The structure $F$ is called a *complex structure*.

The spaces $K_n$ were first considered by Shirokov [18]. Independently, in complex form, these spaces were studied by Kähler [62]. In the available literature, these spaces are also called *Kählerian*. We present the notation that is used in Mikeš's dissertations [4,8] and in many articles, for example, [10–14,16,22,23].

In the Kähler spaces $K_n$, we introduce the operation of the conjugation of indices as follows:

$$A^{\cdots}_{\cdots\,\bar{i}\,\cdots} \equiv A^{\cdots}_{\cdots\,\alpha\,\cdots}\, F^{\alpha}_{i}\,; \qquad A^{\cdots\,\bar{j}\,\cdots}_{\cdots} \equiv A^{\cdots\,\alpha\,\cdots}_{\cdots}\, F^{j}_{\alpha}\,.$$

According to the definition of a tensor $F$, this operation has the following properties:

$$A_{\bar{\bar{i}}} = -A_i; \quad B^{\bar{\bar{i}}} = -B^i; \quad A_{\bar{\alpha}}B^{\alpha} = A_{\alpha}B^{\bar{\alpha}}; \quad A_{\bar{\alpha}}B^{\bar{\alpha}} = -A_{\alpha}B^{\alpha}; \quad (A_{\bar{i}})_{,j} = A_{\bar{i},j}; \quad (B^{\bar{i}})_{,j} = B^{\bar{i}}_{,j}.$$

For the Kronecker symbol, metric, and and its inverse tensors it holds that

$$\delta^{\bar{h}}_{i} = \delta^{h}_{\bar{i}} = F^{h}_{i}; \quad g_{\bar{i}j} + g_{i\bar{j}} = 0; \quad g_{\bar{i}\,\bar{j}} = g_{ij}; \quad g^{\bar{i}j} + g^{i\bar{j}} = 0; \quad g^{\bar{i}\,\bar{j}} = g^{ij}.$$

For the Riemann and Ricci tensors, $R^h_{ijk} = \partial_j\Gamma^h_{ik} - \partial_k\Gamma^h_{ij} + \Gamma^h_{\alpha j}\Gamma^{\alpha}_{ik} - \Gamma^h_{\alpha k}\Gamma^{\alpha}_{ij}$, $\partial_j = \partial/\partial x^j$, and $R_{ik} = R^{\alpha}_{i\alpha k}$, the following formulas hold:

$$R_{hi\bar{j}\,\bar{k}} = R_{hijk} \equiv g_{h\alpha}R^{\alpha}_{ijk}; \quad R_{\bar{i}\,\bar{j}} = R_{ij}; \quad R^{\bar{\alpha}}_{\alpha jk} = 2R_{\bar{j}k}.$$

In the Kähler spaces $K_n$, we can consider the *holomorphically projective curvature tensor*

$$P^h_{ijk} \equiv R^h_{ijk} + \frac{1}{n+2}(\delta^h_k R_{ji} - \delta^h_j R_{ki} + \delta^h_{\bar{k}} R_{\bar{j}i} - \delta^h_{\bar{j}} R_{\bar{k}i} - 2\delta^h_{\bar{i}} R_{\bar{j}k}).$$

When specific maps $f$ of spaces are considered, say, $K_n \xrightarrow{f} \overline{K}_n$, both spaces are assigned to the coordinate system $x$, in general, with respect to these mappings. In this coordinate system, the corresponding points $x \in K_n$ and $f(x) \in \overline{K}_n$ have the same coordinates $x \equiv (x^1, x^2, \ldots, x^n)$.

In this case, we denote the corresponding geometric objects in $\overline{A}_n$ with a bar; for instance, $\overline{R}^h_{ijk}$ and $\overline{R}_{ij}$ are the Riemannian and Ricci tensors.

## 3. General Questions Concerning Holomorphically Projective Mappings of Kähler Spaces

Natural generalizations of geodesic mappings are the holomorphically projective mappings (HP-mappings) of Kähler spaces $K_n$. Naturally, similar problems appear within the HP-mappings theory as in the geodesic mappings theory. Interestingly, numerous findings and results valid for geodesic mappings seamlessly extend to HP-mappings as well, indicating a high degree of compatibility between the two. Note that HP-mappings were considered, as a rule, under the condition of the preservation of the structure. It turned out that in the case of HP-mappings, the structure is necessarily preserved.

The works by Tashiro [63], Ishihara [50], Otsuki and Tashiro [21], Domashev and Mikeš [25], and Mikeš [6,7,26,64,65] are devoted to general questions concerning the theory of holomorphically projective mappings of the Kähler spaces $K_n$.

Problems related to integrating the fundamental equations of HPM theory and other related questions have been examined, for example, in the works by Aminova and Kalinin [27–30]. Unfortunately, many of the questions that the authors present are not their own originally.

The fundamentals of the theory of holomorphically projective mappings can be found in [22] by Beklemishev, [23,24] by Yano, [16] by Sinyukov, and [10–13] by Mikeš. In the monograph ([16], fifth chapter), Sinyukov presented classical results of holomorphically projective mappings, and results were obtained Mikeš and Domashev [25] and Mikeš [4,26].

### *Definitions and the Basic Equations*

Below, the terms related to holomorphically projective mappings and transformations are given in detail, e.g., [10–13,16,21–23].

An *analytically planar curve* $\gamma$ of the Kähler space $K_n$ is a curve defined by the equations $x = x(t)$, whose tangent vector $\lambda = d\gamma(t)/dt$, being translated, remains in the area element formed by the tangent vector $\lambda$ and its conjugate $F\lambda$; i.e., the conditions $\nabla_t \lambda = \rho_1(t)\lambda + \rho_2(t)F\lambda$, where $\rho_1, \rho_2$ are functions of the argument $t$, are fulfilled [21].

If $\rho_2(t) \equiv 0$, then $\ell$ is a *geodesic*. We note that if the tangent vector $\lambda$ of an analytically planar curve $\gamma$ is isotropic (null-vector) in one of its points, then it is isotropic in all its points $\gamma$, which is analogous to geodesics. The physical meaning of these curves is given, for example, in [66,67].

The diffeomorphism of $K_n$ onto $\overline{K}_n$ is a *holomorphically projective mapping* if it transforms all the analytically planar curves of $K_n$ onto analytically planar curves of $\overline{K}_n$.

Under the HP-mapping, the structure of the spaces $K_n$ and $\overline{K}_n$ is preserved; i.e., in the coordinate system $x$, in general, with respect to the mapping, the conditions $\overline{F}_i^h(x) = F_i^h(x)$ are satisfied. To be more precise, $\overline{F}_i^h(x) = \pm F_i^h(x)$ for $K_n$, since the structure in $K_n$ is defined with an accuracy within the sign (see [14]).

The holomorphically projective mappings were introduced by Otsuki and Tashiro [21] for $K_n$ under the a priori assumption that the structure was preserved. Note that HP-mappings are special *F-planar* mappings introduced by Mikeš and Sinyukov [38]. Questions about the preservation of the structure for the above mappings are studied in detail in [14,38,68].

The necessary and sufficient conditions for the holomorphically projective mappings of $K_n$ onto $\overline{K}_n$ fulfill the following conditions in the general (with respect to the mapping) coordinate system (Tashiro [63]),

$$\overline{\Gamma}_{ij}^h(x) = \Gamma_{ij}^h(x) + \psi_i \delta_j^h + \psi_j \delta_i^h - \psi_{\bar{\imath}} \delta_{\bar{j}}^h - \psi_{\bar{j}} \delta_{\bar{\imath}}^h, \tag{1}$$

where $\psi_i$ is a vector, and $\Gamma_{ij}^h$ and $\overline{\Gamma}_{ij}^h$ are the Christoffel symbols of $K_n$ and $\overline{K}_n$. Relation (1) is equivalent to the equation

$$\overline{g}_{ij,k} = 2\,\psi_k \overline{g}_{ij} + \psi_i \overline{g}_{jk} + \psi_j \overline{g}_{ik} + \psi_{\bar{\imath}} \overline{g}_{\bar{j}k} + \psi_{\bar{j}} \overline{g}_{\bar{\imath}k}, \tag{2}$$

where $\overline{g}_{ij}$ are the components of metric $\overline{g}$ on $\overline{K}_n$. When $\psi_i \not\equiv 0$, we say that the holomorphically projective mapping is *nontrivial* or *affine*. After contracting (1), it is valid that $\psi_i$ is necessarily a gradient; moreover,

$$\psi_i = \partial_i \Psi, \quad \text{where } \Psi = \frac{1}{n+2} \ln \sqrt{\left| \frac{\det \overline{g}}{\det g} \right|}.$$

The Riemannian and Ricci tensors $K_n$ and $\overline{K}_n$ are connected by the conditions

$$\overline{R}_{ijk}^h = R_{ijk}^h + \delta_k^h \psi_{ij} - \delta_j^h \psi_{ik} + \delta_{\bar{k}}^h \psi_{i\bar{j}} - \delta_{\bar{j}}^h \psi_{i\bar{k}} + 2\delta_{\bar{\imath}}^h \psi_{\bar{j}k}; \qquad \overline{R}_{ij} = R_{ij} - (n+2)\,\psi_{ij},$$

where $\psi_{ij} \equiv \psi_{i,j} - \psi_i \psi_j + \psi_{\bar{\imath}} \psi_{\bar{j}}$ is a symmetric tensor, for which $\psi_{ij} = \psi_{\bar{\imath}\bar{j}}$.

The holomorphically projective curvature tensor $P_{ijk}^h$ is invariant relative to the holomorphically projective mapping. Its identical vanishing is necessary and sufficient for $K_n$ to be a space of constant holomorphic curvature and for these spaces to admit holomorphically projective mapping onto a flat space (Tashiro [63], Ishihara [50]). It has been proven that non-trivial holomorphically projective mapping can be established between any $K_n$ of constant holomorphic curvature [14].

Mikeš [26] has found that the Kähler space $K_n$ admits a holomorphically projective mapping if and only if the system of the following equations,

$$\text{(a)}\ a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik} + \lambda_{\bar{\imath}} g_{\bar{j}k} + \lambda_{\bar{j}} g_{\bar{\imath}k}; \quad \text{(b)}\ n\lambda_{i,j} = \mu g_{ij} + a_{i\alpha} R_j^\alpha - a_{\alpha\beta} R^{\alpha}{}_{ij}{}^{\beta}; \quad \text{(c)}\ \mu_{,i} = 2\lambda_\alpha R_i^\alpha, \tag{3}$$

has a solution for the unknown tensors $a_{ij}\ (= a_{ji} = a_{\bar{\imath}\bar{j}},\ |a_{ij}| \neq 0)$, $\lambda_i$ and $\mu$. The solutions of (2) and (3) are connected by the relations $a_{ij} = e^{2\psi} \overline{g}^{\alpha\beta} g_{\alpha i} g_{\beta j}$, $\lambda_i = -e^{2\psi} \overline{g}^{\alpha\beta} g_{\alpha i} \psi_\beta$. Evidently, $\lambda_i = \partial_i (2\, a_{\alpha\beta} g^{\alpha\beta})$ is the gradient, and the mapping is trivial if and only if $\lambda_i = 0$. For vector

$\lambda_i$, it holds that $\lambda_{\bar{i},\bar{j}} = \lambda_{i,j} = \lambda_{j,i}$; therefore, $\lambda_{\bar{i},j} + \lambda_{\bar{j},i} = 0$. From this, it follows that the vector $\lambda_{\bar{i}}$ is the Killing vector.

Condition (3)a is necessary and sufficient for the existence of the holomorphically projective mapping $K_n$; this result was obtained by Domashev and Mikeš [25].

Equation (3) forms a linear system of the Cauchy type with respect to the components of the unknown tensors $a_{ij}$, $\lambda_i$, and $\mu$. Consequently, the general solution of this system depends on $r_{hpm} \leq (n/2+1)^2$ parameters [25]. For $r_{hpm} > 2$, Equations (4) and (5) hold; see [10–13,64] and [32].

The solution of Equation (3) in $K_n$ reduces to the study of the integrability conditions for (3) and their differential continuations, which, in turn, constitute a system of linear algebraic equations for the unknows $a_{ij}$, $\lambda_i$, and $\mu$. Thus, we can determine whether the given space $K_n$ admits holomorphically projective mapping, and if it does, then with what arbitrariness.

Holomorphically projective transformations of Kähler spaces are closely related to HP-mappings (see [50,51,59,69,70]). It is obvious that $K_n$, in which NHPT exist, admits NHPM, and conversely, there are no NHPT in the spaces $K_n$ that do not admit NHPM.

Mikeš [71] obtained the inequality $r_{hpt} \leq r_{hpm} + r^*_m$, where $r_{hpt}$ is the order of the complete group HPT, and $r^*_m$ is the order of the complete group of motions that preserves the analytic planar curves. The spaces in which conditions $h_{ij,k} = \psi_i g_{jk} + \psi_j g_{ik} + \psi_{\bar{i}} g_{\bar{j}k} + \psi_{\bar{j}} g_{\bar{i}k}$ are fulfilled necessarily admit HPT and, for $B \neq 0$, NHPT. A more detailed investigation of these regularities was carried out in [71].

Yamaguchi [72] studied a *K-torse-forming vector* $\xi$, for which $\xi^h_{,i} = a\delta^h_i + bF^h_i + \alpha_i \xi^h + \beta_i \xi^\alpha F^h_\alpha$. Esenov's works [73,74] are devoted to the study of $K_n$ in which there exist vector fields of this kind. He showed that $K$-torse-forming vector fields were HPM-invariant. In his works, he studied $K_n$ in which the conditions $\lambda_{i,j} = a g_{ij} + c(\lambda_i \lambda_j - e\lambda_{\bar{i}} \lambda_{\bar{j}})$, where $a$, $c$ are invariants, were satisfied.

These spaces admit NHPM. The metric of the holomorphically projectively corresponding spaces $K_n$ that contain $K$-concircular fields has been found in explicit form. These fields exist in spaces of constant holomorphic curvature.

## 4. Holomorphically Projective Mappings of the Spaces $K_n[B]$

We denote the Kähler space $K_n$ by $K_n[B]$ if it admits a holomorphically projective mapping under which the relations (for details, see Mikeš' dissertation [4], also, see [10,11])

$$\text{(a)}\ a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik} + \lambda_{\bar{i}} g_{\bar{j}k} + \lambda_{\bar{j}} g_{\bar{i}k}; \quad \text{(b)}\ \lambda_{i,j} = \mu g_{ij} + B\, a_{ij} \tag{4}$$

are satisfied, where $a_{ij}$, $(= a_{ji} = a_{\bar{i}\bar{j}},\ |a_{ij}| \neq 0)$, $\lambda_i$ $(\not\equiv 0)$, $\mu$, $B$ are tensors, while $B$ is uniquely determined by the space $K_n$. When $B$ is a constant, then $\mu_{,i} = 2B\lambda_i$, and when $B \equiv 0$, then $\mu$ is a constant. Relations (4) are equivalent to relations (2), and

$$\psi_{ij} = \overline{B}\, \overline{g}_{ij} - B\, g_{ij}. \tag{5}$$

These conditions are fulfilled, in particular, under the holomorphically projective mappings of spaces of a constant holomorphic curvature [14] and for HP-mappings between Einstein spaces, while $B = -\frac{R}{n(n+2)}$ and $\overline{B} = -\frac{\overline{R}}{n(n+2)}$, where $R$ and $\overline{R}$ are scalar curvatures of $K_n$ and $\overline{K}_n$, respectively.

Spaces in which there are $K$-concircular fields are spaces $K_n[B]$. In the spaces $K_n[0]$ and $K_n[B]$, $B \not\equiv$ const, fields of this kind necessarily exist. The spaces $K_n[B]$ admit NHPM only on $\overline{K}_n[\overline{B}]$, with $B$ and $\overline{B}$ being simultaneously constant or nonconstant. The spaces $K_n[B]$, $B =$ const, admit holomorphically projective transformation (nontrivial for $B \neq 0$).

Under the holomorphically projective mapping of $K_n[B]$ onto $\overline{K}_n[\overline{B}]$, the tensors $\overset{*}{Z}{}^h_{ijk}$ and $\overset{*}{Z}_{ij}$ are invariant, where

$$\overset{*}{Z}{}^h_{ijk} \equiv R^h_{ijk} - B\Big(\delta^h_k g_{ij} - \delta^h_j g_{ik} + \delta^h_{\bar{k}} g_{i\bar{j}} - \delta^h_{\bar{j}} g_{i\bar{k}} + 2\delta^h_{\bar{i}} g_{j\bar{k}}\Big); \qquad \overset{*}{Z}_{ij} \equiv \overset{*}{Z}{}^\alpha_{ij\alpha}.$$

The set of solutions of system (4) forms, for $B = \text{const}$, a special Jordan algebra relative to the multiplication operation (see [65,75]):

$$\left(\overset{1}{a},\overset{1}{\lambda},\overset{1}{\mu}\right) \times \left(\overset{2}{a},\overset{2}{\lambda},\overset{2}{\mu}\right) = \left(\overset{3}{a},\overset{3}{\lambda},\overset{3}{\mu}\right),$$

with

$$2\overset{3}{a}_{ij} = B\,\overset{1}{a}{}^{s}_{\cdot(i}\,\overset{2}{a}_{j)s} - \overset{1}{\lambda}_{(i}\,\overset{2}{\lambda}_{j)} - \overset{1}{\lambda}_{(\bar{i}}\,\overset{2}{\lambda}_{\bar{j})}\,; \quad 2\overset{3}{\lambda}_i = B\,(\overset{1}{\lambda}{}^{\alpha}\overset{2}{a}_{i\alpha} + \overset{2}{\lambda}{}^{\alpha}\overset{1}{a}_{i\alpha}) - \overset{1}{\mu}\overset{2}{\lambda}_i - \overset{2}{\mu}\overset{1}{\lambda}_i\,; \quad \overset{3}{\mu} = B\overset{1}{\lambda}{}^{\alpha}\overset{2}{\lambda}_{\alpha} - \overset{1}{\mu}\,\overset{2}{\mu}\,.$$

A similar multiplication operation for solutions in $V_n(B)$ was obtained by Mikeš and Shandra [76].

It was established [75] that every solution $a_{ij}$ of (4) in $K_n[B]$, $B = \text{const} \neq 0$, is associated with a covariantly constant field $A_{ab}$ in the Riemannian space $\overline{V}_{n+2}$, whose metric tensor has the structure

$$G_{ab} = \frac{1}{B}\,e^{2Bx^0}\begin{pmatrix} -B & 0 & 0 \\ 0 & g_{ij} - B\tau_i\tau_j & -B\tau_i \\ 0 & -B\tau_j & -B \end{pmatrix},$$

where $g_{ij}(x^1,\ldots,x^n)$ is the metric tensor of $K_n[B]$, $B = \text{const} \neq 0$, and $\tau_i(x^1,\ldots,x^n)$ is a covector potential; i.e., $F_{ij} = \partial_{[j}\tau_{i]}$ (the form $F_{ij} \equiv g_{i\alpha}F^{\alpha}_{j}$ is exact), $a,\ b = 0,1,\ldots,n,n+1$, with

$$A_{ab} = \begin{pmatrix} \mu & \lambda_i & 0 \\ \lambda_j & a_{ij} + \tau_{(i}F^{\alpha}_{j)}\lambda_\alpha + \mu\tau_i\tau_j & \lambda_\alpha F^{\alpha}_{i} - \mu\tau_i \\ 0 & \lambda_\alpha F^{\alpha}_{j} - \mu\tau_j & \mu \end{pmatrix}.$$

*Holomorphically Projective Mappings of T-Quasi-Semisymmetric Spaces*

The following terms and results, unless otherwise stated, were introduced in Mikeš's dissertation [4] and publications [10–13].

By means of $\overset{*}{\mathcal{Z}}{}^{h}_{ijk}$, we introduce into consideration the operation $\langle\langle lm\rangle\rangle$ as follows:

$$T^{h_1\ldots h_p}_{i_1\ldots i_q\langle\langle lm\rangle\rangle} \equiv \sum_{s=1}^{q} T^{h_1\ldots h_p}_{i_1\ldots i_{s-1}\alpha i_{s+1}\ldots i_q}\overset{*}{Z}{}^{\alpha}_{i_s lm} - T^{h_1\ldots h_{s-1}\alpha h_{s+1}\ldots h_p}_{i_1\ldots i_q}\ \overset{*}{Z}{}^{h_s}_{\alpha lm},$$

where $T$ is a tensor of the type $\binom{p}{q}$. When $B = 0$, $T_{\langle\langle lm\rangle\rangle} = T_{,[lm]}$.

For tensors $u$ and $v$, this operation possesses the properties

$$(u \pm v)_{\langle\langle lm\rangle\rangle} = u_{\langle\langle lm\rangle\rangle} \pm v_{\langle\langle lm\rangle\rangle};\ (uv)_{\langle\langle lm\rangle\rangle} = u_{\langle\langle lm\rangle\rangle}v + uv_{\langle\langle lm\rangle\rangle};\ g_{ij\langle\langle lm\rangle\rangle} = 0;\ g^{ij}_{\langle\langle lm\rangle\rangle} = 0;\ \delta^{i}_{j\langle\langle lm\rangle\rangle} = 0.$$

The analog of the Walker identities [77] is valid:

$$R_{hijk\langle\langle lm\rangle\rangle} + R_{jklm\langle\langle hi\rangle\rangle} + R_{lmhi\langle\langle jk\rangle\rangle} = 0.$$

We say [4,8,10] that the Kähler space $K_n$ is *T-quasi-semisymmetric* ($TPs_n$[B]) if the condition $T_{\langle\langle lm\rangle\rangle} = 0$ is fulfilled in it. Many results regarding HP-mappings of these spaces can be found, for example, in [4,8,10,78–80]. Here, it was proved that HP-mappings of these spaces fulfill Equation (4). Spaces for which $R^{h}_{ijk\langle\langle lm\rangle\rangle} = 0$ and $R_{ij\langle\langle lm\rangle\rangle} = 0$ (see [4,8,78]) were studied by Luczyszyn and Olszak, respectively [81–83].

In the works by Sinyukov, Sinyukova [55], and Mikeš [10,56], a series of results for global geodesic mappings of compact semisymmetric and Ricci-semisymmetric Kähler manifolds with additional conditions was obtained. Haddad proved that the four-dimensional Einsteinian $K_n$ spaces do not admit NHPM onto the Einsteinian spaces of nonconstant holomorphic curvature and do not admit nontrivial holomorphically projective transformations. The investigation of the NHPM of complete Einsteinian $K_n$ was carried out in [54] by Akbar-Zadeh.

## 5. Rigidity of the Kähler Spaces' Respective Holomorphically Projective Mappings

### *5.1. Spaces That Do Not Admit Nontrivial HPM Locally*

Many authors isolated Kähler spaces that do not admit either nontrivial holomorphically projective mappings (NHPM) or nontrivial holomorphically projective transformations (NHPT).

Note that the Kähler spaces $K_n$, which do not admit NHPM, do not admit NHPT either, as well as nontrivial geodesic mappings or nontrivial projective transformations. In these spaces, there are no nonconstant concircular and $K$-concircular vector fields. In this section, this is not specifically stipulated.

In 1974, Sakaguchi [61] proved that proper Kähler symmetric spaces $K_n$ of nonconstant holomorphic curvature do not admit NHPM. For symmetric $K_n$ with a metric of arbitrary signature, Sakaguchi's result was proved by Domashev and Mikeš [25] (see also [10–13,16]). In [8,10], Mikeš indicated more general conditions for recurence under which $K_n$ does not admit NHPM. In particular, recurrent, $m$-recurrent, two-symmetric, and generalized recurrent $D_n^2$ Kähler spaces $K_n^{\pm}$ do not admit NHPM.

The abovementioned results for holomorphically projective mappings of semisymmetric and generalized recurrent manifolds with affine connection were generalized in papers [10,84–86] by al Lamy, Mikeš, Škodová, etc.

### *5.2. Holomorphically Complete Manifolds $K_n[B]$*

I. Hasegawa and K. Yamauchi in [52] proved that an infinitesimal holomorphically projective transformation has infinitesimal isometry on a compact classic Kähler manifold $K_n$ with non-positive constant scalar curvatur. Additionally, they proved that a compact classical Kähler manifold with constant scalar curvature is holomorphically isometric to a complex projective space with the Fubini–Study metric (i.e., manifold with constant holomorphic curvature), provided $K_n$ admits a non-isometric infinitesimal holomorphically projective transformation.

The investigation of the holomorphically projective mappings of the complete Einstein Kähler manifold $K_n$ was carried out by H. Akbar-Zadeh and R. Couty in [53,54,87].

We prove the following theorem ([13], p. 502).

**Theorem 1.** *Let a Kähler manifold $K_n[B]$, $B = \text{const}$, admit a holomorphically projective mapping $f$ onto a complete manifold $\overline{K}_n$.*

1. *If $K_n[B]$ has an indefinite metric, then $f$ is affine.*
2. *If $B \geq 0$, then $f$ is affine.*

Please note that the proof presented there is not correct. Below, we prove more general facts from which this Theorem follows.

### *5.3. Holomorphically Projective Mappings and Fundamental Functions along Geodesics*

Let us suppose that $f\colon K_n \to \overline{K}_n$ is a holomorphically projective mapping and Equation (4) holds with $\psi_i = \partial_i\Psi$; $B$ and $\overline{B}$ are constants. Let $\gamma(s)$ be a geodesic on $K_n$ and a corresponding analytically planar curve $\overline{\gamma}(\tau(s))$ on $\overline{K}_n$ with natural parameter $s$ and with canonical parameter $\tau$, respectively. Assume $\dot{\tau} = d\tau(s)/ds > 0$ for the parameter transformation $\tau = \tau(s)$.

Because $g$ and $e^{-4\Psi}\,\overline{g}$ are first integrals of geodesics, the following holds:

$$g_{ij}\dot{\gamma}^i\dot{\gamma}^j = \varepsilon = \pm 1, 0 \;\text{ and }\; \overline{g}_{ij}\dot{\gamma}^i\dot{\gamma}^j = c\,e^{4\Psi(t)},\; c = \text{const}. \tag{6}$$

The first equality is generally known, and the second follows from the contraction of (2) with $\dot{\gamma}^i\dot{\gamma}^j\dot{\gamma}^k$.

By differentiating $\gamma(s) = \overline{\gamma}(\tau(s))$ with respect to parameter $s$, we obtain

$$\dot{\gamma}(s) = \overset{\circ}{\overline{\gamma}}(\tau(s)) \cdot \dot{\tau}(s), \quad \text{where} \quad \overset{\circ}{\overline{\gamma}} = \frac{d\overline{\gamma}(\tau)}{d\tau}, \tag{7}$$

and, naturally, we suppose that $\dot{\tau}(s) > 0$.

Since $\tau$ is canonical of $\overline{\gamma}$ it holds that $\overline{g}(\overset{\circ}{\overline{\gamma}}, \overset{\circ}{\overline{\gamma}}) = \overline{c} \quad (= \text{ const})$, and from this, it follows that $\overline{g}(\dot{\gamma}, \dot{\gamma}) = \overline{c} \cdot \dot{\tau}^2$. Then, from (7), in the case where $c \neq 0$ (and $\overline{c} \neq 0$), the following holds:

$$\dot{\tau}(s) = \widetilde{c} \cdot e^{2\Psi}, \quad \widetilde{c} > 0; \tag{8}$$

i.e., $\overline{\gamma}$ is a non-isotropic analytical planar curve on $\overline{K}_n$. In the case where $c = 0$ (and $c \neq 0$), this formula may not apply.

Along the geodesic $\gamma(s)$, we put $\Psi(s) = \Psi(\gamma(s))$, and from this, $\dot{\Psi}(s) = \psi_\alpha \dot{\gamma}^\alpha$. For the tensor $\psi_{ij}$ ($\equiv \psi_{i,j} - \psi_i \psi_j + \psi_{\bar{i}} \psi_{\bar{j}}$), the equality $\psi_{ij} = \psi_{ji} = \psi_{\bar{i}\,\bar{j}}$ holds. It follows from this that tensor $\psi_{\bar{i}\,j}$ is skew, and $\psi_{\bar{\alpha}\,\beta} \dot{\gamma}^\alpha \dot{\gamma}^\beta = 0$. From the definition of $\psi_{ij}$, evidently,

$$\psi_{\bar{i},j} = \psi_{\bar{i}} \psi_j + \psi_i \psi_{\bar{j}} + \psi_{\bar{i}\,j}\,.$$

By differentiating the expression $\psi_{\bar{\alpha}}\, \dot{\gamma}^\alpha$ with respect to $s$, from the above, we make sure that $(\psi_{\bar{\alpha}} \dot{\gamma}^\alpha)^{\cdot} = 2\dot{\Psi} \cdot (\psi_{\bar{\alpha}} \dot{\gamma}^\alpha)$, and, after integrating, it is obvious that

$$\psi_{\bar{\alpha}} \dot{\gamma}^\alpha = \chi \cdot e^{2\Psi}, \quad \text{where } \chi \text{ is constant.}$$

Next, we study the holomorphically projective mapping where the condition (4) is valid, i.e., $\psi_{ij} = \overline{B}\,\overline{g}_{ij} - B\,g_{ij}$, where $B$ and $\overline{B}$ are constants. If this mapping is non-trivial, then the spaces $K_n$ and $\overline{K}_n$ will be $K_n[B]$ and $\overline{K}_n[\overline{B}]$, respectively.

We can write the condition (4) in expanded form

$$\psi_{i,j} = \psi_i \psi_j - \psi_{\bar{i}} \psi_{\bar{j}} + \overline{B}\,\overline{g}_{ij} - B\,g_{ij}\,. \tag{9}$$

We calculate $\ddot{\Psi}(s)$ according to geodesics $\gamma(s)$: $\ddot{\Psi}(s) = (\dot{\Psi})^{\cdot} = (\psi_\alpha \dot{\gamma}^\alpha)^{\cdot} = \psi_{\alpha,\beta} \dot{\gamma}^\alpha \dot{\gamma}^\beta$, and after using (9), we obtain

$$\ddot{\Psi} = (\dot{\Psi})^2 + b \cdot e^{4\,\Psi} - a, \tag{10}$$

where $a = \varepsilon\, B$, and $b = c\,\overline{B} - \chi^2$.

We substitute $q = e^{-2\Psi(s)}$; then, Equation (9) is equivalent to

$$2q\,\ddot{q} = \dot{q}^2 - 4\,b + 4a\,q^2\,. \tag{11}$$

The derivative of (11) gives the following equation $\dddot{q} = 4\,a\,\,\dot{q}$, which has a solution

$$\begin{array}{lll} \text{(a)} & q = c_0 + c_1\,s + c_2\,s^2, & \text{if } a = 0, \\ \text{(b)} & q = c_0 + c_1\,\cosh(\alpha\,s) + c_2\,\sinh(\alpha\,s), & \text{if } a > 0, \\ \text{(c)} & q = c_0 + c_1\,\cos(\alpha\,s) \;+\; c_2\,\sin(\alpha\,s), & \text{if } a < 0, \end{array} \tag{12}$$

where $\alpha = 2\sqrt{|a|}$, and $c_0, c_1, c_2$ are constants. Since the function $q$ must satisfy Equation (11), the coefficients $c_i$ are tied to each other.

We analyze the obtained results in terms of the compactness and completeness of the studied geodesics $\gamma$ and their image $\overline{\gamma} = f(\gamma)$.

**Lemma 1.** *If geodesic $\gamma$ on $K_n$ is compact, then, for $a \equiv B \cdot g(\dot{\gamma}, \dot{\gamma}) \geq 0$, the function $\Psi(s)$ is constant.*

**Lemma 2.** *If geodesic $\gamma$ on $K_n$ and its non-isotropic images $\overline{\gamma}$ on $\overline{K}_n$ are complete, then, for $a \equiv B \cdot g(\dot{\gamma}, \dot{\gamma}) \geq 0$, the function $\Psi(s)$ is constant.*

**Lemma 3.** *If geodesic $\gamma$ on $K_n$ is complete, then, for $a \equiv B \cdot g(\dot{\gamma}, \dot{\gamma}) = 0$ and $b \equiv \overline{B}\, c - \chi^2 = 0$, the function $\Psi(s)$ is constant.*

**Proof.** The proof of Lemma 1 is trivial, because the non-constant function $\Psi(s)$ is not bounded for $s \in \mathbb{R}$.

The proof of the analogue of Lemma 2 has been shown by many authors and relies on ideas by Couty [87] in an investigation of projective transformations of Einstein manifolds and by Shen [88] in an investigation of Finsler Einstein geodesically equivalent metrics.

Since $q = e^{-2\Psi}$, from (8), it follows that $\dot{\tau}(s) = \tilde{c}\, e^{2\Psi} = \tilde{c}/q(s) \ > 0$. We mean $\tau(s) = \int_{s_0}^{s} \tilde{c}/q(t)\, dt$.

For functions $q = c_0 + c_1\, s + c_2\, s^2$ and $q = c_0 + c_1 \cosh(\alpha\, s) + c_2 \sinh(\alpha\, s)$, ($c_1 \neq 0$ or $c_2 \neq 0$), this integral diverges (goes to infinity in finite time $s$). Then, $c_1 = c_2 = 0$ and $\tau = \text{const} \cdot s + s_0$, and it follows that $\dot{\tau} = \text{const}$. Evidently, the function $\Psi(s)$ is constant along geodesic $\gamma(s)$.

The proof of Lemma 3 is trivial, because for non-constant function $q(s)$, there exists $s_0$, for which $q(s_0) = 0$; this is the contradiction with $q(s) > 0$ for $s \in \mathbb{R}$. □

### 5.4. Holomorphically Projective Mappings of $K_n$[0] with n Complete Geodesics

Holomorphically projective mapping $K_n[0]$ onto $\overline{K}_n[\overline{B}]$ is characterized by Equation (2) and

$$\psi_{ij} \equiv \psi_{i,j} - \psi_i \psi_j + \psi_{\bar{i}} \psi_{\bar{j}} = \overline{B}\, \overline{g}_{ij}\,, \tag{13}$$

which are equivalent to the equations (see Formulae (4)a and (5))

$$\lambda_{i,j} = \mu\, g_{ij}\,, \quad \mu = \text{const}\,. \tag{14}$$

**Theorem 2.** *Let $g$ be a (pseudo-) Riemannian Kähler metric, its complex structure $F$ on a domain $V$ of $n$-dimensional manifold $M$ (shortly Kähler space $K_n$=$(M, g, F)$), and their holomorphically projective mapping of $K_n$ onto Kähler space $\overline{K}_n$ with Equation (13). Further, assume that there is a point at which not all sectional curvatures are vanishing and through which in linearly independent directions pass n/2 complete geodesics, for which the condition of at least one of Lemmas 1–3 applies. These directions with their complex united vectors form an n-dimension base. Then, this mapping is trivial (affine).*

**Proof.** Let the conditions of the theorem be satisfied. Then, according to the given geodesics, the function $\Psi(s)$ is constant; thus, at point $x_0$ in the direction of these geodesics, $\partial_\alpha \Psi(x) \dot{\gamma}^\alpha = 0$ is vanishing in the tangent directions. Since this also applies to complex united directions at point $x_0$: $\partial_\alpha \Psi(x) \dot{\gamma}^{\bar{\alpha}} = 0$, then $\psi_i(x_0) = 0$ must apply. This is equivalent with $\lambda_i(x_0) = 0$.

The integrability conditions of Equation (14) have the form $\lambda_\alpha R^\alpha_{ijk}$=0. We covariantly differentiate them and use (14): $\mu R_{lijk} + \lambda_\alpha R^\alpha_{ijk,l} = 0$. It follows that $\mu = 0$ to the extent that $R_{hijk}(x_0) \neq 0$. Therefore, equations $\lambda_{i,j} = 0$ for initial conditions $\lambda_i(x_0) = 0$ have a trivial solution $\lambda_i(x) = 0$. It follows that $\psi_i(x)$ is vanishing, and holomorphically projective mapping is trivial (in other words affine). □

Note that, in the above assumption, there do not exist $K_n[0]$ and $\overline{K}_n[\overline{B}]$, which are holomorphically projective correspondent.

### 5.5. Holomorphically Projective Mappings of $K_n$[B] with Finite Complete Geodesics

For spaces $K_n$[B], $B \neq 0$ the similar condition is weak. Therefore, we recall some aspects of matrix theory.

Let the symmetric matrix $A$ be a bilinear mapping $A\colon T_x \times T_x \to R$, where $T_x$ is the tangent space at $x$, $\dim T_x = n$. On the other hand $A\colon S^2T_x \to R$, where $S^2T_x$ is the second symmetric power of $T_x$. Evidently, $\dim S^2T_x = N = \frac{1}{2}n(n+1)$. We choose vectors $v_1, v_2, \dots, v_N \in V$ in such a way that $v_1 \circ v_1, v_2 \circ v_2, \dots, v_N \circ v_N$ is a basis of $S^2T_x$.

Evidently, it follows that $A(v_i, v_i) = 0,\ \text{for}\ \forall i = 1,2,\dots,N \iff A = 0.$

If the symmetric matrix $A$ satisfies the following condition $A_{\bar{i}\bar{j}} = A_{ij}$, the set of vectors $S^2T_x$ can be reduced to $S^{2*}T_x$, and the number of these vectors is $N = (n/2)^2$.

**Theorem 3.** *Let $g$ be a (pseudo-) Riemannian Kähler metric, its complex structure $F$ on a domain $V$ of $n$-dimensional manifold $M$ (shortly Kähler space $K_n$=$(M, g, F)$), and their holomorphically projective mapping of $K_n$ onto Kähler space $\overline{K}_n$ with Equation (9). Further, assume that there is a point $x_0$ through which in directions $v_i \in S^{2*}$ pass $(n/2)^2$ complete geodesics, for which the condition of at least one of Lemmas 1–3 applies. Then, this mapping is homothetic; i.e., the metrics are proportional with a constant coefficient.*

**Proof.** Let the conditions of the theorem be satisfied. We construct $N$ geodesics $\gamma_\alpha(s)$, $\alpha = 1,\dots,N$, for which $x_0 \in \gamma_\alpha$ and the vectors $v_\alpha$ are the tangent vectors of $\gamma_\alpha$ at the point $x_0$. Then, according to the given geodesics, the function $\Psi(s)$ is constant, and thus at point $x_0$ in the direction of these geodesics, $\partial_i\Psi(x_0)\dot\gamma^i_\alpha = 0$ is vanishing in the tangent directions. Since this also applies to complex united directions at point $x_0$: $\partial_i\Psi(x_0)\dot\gamma^{\bar{i}}_\alpha = 0$, $\psi_i(x_0) = 0$ must apply.

From (9), in contraction with $\gamma^i_\alpha(x_0)\gamma^j_\alpha(x_0)$, we obtain $\overline{B}\,\overline{g}(v_\alpha, v_\alpha) - B\,g(v_\alpha, v_\alpha) = 0$ for any vector $v_\alpha \in S^{2*}T$, $\alpha = 1,2,\dots,(n/2)^2$. Therefore, $\overline{B}\,\overline{g} = B\,g$ at point $x_0$; so, at point $x_0$, we have $\overline{g} = \kappa\, g$. Evidently,

$$\overline{g}_{ij}(x_0) = \kappa \cdot g_{ij}(x_0), \quad \text{and} \quad \psi_i(x_0) = 0. \tag{15}$$

As we know, the system of Equations (2) and (9) has only one solution with respect to the unknown functions $\overline{g}_{ij}(x)$ and $\psi_i(x)$ for the initial conditions $\overline{g}_{ij}(x_0) = \overline{g}^*_{ij}$ and $\psi_i(x_0) = \psi^*_i$.

Solution $\overline{g}_{ij}(x) = \kappa \cdot g_{ij}(x)$ and $\psi_i(x) = 0$ satisfy the initial conditions (15), which is unique. This theorem is proven. □

Theorems 2 and 3 imply the validity of Theorem 1. The Kähler space $K_n$ is complete if every geodesic is complete. In this case, $K_n$ in the definition are Kähler space under the Equations (2) and (4).

## 6. Summary

The main results of our study are Theorems 2 and 3. They clearly state that in order for the mapping to be rigid the space does not have to be complete. It suffices that there exist a finite number of geodesics and their images that are complete.

Practically speaking, the space is uniquely defined by the given geodetics, which are the supporting skeleton (reinforcement) of the surface.

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