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Article

Spatial Decay Estimates and Continuous Dependence for the Oldroyd Fluid

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Abstract: This article investigates the Oldroyd fluid, which is widely used in industrial and engineering environments. When the Oldroyd fluid passes through a three-dimensional semi-infinite cylinder, the asymptotic properties of the solutions are established. On this basis, we also studied the continuous dependence of the viscosity coefficient.

Keywords: spatial decay estimates; Oldroyd fluid; structural stability

MSC: 35B40; 35Q30; 76D05

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1. Introduction

In this paper, we let R denote that

$$R = \Big\{(x_1, x_2, x_3)\Big|(x_1, x_2) \in D,\ x_3 > 0\Big\},$$

where $D \subset (x_1, x_2)$-plane, and D is bounded (see Figure 1). We also require that D has a smooth boundary ∂D.

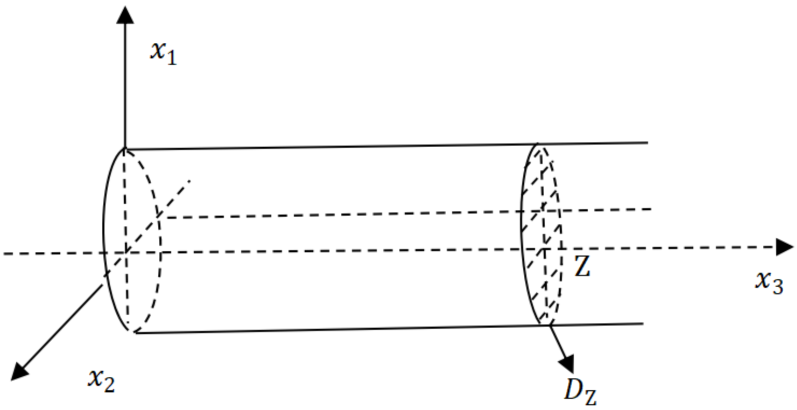

Figure 1. Cylindrical pipe R.

Let u_i, p, and $q_i (i = 1, 2, 3)$ denote the velocity, the pressure, and viscoelastic variables of the fluid, respectively. These variables satisfy the following Oldroyd fluid equations (see [1]):

$$u_{i,t} - \mu\Delta u_i - \lambda\Delta q_i + p_{,i} = 0,\ \boldsymbol{x} \in R,\ t > 0, \quad (1)$$

$$u_{i,i} = 0,\ \boldsymbol{x} \in R, t > 0, \quad (2)$$

$$q_{i,t} + \gamma q_i - u_i = 0,\ \boldsymbol{x} \in R, t > 0, \quad (3)$$

$$u_i, u_{i,j}, q_i, q_{i,j} \to 0,\ \text{as } x_3 \to \infty. \quad (4)$$

In (1)–(4) and the following, we use commas for derivation, repeated English subscripts for summation from 1 to 3, and repeated Greek subscripts for summation from 1 to 2, e.g., $u_{i,i} = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i}$ and $u_{\alpha,\alpha} = \sum_{i=1}^{2} \frac{\partial u_\alpha}{\partial x_\alpha}$.

On the boundary, Equations (1)–(4) satisfy

$$u_i(\boldsymbol{x}, 0) = q_i(\boldsymbol{x}, 0) = 0, \boldsymbol{x} \in R, \tag{5}$$

$$u_i(x_1, x_2, 0, t) = g_i(x_1, x_2, t),\ (x_1, x_2) \in D, t > 0, \tag{6}$$

$$q_i(x_1, x_2, 0, t) = h_i(x_1, x_2, t),\ (x_1, x_2) \in D, t > 0, \tag{7}$$

$$u_i = 0, \frac{\partial q_i}{\partial \boldsymbol{n}} = 0, \boldsymbol{x} \in \partial D \times \{x_3 \geq 0\}, t > 0, \tag{8}$$

where g_i and h_i are given functions, and μ, γ, and λ are positive constants.

Because viscoelastic fluid is widely used in real life, this type of fluid has attracted more and more attention. The Oldroyd fluid Equations (1)–(3) are not only clearly recorded in [2], but also discussed in [1]. Meanwhile, Oskolkov and Shadiev [2] and Christov and Jordan [3] have proved the existence of the solution to Equations (1)–(3) under different conditions. In the context of industrial and engineering applications, Oldroyd fluids always pass through a cylinder. Therefore, it is necessary to study the properties and continuous dependence of solutions to Oldroyd fluids on a cylinder.

The first purpose of the present paper is to study the spatial decay results of the solutions to Equations (1)–(4) when $x_3 \to \infty$. In fact, Keiller [4] studied the spatial decay of steady perturbations of plane Poiseuille flow for the Oldroyd-B equations as $t \to \infty$. For such a type of study, one can also see [5]. After the earlier work of Boley [6], the spatial decay results of fluid equations with spatial variables in a cylinder, which can be thought of as decay results of the Saint-Venant type, have been paid full attention. For more of such Saint-Venant type results, one can see [7–16]. Compared with references [4,5], the innovation of this paper is the use of the methods of references [9,11] to further extend the attenuation results to a semi-infinite cylinder. We demonstrate that the solution decays exponentially on the semi-infinite cylinder, indicating that the velocity and viscoelastic variables of the fluid decay exponentially with distance from the finite end to the infinite. This is the first result that we establish in the present paper.

The second purpose is to study the continuous dependence of the solutions to Equations (1)–(4) on the viscosity coefficient. When a small disturbance occurs in the viscosity coefficient, we study what kind of disturbance will occur in the solutions of the equation. In past decades, studies on structural stability have received a lot of attention, and a large number of papers have emerged, including those of Liu and Zheng [17], Avalos and Lesiecka [18], Meyvacı [19], Quintanilla [1,20,21], Liu et al. [22–26], Hameed and Harfash [27], Scott et al. [28,29], Li et al. [30–33], Ciarletta and Straughan [34], and Franchi and Straughan [35]. Straughan [1] proved that Kelvin–Voigt fluid depended continuously on the coefficients of the fluid. Research results have been focused on bounded regions in both two-dimensional and three-dimensional spaces.

In fact, the study of structural stability in the cylindrical region is equally important. Due to its widespread application in practice, it has gradually begun to receive attention (see [36–38]). This paper will continue the research in this field. We want to extend the results of [1] to the case of the semi-infinite cylinder. Taking the viscosity coefficient as an example, we demonstrate how to derive the continuous dependence results of the solutions on the other coefficients in Equations (1)–(4). In this process, we adopted methods of energy estimation and prior estimates.

2. The Main Theorems

We first list some lemmas.

Lemma 1 (See [7]). *If* $\chi\Big|_{\partial D} = 0$, *then*

$$\lambda_1 \int_D \chi^2 dA \le \int_D \chi_{,\alpha}\chi_{,\alpha} dA,$$

where λ_1 *is the smallest positive eigenvalue of*

$$\Delta\varphi + \lambda\varphi = 0, in\ D,\ \varphi = 0,\ on\ \partial D.$$

Lemma 2 (See [39]). *Assume that* ∂R *is the Lipschitz boundary of* R. *If* $\int_R v dx = 0$, *then* $\exists\ \varphi = (\varphi_1, \varphi_2, \varphi_3)$ *and* $\varphi|_{\partial R} = 0$ *such that*

$$\varphi_{i,i} = v,\ in\ R,$$

and

$$\int_R \varphi_{i,j}\varphi_{i,j} dx \le k_1 \int_R [\varphi_{j,j}]^2 dx,\ k_1 > 0.$$

This lemma in the case of two dimensions has been established by [40].

We consider the identity

$$\int_0^t \int_{R_z} e^{-\omega\eta}(\xi - z)\Big[u_{i,\eta} - \mu\Delta u_i - \lambda\Delta q_i + p_{,i}\Big]u_{i,\eta} d\mathbf{x}d\eta = 0, \tag{9}$$

where $R_z = \Big\{(x_1, x_2, x_3)\Big|(x_1, x_2) \in D,\ x_3 \ge z > 0\Big\}$. *By using the divergence theorem and* (5), (8), *we have*

$$\begin{aligned}
&\frac{1}{2}e^{-\omega t}\mu \int_{R_z}(\xi - z)u_{i,j}u_{i,j} d\mathbf{x} + \int_0^t \int_{R_z} e^{-\omega\eta}(\xi - z)\Big[u_{i,\eta}u_{i,\eta} + \frac{1}{2}\omega\mu u_{i,j}u_{i,j}\Big] d\mathbf{x}d\eta \\
&= -\mu \int_0^t \int_{R_z} e^{-\omega\eta} u_{i,\eta}u_{i,3} d\mathbf{x}d\eta - \lambda \int_0^t \int_{R_z} e^{-\omega\eta} u_{i,\eta}q_{i,3} d\mathbf{x}d\eta \\
&+ \int_0^t \int_{R_z} e^{-\omega\eta} u_{3,\eta} p d\mathbf{x}d\eta - \lambda \int_0^t \int_{R_z} e^{-\omega\eta}(\xi - z)q_{i,j}u_{i,j\eta} d\mathbf{x}d\eta \\
&= -\mu \int_0^t \int_{R_z} e^{-\omega\eta} u_{i,\eta}u_{i,3} d\mathbf{x}d\eta - \lambda \int_0^t \int_{R_z} e^{-\omega\eta} u_{i,\eta}q_{i,3} d\mathbf{x}d\eta \\
&+ \int_0^t \int_{R_z} e^{-\omega\eta} u_{3,\eta} p d\mathbf{x}d\eta - \lambda e^{-\omega t} \int_{R_z}(\xi - z)q_{i,j}u_{i,j} d\mathbf{x} \\
&- \lambda\omega \int_0^t \int_{R_z} e^{-\omega\eta}(\xi - z)q_{i,j}u_{i,j} d\mathbf{x}d\eta \\
&+ \lambda \int_0^t \int_{R_z} e^{-\omega\eta}(\xi - z)q_{i,j\eta}u_{i,j} d\mathbf{x}d\eta,
\end{aligned} \tag{10}$$

where ω *is a positive constant.*

By (2), *we have the following identity:*

$$\int_0^t \int_{R_z} e^{-\omega\eta}(\xi - z)\Big[q_{i,j\eta} + \gamma q_{i,j} - u_{i,j}\Big]q_{i,j\eta} d\mathbf{x}d\eta = 0. \tag{11}$$

From (11), *it follows that*

$$\begin{aligned}
&\frac{1}{2}e^{-\omega t}\gamma \int_{R_z}(\xi - z)q_{i,j}q_{i,j} d\mathbf{x} + \int_0^t \int_{R_z} e^{-\omega\eta}(\xi - z)\Big[\frac{1}{2}\omega\gamma q_{i,j}q_{i,j} + q_{i,j\eta}q_{i,j\eta}\Big] d\mathbf{x}d\eta \\
&= \int_0^t \int_{R_z} e^{-\omega\eta}(\xi - z)u_{i,j}q_{i,j\eta} d\mathbf{x}d\eta.
\end{aligned} \tag{12}$$

If we let

$$E(z,t) = \frac{1}{2}e^{-\omega t}\int_{R_z}(\xi - z)\Big[\mu u_{i,j}u_{i,j} + \delta_1\gamma q_{i,j}q_{i,j}\Big]d\mathbf{x} + \int_0^t\int_{R_z}e^{-\omega\eta}(\xi - z)\Big[u_{i,\eta}u_{i,\eta} + \frac{1}{2}\omega\mu u_{i,j}u_{i,j} + \frac{1}{2}\omega\gamma\delta_1 q_{i,j}q_{i,j} + \delta_1 q_{i,j\eta}q_{i,j\eta}\Big]d\mathbf{x}d\eta, \quad (13)$$

then from (10) *and* (12)*, we have*

$$\begin{aligned} E(z,t) = &-\mu\int_0^t\int_{R_z}e^{-\omega\eta}u_{i,\eta}u_{i,3}d\mathbf{x}d\eta - \lambda\int_0^t\int_{R_z}e^{-\omega\eta}u_{i,\eta}q_{i,3}d\mathbf{x}d\eta \\ &+ \int_0^t\int_{R_z}e^{-\omega\eta}u_{3,\eta}p\,d\mathbf{x}d\eta - \lambda e^{-\omega t}\int_{R_z}(\xi - z)q_{i,j}u_{i,j}d\mathbf{x} \\ &- \lambda\omega\int_0^t\int_{R_z}e^{-\omega\eta}(\xi - z)q_{i,j}u_{i,j}d\mathbf{x}d\eta + (\lambda + \delta_1)\int_0^t\int_{R_z}e^{-\omega\eta}(\xi - z)q_{i,j\eta}u_{i,j}d\mathbf{x}d\eta \\ &\doteq \sum_{i=1}^{6}A_i(z,t), \end{aligned} \quad (14)$$

where $\delta_1 > 0$.

Based on the energy function $E(z,t)$*, we can obtain the spatial decay theorem.*

Theorem 1. *Let* (u_i, q_i, p) *be the solutions of Equations* (1)–(8)*. If* $g_i \in C(\partial D \times [0,\infty))$ *and* $\int_D g_3 dA = 0$*, then the following inequality holds:*

$$\begin{aligned} &\frac{1}{2}e^{-\omega t}\int_{R_z}(\xi - z)\Big[\mu u_{i,j}u_{i,j} + \delta_1\gamma q_{i,j}q_{i,j}\Big]d\mathbf{x} \\ &+ \int_0^t\int_{R_z}e^{-\omega\eta}(\xi - z)\Big[u_{i,\eta}u_{i,\eta} + \frac{1}{2}\omega\mu u_{i,j}u_{i,j} + \frac{1}{2}\omega\gamma\delta_1 q_{i,j}q_{i,j} + \delta_1 q_{i,j\eta}q_{i,j\eta}\Big]d\mathbf{x}d\eta \\ &\leq \frac{1}{m_1}Q(0,t)e^{-m_1 z} \end{aligned}$$

where $m_1 > 0$*, and* $Q(0,t) > 0$ *is a computable function that depends only on* t.

Remark 1. *Obviously,* $e^{-m_1 z} \to 0$ *as* $z \to \infty$*. Therefore Theorem 1 shows that the solutions of* (1)–(8) *decay exponentially to zero as* $z \to \infty$.

We suppose that $u_i, q_i,$ *and* p *are the solutions of* (1)–(8)*, and* u_i^*, q_i^**, and* p^* *are the solutions of* (1)–(8) *with* $\gamma = \gamma^*$*. Let*

$$v_i = u_i - u_i^*,\ \Sigma_i = q_i - q_i^*,\ \pi = p - p^*,\ \widetilde{\gamma} = \gamma - \gamma^*,$$

Then, $v_i, \Sigma_i,$ *and* π *satisfy*

$$v_{i,t} - \mu\Delta v_i - \lambda\Delta\Sigma_i + \pi_{,i} = 0,\ \mathbf{x} \in R,\ t > 0, \quad (15)$$

$$v_{i,i} = 0,\ \mathbf{x} \in R, t > 0, \quad (16)$$

$$\Sigma_{i,t} + \widetilde{\gamma}q_i + \gamma^*\Sigma_i - v_i = 0,\ \mathbf{x} \in R, t > 0, \quad (17)$$

$$v_i, v_{i,j}, \Sigma_i, \Sigma_{i,j}, \to 0, as\ x_3 \to \infty \quad (18)$$

and the following conditions

$$v_i(\mathbf{x},0) = \Sigma_i(\mathbf{x},0) = 0, \mathbf{x} \in R, \quad (19)$$

$$v_i(x_1,x_2,0,t) = 0,\ (x_1,x_2) \in D, t > 0, \quad (20)$$

$$\Sigma_i(x_1,x_2,0,t) = 0,\ (x_1,x_2) \in D, t > 0, \quad (21)$$

$$v_i = 0, \frac{\partial\Sigma_i}{\partial\mathbf{n}} = 0, \mathbf{x} \in \partial D \times \{x_3 \geq 0\}, t > 0. \quad (22)$$

The continuous dependence theorem can be written as follows.

Theorem 2. *Let u_i and q_i be the solutions of (1)–(8) and u_i^* and q_i^* be the solutions of (1)–(8) with $\gamma = \gamma^*$. If $\int_D g_3 dA = 0$, then*

$$(u_i, q_i) \to (u_i^*, q_i^*),\ \text{as}\ \gamma \to \gamma^*.$$

Specifically,

$$\begin{aligned}&\frac{1}{4}\omega\mu\int_0^t\int_{R_z}(\xi - z)e^{-\omega\eta}v_{i,j}v_{i,j}d\mathbf{x}d\eta + \int_0^t\int_{R_z}(\xi - z)e^{-\omega\eta}v_{i,\eta}v_{i,\eta}d\mathbf{x}d\eta \\ &\quad + \frac{1}{4}\delta_2\omega\gamma^*\int_0^t\int_{R_z}(\xi - z)e^{-\omega\eta}\Sigma_{i,j}\Sigma_{i,j}d\mathbf{x}d\eta + \frac{1}{2}\delta_2\int_0^t\int_{R_z}(\xi - z)e^{-\omega\eta}\Sigma_{i,j\eta}\Sigma_{i,j\eta}d\mathbf{x}d\eta \\ &\quad \leq \frac{2\delta_2}{\omega\gamma^*\delta_1}Q(0,t)\tilde{\gamma}^2\Big[\frac{1}{m_2} + \frac{2}{m_1}\Big]e^{-m_2 z} + \frac{4m_2\delta_2}{\omega\gamma^* m_1(m_1 - m_2)\delta_1}Q(0,t)\tilde{\gamma}^2\Big[e^{-m_1 z} - e^{-m_2 z}\Big],\end{aligned}$$

or

$$\begin{aligned}&\frac{1}{4}\omega\mu\int_0^t\int_{R_z}(\xi - z)e^{-\omega\eta}v_{i,j}v_{i,j}d\mathbf{x}d\eta + \int_0^t\int_{R_z}(\xi - z)e^{-\omega\eta}v_{i,\eta}v_{i,\eta}d\mathbf{x}d\eta \\ &\quad + \frac{1}{4}\delta_2\omega\gamma^*\int_0^t\int_{R_z}(\xi - z)e^{-\omega\eta}\Sigma_{i,j}\Sigma_{i,j}d\mathbf{x}d\eta + \frac{1}{2}\delta_2\int_0^t\int_{R_z}(\xi - z)e^{-\omega\eta}\Sigma_{i,j\eta}\Sigma_{i,j\eta}d\mathbf{x}d\eta \\ &\quad \leq \frac{2\delta_2}{\omega\gamma^*\delta_1}Q(0,t)\tilde{\gamma}^2\Big[\frac{1}{m_2} + \frac{2}{m_1}\Big]e^{-m_2 z} + \frac{4m_2\delta_2}{\omega\gamma^* m_1\delta_1}Q(0,t)\tilde{\gamma}^2 z e^{-m_2 z},\end{aligned}$$

where m_2 and δ_2 are positive constants.

3. The Proof of Theorem 1

In this section, we first state some lemmas that are related to (1)–(8). Using these lemmas, we can obtain Theorem 1.

Lemma 3. *Suppose that u_i, q_i, and p are the solutions of (1)–(8) with $\int_D g_3 dA = 0$. Then,*

$$E(z,t) \leq \frac{1}{m_1}\Big[-\frac{\partial}{\partial z}E(z,t)\Big],$$

where $\delta_1 = \frac{16\lambda^2}{\gamma\mu}$, $\omega \geq \frac{8(\lambda+\delta_1)^2}{\mu\delta_1}$, and

$$\frac{1}{m_1} = 2\Big[\frac{\sqrt{k_2}}{\sqrt{\lambda_1}} + \frac{\sqrt{k_2\mu}}{\sqrt{2\omega}} + \frac{\sqrt{k_2}\lambda}{\sqrt{2\gamma\delta_1\omega}} + \frac{\sqrt{k_2\mu}}{2\sqrt{\lambda_1}} + \frac{\sqrt{k_2\lambda}}{2\sqrt{\lambda_1\gamma}} + \frac{\sqrt{\mu}}{\sqrt{2\omega}} + \frac{\lambda}{\sqrt{2\gamma\delta_1\omega}}\Big]$$

Proof. From (13), we also have

$$\begin{aligned}-\frac{\partial}{\partial z}E(z,t) &= \frac{1}{2}e^{-\omega t}\int_{R_z}\Big[\mu u_{i,j}u_{i,j} + \delta_1\gamma q_{i,j}q_{i,j}\Big]d\boldsymbol{x} \\ &\quad + \int_0^t\int_{R_z}e^{-\omega\eta}\Big[u_{i,\eta}u_{i,\eta} + \frac{1}{2}\omega\mu u_{i,j}u_{i,j} + \frac{1}{2}\omega\gamma\delta_1 q_{i,j}q_{i,j} + \delta_1 q_{i,j\eta}q_{i,j\eta}\Big]d\boldsymbol{x}d\eta. \quad (23)\end{aligned}$$

Next, we bound each $A_i (i = 1, 2, \ldots, 6)$ in terms of $-\frac{\partial}{\partial z}E(z,t)$. By Hölder's and Young's inequalities, we have

$$A_1(z,t) \le \mu\Big[\int_0^t\int_{R_z} e^{-\omega\eta}u_{i,3}u_{i,3}d\boldsymbol{x}d\eta\int_0^t\int_{R_z} e^{-\omega\eta}u_{i,\eta}u_{i,\eta}d\boldsymbol{x}d\eta\Big]^{\frac{1}{2}}$$
$$\le \frac{\sqrt{\mu}}{\sqrt{2\omega}}\Big[\frac{1}{2}\omega\mu\int_0^t\int_{R_z} e^{-\omega\eta}u_{i,3}u_{i,3}d\boldsymbol{x}d\eta + \int_0^t\int_{R_z} e^{-\omega\eta}u_{i,\eta}u_{i,\eta}d\boldsymbol{x}d\eta\Big], \quad (24)$$

$$A_2(z,t) \le \frac{\lambda}{\sqrt{2\gamma\delta_1\omega}}\Big[\frac{1}{2}\omega\gamma\delta_1\int_0^t\int_{R_z} e^{-\omega\eta}q_{i,3}q_{i,3}d\boldsymbol{x}d\eta + \int_0^t\int_{R_z} e^{-\omega\eta}u_{i,\eta}u_{i,\eta}d\boldsymbol{x}d\eta\Big], \quad (25)$$

$$A_4(z,t) \le \frac{\lambda}{\sqrt{\gamma\mu\delta_1}}\Big[\frac{1}{2}\mu e^{-\omega t}\int_{R_z}(\xi-z)u_{i,j}u_{i,j}d\boldsymbol{x} + \frac{1}{2}\gamma\delta_1 e^{-\omega t}\int_{R_z}(\xi-z)q_{i,j}q_{i,j}d\boldsymbol{x}\Big], \quad (26)$$

$$A_5(z,t) \le \frac{\lambda}{\sqrt{\gamma\mu\delta_1}}\Big[\frac{1}{2}\omega\mu\int_0^t\int_{R_z} e^{-\omega\eta}(\xi-z)u_{i,j}u_{i,j}d\boldsymbol{x}d\eta$$
$$+ \frac{1}{2}\omega\gamma\delta_1\int_0^t\int_{R_z} e^{-\omega\eta}(\xi-z)q_{i,j}q_{i,j}d\boldsymbol{x}d\eta\Big], \quad (27)$$

$$A_6(z,t) \le \frac{\lambda+\delta_1}{\sqrt{2\omega\mu\delta_1}}\Big[\delta_1\int_0^t\int_{R_z} e^{-\omega\eta}(\xi-z)q_{i,j\eta}q_{i,j\eta}d\boldsymbol{x}d\eta$$
$$+ \frac{1}{2}\omega\mu\int_0^t\int_{R_z} e^{-\omega\eta}(\xi-z)u_{i,j}u_{i,j}d\boldsymbol{x}d\eta\Big]. \quad (28)$$

Next, we will bound $A_3(z,t)$. To do this, we note that

$$\begin{aligned}\int_{D_z} u_{3,t}dA &= \int_D u_{3,t}dA + \int_0^z\int_{D_\xi} u_{3,3t}dAd\xi \\ &= \int_D u_{3,t}dA - \int_0^z\int_{D_\xi} u_{\alpha,\alpha t}dAd\xi \\ &= \int_D g_{3,t}dA.\end{aligned} \quad (29)$$

In view of $\int_D g_3 dA = 0$, we have $\int_{D_z} u_{3,t}dA = 0$. Through using Lemma 2, $\exists\ \varphi = (\varphi_1, \varphi_2, \varphi_3)$ such that

$$\varphi_{i,i} = u_{3,t}, \text{ in } R, \varphi_i = 0, \text{on } \partial R.$$

Therefore, we have

$$\begin{aligned}A_3(z,t) &= \int_0^t\int_z^\infty\int_{D_\xi} e^{-\omega\eta}\varphi_{i,i}p\,d\boldsymbol{x}d\eta = -\int_0^t\int_z^\infty\int_{D_\xi} e^{-\omega\eta}\varphi_i p_{,i}d\boldsymbol{x}d\eta \\ &= \int_0^t\int_z^\infty\int_{D_\xi} e^{-\omega\eta}\varphi_i\Big[u_{i,\eta} - \mu\Delta u_i - \lambda\Delta q_i\Big]d\boldsymbol{x}d\eta \\ &= \int_0^t\int_z^\infty\int_{D_\xi} e^{-\omega\eta}\varphi_i u_{i,\eta}d\boldsymbol{x}d\eta + \mu\int_0^t\int_z^\infty\int_{D_\xi} e^{-\omega\eta}\varphi_{i,j}u_{i,j}d\boldsymbol{x}d\eta \\ &+ \lambda\int_0^t\int_z^\infty\int_{D_\xi} e^{-\omega\eta}\varphi_{i,j}q_{i,j}d\boldsymbol{x}d\eta \\ &\doteq \sum_{i=1}^3 A_{3i}(z,t).\end{aligned} \quad (30)$$

Using the Schwarz inequality and Lemmas 1 and 2, we have

$$
\begin{aligned}
A_{31}(z,t) &\leq \Big[\int_0^t\int_{R_z} e^{-\omega\eta}u_{i,\eta}u_{i,\eta}d\mathbf{x}d\eta\int_0^t\int_{R_z} e^{-\omega\eta}\varphi_i\varphi_i d\mathbf{x}d\eta\Big]^{\frac{1}{2}}\\
&\leq \frac{1}{\sqrt{\lambda_1}}\Big[\int_0^t\int_{R_z} e^{-\omega\eta}u_{i,\eta}u_{i,\eta}d\mathbf{x}d\eta\int_0^t\int_{R_z} e^{-\omega\eta}\varphi_{i,\beta}\varphi_{i,\beta}d\mathbf{x}d\eta\Big]^{\frac{1}{2}}\\
&\leq \frac{\sqrt{k_1}}{\sqrt{\lambda_1}}\Big[\int_0^t\int_{R_z} e^{-\omega\eta}u_{i,\eta}u_{i,\eta}d\mathbf{x}d\eta\int_0^t\int_{R_z} e^{-\omega\eta}\varphi_{i,i}^2 d\mathbf{x}d\eta\Big]^{\frac{1}{2}}\\
&\leq \frac{\sqrt{k_1}}{\sqrt{\lambda_1}}\Big[\int_0^t\int_{R_z} e^{-\omega\eta}u_{i,\eta}u_{i,\eta}d\mathbf{x}d\eta\int_0^t\int_{R_z} e^{-\omega\eta}u_{3,\eta}^2 d\mathbf{x}d\eta\Big]^{\frac{1}{2}}\\
&\leq \frac{\sqrt{k_1}}{\sqrt{\lambda_1}}\int_0^t\int_{R_z} e^{-\omega\eta}u_{i,\eta}u_{i,\eta}d\mathbf{x}d\eta,
\end{aligned} \tag{31}
$$

$$
\begin{aligned}
A_{32}(z,t) &\leq \mu\Big[\int_0^t\int_{R_z} e^{-\omega\eta}u_{i,j}u_{i,j}d\mathbf{x}d\eta\int_0^t\int_{R_z} e^{-\omega\eta}\varphi_{i,j}\varphi_{i,j}d\mathbf{x}d\eta\Big]^{\frac{1}{2}}\\
&\leq \frac{\sqrt{k_1\mu}}{\sqrt{2\omega}}\Big[\frac{1}{2}\omega\mu\int_0^t\int_{R_z} e^{-\omega\eta}u_{i,j}u_{i,j}d\mathbf{x}d\eta+\int_0^t\int_{R_z} e^{-\omega\eta}u_{3,\eta}^2 d\mathbf{x}d\eta\Big],
\end{aligned} \tag{32}
$$

$$
A_{33}(z,t) \leq \frac{\sqrt{k_1}\lambda}{\sqrt{2\gamma\delta_1\omega}}\Big[\frac{1}{2}\omega\delta_1\gamma\int_0^t\int_{R_z} e^{-\omega\eta}q_{i,j}q_{i,j}d\mathbf{x}d\eta+\int_0^t\int_{R_z} e^{-\omega\eta}u_{3,\eta}^2 d\mathbf{x}d\eta\Big]. \tag{33}
$$

Inserting (31)–(33) into (30) and in view of (23), we obtain

$$
A_3(z,t) \leq \Big[\frac{\sqrt{k_1}}{\sqrt{\lambda_1}}+\frac{\sqrt{k_1\mu}}{\sqrt{2\omega}}+\frac{\sqrt{k_1}\lambda}{\sqrt{2\gamma\delta_1\omega}}\Big]\Big[-\frac{\partial}{\partial z}E(z,t)\Big]. \tag{34}
$$

Since

$$
\delta_1=\frac{16\lambda^2}{\gamma\mu},\ \omega\geq\frac{8(\lambda+\delta_1)^2}{\mu\delta_1},
$$

by inserting (24)–(28) and (34) into (14), we can obtain Lemma 3.

We also need the bound of $E(0,t)$. Therefore, we bound $E(0,t)$ using known data. □

Lemma 4. *Assume that u_i, q_i, and p are the solutions of (1)–(8) with $\int_D g_3 dA=0$. Then,*

$$
\begin{aligned}
&\frac{1}{2}e^{-\omega t}\int_R\Big[\mu u_{i,j}u_{i,j}+\delta_1\gamma q_{i,j}q_{i,j}\Big]d\mathbf{x}\\
&\quad+\int_0^t\int_R e^{-\omega\eta}\Big[u_{i,\eta}u_{i,\eta}+\frac{1}{2}\omega\mu u_{i,j}u_{i,j}+\frac{1}{2}\omega\gamma\delta_1 q_{i,j}q_{i,j}+\delta_1 q_{i,j\eta}q_{i,j\eta}\Big]d\mathbf{x}d\eta\\
&\quad\leq Q(0,t),
\end{aligned}
$$

and

$$
E(0,t)\leq\frac{1}{m_1}Q(0,t),
$$

where $Q(0,t)$ is a positive computable function.

Proof. We let $z=0$ in Lemma 3, and then we can have

$$
E(0,t)\leq\frac{1}{m_1}\Big[-\frac{\partial}{\partial z}E(0,t)\Big]. \tag{35}
$$

Therefore, to bound $E(0,t)$, it is necessary to seek a bound for $-\frac{\partial}{\partial z}E(0,t)$. In (23), by choosing $z=0$, we have

$$-\frac{\partial}{\partial z}E(0,t)=\frac{1}{2}e^{-\omega t}\int_R\left[\mu u_{i,j}u_{i,j}+\delta_1\gamma q_{i,j}q_{i,j}\right]d\boldsymbol{x}$$
$$+\int_0^t\int_R e^{-\omega\eta}\left[u_{i,\eta}u_{i,\eta}+\frac{1}{2}\omega\mu u_{i,j}u_{i,j}+\frac{1}{2}\omega\gamma\delta_1 q_{i,j}q_{i,j}+\delta_1 q_{i,j\eta}q_{i,j\eta}\right]d\boldsymbol{x}d\eta. \tag{36}$$

From (14), we have

$$\begin{aligned}-\frac{\partial}{\partial z}E(0,t)=&-\mu\int_0^t\int_D e^{-\omega\eta}u_{i,\eta}u_{i,3}dAd\eta-\lambda\int_0^t\int_D e^{-\omega\eta}u_{i,\eta}q_{i,3}dAd\eta\\&+\int_0^t\int_D e^{-\omega\eta}u_{3,\eta}pdAd\eta-\lambda e^{-\omega t}\int_R q_{i,j}u_{i,j}d\boldsymbol{x}\\&-\lambda\omega\int_0^t\int_R e^{-\omega\eta}q_{i,j}u_{i,j}d\boldsymbol{x}d\eta+(\lambda+\delta_1)\int_0^t\int_R e^{-\omega\eta}q_{i,j\eta}u_{i,j}d\boldsymbol{x}d\eta\\&\doteq\sum_{i=1}^{6}\Theta_i(0,t),\end{aligned} \tag{37}$$

Now, we let $\mathbf{G}$ denote

$$\mathbf{G}(\boldsymbol{x},t)=\boldsymbol{g}(x_1,x_2,0,t)e^{-\sigma_1x_3}, \tag{38}$$

where σ_1 is a positive constant, and $g_{\alpha,\alpha}-\sigma_1 g_3=0$.

Using Equation (1) and the divergence theorem, we have

$$\begin{aligned}&\Theta_1(0,t)+\Theta_2(0,t)+\Theta_3(0,t)\\&=-\mu\int_0^t\int_D e^{-\omega\eta}G_{i,\eta}u_{i,3}dAd\eta-\lambda\int_0^t\int_D e^{-\omega\eta}G_{i,\eta}q_{i,3}dAd\eta\\&+\int_0^t\int_D e^{-\omega\eta}G_{3,\eta}pdAd\eta\\&=\lambda\int_0^t\int_R e^{-\omega\eta}(q_{i,j}G_{i,\eta})_{,j}d\boldsymbol{x}d\eta+\mu\int_0^t\int_R e^{-\omega\eta}(u_{i,j}G_{i,\eta})_{,j}d\boldsymbol{x}d\eta\\&-\int_0^t\int_R e^{-\omega\eta}(pG_{i,\eta})_{,i}d\boldsymbol{x}d\eta\\&=\mu\int_0^t\int_R e^{-\omega\eta}u_{i,j}G_{i,j\eta}d\boldsymbol{x}d\eta+\lambda\int_0^t\int_R e^{-\omega\eta}q_{i,j}G_{i,j\eta}d\boldsymbol{x}d\eta\\&-\int_0^t\int_R e^{-\omega\eta}u_{i,\eta}G_{i,\eta}d\boldsymbol{x}d\eta\\&\doteq\varrho_1(0,t)+\varrho_2(0,t)+\varrho_3(0,t).\end{aligned} \tag{39}$$

Combining Hölder's inequality and Young's inequality in (39), we obtain

$$\varrho_1(0,t)\le\frac{1}{8}\mu\omega\int_0^t\int_R e^{-\omega\eta}u_{i,j}u_{i,j}d\boldsymbol{x}d\eta+\frac{2}{\omega}\mu\int_0^t\int_R e^{-\omega\eta}G_{i,j\eta}G_{i,j\eta}d\boldsymbol{x}d\eta, \tag{40}$$

$$\varrho_2(0,t)\le\frac{1}{8}\omega\gamma\delta_1\int_0^t\int_R e^{-\omega\eta}q_{i,j}q_{i,j}d\boldsymbol{x}d\eta+\frac{2}{\gamma\omega\delta_1}\lambda^2\int_0^t\int_R e^{-\omega\eta}G_{i,j\eta}G_{i,j\eta}d\boldsymbol{x}d\eta, \tag{41}$$

$$\varrho_3(0,t)\le\frac{1}{4}\int_0^t\int_R e^{-\omega\eta}u_{i,\eta}u_{i,\eta}d\boldsymbol{x}d\eta+\int_0^t\int_R e^{-\omega\eta}G_{i,\eta}G_{i,\eta}d\boldsymbol{x}d\eta. \tag{42}$$

Inserting (40)–(42) into (39), we have

$$\Theta_1(0,t)+\Theta_2(0,t)+\Theta_3(0,t)\le\frac{1}{4}\left[-\frac{\partial}{\partial z}E(0,t)\right]+\frac{1}{4}Q(0,t), \tag{43}$$

where

$$\frac{1}{4}Q(0,t) = \Big[\frac{2}{\omega}\mu + \frac{2}{\gamma\omega\delta_1}\lambda^2\Big]\int_0^t\int_R e^{-\omega\eta}G_{i,j\eta}G_{i,j\eta}d\boldsymbol{x}d\eta + \int_0^t\int_R e^{-\omega\eta}G_{i,\eta}G_{i,\eta}d\boldsymbol{x}d\eta.$$

Combining Hölder's inequality and Young's inequality again, we obtain

$$\Theta_4(0,t) \le \frac{\lambda}{\sqrt{\gamma\mu\delta_1}}\Big[\frac{1}{2}\mu e^{-\omega t}\int_R u_{i,j}u_{i,j}d\boldsymbol{x} + \frac{1}{2}\gamma\delta_1 e^{-\omega t}\int_R q_{i,j}q_{i,j}d\boldsymbol{x}\Big], \tag{44}$$

$$\Theta_5(0,t) \le \frac{\lambda}{\sqrt{\gamma\mu\delta_1}}\Big[\frac{1}{2}\omega\mu\int_0^t\int_R e^{-\omega\eta}u_{i,j}u_{i,j}d\boldsymbol{x}d\eta + \frac{1}{2}\omega\gamma\delta_1\int_0^t\int_R e^{-\omega\eta}q_{i,j}q_{i,j}d\boldsymbol{x}d\eta\Big], \tag{45}$$

$$\Theta_6(0,t) \le \frac{\lambda+\delta_1}{\sqrt{2\omega\mu\delta_1}}\Big[\delta_1\int_0^t\int_R e^{-\omega\eta}q_{i,j\eta}q_{i,j\eta}d\boldsymbol{x}d\eta + \frac{1}{2}\omega\mu\int_0^t\int_R e^{-\omega\eta}u_{i,j}u_{i,j}d\boldsymbol{x}d\eta\Big]. \tag{46}$$

Since

$$\delta_1 = \frac{16\lambda^2}{\gamma\mu},\ \omega \ge \frac{8(\lambda+\delta_1)^2}{\mu\delta_1},$$

we have

$$\Theta_4(0,t) + \Theta_5(0,t) + \Theta_6(0,t) \le \frac{1}{2}\Big[-\frac{\partial}{\partial z}E(0,t)\Big]. \tag{47}$$

Inserting (43) and (47) into (37), we have

$$-\frac{\partial}{\partial z}E(0,t) \le \frac{3}{4}\Big[-\frac{\partial}{\partial z}E(0,t)\Big] + \frac{1}{4}Q(0,t). \tag{48}$$

Combining (35), (36), and (48), we can obtain Lemma 4.
From Lemma 3, it follows that

$$\frac{\partial}{\partial z}\Big\{E(z,t)e^{m_1 z}\Big\} \le 0. \tag{49}$$

Integrating (49) from 0 to z, we have

$$E(z,t) \le E(0,t)e^{-m_1 z}. \tag{50}$$

Combining (50) and Lemma 4, we can obtain Theorem 1. □

4. The Proof of Theorem 2

From Theorem 1, we have the following result

$$\int_0^t\int_{R_z} e^{-\omega\eta}(\xi - z)q_{i,j}q_{i,j}d\boldsymbol{x}d\eta \le \frac{2}{\omega\gamma^* m_1\delta_1}Q(0,t)e^{-m_1 z}. \tag{51}$$

From Lemma 4, it can be followed that

$$\int_0^t\int_R e^{-\omega\eta}q_{i,j}q_{i,j}d\boldsymbol{x}d\eta \le \frac{1}{\omega\gamma^*\delta_1}Q(0,t). \tag{52}$$

We let $F(z,t)$ denote

$$F(z,t) = -\mu\int_0^t\int_{R_z} e^{-\omega\eta}v_{i,\eta}v_{i,3}d\boldsymbol{x}d\eta - \lambda\int_0^t\int_{R_z} e^{-\omega\eta}v_{i,\eta}\Sigma_{i,3}d\boldsymbol{x}d\eta + \int_0^t\int_{R_z} e^{-\omega\eta}\pi v_{3,\eta}d\boldsymbol{x}d\eta. \tag{53}$$

By using the divergence theorem and (15)–(18), (22), we obtain

$$
\begin{aligned}
F(z,t) = {} & -\lambda\int_0^t\int_{R_z} e^{-\omega\eta}(\xi-z)_{,j}v_{i,\eta}\Sigma_{i,j}d\boldsymbol{x}d\eta - \mu\int_0^t\int_{R_z} e^{-\omega\eta}(\xi-z)_{,j}v_{i,\eta}v_{i,j}d\boldsymbol{x}d\eta \\
& + \int_0^t\int_{R_z} e^{-\omega\eta}(\xi-z)_{,i}\pi v_i d\boldsymbol{x}d\eta \\
= {} & \int_0^t\int_{R_z} e^{-\omega\eta}(\xi-z)v_{i,\eta}v_{i,\eta}d\boldsymbol{x}d\eta + \mu\int_0^t\int_{R_z} e^{-\omega\eta}(\xi-z)v_{i,j\eta}v_{i,j}d\boldsymbol{x}d\eta \\
& + \lambda\int_0^t\int_{R_z} e^{-\omega\eta}(\xi-z)v_{i,j\eta}\Sigma_{i,j}d\boldsymbol{x}d\eta \\
= {} & \frac{1}{2}\omega\mu\int_0^t\int_{R_z} e^{-\omega\eta}(\xi-z)v_{i,j}v_{i,j}d\boldsymbol{x}d\eta + \frac{1}{2}e^{-\omega t}\mu\int_{R_z}(\xi-z)v_{i,j}v_{i,j}d\boldsymbol{x} \\
& + \lambda e^{-\omega t}\int_{R_z}(\xi-z)\Sigma_{i,j}v_{i,j}d\boldsymbol{x} + \int_0^t\int_{R_z} e^{-\omega\eta}(\xi-z)v_{i,\eta}v_{i,\eta}d\boldsymbol{x}d\eta \\
& - \lambda\int_0^t\int_{R_z} e^{-\omega\eta}(\xi-z)\Sigma_{i,j\eta}v_{i,j}d\boldsymbol{x}d\eta + \omega\lambda\int_0^t\int_{R_z} e^{-\omega\eta}(\xi-z)\Sigma_{i,j}v_{i,j}d\boldsymbol{x}d\eta.
\end{aligned} \tag{54}
$$

We begin with

$$
\int_0^t\int_{R_z}(\xi-z)e^{-\omega\eta}\Big[\Sigma_{i,j\eta} + \widetilde{\gamma}q_{i,j} + \gamma^*\Sigma_{i,j} - v_{i,j}\Big]\Sigma_{i,j\eta}d\boldsymbol{x}d\eta = 0. \tag{55}
$$

Using (17), we have from (55) that

$$
\begin{aligned}
& \int_0^t\int_{R_z}(\xi-z)e^{-\omega\eta}\Sigma_{i,j\eta}\Sigma_{i,j\eta}d\boldsymbol{x}d\eta + \frac{1}{2}e^{-\omega t}\gamma^*\int_{R_z}(\xi-z)\Sigma_{i,j}\Sigma_{i,j}d\boldsymbol{x} \\
& - \int_0^t\int_{R_z}(\xi-z)e^{-\omega\eta}\Sigma_{i,j\eta}v_{i,j}d\boldsymbol{x}d\eta + \frac{1}{2}\omega\gamma^*\int_0^t\int_{R_z}(\xi-z)e^{-\omega\eta}\Sigma_{i,j}\Sigma_{i,j}d\boldsymbol{x}d\eta \\
& = -\widetilde{\gamma}\int_0^t\int_{R_z}(\xi-z)e^{-\omega\eta}q_{i,j}\Sigma_{i,j\eta}d\boldsymbol{x}d\eta.
\end{aligned} \tag{56}
$$

Now, we let

$$
\begin{aligned}
\mathcal{H}(z,t) = {} & \frac{1}{2}e^{-\omega t}\mu\int_{R_z}(\xi-z)v_{i,j}v_{i,j}d\boldsymbol{x} + \frac{1}{2}\omega\mu\int_0^t\int_{R_z} e^{-\omega\eta}(\xi-z)v_{i,j}v_{i,j}d\boldsymbol{x}d\eta \\
& + \int_0^t\int_{R_z} e^{-\omega\eta}(\xi-z)v_{i,\eta}v_{i,\eta}d\boldsymbol{x}d\eta + \int_0^t\int_{R_z}(\xi-z)e^{-\omega\eta}\Sigma_{i,j\eta}\Sigma_{i,j\eta}d\boldsymbol{x}d\eta \\
& + \frac{1}{2}e^{-\omega t}\gamma^*\int_{R_z}(\xi-z)\Sigma_{i,j}\Sigma_{i,j}d\boldsymbol{x} + \frac{1}{2}\omega\gamma^*\int_0^t\int_{R_z}(\xi-z)e^{-\omega\eta}\Sigma_{i,j}\Sigma_{i,j}d\boldsymbol{x}d\eta \\
& + \lambda e^{-\omega t}\int_{R_z}(\xi-z)\Sigma_{i,j}v_{i,j}d\boldsymbol{x} + \omega\lambda\int_0^t\int_{R_z} e^{-\omega\eta}(\xi-z)\Sigma_{i,j}v_{i,j}d\boldsymbol{x}d\eta \\
& - \lambda\int_0^t\int_{R_z} e^{-\omega\eta}(\xi-z)\Sigma_{i,j\eta}v_{i,j}d\boldsymbol{x}d\eta - \int_0^t\int_{R_z}(\xi-z)e^{-\omega\eta}\Sigma_{i,j\eta}v_{i,j}d\boldsymbol{x}d\eta.
\end{aligned} \tag{57}
$$

Combining (53) and (56), we can have

$$
\begin{aligned}
\mathcal{H}(z,t) = {} & -\lambda\int_0^t\int_{R_z} e^{-\omega\eta}v_{i,\eta}\Sigma_{i,3}d\boldsymbol{x}d\eta - \mu\int_0^t\int_{R_z} e^{-\omega\eta}v_{i,\eta}v_{i,3}d\boldsymbol{x}d\eta \\
& + \int_0^t\int_{R_z} e^{-\omega\eta}\pi v_{3,\eta}d\boldsymbol{x}d\eta - \widetilde{\gamma}\delta_2\int_0^t\int_{R_z}(\xi-z)e^{-\omega\eta}q_{i,j}\Sigma_{i,j\eta}d\boldsymbol{x}d\eta \\
\doteq {} & \sum_{i=1}^{4} I_i(z,t).
\end{aligned} \tag{58}
$$

It follows from (57) that

$$
\begin{aligned}
-\frac{\partial}{\partial z}\mathcal{H}(z,t) &= \frac{1}{2}e^{-\omega t}\mu\int_{R_z} v_{i,j}v_{i,j}d\boldsymbol{x} + \frac{1}{2}\omega\mu\int_0^t\int_{R_z} e^{-\omega\eta}v_{i,j}v_{i,j}d\boldsymbol{x}d\eta \\
&+ \int_0^t\int_{R_z} e^{-\omega\eta}v_{i,\eta}v_{i,\eta}d\boldsymbol{x}d\eta + \frac{1}{2}\delta_2\omega\gamma^*\int_0^t\int_{R_z} e^{-\omega\eta}\Sigma_{i,j}\Sigma_{i,j}d\boldsymbol{x}d\eta \\
&+ \delta_2\int_0^t\int_{R_z} e^{-\omega\eta}\Sigma_{i,j\eta}\Sigma_{i,j\eta}d\boldsymbol{x}d\eta + \frac{1}{2}\delta_2 e^{-\omega t}\gamma^*\int_{R_z}\Sigma_{i,j}\Sigma_{i,j}d\boldsymbol{x} \\
&+ \lambda e^{-\omega t}\int_{R_z}\Sigma_{i,j}v_{i,j}d\boldsymbol{x} + \omega\lambda\int_0^t\int_{R_z} e^{-\omega\eta}\Sigma_{i,j}v_{i,j}d\boldsymbol{x}d\eta \\
&- (\lambda+\delta_2)\int_0^t\int_{R_z} e^{-\omega\eta}\Sigma_{i,j\eta}v_{i,j}d\boldsymbol{x}d\eta,
\end{aligned}
\tag{59}
$$

where δ_2 is a positive constant.

In view of Hölder's and Young's inequalities, we obtain

$$
\begin{aligned}
\lambda e^{-\omega t}\int_{R_z}\Sigma_{i,j}v_{i,j}d\boldsymbol{x} &\geq -\frac{1}{2}e^{-\omega t}\mu\int_{R_z} v_{i,j}v_{i,j}d\boldsymbol{x} - \frac{\lambda^2}{2\mu}e^{-\omega t}\int_{R_z}\Sigma_{i,j}\Sigma_{i,j}d\boldsymbol{x}, \\
\omega\lambda\int_0^t\int_{R_z} e^{-\omega\eta}\Sigma_{i,j}v_{i,j}d\boldsymbol{x}d\eta &\geq -\frac{1}{8}\omega\mu\int_0^t\int_{R_z} e^{-\omega\eta}v_{i,j}v_{i,j}d\boldsymbol{x}d\eta \\
&\quad - \frac{2\lambda^2\omega}{\mu}\int_0^t\int_{R_z} e^{-\omega\eta}\Sigma_{i,j}\Sigma_{i,j}d\boldsymbol{x}d\eta, \\
(\lambda+\delta_2)\int_0^t\int_{R_z} e^{-\omega\eta}\Sigma_{i,j\eta}v_{i,j}d\boldsymbol{x}d\eta &\geq -\frac{1}{8}\omega\mu\int_0^t\int_{R_z} e^{-\omega\eta}v_{i,j}v_{i,j}d\boldsymbol{x}d\eta \\
&\quad - \frac{2(\lambda+\delta_2)^2}{\omega\mu}\int_0^t\int_{R_z} e^{-\omega\eta}\Sigma_{i,j\eta}\Sigma_{i,j\eta}d\boldsymbol{x}d\eta.
\end{aligned}
$$

Inserting the above inequalities into (56) and choosing $\delta_2 \geq \max\{\frac{\lambda}{\gamma^*\mu}, \frac{4\lambda^2}{\gamma^*\mu}\}, \omega = \frac{4(\lambda+\delta_2)^2}{\mu\delta_2}$, we obtain

$$
\begin{aligned}
-\frac{\partial}{\partial z}\mathcal{H}(z,t) &\geq \frac{1}{4}\omega\mu\int_0^t\int_{R_z} e^{-\omega\eta}v_{i,j}v_{i,j}d\boldsymbol{x}d\eta + \int_0^t\int_{R_z} e^{-\omega\eta}v_{i,\eta}v_{i,\eta}d\boldsymbol{x}d\eta \\
&+ \frac{1}{4}\delta_2\omega\gamma^*\int_0^t\int_{R_z} e^{-\omega\eta}\Sigma_{i,j}\Sigma_{i,j}d\boldsymbol{x}d\eta + \frac{1}{2}\delta_2\int_0^t\int_{R_z} e^{-\omega\eta}\Sigma_{i,j\eta}\Sigma_{i,j\eta}d\boldsymbol{x}d\eta.
\end{aligned}
\tag{60}
$$

Next, we use similar methods to those in (24), (25), and (34) to obtain

$$
I_1(z,t) \leq \frac{\sqrt{\mu}}{\sqrt{\omega}}\Big[\frac{1}{4}\omega\mu\int_0^t\int_{R_z} e^{-\omega\eta}v_{i,3}v_{i,3}d\boldsymbol{x}d\eta + \int_0^t\int_{R_z} e^{-\omega\eta}v_{i,\eta}v_{i,\eta}d\boldsymbol{x}d\eta\Big], \tag{61}
$$

$$
I_2(z,t) \leq \frac{\lambda}{\sqrt{\gamma^*\delta_2\omega}}\Big[\frac{1}{4}\omega\gamma^*\delta_2\int_0^t\int_{R_z} e^{-\omega\eta}\Sigma_{i,3}\Sigma_{i,3}d\boldsymbol{x}d\eta + \int_0^t\int_{R_z} e^{-\omega\eta}v_{i,\eta}v_{i,\eta}d\boldsymbol{x}d\eta\Big], \tag{62}
$$

$$
\begin{aligned}
I_3(z,t) &\leq \Big[\frac{\sqrt{k_2}}{\sqrt{\lambda_1}} + \frac{\sqrt{k_2\mu}}{\sqrt{\omega}} + \frac{\sqrt{k_2}\lambda}{\sqrt{\gamma^*\delta_1\omega}}\Big]\Big[-\frac{\partial}{\partial z}F(z,t)\Big], \\
I_4(z,t) &\leq -\frac{1}{4}\delta_2\int_0^t\int_{R_z}(\xi-z)e^{-\omega\eta}\Sigma_{i,j\eta}\Sigma_{i,j\eta}d\boldsymbol{x}d\eta + \delta_2\widetilde{\gamma}^2\int_0^t\int_{R_z}(\xi-z)e^{-\omega\eta}q_{i,j}q_{i,j}d\boldsymbol{x}d\eta.
\end{aligned}
\tag{63}
$$

Combining (60)–(63), we have

$$
\mathcal{H}(z,t) \leq \frac{1}{2m_2}\Big[-\frac{\partial}{\partial z}\mathcal{H}(z,t)\Big] + \frac{1}{2}\mathcal{H}(z,t) + \delta_2\widetilde{\gamma}^2\int_0^t\int_{R_z}(\xi-z)e^{-\omega\eta}q_{i,j}q_{i,j}d\boldsymbol{x}d\eta, \tag{64}
$$

where
$$\frac{1}{2m_2}=\frac{\sqrt{\mu}}{\sqrt{\omega}}+\frac{\lambda}{\sqrt{\gamma^*\delta_2\omega}}+\frac{\sqrt{k_2}}{\sqrt{\lambda_1}}+\frac{\sqrt{k_2\mu}}{\sqrt{\omega}}+\frac{\sqrt{k_2}\lambda}{\sqrt{\gamma^*\delta_1\omega}}.$$

Combining (51) and (64), we have
$$\mathcal{H}(z,t)\le\frac{1}{m_2}\Big[-\frac{\partial}{\partial z}\mathcal{H}(z,t)\Big]+\frac{4\delta_2}{\omega\gamma^*m_1\delta_1}Q(0,t)\widetilde{\gamma}^2e^{-m_1z}, \tag{65}$$
or
$$\frac{\partial}{\partial z}\Big[\mathcal{H}(z,t)e^{m_2z}\Big]\le\frac{4m_2\delta_2}{\omega\gamma^*m_1\delta_1}Q(0,t)\widetilde{\gamma}^2e^{-(m_1-m_2)z}, \tag{66}$$

Integrating (61) from 0 to z, we have
$$\mathcal{H}(z,t)\le\mathcal{H}(0,t)e^{-m_2z}+\frac{4m_2\delta_2}{\omega\gamma^*m_1(m_1-m_2)\delta_1}Q(0,t)\widetilde{\gamma}^2\Big[e^{-m_1z}-e^{-m_2z}\Big], if\ m_1\neq m_2, \tag{67}$$
$$\mathcal{H}(z,t)\le\mathcal{H}(0,t)e^{-m_2z}+\frac{4m_2\delta_2}{\omega\gamma^*m_1\delta_1}Q(0,t)\widetilde{\gamma}^2ze^{-m_2z},\qquad if\ m_1=m_2. \tag{68}$$

Now, we choose $z=0$ in (65) to obtain
$$\mathcal{H}(0,t)\le\frac{1}{m_2}\Big[-\frac{\partial}{\partial z}\mathcal{H}(0,t)\Big]+\frac{4\delta_2}{\omega\gamma^*m_1\delta_1}Q(0,t)\widetilde{\gamma}^2, \tag{69}$$

From (68), we have
$$\begin{aligned}-\frac{\partial}{\partial z}\mathcal{H}(0,t)=&-\mu\int_0^t\int_D e^{-\omega\eta}v_{i,\eta}v_{i,3}dAd\eta-\lambda\int_0^t\int_D e^{-\omega\eta}v_{i,\eta}\Sigma_{i,3}dAd\eta\\&+\int_0^t\int_D e^{-\omega\eta}\pi v_{3,\eta}dAd\eta-\widetilde{\gamma}\delta_2\int_0^t\int_R e^{-\omega\eta}q_{i,j}\Sigma_{i,j\eta}dxd\eta.\end{aligned} \tag{70}$$

From (60), we obtain
$$\begin{aligned}-\frac{\partial}{\partial z}\mathcal{H}(0,t)\ge&\frac{1}{4}\omega\mu\int_0^t\int_R e^{-\omega\eta}v_{i,j}v_{i,j}dxd\eta+\int_0^t\int_R e^{-\omega\eta}v_{i,\eta}v_{i,\eta}dxd\eta\\&+\frac{1}{4}\delta_2\omega\gamma^*\int_0^t\int_R e^{-\omega\eta}\Sigma_{i,j}\Sigma_{i,j}dxd\eta+\frac{1}{2}\delta_2\int_0^t\int_R e^{-\omega\eta}\Sigma_{i,j\eta}\Sigma_{i,j\eta}dxd\eta.\end{aligned} \tag{71}$$

In view of (20) and (21), we can conclude from (70) that
$$\begin{aligned}-\frac{\partial}{\partial z}\mathcal{H}(0,t)=&-\widetilde{\gamma}\int_0^t\int_R e^{-\omega\eta}q_{i,j}\Sigma_{i,j\eta}dxd\eta\\\le&\frac{1}{4}\delta_2\int_0^t\int_R e^{-\omega\eta}\Sigma_{i,j\eta}\Sigma_{i,j\eta}dxd\eta+\widetilde{\gamma}^2\delta_2\int_0^t\int_R e^{-\omega\eta}q_{i,j}q_{i,j}dxd\eta.\end{aligned} \tag{72}$$

Using (52) and (71), we have from (72) that
$$-\frac{\partial}{\partial z}\mathcal{H}(0,t)\le\frac{1}{2}\Big[-\frac{\partial}{\partial z}H(0,t)\Big]+\frac{1}{\omega\gamma^*\delta_1\delta_2}Q(0,t)\widetilde{\gamma}^2,$$
or
$$-\frac{\partial}{\partial z}\mathcal{H}(0,t)\le\frac{2\delta_2}{\omega\gamma^*\delta_1}Q(0,t)\widetilde{\gamma}^2. \tag{73}$$

Inserting (73) into (69), we have
$$\mathcal{H}(0,t)\le\frac{2\delta_2}{\omega\gamma^*\delta_1}Q(0,t)\widetilde{\gamma}^2\Big[\frac{1}{m_2}+\frac{2}{m_1}\Big]. \tag{74}$$

Inserting (74) into (67) and (68), we have

$$\mathcal{H}(z,t) \leq \frac{2\delta_2}{\omega\gamma^*\delta_1}Q(0,t)\widetilde{\gamma}^2\Big[\frac{1}{m_2}+\frac{2}{m_1}\Big]e^{-m_2 z} + \frac{4m_2\delta_2}{\omega\gamma^* m_1(m_1-m_2)\delta_1}Q(0,t)\widetilde{\gamma}^2\Big[e^{-m_1 z}-e^{-m_2 z}\Big], if\ m_1 \neq m_2, \quad (75)$$

$$\mathcal{H}(z,t) \leq \frac{2\delta_2}{\omega\gamma^*\delta_1}Q(0,t)\widetilde{\gamma}^2\Big[\frac{1}{m_2}+\frac{2}{m_1}\Big]e^{-m_2 z} + \frac{4m_2\delta_2}{\omega\gamma^* m_1\delta_1}Q(0,t)\widetilde{\gamma}^2 z e^{-m_2 z},\ if\ m_1 = m_2. \quad (76)$$

Inequalities (75) and (76) show that $\mathcal{H}(z,t)$ decays exponentially as $z \to \infty$. Therefore, integrating (60) from z to ∞, we obtain

$$F(z,t) \geq \frac{1}{4}\omega\mu\int_0^t\int_{R_z}(\xi-z)e^{-\omega\eta}v_{i,j}v_{i,j}dxd\eta + \int_0^t\int_{R_z}(\xi-z)e^{-\omega\eta}v_{i,\eta}v_{i,\eta}dxd\eta + \frac{1}{4}\delta_2\omega\gamma^*\int_0^t\int_{R_z}(\xi-z)e^{-\omega\eta}\Sigma_{i,j}\Sigma_{i,j}dxd\eta + \frac{1}{2}\delta_2\int_0^t\int_{R_z}(\xi-z)e^{-\omega\eta}\Sigma_{i,j\eta}\Sigma_{i,j\eta}dxd\eta. \quad (77)$$

Combining (75), (76), and (77), we can obtain Theorem 2.

5. Conclusions

In this article, we used energy estimations and prior estimations to obtain the properties and structural stability of the solution on a cylinder. This can also provide reference for further research on other fluid equation systems.

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