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Article

Bridging the p-Special Functions between the Generalized Hyperbolic and Trigonometric Families

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Abstract: Here, we study the extension of p-trigonometric functions sinp and cosp family in complex domains and p-hyperbolic functions sinhp and the coshp family in hyperbolic complex domains. These functions satisfy analogous relations as their classical counterparts with some unknown properties. We show the relationship of these two classes of special functions viz. p-trigonometric and p-hyperbolic functions with imaginary arguments. We also show many properties and identities related to the analogy between these two groups of functions. Further, we extend the research bridging the concepts of hyperbolic and elliptical complex numbers to show the properties of logarithmic functions with complex arguments.

Keywords: p-trigonometric functions; p-hyperbolic functions; p-complex logarithm; special functions

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1. Introduction

The generalized complex numbers were introduced in [1,2] as follows:

$$\mathbb{C}_p := \{\mu + i\gamma : \mu;\gamma \in \mathbb{R}; i^2 = p; p \in \mathbb{R}_-\}.$$

It was observed that $\mathbb{C}_p$ corresponds to the set of elliptical complex numbers. For $\xi_1 = \mu_1 + i\gamma_1$ and $\xi_2 = \mu_2 + i\gamma_2 \in \mathbb{C}_p$, addition and multiplication are defined by:

$$\xi_1 + \xi_2 = (\mu_1 + i\gamma_1) + (\mu_2 + i\gamma_2) = (\mu_1 + \mu_2) + i(\gamma_1 + \gamma_2),$$

and

$$\xi_1\xi_2 = (\mu_1\mu_2 + p\gamma_1\gamma_2) + i(\mu_1\gamma_2 + \mu_2\gamma_1).$$

As is well known, $\mathbb{C}_p$ is a field under these two operations [1]. On the other hand, the p-magnitude of $\xi = \mu + i\gamma \in \mathbb{C}_p$ is $\|\xi\|_p^2 = \mu^2 - p\gamma^2$. The unit circle in $\mathbb{C}_p$ is an Euclidean ellipse, which is given by the equation $\mu^2 - p\gamma^2 = 1$. Specially, if $p = -1$, this ellipse matches the Euclidean unit.

Let $\xi = \mu + i\gamma \in \mathbb{C}_p$; it was observed in [1] that the number ξ can be expressed with a position vector (see [1]). The arc of ellipse between this vector and the real axis determines an elliptic angle θ_p. This angle is called p-argument of ξ. On the generalized complex numbers and elliptical complex numbers in the literature, we invite the interest of the readers to some interesting studies, namely [3–8] and the reference therein. The authors of [1] introduced in $\mathbb{C}_p$ the p-trigonometric functions p-cosine, p-sine and p-tangent as follows:

$$\operatorname{cosp}(\theta_p) = \cos\left(\sqrt{|p|}\theta_p\right), \tag{1}$$

$$\operatorname{sinp}(\theta_p) = \frac{1}{\sqrt{|p|}} \sin\left(\sqrt{|p|}\theta_p\right), \tag{2}$$

$$\operatorname{tanp}(\theta_p) = \frac{\operatorname{sinp}(\theta_p)}{\operatorname{cosp}(\theta_p)}. \tag{3}$$

We may then define the other p-trigonometric functions as:

$$\operatorname{cotp}(\theta_p) = \frac{\operatorname{cosp}(\theta_p)}{\operatorname{sinp}(\theta_p)} = \frac{\sqrt{|p|}\cos(\theta_p\sqrt{|p|})}{\sin(\theta_p\sqrt{|p|})} = \sqrt{|p|}\cot(\theta_p\sqrt{|p|}), \tag{4}$$

$$\operatorname{secp}(\theta_p) = \frac{1}{\operatorname{cosp}(\theta_p)} = \frac{1}{\cos(\theta_p\sqrt{|p|})} = \sec(\theta_p\sqrt{|p|}), \tag{5}$$

$$\operatorname{cosecp}(\theta_p) = \frac{1}{\operatorname{sinp}(\theta_p)} = \frac{\sqrt{|p|}}{\sin(\theta_p\sqrt{|p|})} = \sqrt{|p|}\operatorname{cosec}(\theta_p\sqrt{|p|}). \tag{6}$$

According to the generalized hyperbolic number system [9–13]:

$$\mathbb{H}_p = \{\xi = \mu + i\gamma : \mu, \gamma \in \mathbb{R},\ i^2 = p,\ p \in \mathbb{R}^+\}.$$

When $p = 1$, we get the hyperbolic numbers system:

$$\mathbb{H}_1 = \{\xi = \mu + i\gamma : \mu, \gamma \in \mathbb{R},\ i^2 = 1\}.$$

We have introduced the new concept of generalized p-hyperbolic functions related to the generalized hyperbolic number systems. We start by defining coshp, sinhp, tanhp, cothp, sechp and cosecp functions, which generalize the standard hyperbolic functions. These definitions run parallel to the definitions of generalization of p-trigonometric functions. For $p > 0$, we define the following p-hyperbolic functions as:

$$\operatorname{coshp}(\mu) = \cosh(\sqrt{p}\mu), \tag{7}$$

$$\operatorname{sinhp}(\mu) = \frac{1}{\sqrt{p}}\sinh(\sqrt{p}\mu), \tag{8}$$

$$\operatorname{tanhp}(\mu) = \frac{\operatorname{sinhp}(\mu)}{\operatorname{coshp}(\mu)} = \frac{\sinh(\sqrt{p}\mu)}{\sqrt{p}\cosh(\sqrt{p}\mu)}, \tag{9}$$

$$\operatorname{cothp}(\mu) = \frac{\operatorname{coshp}(\mu)}{\operatorname{sinhp}(\mu)} = \frac{\sqrt{p}\cosh(\sqrt{p}\mu)}{\sinh(\sqrt{p}\mu)}, \tag{10}$$

$$\operatorname{sechp}(\mu) = \frac{1}{\operatorname{coshp}(\mu)} = \frac{1}{\cosh(\sqrt{p}\mu)}, \tag{11}$$

$$\operatorname{cosechp}(\mu) = \frac{1}{\operatorname{sinhp}(\mu)} = \frac{\sqrt{p}}{\sinh(\sqrt{p}\mu)}. \tag{12}$$

In recent times, properties involving p-trigonometric and hyperbolic functions have become a subject of intense discussion, and there exists vast literature on such functions. For more information on this topic, one may refer to [14] and the references therein. The purpose of this paper is twofold. We begin with a short survey of results from [3,7]. Then, we extend the ideas from [14] to define corresponding generalization of hyperbolic functions and study relations of p-trigonometric and p-hyperbolic functions on a complex domain. The connection between the p-trigonometric and p-hyperbolic functions is established by the definition of such functions of a generalized complex number. We have developed a generalization of the usual logarithm and power of complex functions, based on the properties of p-generalized complex numbers. We have established some basic relations for the proposed p-logarithmic functions. For example, the p-logarithm of product and quotient of members of $\mathbb{C}_p$.

The use generalized trigonometric functions as the basis has already been studied by Harkin and Harkin [1]. However, many formal proofs on orthogonality and series expansions, etc., do not exist in the literature, unlike for other special functions. The main contribution of this paper is to show the duality between p-trigonometric and p-hyperbolic functions. Once we develop the relationship between the p-trigonometric and p-hyperbolic functions, this can lead to the solution of complex differential equation problems involving p-complex numbers. It is well-known that standard hyperbolic and trigonometric functions are solution of certain class of ODEs. Orthogonality of these basis functions can only be developed by first investigating the duality between the p-hyperbolic and p-trigonometric functions with complex arguments which is the main motivation of this paper.

2. The *p*-Trigonometric Functions with Generalized Complex Variables

In the following definitions, we introduce the concepts of p-trigonometric functions with a generalized complex variable.

Definition 1. *Following [3], for $\xi = \mu + i\gamma \in \mathbb{C}_P$, where $i^2 = p < 0$,*

$$\text{cosp}(\xi) = \frac{e^{i\xi} + e^{-i\xi}}{2}, \tag{13}$$

$$\text{sinp}(\xi) = \frac{e^{i\xi} - e^{-i\xi}}{2i}. \tag{14}$$

Remark 1. *When $\xi = \mu \in \mathbb{R}$,*

$$\text{cosp}(\mu) = \frac{e^{i\mu} + e^{-i\mu}}{2}, \tag{15}$$

$$\text{sinp}(\mu) = \frac{e^{i\mu} - e^{-i\mu}}{2i}. \tag{16}$$

Remark 2. *When $\xi = \mu \in \mathbb{R}$ and $p = -1$ ($i^2 = -1$), we obtain the classical relations:*

$$\cos(\mu) = \frac{e^{i\mu} + e^{-i\mu}}{2}, \tag{17}$$

$$\sin(\mu) = \frac{e^{i\mu} - e^{-i\mu}}{2i}. \tag{18}$$

Lemma 1. *For all $\xi \in \mathbb{C}_p$ with $p < 0$, the following identity holds:*

$$\text{cosp}^2(\xi) - p\,\text{sinp}^2(\xi) = 1.$$

Definition 2. *For $\xi \in \mathbb{C}_p$, we define the p-trigonometric functions with a generalized complex variable:*

$$\text{tanp}(\xi) = \frac{\text{sinp}(\xi)}{\text{cosp}(\xi)}, \tag{19}$$

$$\text{cotp}(\xi) = \frac{\text{cosp}(\xi)}{\text{sinp}(\xi)} = \frac{1}{\text{tanp}(\xi)}, \tag{20}$$

$$\text{secp}(\xi) = \frac{1}{\text{cosp}(\xi)}, \tag{21}$$

$$\text{cosecp}(\xi) = \frac{1}{\text{sinp}(\xi)}. \tag{22}$$

3. The p-Hyperbolic Functions with Generalized Hyperbolic Complex Variables

In the following definition, we introduce the concepts of p-hyperbolic functions with hyperbolic complex variable [15,16].

Definition 3. *For $\xi = \mu + i\gamma \in \mathbb{H}_p$ where $i^2 = p > 0$,*

$$\text{coshp}(\xi) = \frac{e^{(\sqrt{p}\xi)} + e^{(-\sqrt{p}\xi)}}{2}, \tag{23}$$

$$\text{sinhp}(\xi) = \frac{e^{(\sqrt{p}\xi)} - e^{(-\sqrt{p}\xi)}}{2\sqrt{p}}. \tag{24}$$

Remark 3. *When $\xi = \mu \in \mathbb{R}$,*

$$\text{coshp}(\mu) = \frac{e^{\sqrt{p}\mu} + e^{-\sqrt{p}\mu}}{2}, \tag{25}$$

$$\text{sinhp}(\mu) = \frac{e^{\sqrt{p}\mu} - e^{-\sqrt{p}\mu}}{2\sqrt{p}}. \tag{26}$$

Remark 4. *When $\xi = \mu \in \mathbb{R}$ and $p = 1$ $(i^2 = 1)$, we obtain the classical relations:*

$$\cosh(\mu) = \frac{e^{\mu} + e^{-\mu}}{2}, \tag{27}$$

$$\sinh(\mu) = \frac{e^{\mu} - e^{-\mu}}{2}. \tag{28}$$

Remark 5. *When $p = 1$, we obtain:*

$$\text{coshp}(\mu) = \cosh(\mu), \tag{29}$$

$$\text{sinhp}(\mu) = \sinh(\mu), \tag{30}$$

$$\text{tanhp}(\mu) = \tanh(\mu), \tag{31}$$

$$\text{cothp}(\mu) = \coth(\mu). \tag{32}$$

Proposition 1. *For $p > 0$, the following identities hold:*

$$\text{coshp}(\mu) = \frac{e^{\sqrt{p}\mu} + e^{-\sqrt{p}\mu}}{2}, \tag{33}$$

$$\text{sinhp}(\mu) = \frac{e^{\sqrt{p}\mu} - e^{-\sqrt{p}\mu}}{2\sqrt{p}}, \tag{34}$$

$$\text{tanhp}(\mu) = \frac{e^{\sqrt{p}\mu} - e^{-\sqrt{p}\mu}}{\sqrt{p}\left(e^{\sqrt{p}\mu} + e^{-\sqrt{p}\mu}\right)}, \tag{35}$$

$$\text{cothp}(\mu) = \frac{\sqrt{p}\left(e^{\sqrt{p}\mu} + e^{-\sqrt{p}\mu}\right)}{e^{\sqrt{p}\mu} - e^{-\sqrt{p}\mu}}. \tag{36}$$

Proof.

$$\text{coshp}(\mu) = \cosh(\sqrt{p}\mu) = \frac{e^{\sqrt{p}\mu} + e^{-\sqrt{p}\mu}}{2}, \tag{37}$$

$$\mathrm{sinhp}(\mu) = \frac{1}{\sqrt{p}}\sinh(\sqrt{p}\mu) = \frac{e^{\sqrt{p}\mu} - e^{-\sqrt{p}\mu}}{2\sqrt{p}}, \tag{38}$$

$$\mathrm{tanhp}(\mu) = \frac{\mathrm{sinhp}(\mu)}{\mathrm{coshp}(\mu)} = \frac{e^{\sqrt{p}\mu} - e^{-\sqrt{p}\mu}}{\sqrt{p}\left(e^{\sqrt{p}\mu} + e^{-\sqrt{p}\mu}\right)}, \tag{39}$$

$$\mathrm{cothp}(\mu) = \frac{\mathrm{coshp}(\mu)}{\mathrm{sinhp}(\mu)} = \frac{\sqrt{p}\left(e^{\sqrt{p}\mu} + e^{-\sqrt{p}\mu}\right)}{e^{\sqrt{p}\mu} - e^{-\sqrt{p}\mu}}. \tag{40}$$

□

Remark 6. *When $p = 1$, we obtain the following classical identities:*

$$\cosh(\mu) = \frac{e^{\mu} + e^{-\mu}}{2}, \tag{41}$$

$$\sinh(\mu) = \frac{e^{\mu} - e^{-\mu}}{2}, \tag{42}$$

$$\tanh(x) = \frac{e^{\mu} - e^{-\mu}}{e^{\mu} + e^{-\mu}}, \tag{43}$$

$$\coth(\mu) = \frac{e^{\mu} + e^{-\mu}}{e^{\mu} - e^{-\mu}}. \tag{44}$$

Proposition 2. *For $p < 0$, the following identities hold:*

$$\mathrm{cosp}(i\mu) = \frac{e^{p\mu} + e^{-p\mu}}{2}, \tag{45}$$

$$\mathrm{cosp}(i\mu) = \cosh|p|\left(\sqrt{|p|}\mu\right), \tag{46}$$

$$\cosh|p|(i\mu) = \mathrm{cosp}\left(\sqrt{|p|}\mu\right). \tag{47}$$

Proof. From identity (13), we have:

$$\mathrm{cosp}(i\mu) = \frac{e^{i^2\mu} + e^{-i^2\mu}}{2} = \frac{e^{p\mu} + e^{-p\mu}}{2}.$$

On the other hand, since $p = -\mid p \mid$ for $p < 0$, we my write:

$$\begin{aligned} \mathrm{cosp}(i\mu) &= \frac{e^{p\mu} + e^{-p\mu}}{2} \\ &= \frac{e^{-|p|\mu} + e^{|p|\mu}}{2} \\ &= \frac{e^{\sqrt{|p|}\left(\sqrt{|p|}\mu\right)} + e^{-\sqrt{|p|}\left(\sqrt{|p|}\mu\right)}}{2} \\ &= \cosh|p|\left(\sqrt{|p|}\mu\right) \quad (\text{by } (25)). \end{aligned}$$

Similarly, from identity (23), we have:

$$\begin{aligned}\cosh|p|(i\mu) &= \frac{e^{\sqrt{|p|}i\mu} + e^{-\sqrt{|p|}i\mu}}{2} \\ &= \frac{e^{i\sqrt{|p|}\mu} + e^{-i\sqrt{|p|}\mu}}{2} \\ &= \operatorname{cosp}\left(\sqrt{|p|}\mu\right) \quad (by\ (13)).\end{aligned}$$

□

Proposition 3. *For $p < 0$, the following statements are true:*

$$\operatorname{sinp}(i\mu) = \frac{e^{p\mu} - e^{-p\mu}}{2i}, \tag{48}$$

$$i\operatorname{sinp}(i\mu) = -\sqrt{|p|}\,\sinh|p|\left(\sqrt{|p|}\mu\right), \tag{49}$$

$$\sqrt{|p|}\,\sinh|p|(i\mu) = i\,\operatorname{sinp}\left(\sqrt{|p|}\,\mu\right). \tag{50}$$

Proof. From identity (14), we have:

$$\operatorname{sinp}(i\mu) = \frac{e^{i^2\mu} - e^{-i^2\mu}}{2i} = \frac{e^{p\mu} - e^{-p\mu}}{2i}.$$

On the other hand, since $p = -\mid p \mid$ for $p < 0$, we may write:

$$\begin{aligned}\operatorname{sinp}(i\mu) &= \frac{e^{p\mu} - e^{-p\mu}}{2i} \\ &= \frac{e^{-|p|\mu} - e^{|p|\mu}}{2i}.\end{aligned}$$

From which it follows that

$$\begin{aligned}i\operatorname{sinp}(i\mu) &= \frac{e^{-\left(\sqrt{|p|}\sqrt{|p|}\mu\right)} - e^{\left(\sqrt{|p|}\sqrt{|p|}\mu\right)}}{2} \\ &= \frac{\sqrt{|p|}\left(e^{-(\sqrt{|p|}\sqrt{|p|}\mu)} - e^{(\sqrt{|p|}\sqrt{|p|}\mu)}\right)}{2\sqrt{|p|}} \\ &= -\sqrt{|p|}\sinh|p|\left(\sqrt{|p|}\mu\right) \qquad (\text{by } (26)).\end{aligned}$$

From identity (24), we have:

$$\begin{aligned}\sqrt{|p|}\,\sinh|p|(i\mu) &= \frac{e^{\sqrt{|p|}\,i\mu} - e^{-\sqrt{|p|}\,i\mu}}{2} \\ &= \frac{e^{i\,\sqrt{|p|}\,\mu} - e^{-i\,\sqrt{|p|}\,\mu}}{2} \\ &= i\,\frac{e^{i\,\sqrt{|p|}\,\mu} - e^{-i\,\sqrt{|p|}\,\mu}}{2i} \\ &= i\operatorname{sinp}\left(\sqrt{|p|}\,\mu\right) \qquad (\text{by } (14)).\end{aligned}$$

□

Definition 4. *For $\xi \in \mathbb{H}_p$, we define the p-hyperbolic functions:*

$$\operatorname{tanhp}(\xi) = \frac{\operatorname{sinhp}(\xi)}{\operatorname{coshp}(\xi)}, \tag{51}$$

$$\operatorname{cothp}(\xi) = \frac{\operatorname{coshp}(\xi)}{\operatorname{sinhp}(\xi)} = \frac{1}{\operatorname{tanhp}(\xi)}, \tag{52}$$

$$\operatorname{sechp}(\xi) = \frac{1}{\operatorname{coshp}(\xi)}, \tag{53}$$

$$\operatorname{cosechp}(\xi) = \frac{1}{\operatorname{sinhp}(\xi)}. \tag{54}$$

Proposition 4. *The following identities hold for $\mu \in \mathbb{R}$ and $p < 0$:*

$$\operatorname{tanp}(i\mu) = \frac{i}{\sqrt{|p|}} \tanh|p|\left(\sqrt{|p|}\mu\right), \tag{55}$$

$$\operatorname{cotp}(i\mu) = -\frac{i}{\sqrt{|p|}} \coth|p|\left(\sqrt{|p|}\mu\right). \tag{56}$$

Proof. By taking into account the identities (46) and (49), we obtain:

$$\begin{aligned} \operatorname{tanp}(i\mu) &= \frac{\operatorname{sinp}(i\mu)}{\operatorname{cosp}(i\mu)} \\ &= \frac{i}{\sqrt{|p|}} \frac{\sinh|p|\left(\sqrt{|p|}\mu\right)}{\cosh|p|\left(\sqrt{|p|}\mu\right)} \\ &= \frac{i}{\sqrt{|p|}} \tanh|p|\left(\sqrt{|p|}\mu\right). \end{aligned}$$

A similar calculation based on identities (46) and (49) yields:

$$\operatorname{cotp}(i\mu) = -\frac{i}{\sqrt{|p|}} \coth|p|\left(\sqrt{|p|}\mu\right).$$

□

Proposition 5. *The following identities hold for $\mu \in \mathbb{R}$ and $p < 0$:*

$$\operatorname{secp}(i\mu) = \operatorname{sech}|p|\left(\sqrt{|p|}\mu\right), \tag{57}$$

$$\operatorname{cosecp}(i\mu) = -i \operatorname{cosech}|p|\left(\sqrt{|p|}\mu\right). \tag{58}$$

Proof.

$$\operatorname{secp}(i\mu) = \frac{1}{\operatorname{cosp}(i\mu)} = \frac{1}{\cosh|p|(\mu)} = \operatorname{sech}|p|\left(\sqrt{|p|}\mu\right).$$

Similarly,

$$\operatorname{cosecp}(i\mu) = \frac{1}{\operatorname{sinp}(i\mu)} = \frac{\sqrt{|p|}}{i \sinh|p|\left(\sqrt{|p|}\mu\right)} = \frac{\sqrt{|p|}}{i} \operatorname{cosech}|p|\left(\sqrt{|p|}\mu\right).$$

□

4. The *p*-Complex Logarithmic Functions and *p*-Complex Powers of Generalized Complex Numbers

The multi-valued function log is defined by:

$$\log(\xi) = \ln(|\xi|) + i.\arg(\xi), \quad \xi \in \mathbb{C}, \tag{59}$$

where $\xi \neq 0$ is called the complex logarithm.

Remark 7. *For $\xi \in \mathbb{C}$ with $\xi \neq 0$, it is well known that*

$$log(\xi) = \log|\xi| + i\big(\theta + 2k\pi\big),\ k \in \mathbb{Z} \tag{60}$$

where, $-\pi \leq \theta < \pi$.

Moreover,

$$\xi^a = e^{a \log(\xi)} = e^{a\left(\ln|\xi| + i\left(\theta + 2k\pi\right)\right)}. \tag{61}$$

Let $\xi = \mu + i\gamma$ be a number in $\mathbb{C}_p^*$, where:

$$\mathbb{C}_p^* = \Big\{\mu + i\gamma,\ ;\mu,\gamma \in \mathbb{R},\ i^2 = p,\ p \in \mathbb{R}_-\ \ (p < 0)\Big\}.$$

The p-magnitude of $\xi = \mu + i\gamma \in \mathbb{C}_p^*$ is given by $\|\xi\|_p = \sqrt{|\mu^2 - p\gamma^2|}$.
For $i^2 = p < 0$ we have $e^{i\mu} = \operatorname{cosp}(\mu) + i\operatorname{sinp}(\mu)$ and

$$\begin{aligned} e^{i\left(\theta_p + \frac{2\pi}{\sqrt{|p|}}\right)} &= \operatorname{cosp}\left(\theta_p + \frac{2\pi}{\sqrt{|p|}}\right) + i \cdot \operatorname{sinp}\left(\theta_p + \frac{2\pi}{\sqrt{|p|}}\right) \\ &= \operatorname{cosp}\big(\theta_p\big) + i \cdot \operatorname{sinp}\big(\theta_p\big) \\ &= e^{i\theta_p}. \end{aligned}$$

Remark 8. *We observe that*

$$e^{i\frac{2\pi k}{\sqrt{|p|}}} = 1, \quad k = 0, \pm 1,\ \pm 2,\ \cdots. \tag{62}$$

According to [1], it is well known for $\xi = \varphi + i\psi \in \mathbb{C}_p$ that we have:

$$\begin{aligned} e^{\xi} &= e^{\varphi + i\psi} \\ &= e^{\varphi} \cdot e^{i\psi} \\ &= e^{\varphi}\big(\operatorname{cosp}(\psi) + i\operatorname{sinp}(\psi)\big). \end{aligned}$$

For $\xi \in \mathbb{C}_p \neq 0$, we need to define $\omega = \log_p(\xi)$ for which $p^{\omega} = \xi$.

Definition 5. *Let $\xi = \mu + i\gamma \in \mathbb{C}_p^*$ with $\xi \neq 0$. The p-complex logarithm of ξ is defined by:*

$$\begin{aligned} \log_p(\xi) &= \log(||\xi||_p) + i \cdot arg_p(\xi) \quad (63)\\ &= \log\left(\sqrt{|\mu^2 - p\gamma^2|}\right) + i\left(\theta_p + \frac{2\pi k}{\sqrt{|p|}}\right). \quad (64) \end{aligned}$$

Definition 6. *For $\xi \in \mathbb{C}_p$, with $\xi \neq 0$ the principal value of p-complex logarithm is defined by:*

$$\mathrm{Log}_p(\xi) = ln(||\xi||_p) + i\theta_p, \quad (65)$$

where, $\theta_p \in \left(\frac{-\pi}{\sqrt{|p|}}, \frac{\pi}{\sqrt{|p|}}\right]$.

Remark 9. *We observe that for $\xi \in \mathbb{C}_p$ with $\xi \neq 0$, we have:*

$$\log_p(\xi) = \mathrm{Log}_p(\xi) + i\frac{2k\pi}{\sqrt{|p|}},\ k \in \mathbb{Z}. \quad (66)$$

Proposition 6. *Let $\xi_1, \xi_2 \in \mathbb{C}_p$ with $\xi_1, \xi_2 \neq 0$, then the following identities hold:*

$$\log_p(\xi_1.\xi_2) = \log_p(\xi_1) + \log_p(\xi_2). \quad (67)$$

Proof. According to [1] we may write

$$\xi_1 = ||\xi_1||_p\big(\mathrm{cosp}(\theta_p) + i\mathrm{sinp}(\theta_p)\big),$$

and

$$\xi_2 = ||\xi_2||_p\big(\mathrm{cosp}(\theta'_p) + i\mathrm{sinp}(\theta'_p)\big).$$

Then,

$$\xi_1.\xi_2 = ||\xi_1||_p \,.\, ||\xi_2||_p\left(\mathrm{cosp}(\theta_p + \theta'_p) + i\mathrm{sinp}(\theta_p + \theta'_p)\right).$$

From which we obtain:

$$\begin{aligned} \log_p(\xi_1 \cdot \xi_2) &= \ln\big(||\xi_1||_p \cdot ||\xi_2||_p\big) + i(\theta_p + \theta'_p)\\ &= \ln\big(||\xi_1||_p\big) + \ln\big(||\xi_2||_p\big) + i(\theta_p) + i(\theta'_p)\\ &= \ln\big(||\xi_1||_p\big) + i\theta_p + \ln\big(||\xi_2||_p\big) + i\,\theta'_p\\ &= \log_p(\xi_1) + \log_p(\xi_2). \end{aligned}$$

Therefore, the proof is complete. □

Remark 10. *In general, for $\xi_1, \xi_2 \in \mathbb{C}_p$ with $\xi_1.\xi_2 \neq 0$, the following identity does not hold:*

$$\mathrm{Log}_p(\xi_1.\xi_2) \neq \mathrm{Log}_p(\xi_1) + \mathrm{Log}_p(\xi_2), \quad (68)$$

as shown in the following example.

Example 1. *Consider $\xi_1 = \xi_2 = p < 0$; we have:*

$$\begin{aligned} \mathrm{Log}_p(\xi_1 \cdot \xi_2) = \mathrm{Log}_p(p^2) &= ln(|p^2|) + i0\\ &= 2\,ln(|p|). \end{aligned}$$

However,

$$\begin{aligned} \mathrm{Log}_p(\xi_1) &= \mathrm{Log}_p(p) \\ &= ln(|p|) + i\frac{\pi}{\sqrt{|p|}}. \end{aligned}$$

Similarly,

$$\mathrm{Log}_p(\xi_2) = ln(|p|) + i\frac{\pi}{\sqrt{|p|}}.$$

Therefore,

$$\mathrm{Log}_p(\xi_1) + \mathrm{Log}_p(\xi_2) = 2ln(|p|) + i\frac{2\pi}{\sqrt{|p|}}.$$

From the above calculation, we obtain:

$$\mathrm{Log}_p(\xi_1 \, .\xi_2) \neq \mathrm{Log}_p(\xi_1) + \mathrm{Log}_p(\xi_2).$$

Proposition 7. *Let $\xi_1, \xi_2 \in \mathbb{C}_p$, such that $\xi_1 = ||\xi_1||_p \, e^{i\theta_p}$ and $\xi_2 = ||\xi_2||_p \, e^{i\varphi_p}$ for which $\{\theta_p, \varphi_p\} \in \left(\frac{-\pi}{2\sqrt{|p|}}, \frac{\pi}{2\sqrt{|p|}}\right]$. Then,*

$$\mathrm{Log}_p(\xi_1 \cdot \xi_2) = \mathrm{Log}_p(\xi_1) + \mathrm{Log}_p(\xi_2). \tag{69}$$

Proof. Since,

$$\mathrm{Log}_p(\xi_1) = \ln||\xi_1||_p + i\theta_p,$$

and

$$\mathrm{Log}_p(\xi_2) = \ln||\xi_2||_p + i\varphi_p,$$

we obtain

$$\begin{aligned} \mathrm{Log}_p(\xi_1) + \mathrm{Log}_p(\xi_2) &= \ln||\xi_1||_p + \ln||\xi_2||_p + i(\theta_p + \varphi_p) \\ &= \ln(||\xi_1||_p \, \cdot ||\xi_2||_p) + i(\theta_p + \varphi_p). \end{aligned}$$

On the other hand, according to [1], we have:

$$\xi_1 \cdot \xi_2 = ||\xi_1||_p \, \cdot ||\xi_2||_p \, e^{i(\theta_p + \varphi_p)},$$

and, consequently,

$$\mathrm{Log}_p(\xi_1 \cdot \xi_2) = \ln(||\xi_1||_p \, \cdot ||\xi_2||_p) + i(\theta_p + \varphi_p).$$

Under the condition for

$$\{\theta_p, \varphi_p\} \in \left(\frac{-\pi}{2\sqrt{|p|}}, \frac{\pi}{2\sqrt{|p|}}\right],$$

we have

$$\{\theta_p + \varphi_p\} \in \left(\frac{-\pi}{\sqrt{|p|}}, \frac{\pi}{\sqrt{|p|}}\right].$$

Therefore,

$$\text{Log}_p(\xi_1.\xi_2) = \text{Log}_p(\xi_1) + \text{Log}_p(\xi_2).$$

□

Proposition 8. *Let $\xi_1, \xi_2 \in \mathbb{C}_p$ such that $\xi_1 \neq 0$ and $\xi_2 \neq 0$. Then,*

$$\log_p\left(\frac{\xi_1}{\xi_2}\right) = \log_p(\xi_1) - \log_p(\xi_2). \quad (70)$$

Proof.

$$\log_p(\xi_1\xi_2) = \ln\left(\frac{||\xi_1||_p}{||\xi_2||_p}\right) + i\left(\theta_p - \varphi_p + \frac{2k\pi}{\sqrt{|p|}}\right)$$

$$\log_p(\xi_1) = \ln||\xi_1||_p + i\left(\theta_p + \frac{2k\pi}{\sqrt{|p|}}\right)$$

and

$$\log_p(\xi_2) = \ln||\xi_2||_p + i\left(\varphi_p + \frac{2k'\pi}{\sqrt{|p|}}\right)$$

$$\begin{aligned} \log_p(\xi_1) - \log_p(\xi_2) &= \ln||\xi_1||_p - \ln||\xi_2||_p + i\left(\theta_p - \varphi_p + \frac{2(k-k')\pi}{\sqrt{|p|}}\right) \\ &= \ln\left(\frac{||\xi_1||_p}{||\xi_2||_p}\right) + i\left(\theta_p - \varphi_p + \frac{2r\pi}{\sqrt{|p|}}\right). \end{aligned}$$

□

Remark 11. *In general, $\xi_1, \xi_2 \in \mathbb{C}_p - \{0\}$, then*

$$\text{Log}_p\left(\frac{\xi_1}{\xi_2}\right) \neq \text{Log}_p(\xi_1) - \text{Log}_p(\xi_2). \quad (71)$$

Definition 7. *Let ξ and $a \in \mathbb{C}_p$ with $\xi \neq 0$; we define:*

$$\xi^a = e^{a\ \log_p(\xi)}. \quad (72)$$

The principal determination of ξ^a is given by:

$$\xi^a = e^{a\ \text{Log}_p(\xi)}. \quad (73)$$

Remark 12. *For $\{\xi, a\} \in \mathbb{C}_p$ with $\xi \neq 0$, we have:*

$$\begin{aligned} \xi^a &= e^{a \log_p(\xi)} \\ &= e^{a\left(\log\left(||\xi||_p\right) + iarg_p(\xi)\right)} \\ &= e^{a\left(\log\left(\sqrt{|\mu^2 - p\gamma^2|}\right) + i\left(\theta_p + \frac{2\pi k}{\sqrt{|p|}}\right)\right)}. \end{aligned}$$

Remark 13 (Branches of Logarithms)**.** *From the identity:*

$$\log_p(\xi) = \log(||\xi||_p) + i\left(\theta_p + \frac{2\pi k}{\sqrt{|p|}}\right),$$

and by assuming $\Theta_p = \theta_p + \frac{2\pi k}{\sqrt{|p|}}$*, we can write:*

$$\log(\xi) = \ln(\|\xi\|_p) + i\Theta_p.$$

Now, let α *be any real number. If we restrict the value of* α *so that* $\alpha < \Theta_p < \left(\alpha + \frac{2\pi k}{\sqrt{|p|}}\right)$*, then the function*

$$\log(\xi) = \ln(\|\xi\|_p) + i\Theta_p,$$

is a single-valued function in the above stated domain.

Observe that for each fixed α*, the single-valued function* $\log(\xi) = \ln(\|\xi\|_p) + i\Theta_p$ *is a branch of the multiple-valued function* $\log$*. The function* $Log(\xi) = \ln(\|\xi\|_p) + i\Theta_p$*, where* $-\frac{\pi}{\sqrt{|p|}} < \Theta_p \leq \frac{\pi}{\sqrt{|p|}}$ *is called the principal branch.*

Example 2. *If* i *is generalized imaginary number, find* i^i*.*

According to (72), we may write

$$\begin{aligned} i^i &= e^{i(\log||i||_p + i\cdot arg_p(i))} \\ &= e^{i\left(\log\left(\sqrt{|p|}\right) + i\left(\theta_p + \frac{2\pi k}{\sqrt{|p|}}\right)\right)} \quad \left(since,\ \log\|i\|_p = \sqrt{|p|}\right) \\ &= e^{i\log(\sqrt{|p|}) + i^2\left(\theta_p + \frac{2k\pi}{\sqrt{|p|}}\right)} \quad where, \quad i^2 = p \\ &= e^{i\log\sqrt{|p|} + p\left(\theta_p + \frac{2k\pi}{\sqrt{|p|}}\right)}. \end{aligned}$$

Remark 14. *Now, the question is when* $p = -1$ *what happens? We obtain*

$$\begin{aligned} i^i &= e^{i(\log(i)} \\ &= e^{i\log\sqrt{1} + i\left(\theta_{-1} + 2k\pi\right))} \\ &= e^{-\left(\theta_{-1} + 2k\pi\right)} \\ &= e^{-\frac{\pi}{2}} e^{-2k\pi} \ ;\ k = 0,\ \pm 1\ \pm 2\ \cdots. \end{aligned}$$

The principal branch of i^i *is* $e^{-\frac{\pi}{2}}$*.*

Remark 15. *According to Example 2 and Remark 14, we observe that for* $p = -1$*,* i^i *is real number; however, for* $p \neq -1$*,* i^i *is general complex number.*

5. Conclusions

In this paper, we provide rigorous proofs for some important identities related to bridging the family of p-trigonometric and p-hyperbolic functions, involving the p-complex numbers. We also extend these properties to the logarithmic functions with complex arguments. The study of these special functions will also help in the development of the unknown properties and identities involving other classes of p-special functions.

In future, study can be extended to similar relationships between the inverse p-trigonometric functions and inverse p-hyperbolic functions [17,18]. The study of ordinary differential equations (ODEs) involving complex numbers and their solutions in the generalised p-trigonometric and hyperbolic function basis can also be explored in the future.

This may also involve the study of the orthogonality properties of the basis of complex ODEs and their solution using various integral transforms.

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