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Lie Modules of Banach Space Nest Algebras

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Abstract: In the present work, we extend to Lie modules of Banach space nest algebras a well-known characterisation of Lie ideals of (Hilbert space) nest algebras. Let $\mathcal{A}$ be a Banach space nest algebra and $\mathcal{L}$ be a weakly closed Lie $\mathcal{A}$-module. We show that there exist a weakly closed $\mathcal{A}$-bimodule $\mathcal{K}$, a weakly closed subalgebra $\mathcal{D}_{\mathcal{K}}$ of $\mathcal{A}$, and a largest weakly closed $\mathcal{A}$-bimodule $\mathcal{J}$ contained in $\mathcal{L}$, such that $\mathcal{J} \subseteq \mathcal{L} \subseteq \mathcal{K} + \mathcal{D}_{\mathcal{K}}$, with $[\mathcal{K}, \mathcal{A}] \subseteq \mathcal{L}$. The first inclusion holds in general, whilst the second is shown to be valid in a class of nest algebras.

Keywords: Lie module; bimodule; nest algebra; Banach space

MSC: 46K50; 47A15; 47L35; 17B60

1. Introduction

Associative, Jordan, and Lie ideals of nest algebras on Hilbert space have been substantially investigated in the literature (e.g., [1–4]), whilst in what concerns their module structure, or the corresponding theory in the Banach space setting, the literature is not so abundant. Notwithstanding, some headway in that direction has been made, as in the case of [5–9], for example.

In particular, the Lie structure of Hilbert space nest algebras seems to be substantially more difficult to unveil than its associative counterpart. In fact, bimodules of Hilbert space nest algebras were fully characterised in the early eighties ([1]), whilst Lie ideals had to wait for almost twenty years to be given some insight ([2]). The existing characterisation of Lie ideals, provided by Hudson, Marcoux, and Sourour in ([2], Theorem 12), is given in terms of inclusions: it states that any weakly closed Lie ideal of a Hilbert space nest algebra must contain a weakly closed ideal and be contained in the sum of that ideal with a von Neumann subalgebra of the diagonal of the nest algebra.

Almost two decades later, this characterisation of Lie ideals of nest algebras was shown to admit an extension to Lie modules ([8]). It was proved in [8] that for any weakly closed Lie module $\mathcal{L}$ of a nest algebra $\mathcal{A}$, similarly to ([2], Theorem 12), there exist weakly closed $\mathcal{A}$-bimodules $\mathcal{J}$ and $\mathcal{K}$, and a von Neumann subalgebra $\mathcal{D}_{\mathcal{K}}$ of the diagonal of the nest algebra, such that:

$$\mathcal{J} \subseteq \mathcal{L} \subseteq \mathcal{K} + \mathcal{D}_{\mathcal{K}}, \tag{1}$$

where $\mathcal{J}$ is the largest weakly closed $\mathcal{A}$-bimodule contained in $\mathcal{L}$ and $[\mathcal{K}, \mathcal{A}] \subseteq \mathcal{L}$ (cf. ([8], Theorem 1)).

The present work is an extension of ([8], Theorem 1), and consequently, of ([2], Theorem 12) both in what concerns the setting and the structures considered: here, we go from Hilbert to Banach spaces and from Lie ideals to Lie modules. It is also the case that, with respect to the bimodule $\mathcal{J}$, we obtain a stronger result inasmuch as we are able to characterise the largest $\mathcal{A}$-bimodule contained in a weakly closed subspace $\mathcal{L}$ of $B(X)$, not necessarily a Lie $\mathcal{A}$-module (see Theorem 1). In this regard, we also characterise the weakly

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closed bimodule generated by a given subspace of $B(X)$, which needs not be closed in any topology (see Proposition 1).

Some challenges arise when attempting to extend the second inclusion in (1) to the Banach space setting, mainly due to the absence of orthogonal projections intrinsically associated with the structure of Banach spaces, whereas, in the case of Hilbert spaces, orthogonality is a key tool to obtain the results. However, we show that an identical result still holds for a class of Banach space nest algebras (see Section 3).

The main results are Theorems 1–3, the latter two appearing in Section 3 and the first one appearing in Section 2.

This work is organised as follows. Section 2 is concerned with the first inclusion in (1) and culminates with its main result, i.e., Theorem 1. In this theorem, we obtain an explicit description of the largest $\mathcal{A}$-bimodule $\mathcal{J}$ similar to that of [8], but now holding for any weakly closed subspace $\mathcal{L}$ of $B(X)$, be it a Lie $\mathcal{A}$-module or not. Moreover, we also characterise the smallest weakly closed $\mathcal{A}$-bimodule that contains a subspace $\mathcal{L}$, which is not assumed to be closed in any topology (Proposition 1). In Section 3, we construct a weakly closed $\mathcal{A}$-bimodule $\mathcal{K}$ and a weakly closed subalgebra $\mathcal{D}_{\mathcal{K}}$ of $\mathcal{A}$, determined by $\mathcal{K}$, such that the second inclusion in (1) can be extended to an appropriate class of Banach space nest algebras (those satisfying the so-called π-property defined in Section 3) and their modules (Theorem 2). Section 3 ends with Theorem 3, which integrates all the main results for the class of nest algebras satisfying the π-property. Section 3 also includes two examples: one example of a nest algebra satisfying the π-property (Example 1), and another giving the explicit form of a weakly closed Lie module, which was obtained using the three main theorems of this work (Example 2).

We end this section by establishing some notation and recalling a few facts needed in what follows. Let X be a complex Banach space, and let $B(X)$ denote the algebra of all bounded linear operators on X. The set of closed subspaces of X is partially ordered by set inclusion. A family $\mathcal{M}$ of closed subspaces of X is said to be a *subspace lattice* if it contains arbitrary infima and suprema, that is, if it is closed under intersections ($\wedge$) and norm closure of linear spans ($\vee$). A *nest* is a totally ordered subspace lattice containing $\{0\}, X$. The *nest algebra* $\mathcal{A}$ associated with the nest $\mathcal{N}$ is the weakly closed subalgebra of $B(X)$ defined by:

$$\mathcal{A} = \{T \in \mathcal{B}(X) : T(N) \subseteq N, \text{ for all } N \in \mathcal{N}\}.$$

The space $B(X)$ together with the product defined, for all $T, S \in B(X)$, by $[T, S] = TS - ST$ is a Lie algebra. A linear subspace $\mathcal{V}$ of $B(X)$ is called an $\mathcal{A}$-*bimodule* if $\mathcal{V}\mathcal{A}$, $\mathcal{A}\mathcal{V} \subseteq \mathcal{V}$, and a *Lie* $\mathcal{A}$-*module* if $[\mathcal{V}, \mathcal{A}] \subseteq \mathcal{V}$.

For $N \in \mathcal{N}$, we define N_- and N_+ by:

$$N_- = \vee\{M \in \mathcal{N} : M < N\}, \quad N_+ = \wedge\{M \in \mathcal{N} : M > N\},$$

respectively.

Let f lie in the dual space X^* of X, and let $y \in X$. The rank one operator $f \otimes y$ on X is defined by $x \mapsto f(x)y$. Following [3,7], define N_y and $\hat{N}_f$ by:

$$N_y = \wedge\{N \in \mathcal{N} : y \in N\}, \quad \hat{N}_f = \vee\{N \in \mathcal{N} : f \in N^{\perp}\},$$

where the annihilator of a subspace $N \subseteq X$ is $N^{\perp} = \{g \in X^* : g(x) = 0, \text{ for all } x \in N\}$. The closure, in the weak operator topology, of a set $\mathcal{U} \subseteq B(X)$ is denoted by $\overline{\mathcal{U}}^{w}$. Recall that the *strong operator topology* and the *weak operator topology* on $B(X)$ are defined, respectively, by means of convergence of nets as:

(i) A net $\{T_\alpha\}$ in $\mathcal{B}(X)$ converges to the operator T in the strong operator topology if $\{T_\alpha x\}$ converges to Tx, for all $x \in X$.
(ii) A net $\{T_\alpha\}$ in $\mathcal{B}(X)$ converges to the operator T in the weak operator topology if $\{f(T_\alpha x)\}$ converges to $f(Tx)$, for all $x \in X$ and $f \in X^*$.

The next two lemmas make a note of some facts, whose proofs are included for the reader's convenience.

Lemma 1. *Let $\mathcal{N}$ be a nest on a Banach space X, and let $N \in \mathcal{N}$. Then, the following hold:*

(i) $N = \vee\{L \in \mathcal{N} : L_- < N\}$;

(ii) $\mathrm{span}\{M^\perp : M \in \mathcal{N}, M_+ > N\}$ *is dense in* $N^\perp$ *in the weak*-topology of* X^*.

Proof. (i) If $L \in \mathcal{N}$ is such that $L_- < N$, then $L \leq N$. Hence, $N \geq \vee\{L \in \mathcal{N} : L_- < N\}$. For any $M \in \mathcal{N}$ such that $M < N$, we have $M_- < N$, and therefore, $M \leq \vee\{L \in \mathcal{N} : L_- < N\}$. Hence, $\vee\{L \in \mathcal{N} : L_- < N\}$ is equal to N or N_-. However, if $N_- \neq N$, then $N \in \{L \in \mathcal{N} : L_- < N\}$, because $N_- < N$. Therefore, in both cases, $N = \vee\{L \in \mathcal{N} : L_- < N\}$.

(ii) The assertion holds trivially for $N = X$. Suppose now that $N \neq X$. Observe that $N^\perp$ is weak*-closed and that, for $M \geq N$, we have $M^\perp \subseteq N^\perp$. Let $Y = \mathrm{span}\{M^\perp : M \in \mathcal{N}, M_+ > N\}$. If $N_+ > N$, then $N \in \{M \in \mathcal{N} : M_+ > N\}$, and consequently, $N^\perp = Y$.

Suppose now that $N_+ = N$. Then, the pre-annihilator of Y is:

$$Y_\perp = \left(\bigcup_{M_+ > N} M^\perp\right)_\perp = \bigcap_{M_+ > N} (M^\perp)_\perp = \bigcap_{M_+ > N} M = N.$$

In the third equality, we make use of the fact that all $M \in \mathcal{N}$ are closed and, in the last equality, of the fact that $N = \wedge\{M \in \mathcal{N} : M_+ > N\}$, whose proof is similar to that of Lemma 1 (i). It follows that the weak*- closure of Y is $(Y_\perp)^\perp = N^\perp$. $\square$

The next lemma is essentially in [7].

Lemma 2. *Let $\mathcal{N}$ be a nest on a Banach space X, let $\mathcal{A}$ be the corresponding nest algebra, let $\mathcal{U}$ be a norm closed $\mathcal{A}$-bimodule, and let $f \otimes y$ be a rank one operator. Then, the following hold:*

(i) *The operator $f \otimes y$ lies in $\mathcal{A}$ if and only if there exists $N \in \mathcal{N}$ such that $y \in N$ and $f \in N_-^\perp$;*

(ii) *If $f \otimes y$ lies in $\mathcal{U}$, then, for all $z \in N_y$, the rank one operator $f \otimes z$ lies in $\mathcal{U}$;*

(iii) *If $f \otimes y$ lies in $\mathcal{U}$, then, for all $g \in \hat{N}_f^\perp$, the rank one operator $g \otimes y$ lies in $\overline{\mathcal{U}}^w$;*

(iv) *If $\mathcal{U}$ is weakly closed and $f \otimes y$ lies in $\mathcal{U}$, then, for all $g \in \hat{N}_f^\perp$, $z \in N_y$, the rank one operator $g \otimes z$ lies in $\mathcal{U}$.*

Proof. (i) See [7], Lemma 1.1.

(ii) We begin by proving a claim.

Claim. If $N \in \mathcal{N}$ is such that $N_- < N_y$, then, for all $z \in N$, the operator $f \otimes z$ lies in $\mathcal{U}$.

Let $z \in N$. Since $N_- < N_y$, $y \notin N_-$. Thus:

$$\inf_{w \in N_-} \|y - w\| > 0,$$

because N_- is closed. By the Hahn–Banach Theorem, there exists $g \in N_-^\perp$ such that $g(y) = 1$. Thus, by (i), $g \otimes z \in \mathcal{A}$. Since $\mathcal{U}$ is a module, it follows from $f \otimes y \in \mathcal{U}$ and $g \otimes z \in \mathcal{A}$ that $(g \otimes z)(f \otimes y) = f \otimes z$ lies in $\mathcal{U}$, thus establishing the claim.

Now, let $z \in N_y$ be arbitrary. Notice that the case $(N_y)_- < N_y$ has already been addressed. By Lemma 1 (i), $N_y = \vee\{N \in \mathcal{N} : N_- < N_y\}$. Hence, there exists a sequence (z_n) in $\mathrm{span}\{N \in \mathcal{N} : N_- < N_y\}$ that converges to z in the norm topology. Accordingly, let $z_n \in M_n$, where, for all $n \in \mathbb{N}$, $M_n \in \mathcal{N}$ and $(M_n)_- < N_y$. By the above claim, each $f \otimes z_n \in \mathcal{U}$. Since $\mathcal{U}$ is norm closed and $(f \otimes z_n)$ converges in norm to $f \otimes z$, one has that $f \otimes z \in \mathcal{U}$.

(iii) Firstly we prove the following claim.

Claim. If $N \in \mathcal{N}$ is such that $N_+ > \hat{N}_f$, then, for all $g \in N^\perp$, the operator $g \otimes y$ lies in $\mathcal{U}$.

Let $g \in N^\perp$. Since $N_+ > \hat{N}_f$, $f \notin N_+^\perp$, hence, we can choose $z \in N_+$ such that $f(z) = 1$. To show that $g \otimes z \in \mathcal{A}$, we consider the two cases $N < N_+$ and $N = N_+$ separately.

If $N < N_+$, then $N = (N_+)_-$. Since $g \in N^\perp$ and $z \in N_+$, by (i), $g \otimes z \in \mathcal{A}$. If $N = N_+$, then $z \in N$. Additionally, we have $g \in N^\perp$, which implies $g \in N_-^\perp$. Thus, by (i), $g \otimes z \in \mathcal{A}$.

Since $\mathcal{U}$ is a module, $f \otimes y \in \mathcal{U}$ and $g \otimes z \in \mathcal{A}$:

$$(f \otimes y)(g \otimes z) = g \otimes y \in \mathcal{U},$$

which finishes the proof of the claim.

Now let $g \in \hat{N}_f^\perp$ be arbitrary. Notice that the case $\hat{N}_f < (\hat{N}_f)_+$ has already been addressed. By Lemma 1 (ii), there exists a net (g_α) in span$\{N^\perp : N \in \mathcal{N}, N_+ > \hat{N}_f\}$ that converges to g in the weak*-topology. Accordingly, each $g_\alpha \in M_\alpha^\perp$, for some $M_\alpha \in \mathcal{N}$ with $(M_\alpha)_+ > \hat{N}_f$. By the above claim, each $g_\alpha \otimes y$ lies in $\mathcal{U}$. The weak convergence of the net $(g_\alpha \otimes y)$ to $g \otimes y$ yields $g \otimes y \in \overline{\mathcal{U}}^w$.

(iv) Since $z \in N_y$, by (ii) we have $f \otimes z \in \mathcal{U}$. Therefore, since $g \in \hat{N}_f^\perp$, by (iii), it follows that $g \otimes z \in \mathcal{U}$. □

For simplicity, in what follows, $\mathcal{A}$-bimodules and Lie $\mathcal{A}$-modules might be referred to as bimodules and Lie modules, respectively.

2. Bimodules

A function $\phi : \mathcal{N} \to \mathcal{N}$ on a Banach space nest $\mathcal{N}$ is said to be an *order homomorphism* if $\phi(M) \leq \phi(N)$, whenever $M \leq N$. An order homomorphism ϕ is said to be *left continuous* if, for any set $\{M_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{N}$, $\phi\big(\vee(\{M_\lambda\}_{\lambda\in\Lambda})\big) = \vee\{\phi(M_\lambda)\}_{\lambda\in\Lambda}$.

It is well known that, for Banach and Hilbert spaces alike, weakly closed bimodules of nest algebras can be determined by order homomorphisms on the relevant nests (see, e.g., [1,5,6]). In this section, we resume the analysis of these order homomorphisms associated with bimodules. However, in contrast to [1,5,6], we obtain an explicit description of the order homomorphism solely determined by the rank one operators in the bimodule. This approach is akin to that taken in [8] in the Hilbert space setting.

The next lemma, of which we make a note here for future reference, is a consequence of ([6], Theorem 2.10, Corollary 2.13, Theorem 2.15).

Lemma 3. *Let $\mathcal{N}$ be a nest in a Banach space X, let $\mathcal{A}$ be the corresponding nest algebra, and let $\mathcal{U}$ be a weakly closed $\mathcal{A}$-bimodule. Then, there exists a left continuous order homomorphism $\phi : \mathcal{N} \to \mathcal{N}$ such that:*

$$\mathcal{U} = \{T \in \mathcal{B}(X) : TN \subseteq \phi(N), \textit{ for all } N \in \mathcal{N}\}.$$

Moreover, $\mathcal{U}$ coincides with the closure in the weak operator topology of the linear span of its rank one operators.

Remark 1. *Notice that, by Lemma 3, the weak closure $\overline{\mathcal{U}}^w$ of an $\mathcal{A}$-bimodule $\mathcal{U}$ is:*

$$\overline{\mathcal{U}}^w = \{A \in \mathcal{B}(X) : AN \subseteq \phi(N), \textit{ for all } N \in \mathcal{N}\},$$

for some left continuous order homomorphism $\phi : \mathcal{N} \to \mathcal{N}$.

Lemma 4. *Let $\mathcal{N}$ be a nest on a Banach space X, let $\mathcal{A}$ be the corresponding nest algebra, and let $\mathcal{U}$ be a weakly closed $\mathcal{A}$-bimodule. Let $\phi\colon \mathcal{N} \to \mathcal{N}$ be the mapping defined, for all $N \in \mathcal{N}$, by:*

$$\phi(N) = \vee\{N_y \in \mathcal{N} \,|\, \exists f \in X^* \colon f \otimes y \in \mathcal{U},\ \hat{N}_f < N\}. \tag{2}$$

Then, a rank one operator $f \otimes y \in \mathcal{U}$ if and only if, for all $N \in \mathcal{N}$, $f \otimes y(N) \subseteq \phi(N)$. The map ϕ on $\mathcal{N}$ is a left continuous order homomorphism.

Proof. Let $f \otimes y$ be a rank one operator in $\mathcal{U}$, and let $N \in \mathcal{N}$. If $\hat{N}_f < N$, then $N_y \leq \phi(N)$. Since $y \in N_y$, we have that $y \in \phi(N)$, and therefore, $f \otimes y(N) \subseteq \phi(N)$. On the other hand, if $N \leq \hat{N}_f$, then $f \in N^{\perp}$, since $f \in \hat{N}_f^{\perp}$. Hence, $f \otimes y(N) = \{0\} \subseteq \phi(N)$.

Conversely, suppose that, for all $N \in \mathcal{N}$, one has $f \otimes y(N) \subseteq \phi(N)$. We begin by proving a claim.

Claim. If $N \in \mathcal{N}$ is such that $\hat{N}_f < N$, then, for all $g \in N_-^{\perp}$, the operator $g \otimes y$ lies in $\mathcal{U}$.

Since $\hat{N}_f < N$, we know that $f \notin N^{\perp}$. In addition, $f \otimes y(N) = f(N)y \subseteq \phi(N)$, and therefore, $y \in \phi(N)$. Hence, there exists a sequence (y_n) in:

$$\operatorname{span}\{N_z \in \mathcal{N} \,|\, \exists h \in X^* \colon h \otimes z \in \mathcal{U},\ \hat{N}_h < N\}$$

converging to y in the norm topology. Accordingly, for all $n \in \mathbb{N}$, there exists $h_n \otimes z_n \in \mathcal{U}$, with $y_n \in N_{z_n}$, $\hat{N}_{h_n} < N$. Since $\hat{N}_{h_n} \leq N_-$ and $g \in N_-^{\perp}$, it follows that $g \in \hat{N}_{h_n}^{\perp}$. By Lemma 2 (iv), each $g \otimes y_n \in \mathcal{U}$. Since $(g \otimes y_n)$ converges to $g \otimes y$ in the norm topology, $g \otimes y \in \mathcal{U}$, thus establishing the claim.

Now let $g \in \hat{N}_f^{\perp}$ be arbitrary. Observe that the case $\hat{N}_f < (\hat{N}_f)_+$ has already been addressed. By Lemma 1 (ii), there exists a net (g_α) in $\operatorname{span}\{N^{\perp} : N \in \mathcal{N},\ N_+ > \hat{N}_f\}$ which converges to g in the weak*-topology. Accordingly, each $g_\alpha \in M_\alpha^{\perp}$, for some $M_\alpha \in \mathcal{N}$ with $(M_\alpha)_+ > \hat{N}_f$. By the above claim, each $g_\alpha \otimes y$ lies in $\mathcal{U}$. As $(g_\alpha \otimes y)$ weakly converges to $g \otimes y$, and it follows that $g \otimes y \in \mathcal{U}$. In particular, $f \in \hat{N}_f^{\perp}$, yielding $f \otimes y \in \mathcal{U}$.

We will show next that ϕ is a left continuous order homomorphism. Let $M, N \in \mathcal{N}$ be such that $M \leq N$. Then:

$$\{N_y : f \otimes y \in \mathcal{U}, \hat{N}_f < M\} \subseteq \{N_y : f \otimes y \in \mathcal{U}, \hat{N}_f < N\},$$

and consequently, $\phi(M) \leq \phi(N)$. Hence, ϕ is an order homomorphism on $\mathcal{N}$.

Let $\{M_\alpha\}$ be a non-empty family of subspaces of $\mathcal{N}$ and set $N = \vee\{M_\alpha\}$. We must show $\phi(N) = \vee\{\phi(M_\alpha)\}$. If $N \in \{M_\alpha\}$, then, since ϕ is an order homomorphism, $\phi(N) = \vee\{\phi(M_\alpha)\}$. If $N \notin \{M_\alpha\}$, then for each $M \in \mathcal{N}$ satisfying $M < N$ there exists M_α such that $M < M_\alpha < N$. Hence:

$$\{N_y : f \otimes y \in \mathcal{U}, \hat{N}_f < N\} \subseteq \bigcup_\alpha \{N_y : f \otimes y \in \mathcal{U}, \hat{N}_f < M_\alpha\}.$$

By the definition (2) of ϕ, we have $\phi(N) \leq \vee\{\phi(M_\alpha)\}$. Since ϕ is an order homomorphism, the inequality $\vee\{\phi(M_\alpha) \leq \phi(N)\}$ also holds. Thus, $\phi(N) = \vee\{\phi(M_\alpha)\}$, which shows that ϕ is left continuous. □

Corollary 1. *Let $\mathcal{N}$ be a nest in a Banach space X, let $\mathcal{A}$ be the corresponding nest algebra, and let $\mathcal{U}$ be a weakly closed $\mathcal{A}$-bimodule. Let $f \otimes y$ a rank one operator of $\mathcal{B}(X)$, and let*

$$\mathcal{G} = \big\{T \in \mathcal{B}(X) : T(X) \subseteq N_y,\ T\hat{N}_f = \{0\}\big\}.$$

Then, $f \otimes y \in \mathcal{U}$ if and only if $\mathcal{G} \subseteq \mathcal{U}$.

Proof. Suppose $f \otimes y \in \mathcal{U}$. By Lemma 3, there exists an order homomorphism ϕ on $\mathcal{N}$ such that:

$$\mathcal{U} = \{T \in \mathcal{B}(X) : T(N) \subseteq \phi(N), \text{ for all } N \in \mathcal{N}\}.$$

Let $T \in \mathcal{G}$ and $N \in \mathcal{N}$.

If $N \leq \hat{N}_f$, then, since $T\hat{N}_f = \{0\}$, we have $T(N) = \{0\} \subseteq \phi(N)$.

If $N > \hat{N}_f$, then, since $f \otimes y \in \mathcal{U}$, by (2), $N_y \leq \phi(N)$. Hence, since $T(X) \subseteq N_y$, we have $T(N) \subseteq \phi(N)$.

Therefore, for any $T \in \mathcal{G}$, $T(N) \subseteq \phi(N)$ for all $N \in \mathcal{N}$. Thus, $\mathcal{G} \subseteq \mathcal{U}$.

Conversely, suppose $\mathcal{G} \subseteq \mathcal{U}$. Since $f \otimes y \in \mathcal{G}$, it is immediate that $f \otimes y \in \mathcal{U}$. □

We are now able to characterise the largest $\mathcal{A}$-bimodule contained in a weakly closed subspace of $B(X)$.

Theorem 1. *Let $\mathcal{L}$ be a weakly closed subspace of the algebra $B(X)$ of bounded linear operators on a complex Banach space X, let:*

$$\mathcal{C} = \{f \otimes y \in \mathcal{B}(X) | g \otimes z \in \mathcal{L}, \text{ for all, } g \in \hat{N}_f^{\perp},\ z \in N_y\}, \tag{3}$$

and let $\phi : \mathcal{N} \to \mathcal{N}$ be the left continuous order homomorphism defined by:

$$\phi(N) = \vee\{N_y | \ \exists f \in X^* \colon f \otimes y \in \mathcal{C},\ \hat{N}_f < N\}. \tag{4}$$

Then, the largest $\mathcal{A}$-bimodule contained in $\mathcal{L}$ is:

$$\mathcal{J} = \{T \in \mathcal{B}(X) : T(N) \subseteq \phi(N), \text{ for all } N \in \mathcal{N}\}.$$

Notice that, since this theorem holds for general weakly closed subspaces of $B(X)$, it also gives a characterisation of the largest $\mathcal{A}$-bimodule contained in a weakly closed Lie $\mathcal{A}$-module. It is also worth pointing out that the order homomorphism ϕ is uniquely determined amongst those fixing $\{0\}$ (see ([6], Proposition 2.3)).

Proof. We show next that $\mathcal{C}$ coincides with the set of rank one operators in $\mathcal{J}$.

Suppose firstly that $f \otimes y$ is an operator in $\mathcal{C}$, and let $N \in \mathcal{N}$ with $N > \hat{N}_f$. It follows from the definition (4) that $N_y \leq \phi(N)$. Hence, $y \in \phi(N)$, and consequently, $f \otimes y(N) = f(N)y \subseteq \phi(N)$. On the other hand, if $N \leq \hat{N}_f$, then $f \in N^{\perp}$. Hence, $f \otimes y(N) = \{0\} \subseteq \phi(N)$. Therefore, $\mathcal{C} \subseteq \mathcal{J}$.

Conversely, we must show that, if $f \otimes y \in \mathcal{J}$, then $f \otimes y \in \mathcal{C}$, that is, for all $g \in \hat{N}_f^{\perp}, z \in N_y$ the operator $g \otimes z \in \mathcal{L}$.

Fix then a rank one operator $f \otimes y$ lying in $\mathcal{J}$. Firstly, we prove that for $N \in \mathcal{N}$, with $N > \hat{N}_f$, and $g \in N_-^{\perp}$, the operator $g \otimes y$ lies in $\mathcal{C}$.

Observe that $f \otimes y(N) \subseteq \phi(N)$, for $f \otimes y \in \mathcal{J}$. Since $N > \hat{N}_f$, it is also the case that $f \otimes y(N) = f(N)y$ is a non-zero set spanned by y. Therefore, $y \in \phi(N)$, and consequently, $\phi(N) \geq N_y$. Now, we consider two cases: (1) $\phi(N) > N_y$ and (2) $\phi(N) = N_y$.

Case (1) $\phi(N) > N_y$.

By (4), there exists a rank one operator $h \otimes z \in \mathcal{C}$ such that $\hat{N}_h < N$ and $N_z > N_y$. Since $g \in N_-^{\perp}$ and $\hat{N}_h \leq N_-$, one has $\hat{N}_h \leq \hat{N}_g$. Since $N_z > N_y$ and $\hat{N}_h \leq \hat{N}_g$, it follows that:

$$\{e \otimes w \in \mathcal{B}(X) : e \in \hat{N}_g^{\perp},\ w \in N_y\} \subseteq \{e \otimes w \in \mathcal{B}(X) : e \in \hat{N}_h^{\perp},\ w \in N_z\}.$$

Hence, since $h \otimes z \in \mathcal{C}$, we have that $g \otimes y \in \mathcal{C}$.

Case (2). $\phi(N) = N_y$.

In this case, $N_y = \vee\{N_z \in \mathcal{N} | \exists h \in X^* \colon h \otimes z \in \mathcal{C},\ \hat{N}_h < N\}$. For $z \in N_y$, there exists a sequence (z_j) in $\operatorname{span}\{N_w \in \mathcal{N} | \exists h \in X^* \colon h \otimes w \in \mathcal{C},\ \hat{N}_h < N\}$ converging to z in the norm topology. Accordingly, for all j, consider $z_j \in N_{w_j}$, with $h_j \otimes w_j \in \mathcal{C},\ \hat{N}_{h_j} < N$.

It follows from $z_j \in N_{w_j}$ that $N_{z_j} \leq N_{w_j}$. Notice also that $g \in N_-^{\perp}$, $\hat{N}_{h_j} \leq N_-$ yield $\hat{N}_{h_j} \leq \hat{N}_g$.

Since $h_j \otimes w_j \in \mathcal{C}$, $\hat{N}_{h_j} \leq \hat{N}_g$, and $N_{w_j} \geq N_{z_j}$, the operator $h \otimes z_j$ lies in $\mathcal{L}$, for all $h \in \hat{N}_g^{\perp}$. Noticing that $(h \otimes z_j)$ weakly converges to $h \otimes z$, it follows that $h \otimes z \in \mathcal{L}$. That is, for all $h \in \hat{N}_g^{\perp}$ and $z \in N_y$, the operator $h \otimes z$ lies in $\mathcal{L}$. Hence, by (3), one has $g \otimes y \in \mathcal{C}$, which ends the proof of the claim.

Consider again the operator $f \otimes y \in \mathcal{J}$ and let $g \in \hat{N}_f^{\perp}$. Then, by Lemma 1 (ii), there exists a net (g_j) in $\text{span}\{M^{\perp} : M \in \mathcal{N}, M_+ > \hat{N}_f\}$ that converges to g in the weak*-topology. Accordingly, let $g_j \in M_j^{\perp}$, where $M_j \in \mathcal{N}$ and $(M_j)_+ > \hat{N}_f$.

By the above claim, each $g_j \otimes y \in \mathcal{C}$. Hence, by (3), for all $z \in N_y$, the operator $g_j \otimes z$ lies in $\mathcal{L}$. Since $\mathcal{L}$ is weakly closed and $(g_j \otimes z)$ weakly converges to $g \otimes z$, we have that $g \otimes z \in \mathcal{L}$. Therefore, $f \otimes y \in \mathcal{C}$, as required.

We have shown that the set of rank one operators of the weakly closed $\mathcal{A}$-bimodule $\mathcal{J}$ coincides with $\mathcal{C}$. Therefore, by Lemma 3, $\mathcal{J}$ is equal to the closure in the weak operator topology of the linear span of $\mathcal{C}$. Since $\mathcal{C} \subseteq \mathcal{L}$, it follows that $\mathcal{J} \subseteq \mathcal{L}$.

We show now that, if $\mathcal{U}$ is an $\mathcal{A}$-bimodule that is contained in $\mathcal{L}$ and contains $\mathcal{J}$, then $\mathcal{U} = \mathcal{J}$. It suffices to consider the case where $\mathcal{U}$ is weakly closed. Notice that, if this were not the case, then the closure of $\mathcal{U}$ in the weak operator topology is itself an $\mathcal{A}$-bimodule containing $\mathcal{U}$ and contained in $\mathcal{L}$, since $\mathcal{L}$ is weakly closed.

Let $f \otimes y \in \mathcal{U}$. By Lemma 2 (iv), for all $g \in \hat{N}_f^{\perp}$ and $z \in N_y$, the operator $g \otimes z$ lies in $\mathcal{U}$. Hence, since $\mathcal{U} \subseteq \mathcal{L}$, $f \otimes y \in \mathcal{C}$, and consequently, $f \otimes y \in \mathcal{J}$. Since $\mathcal{J} \subseteq \mathcal{U}$, the bimodules $\mathcal{U}$ and $\mathcal{J}$ have the same set of rank one operators. By Lemma 3, both $\mathcal{U}$ and $\mathcal{J}$ coincide with the closure in the weak operator topology of the linear span of $\mathcal{C}$, yielding $\mathcal{U} = \mathcal{J}$. Consequently, $\mathcal{J}$ is the largest $\mathcal{A}$-bimodule contained in $\mathcal{L}$.

We omit the proof that ϕ is a left continuous order homomorphism, as it is very similar to the one presented in Lemma 4. $\square$

As a complement to Theorem 1, which describes the largest $\mathcal{A}$-bimodule contained in a weakly closed subspace $\mathcal{L}$, the following proposition describes the smallest (weakly closed) $\mathcal{A}$-bimodule containing a subspace $\mathcal{L}$.

Proposition 1. *Let $\mathcal{L}$ be a subspace of the algebra $B(X)$ of bounded linear operators on a Banach space X. Let $\mathcal{N}$ be a nest in X and let $\mathcal{A}$ be the corresponding nest algebra. Then, the smallest weakly closed $\mathcal{A}$-bimodule containing $\mathcal{L}$ is:*

$$\mathcal{U} = \{T \in \mathcal{B}(X) : T(N) \subseteq \phi(N), \textit{ for all } N \in \mathcal{N}\},$$

where $\phi : \mathcal{N} \to \mathcal{N}$ is the order homomorphism defined by:

$$\phi(N) = \wedge\{M \in \mathcal{N} : T(N) \subseteq M, \textit{ for all } T \in \mathcal{L}\}.$$

Proof. It is immediate from the definition of ϕ that, for all $T \in \mathcal{L}$ and $N \in \mathcal{N}$, we have $T(N) \subseteq \phi(N)$. Therefore, $\mathcal{L} \subseteq \mathcal{U}$. It is also easy to see that $\mathcal{U}$ is a weakly closed $\mathcal{A}$-bimodule.

Let $\mathcal{V}$ be a weakly closed $\mathcal{A}$-bimodule. By Lemma 3:

$$\mathcal{V} = \{T \in \mathcal{B}(X) : T(N) \subseteq \psi(N) \text{ for all } N \in \mathcal{N}\},$$

for some left continuous order homomorphism $\psi : \mathcal{N} \to \mathcal{N}$. Suppose that there exists $N \in \mathcal{N}$ for which $\psi(N) < \phi(N)$. Then, by the definition of ϕ, there exists $T \in \mathcal{L}$ such that $T(N) \nsubseteq \psi(N)$, yielding that $\mathcal{L} \nsubseteq \mathcal{V}$. Therefore, for the condition $\mathcal{L} \subseteq \mathcal{V}$ to hold, it is necessary that, for all $N \in \mathcal{N}$, we have $\phi(N) \leq \psi(N)$. It follows that $\mathcal{U} \subseteq \mathcal{V}$, and consequently, $\mathcal{U}$ is the smallest weakly closed $\mathcal{A}$-bimodule containing $\mathcal{L}$. $\square$

3. Lie Modules

In this section, we consider nest algebras associated with families of projections. A *projection* P on X is a bounded linear operator such that $P^2 = P$. The set $\mathcal{P}(X)$ of projections on X can be endowed with an ordering: for projections $P, Q \in \mathcal{P}(X)$, define $P \leq Q$ if $PQ = P = QP$ (see [10]). Notice that, in this case, $P(X) \subseteq Q(X)$ and $(I - Q)(X) \subseteq (I - P)(X)$.

Now, we consider nests $\mathcal{N}$ of complemented subspaces. More precisely, we assume that, for every $N \in \mathcal{N}$, there exists a subspace C of X such that $N \oplus C$ is isomorphic to X (as topological vector spaces), and therefore, the canonical projection $P : X \to N$ induced by this isomorphism is a bounded linear operator on X. Nests of finite dimensional or finite co-dimensional closed subspaces are concrete examples satisfying this condition (see [11,12]). We will suppose additionally that the subspaces in the nest $\mathcal{N}$ (and their complements) are such that the corresponding projections commute (see Example 1) and will restrict our investigation to this class of nests in this section. In what follows, $\mathcal{N}$ will denote both the set of ranges and the associated family of projections, to simplify the notation. We will denote elements of $\mathcal{N}$ by the letters N, M when treating them as subspaces and by P, Q when viewing them as projections.

This class of nests will allow for extending the results of [8] in a natural way, as is shown in the remainder of this section. Henceforth, the nest algebra corresponding to a nest of the type above will be described adopting the notation of Section 2 , that is, $\mathcal{A} = \{T \in B(X)\colon T(N) \subseteq N$, for all $N \in \mathcal{N}\}$. In fact, $\mathcal{A}$ can now have an alternative description:

$$\mathcal{A} = \{T \in B(X) : (I - P)TP = 0, \text{ for all } P \in \mathcal{N}\},$$

which will be frequently used. Notice also that $\mathcal{N}$ is contained in the nest algebra $\mathcal{A}$.

Lemma 5. *Let $\mathcal{L}$ be a Lie $\mathcal{A}$-module and let $T \in \mathcal{L}$. Then, the following hold:*

(i) If $P, Q \in \mathcal{A}$ are projections such that $PQ = 0 = QP$, then $PTQ \in \mathcal{L}$;

(ii) If $\mathcal{L}$ is weakly closed and P is a projection such that $(I - P)\mathcal{L}P \neq \{0\}$, then:

$$P\mathcal{L}(I - P) = PB(X)(I - P).$$

Proof. (i) Since $PQ = 0 = QP$, we can write:

$$PTQ = \frac{1}{2}\big([[[T, Q], P], P] - [[T, Q], P]\big).$$

Thus, since $\mathcal{L}$ is a Lie $\mathcal{A}$-module, $PTQ \in \mathcal{L}$.

(ii) Let $P \in \mathcal{N}$ and $T \in \mathcal{L}$ be such that $(I - P)TP \neq 0$. Since $\mathcal{L}$ is a weakly closed subspace of $B(X)$, it suffices to show that, for every rank one operator $f \otimes y \in B(X)$, $P(f \otimes y)(I - P) \in \mathcal{L}$. Note that $P(f \otimes y)(I - P) \in \mathcal{A}$, and by part (i) of this lemma $(I - P)TP \in \mathcal{L}$ (see also Remark 2 below); hence, the following operator:

$$[[P(f \otimes y)(I - P), (I - P)TP], P(f \otimes y)(I - P)] = 2f((I - P)TPy)P(f \otimes y)(I - P)$$

lies in $\mathcal{L}$. Therefore, $P(f \otimes y)(I - P) \in \mathcal{L}$, whenever $f((I - P)TPy) \neq 0$.

We will now consider the case $f((I - P)TPy) = 0$. Suppose firstly that $(I - P)TPy \neq 0$. Then, there exists a functional $g \in X^*$ such that $g((I - P)TPy) \neq 0$. Replacing $f \otimes y$ by $g \otimes Py$ in the above calculation, we obtain that $2g((I - P)TPy)g \otimes Py \in \mathcal{L}$. Hence, $g \otimes Py \in \mathcal{L}$.

By a similar reasoning, since $(g - f)((I - P)TPy) \neq 0$, it follows that $(g - f) \otimes Py \in \mathcal{L}$. Therefore, the following lies in $\mathcal{L}$:

$$P(f \otimes y)(I - P) = g \otimes Py - (g - f) \otimes Py. \tag{5}$$

Suppose now that $(I - P)TPy = 0$. Since $(I - P)TP \neq 0$, there exists $z \in X$ such that $(I - P)TPz \neq 0$, and thus, also $(I - P)TP(z - y) \neq 0$. Applying a reasoning along the lines of that above, leads to two equalities similar to (5), but now replacing $f \otimes y$ by $f \otimes z$ and $f \otimes (z - y)$, respectively. It follows that $P(f \otimes z)(I - P)$ and $P(f \otimes (z - y))(I - P)$ lie in $\mathcal{L}$. Hence, the following lies in $\mathcal{L}$:

$$P(f \otimes y)(I - P) = P(f \otimes z)(I - P) - P(f \otimes (z - y))(I - P).$$

□

Remark 2. *We will often use the fact that, for all $P \in \mathcal{N}, T \in \mathcal{L}$, the operators $(I-P)TP$ and $PT(I-P)$ lie in $\mathcal{L}$, which is a direct consequence of part (i) of this lemma.*

Let $\mathcal{L}$ be a weakly closed Lie $\mathcal{A}$-module. We will now focus on constructing a weakly closed $\mathcal{A}$-bimodule $\mathcal{K}$ and a weakly closed subalgebra $\mathcal{D}_{\mathcal{K}}$ of $\mathcal{A}$ such that $\mathcal{L} \subseteq \mathcal{K} + \mathcal{D}_{\mathcal{K}}$. To this end, we begin with the definition of four subspaces along the lines of [8].

Given a weakly closed Lie $\mathcal{A}$-module $\mathcal{L}$, define the subspaces the subspaces $\mathcal{K}_V$, $\mathcal{K}_L$, $\mathcal{K}_D$, $\mathcal{K}_\Delta$, and $\mathcal{K}$ of $B(X)$ by:

$$\begin{aligned}
\mathcal{K}_V &= \overline{\operatorname{span}}^w\{PT(I-P) : P \in \mathcal{N},\ T \in \mathcal{L}\},\\
\mathcal{K}_L &= \overline{\operatorname{span}}^w\{(I-P)TP : P \in \mathcal{N},\ T \in \mathcal{L}\},\\
\mathcal{K}_D &= \overline{\operatorname{span}}^w\{PS(I-P)TP : P \in \mathcal{N},\ T \in \mathcal{L},\ S \in \mathcal{A}\},\\
\mathcal{K}_\Delta &= \overline{\operatorname{span}}^w\{(I-P)TPS(I-P) : P \in \mathcal{N},\ T \in \mathcal{L},\ S \in \mathcal{A}\},
\end{aligned}$$

$$\mathcal{K} = \mathcal{K}_V + \mathcal{K}_L + \mathcal{K}_D + \mathcal{K}_\Delta. \tag{6}$$

Lemma 6. *Let $\mathcal{L}$ be a weakly closed Lie $\mathcal{A}$-module. Then, the following hold.*

(i) $\mathcal{K}_V$ *is a weakly closed ideal of* $\mathcal{A}$*;*
(ii) $\mathcal{K}$ *is a weakly closed* $\mathcal{A}$*-bimodule;*
(iii) $[\mathcal{K}, \mathcal{A}] \subseteq \mathcal{L}$.

Proof. (i) We start by showing $\mathcal{K}_V$ is an ideal of $\mathcal{A}$. For all $P, Q \in \mathcal{N}$, and $T \in \mathcal{L}$,:

$$(I-Q)PT(I-P)Q = 0,$$

and hence, $PT(I-P) \in \mathcal{A}$. Since $\mathcal{A}$ is weakly closed, it follows that $\mathcal{K}_V \subseteq \mathcal{A}$.

Let $R \in \mathcal{A}$. The operator $(I-P)R(I-P) \in \mathcal{A}$ and, by Lemma 5 (i), $PT(I-P) \in \mathcal{L}$, therefore:

$$[PT(I-P), (I-P)R(I-P)] = PT(I-P)R(I-P) \in \mathcal{L}.$$

Since $(I-P)R(I-P) = (I-P)R$, we have shown that $PT(I-P)R \in \mathcal{L}$. By observing that $PT(I-P)R = PPT(I-P)R(I-P)$, we conclude that this operator lies in $\mathcal{K}_V$.

Analogously:

$$[PRP, PT(I-P)] = PRPT(I-P) \in \mathcal{L},$$

and $RPT(I-P) = PRPT(I-P)(I-P) \in \mathcal{K}_V$.

For every $U \in \mathcal{K}_V$, there is a net (U_α) converging to U in the strong operator topology, with:

$$U_\alpha = \sum_{k=1}^{k_\alpha} P_{\alpha,k} T_{\alpha,k} (I - P_{\alpha,k}), \tag{7}$$

where $P_{\alpha,k} \in \mathcal{N}$ and $T_{\alpha,k} \in \mathcal{L}$ (here, we use the fact that the closure in the weak operator topology of a convex set coincides with its closure in the strong operator topology). Let $R \in \mathcal{A}$. As seen above, $RP_{\alpha,k}T_{\alpha,k}(I-P_{\alpha,k}) \in \mathcal{K}_V$. Hence:

$$RU = R(\lim U_\alpha) = \lim \sum_{k=1}^{k_\alpha} RP_{\alpha,k}T_{\alpha,k}(I-P_{\alpha,k}), \tag{8}$$

yielding that $RU \in \mathcal{K}_V$. Similarly, $UR \in \mathcal{K}_V$. Thus, $\mathcal{K}_V$ is an ideal of $\mathcal{A}$.

(ii) We will now show that $\mathcal{K}_L\mathcal{A}, \mathcal{A}\mathcal{K}_L \subseteq \mathcal{K}$. It is sufficient to show that, for all $T \in \mathcal{L}$, $P \in \mathcal{N}$ and $R \in \mathcal{A}$, the operators $(I-P)TPR$, and $R(I-P)TP$ are in $\mathcal{K}$, as we can then apply a similar argument based on limits. In fact, similarly to (7), for every $U \in \mathcal{K}_L$, there is a net (U_α) converging to U in the strong operator topology, with:

$$U_\alpha = \sum_{k=1}^{k_\alpha} (I - P_{\alpha,k}) T_{\alpha,k} (P_{\alpha,k}),$$

where $P_{\alpha,k} \in \mathcal{N}$ and $T_{\alpha,k} \in \mathcal{L}$. If $R \in \mathcal{A}$ were such that $RP_{\alpha,k}T_{\alpha,k}(I - P_{\alpha,k}) \in \mathcal{K}$, and then:

$$RU = R(\lim U_\alpha) = \lim \sum_{k=1}^{k_\alpha} RP_{\alpha,k}T_{\alpha,k}(I - P_{\alpha,k}), \tag{9}$$

yielding that $RU \in \mathcal{K}$. Similarly, we could argue that $UR \in \mathcal{K}$.

We show now that, for all $T \in \mathcal{L}$, $P \in \mathcal{N}$ and $R \in \mathcal{A}$, we have indeed that the operators $(I - P)TPR$ and $R(I - P)TP$ lie in $\mathcal{K}$. Since, by Lemma 5 (i), $(I - P)TP \in \mathcal{L}$, we may consider only the case where $T = (I - P)TP$.

Let $T = (I - P)TP \in \mathcal{L}$ and $R \in \mathcal{A}$. We can write:

$$TR = (I - P)TPRP + (I - P)TPR(I - P).$$

The operator $(I - P)TPR(I - P)$ is in $\mathcal{K}_\Delta$, and the following is in $\mathcal{K}_L$:

$$(I - P)TPRP = (I - P)[T, PRP]P,$$

because $[T, PRP] \in \mathcal{L}$. Hence, $TR \in \mathcal{K}$.

Analogously, the following is in $\mathcal{K}$:

$$RT = PR(I - P)TP + (I - P)R(I - P)TP,$$

since $PR(I - P)TP \in \mathcal{K}_D$, and the following is in $\mathcal{K}_L$:

$$(I - P)R(I - P)TP = (I - P)[(I - P)R(I - P), T]P.$$

We will show next that $\mathcal{K}_D\mathcal{A}, \mathcal{A}\mathcal{K}_D \subseteq \mathcal{K}$. It suffices to show that, for all $T \in \mathcal{L}$, $P \in \mathcal{N}$ and $S, R \in \mathcal{A}$, the operators $RPS(I - P)TP$ and $PS(I - P)TPR$ lie in $\mathcal{K}$.

Since $(I - P)RP = 0$, we have $RP = PRP$, and thus:

$$RPS(I - P)TP = P(RPS)(I - P)TP,$$

where $RPS \in \mathcal{A}$. It follows that $RPS(I - P)TP$ lies in $\mathcal{K}_D$, and therefore, also in $\mathcal{K}$.

It remains to see that $PS(I - P)TPR \in \mathcal{K}$. If $(I - P)\mathcal{L}P = \{0\}$, then the result holds trivially. Suppose that $(I - P)\mathcal{L}P \neq \{0\}$. Then, by Lemma 5 (ii):

$$P\mathcal{L}(I - P) = PB(X)(I - P).$$

Therefore, $PB(X)(I - P) \subseteq \mathcal{K}_V$. Observe that:

$$\begin{aligned} PS(I - P)TPR &= PS(I - P)TPRP + PS(I - P)TPR(I - P) \\ &= PS(I - P)[(I - P)TP, PRP]P + PS(I - P)TPR(I - P). \end{aligned}$$

Since $PB(X)(I - P) \subseteq \mathcal{K}_V$, the operator $PS(I - P)TPR(I - P) \in \mathcal{K}_V$. By Lemma 5 (i), $(I - P)TP \in \mathcal{L}$, and hence, $[(I - P)TP, PRP] \in \mathcal{L}$. It follows that $PS(I - P)[(I - P)TP, PRP]P \in \mathcal{K}_D$. Therefore, $PS(I - P)TPR \in \mathcal{K}$.

Similarly to what was just shown for $\mathcal{K}_D$, we will now prove that $\mathcal{K}_\Delta\mathcal{A}, \mathcal{A}\mathcal{K}_\Delta \subseteq \mathcal{K}$. It is sufficient to show that, for all $T \in \mathcal{L}$, $P \in \mathcal{N}$, and $S, R \in \mathcal{A}$, the operators $R(I - P)TPS(I - P)$ and $(I - P)TPS(I - P)R$ are in $\mathcal{K}$.

Since $S(I-P)R \in \mathcal{A}$, the following lies in $\mathcal{K}_\Delta$:

$$(I-P)TPS(I-P)R = (I-P)TP(S(I-P)R)(I-P).$$

It only remains to see that $R(I-P)TPS(I-P) \in \mathcal{K}$. If $(I-P)\mathcal{L}P = \{0\}$, the result is immediate. If this is not the case, then $PB(X)(I-P) \subseteq \mathcal{K}_V$, as before. We have:

$$\begin{aligned} R(I-P)TPS(I-P) &= PR(I-P)TPS(I-P) + (I-P)R(I-P)TPS(I-P) \\ &= PR(I-P)TPS(I-P) + (I-P)[(I-P)R(I-P),(I-P)TP]PS(I-P). \end{aligned}$$

Since $[(I-P)R(I-P),(I-P)TP] \in \mathcal{L}$, the following operator lies in $\mathcal{K}_\Delta$:

$$(I-P)[(I-P)R(I-P),(I-P)TP]PS(I-P)$$

Furthermore, $PR(I-P)TPS(I-P) \in \mathcal{K}_V$, since $PB(X)(I-P) \subseteq \mathcal{K}_V$. Hence, $R(I-P)TPS(I-P) \in \mathcal{K}$. This concludes the proof that $\mathcal{K}$ is a weakly closed $\mathcal{A}$-bimodule.

(iii) Finally, we prove that $[\mathcal{K},\mathcal{A}] \subseteq \mathcal{L}$. Since $\mathcal{K}_V, \mathcal{K}_L \subseteq \mathcal{L}$, we have only to show that:

$$[\mathcal{K}_D,\mathcal{A}], [\mathcal{K}_\Delta,\mathcal{A}] \subseteq \mathcal{L}.$$

It is sufficient to show that, for all $P \in \mathcal{N}$, $T \in \mathcal{L}$ and $S, R \in \mathcal{A}$, the operators $[PS(I-P)TP, R]$ and $[(I-P)TPS(I-P), R]$ lie in $\mathcal{L}$.

If $(I-P)\mathcal{L}P = \{0\}$, the result is immediate. If this is not the case, then, by Lemma 5, $PB(X)(I-P) = P\mathcal{L}(I-P) \subseteq \mathcal{L}$. Thus, since $(I-P)TP \in \mathcal{L}$, and noting that $RP = PRP$, the following is in $\mathcal{L}$:

$$\begin{aligned} &[PS(I-P)TP, R] = \\ &= [PS(I-P)TP, RP] + [PS(I-P)TP, PR(I-P)] \\ &\quad + [PS(I-P)TP, (I-P)R(I-P)] \\ &= [PS(I-P)TP - (I-P)TPS(I-P), RP] + PS(I-P)TPR(I-P) \\ &= [[PS(I-P),(I-P)TP], RP] + PS(I-P)TPR(I-P). \end{aligned}$$

Similarly:

$$\begin{aligned} &[(I-P)TPS(I-P), R] = \\ &= [(I-P)TPS(I-P), RP] + [(I-P)TPS(I-P), PR(I-P)] \\ &\quad + [(I-P)TPS(I-P), (I-P)R(I-P)] \\ &= -PR(I-P)TPS(I-P) + [(I-P)TPS(I-P) - PS(I-P)TP, (I-P)R(I-P)] \\ &= -PR(I-P)TPS(I-P) + [[(I-P)TP, PS(I-P)], (I-P)R(I-P)], \end{aligned}$$

which shows that this operator lies also in $\mathcal{L}$. □

Remark 3. *Since $\mathcal{K}$ is a weakly closed $\mathcal{A}$-bimodule, there exists a left continuous order homomorphism $\phi : \mathcal{N} \to \mathcal{N}$ such that:*

$$\mathcal{K} = \{T \in B(X) : T(N) \subseteq \phi(N) \text{ for all } N \in \mathcal{N}\},$$

as pointed out in Section 2 (see Lemma 3). Suppose that $P \in \mathcal{N}$ is a projection such that $\phi(P) < P$. Then, for all projections $Q \in \mathcal{N}$ with $\phi(P) < Q < P$, we have:

$$(Q-\phi(P))T(P-Q) = 0, \tag{10}$$

where $T \in \mathcal{L}$ is any operator in the Lie module $\mathcal{L}$. In fact, since $QP = Q$ and $\phi(P)Q = \phi(P)$:

$$(Q-\phi(P))T(P-Q) = (I-\phi(P))QT(I-Q)P.$$

By the definition of $\mathcal{K}$, $QT(I-Q) \in \mathcal{K}$ and, therefore, $(I-\phi(P))QT(I-Q)P = 0$.

We turn our attention now to the construction of the weakly closed subalgebra $\mathcal{D}_{\mathcal{K}}$ of the nest algebra $\mathcal{A}$. We begin with some definitions.

Let $\mathcal{N}' = \{T \in B(X)\colon [T,P] = 0, \text{ for all } P \in \mathcal{N}\}$ be the commutant of the nest $\mathcal{N}$ in $B(X)$. Observe that $\mathcal{N}'$ is a weakly closed subalgebra of $\mathcal{A}$ and that we also have:

$$\mathcal{N}' = \mathcal{A} \cap \{T \in B(X)\colon PT(I-P) = 0, \text{ for all } P \in \mathcal{N}\}.$$

Define $\mathcal{D}_{\mathcal{K}}$ as the weakly closed algebra of all operators $T \in \mathcal{N}'$ for which there exists $\lambda \in \mathbb{C}$ such that:

$$T(P-\phi(P)) = \lambda(P-\phi(P)), \tag{11}$$

whenever $P \in \mathcal{N}$ is such that $\phi(P) < P_-$. It is understood here that the scalar λ depends both on the operator T and the projection P.

We define now a property that characterises the class of nest algebras to which the main result of this section (Theorem 2) applies, as this property plays a relevant role in its proof. A nest algebra $\mathcal{A}$ associated with a nest $\mathcal{N}$ is said to have the *π-property* if there exists a (surjective) projection $\pi\colon B(X) \to \mathcal{N}'$ such that:

$$\pi(ATB) = A\pi(T)B,$$

for all $A, B \in \mathcal{N}'$, $T \in B(X)$. Any idempotent homomorphism is such a projection. To help motivate this definition, we mention below some examples of nest algebras with the π-property.

Example 1. *Let X have a Schauder basis $\{e_i\}_{i=1}^{\infty}$ and let $\mathcal{N}$ be a nest of the form $\mathcal{N} = \{P_n : n \in \mathbb{N}\} \cup \{0, I\}$, where:*

$$P_n\left(\sum_{i=1}^{\infty} \alpha_i e_i\right) = \sum_{i=1}^{k_n} \alpha_i e_i,$$

for some $k_1, k_2, \ldots \in \mathbb{N}$ with $k_i < k_j$ for $i < j$. Then, there is a (contractive) projection $\pi\colon B(X) \to \mathcal{N}'$ satisfying, for all $A, B \in \mathcal{N}'$, $T \in B(X)$, $\pi(ATB) = A\pi(T)B$.

We now give a proof of this statement. Note that, for $m \le n$, $P_nP_m = P_m = P_mP_n$. It is easy to see that the corresponding nest algebra is:

$$\mathcal{A} = \left\{T \in B(X) : Te_j = \sum_{i=1}^{k_n} \lambda_{i,j} e_i, \text{ for all } k_{n-1}+1 \le j \le k_n,\ n \in \mathbb{N}\right\},$$

where we define $k_0 = 0$. Similarly, $\{T \in B(X)\colon PT(I-P) = 0, \text{ for all } P \in \mathcal{N}\}$ is the set of operators T such that $Te_j = \sum_{i=k_{n-1}+1}^{\infty} \lambda_{i,j} e_i$. Therefore:

$$\begin{aligned}\mathcal{N}' &= \mathcal{A} \cap \{T \in B(X)\colon PT(I-P) = 0, \text{ for all } P \in \mathcal{N}\}\\ &= \left\{T \in B(X) : Te_j = \sum_{i=k_{n-1}+1}^{k_n} \lambda_{i,j} e_i, \text{ for all } k_{n-1}+1 \le j \le k_n,\ n \in \mathbb{N}\right\}.\end{aligned}$$

We define the projection $\pi : B(X) \to \mathcal{N}'$ as follows: for each $T \in B(X)$, with $Te_j = \sum_{i=1}^{\infty} \lambda_{i,j} e_i$, for $j \in \mathbb{N}$, let:

$$\pi(T)e_j = \sum_{i=k_{n-1}+1}^{k_n} \lambda_{i,j} e_i,\ \ k_{n-1}+1 \le j \le k_n,\ n \in \mathbb{N}.$$

Then, for any $A \in \mathcal{N}'$, $T \in B(X)$, with $Ae_j = \sum_{i=k_{n-1}+1}^{k_n} a_{i,j}e_i$ and $k_{n-1} + 1 \leq j \leq k_n$:

$$ATe_j = \sum_{i=1}^{\infty} \lambda_{i,j} Ae_i = \sum_{n=1}^{\infty} \sum_{i=k_{n-1}+1}^{k_n} \lambda_{i,j} \sum_{\ell=k_{n-1}+1}^{k_n} a_{\ell,i} e_\ell.$$

Hence:

$$\pi(AT)e_j = \sum \lambda_{i,j} a_{\ell,i} e_\ell = A\pi(T)e_j,$$

where the sum is taken over $n \in \mathbb{N}$ and $i, \ell \in \{k_{n-1}+1, \cdots, k_n\}$. Therefore:

$$\pi(AT) = A\pi(T).$$

Similarly, $\pi(TA) = \pi(T)A$ holds. It follows that, for all $A, B \in \mathcal{N}'$, $T \in B(X)$,

$$\pi(ATB) = A\pi(T)B.$$

Remark 4. *Any nest in a finite dimensional Banach space X has a projection π as in Example 1. It only suffices to consider a finite basis in the above reasoning.*

The nest algebras in Example 1 and Remark 4 are concrete examples of nest algebras satisfying the π-property, as is any Hilbert space nest algebra (see ([13], Chapter II, Section 8)).

Theorem 2. *Let $\mathcal{A}$ be a nest algebra having the π-property, and let $\mathcal{L}$ be a weakly closed Lie $\mathcal{A}$-module. Then, there exists a weakly closed $\mathcal{A}$-bimodule $\mathcal{K}$, and there exists a weakly closed subalgebra $\mathcal{D}_\mathcal{K}$ of $\mathcal{A}$ such that $\mathcal{L} \subseteq \mathcal{K} + \mathcal{D}_\mathcal{K}$, where $\mathcal{K}$ and $D_\mathcal{K}$ are as in (6) and (11), respectively. Moreover, $[\mathcal{K}, \mathcal{A}] \subseteq \mathcal{L}$.*

Proof. Let $\mathcal{K}$ and $D_\mathcal{K}$ be defined as in (6) and (11), respectively. By Lemma 6 (iii), we have $[\mathcal{K}, \mathcal{A}] \subseteq \mathcal{L}$.

Let $T \in \mathcal{L}$ and define:

$$T_\pi = T - \pi(T).$$

We will show that $T_\pi \in \mathcal{K}$ and $\pi(T) \in \mathcal{D}_\mathcal{K}$, yielding that $T \in \mathcal{K} + \mathcal{D}_\mathcal{K}$. Let $\phi : \mathcal{N} \to \mathcal{N}$ be a left continuous order homomorphism such that:

$$\mathcal{K} = \{S \in B(X) \colon (I - \phi(P))SP = 0 \text{ for all } P \in \mathcal{N}\},$$

which exists since $\mathcal{K}$ is a weakly closed $\mathcal{A}$-bimodule. We will show that, for all $P \in \mathcal{N}$:

$$(I - \phi(P))T_\pi P = 0.$$

For any $Q \in \mathcal{N}$, one has:

$$(I - \phi(P))(I - Q)TQP = 0,$$

since, by the definition of $\mathcal{K}$, the operator $(I - Q)TQ$ lies in $\mathcal{K}$. Hence:

$$\begin{aligned}
(I - Q)(I - \phi(P))T_\pi PQ &= (I - \phi(P))(I - Q)T_\pi QP \\
&= (I - \phi(P))(I - Q)TQP - (I - \phi(P))(I - Q)\pi(T)QP \\
&= -(I - \phi(P))(I - Q)\pi(T)QP.
\end{aligned}$$

Since $(I - \phi(P))(I - Q)$, $QP \in \mathcal{N}'$, by the π-property:

$$(I - \phi(P))(I - Q)\pi(T)QP = \pi\big((I - \phi(P))(I - Q)TQP\big) = 0,$$

from which it follows that $(I-Q)(I-\phi(P))T_\pi PQ = 0$. By a similar reasoning:

$$\begin{aligned} Q(I-\phi(P))T_\pi P(I-Q) &= Q(I-\phi(P))TP(I-Q) - Q(I-\phi(P))\pi(T)P(I-Q) \\ &= -\pi\big(Q(I-\phi(P))TP(I-Q)\big) = 0. \end{aligned}$$

Therefore, for all $Q \in \mathcal{N}$:

$$(I-\phi(P))T_\pi P = Q(I-\phi(P))T_\pi PQ + (I-Q)(I-\phi(P))T_\pi P(I-Q),$$

which implies that $(I-\phi(P))T_\pi P$ commutes with Q. Therefore, $(I-\phi(P))T_\pi P \in \mathcal{N}'$. It follows that:

$$\begin{aligned} (I-\phi(P))T_\pi P &= \pi\big((I-\phi(P))T_\pi P\big) \\ &= (I-\phi(P))\pi(T_\pi)P = 0, \end{aligned}$$

since $\pi(T_\pi) = \pi(T-\pi(T)) = 0$. Therefore, $T_\pi \in \mathcal{K}$.

It remains to show that $\pi(T) \in \mathcal{D}_\mathcal{K}$. Let $P \in \mathcal{N}$ be such that $\phi(P) < P_-$. Then, there exists $Q \in \mathcal{N}$ such that $\phi(P) < Q < P$.

Recall that $[\mathcal{L},\mathcal{A}] \subseteq \mathcal{L}$ and, by Lemma 6, $[\mathcal{K},\mathcal{A}] \subseteq \mathcal{L}$. Since $T \in \mathcal{L}$ and $T_\pi \in \mathcal{K}$, we have also $[\pi(T),\mathcal{A}] \in \mathcal{L}$. Therefore, for any rank one operator $f \otimes y$, the operator:

$$[\pi(T),(Q-\phi(P))(f\otimes y)(P-Q)]$$

lies in $\mathcal{L}$, since $(Q-\phi(P))(f\otimes y)(P-Q) \in \mathcal{A}$. Hence, by (10):

$$(Q-\phi(P))[\pi(T),(Q-\phi(P))(f\otimes y)(P-Q)](P-Q) = 0,$$

that is:

$$f(P-Q)\otimes\pi(T)(Q-\phi(P))y = f\pi(T)(P-Q)\otimes(Q-\phi(P))y,$$

where we have used the fact that $\pi(T)$ commutes with all $P \in \mathcal{N}$. Therefore, there exists $\lambda \in \mathbb{C}$ such that, for all $f \in X^*$, $y \in X$:

$$\pi(T)(Q-\phi(P))y = \lambda(Q-\phi(P))y,$$

and:

$$f\pi(T)(P-Q) = \lambda f(P-Q).$$

Consequently:

$$\pi(T)(Q-\phi(P)) = \lambda(Q-\phi(P)),$$

and:

$$\pi(T)(P-Q) = \lambda(P-Q).$$

It follows that:

$$\pi(T)(P-\phi(P)) = \lambda(P-\phi(P)),$$

yielding $\pi(T) \in \mathcal{D}_\mathcal{K}$, as required. □

An immediate consequence of this theorem and Theorem 1 is that we can find that the inclusions in (1) do hold for nest algebras having the π-property, provided the spaces involved are properly defined. We summarise this in the next theorem.

Theorem 3. *Let $\mathcal{A}$ be a nest algebra having the π-property, and let $\mathcal{L}$ be a weakly closed Lie $\mathcal{A}$-module. Then, there exist weakly closed $\mathcal{A}$-bimodules $\mathcal{J}$ and $\mathcal{K}$, and there exists a weakly closed subalgebra $\mathcal{D}_\mathcal{K}$ of $\mathcal{A}$ such that:*

$$\mathcal{J} \subseteq \mathcal{L} \subseteq \mathcal{K} + \mathcal{D}_\mathcal{K},$$

where:

(i) *$\mathcal{J}$ is the largest weakly closed $\mathcal{A}$-bimodule contained in $\mathcal{L}$ and is given by:*

$$\mathcal{J} = \{T \in \mathcal{B}(X) : T(N) \subseteq \phi(N), \text{ for all } N \in \mathcal{N}\},$$

where $\phi : \mathcal{N} \to \mathcal{N}$ is the left continuous order homomorphism defined in (4),

(ii) *$\mathcal{K}$ and $\mathcal{D}_{\mathcal{K}}$ are as in (6) and (11), respectively, and $[\mathcal{K}, \mathcal{A}] \subseteq \mathcal{L}$.*

Proof. The theorem follows immediately from Theorems 1 and 2. □

The next example applies the main results of this work to determine the form of a Lie module.

Example 2. *Let $\mathcal{N}$ be the nest in the Banach space X of Example 1, and let $\mathcal{A}$ be the corresponding nest algebra. Let $f \otimes y$ be a rank one operator, and let $\mathcal{L}$ be the weakly closed Lie $\mathcal{A}$-module generated by $f \otimes y$ and the identity I. We wish to find $\mathcal{L}$ explicitly.*

Observe that, by Theorem 3, the Lie module $\mathcal{L}$ contains a non-zero largest bimodule. In fact, by Corollary 1 and Theorem 1, this bimodule must contain $f \otimes y$.

Notice also that, if $\mathcal{U}$ is a weakly closed $\mathcal{A}$-bimodule containing $f \otimes y$, then $\mathcal{U} + \operatorname{span}\{I\}$ is the smallest weakly closed Lie $\mathcal{A}$-module containing $\mathcal{U}$ and the identity I.

Let $\mathcal{J}_{f\otimes y}$ be the smallest $\mathcal{A}$-bimodule containing $f \otimes y$, that is, $\mathcal{J}_{f\otimes y}$ is the $\mathcal{A}$-bimodule generated by $f \otimes y$. Since we want to find the smallest Lie module containing $f \otimes y$ and the identity I, then it suffices to consider $\mathcal{L} = \mathcal{J}_{f\otimes y} + \operatorname{span} I$.

Now, by Corollary 1 and Theorem 1, the left continuous order homomorphism $\phi\colon \mathcal{N} \to \mathcal{N}$ associated with $\mathcal{J}_{f\otimes y}$ is:

$$\phi(N) = \begin{cases} 0, & N \leq \hat{N}_f \\ N_y, & \hat{N}_f < N. \end{cases}$$

Hence, $\mathcal{L} = \mathcal{J}_{f\otimes y} + \operatorname{span}\{I\}$, where:

$$\mathcal{J}_{f\otimes y} = \{T \in B(X)\colon T(N) \subseteq \phi(N), \text{ for all } N \in \mathcal{N}\}.$$

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