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Article

On an Anisotropic Logistic Equation

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Abstract: We consider a nonlinear Dirichlet problem driven by the $(p(z), q)$-Laplacian and with a logistic reaction of the equidiffusive type. Under a nonlinearity condition on a quotient map, we show existence and uniqueness of positive solutions and the result is global in parameter λ. If the monotonicity condition on the quotient map is not true, we can no longer guarantee uniqueness, but we can show the existence of a minimal solution u_λ^* and establish the monotonicity of the map $\lambda \longmapsto u_\lambda^*$ and its asymptotic behaviour as the parameter λ decreases to the critical value $\widehat{\lambda}_1(q) > 0$ (the principal eigenvalue of $(-\Delta_q, W_0^{1,q}(\Omega))$).

Keywords: anisotropic operator; equidiffusive logistic reaction; uniqueness; minimal positive solution; anisotropic regularity

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1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following anisotropic $(p.q)$-equation with a logistic reaction

$$(P_\lambda) \qquad \begin{cases} -\Delta_{p(z)}u(z) - \Delta_q u(z) = \lambda u(z)^{q-1} - f(z, u(z)) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \ u > 0, \ \lambda > 0. \end{cases}$$

Here, $p \in C^{0,1}(\overline{\Omega})$ with $1 < q < p_- = \min\limits_{\overline{\Omega}} p$ and by $\Delta_{p(z)}$ we denote the anisotropic p-Laplace differential operator defined by

$$\Delta_{p(z)}u = \operatorname{div}(|Du|^{p(z)-2}Du) \quad \forall u \in W_0^{1,p(z)}(\Omega).$$

In contrast to the isotropic p-Laplacian (that is, the exponent is constant function $p(z) = p > 1$), the anisotropic operator is not homogeneous and this makes anisotropic equations more difficult to deal with. Problem (P_λ) is driven by the sum of an anisotropic and of an isotropic operators. At the end of the paper, after having the complete picture of our method of proof, we comment on why we have the smaller exponent q to be constant (isotropic operator). It is an open problem, whether our work here can be extended to fully anisotropic (p, q) equations. In the reaction (right hand side) of (P_λ), $\lambda > 0$ is a parameter and the perturbation is $-f(z, x)$ with $f(z, x)$ being Carathéodory function (i.e., $z \longmapsto f(z, x)$ measurable for all $x \geqslant 0$, and $x \longmapsto f(z, x)$ is continuous for a.a. $z \in \Omega$). We assume that $f(z, \cdot)$ is $(p(z) - 1)$-superlinear. So, we see that the reaction of (P_λ) is of logistic-type and in particular it is equidiffusive since the power of the parameter term λu^{q-1} is the same as the exponent of the isotropic operator.

Logistic equations are important in mathematical biology. The semilinear parabolic logistic equation describes the evolution and spatial distribution of a biological population when constant rates of reproduction and/or mortality are present (Verhulst's law; see Gurtin-Mac Camy [1]). For this reason, when we consider logistic equations, we are usually interested in positive solutions. More recently, evolution systems with logistic forcing terms have been studied as a model for the biological phenomenon of chemotaxis (see Tello-Winkler [2]). The elliptic equation examined in this paper models an equilibrium distribution (see Costa-Drábek-Tehrani [3]). In the past, most works on elliptic logistic equations deal with isotropic problems with a superdiffusive reaction (that is, a reaction of the form $\lambda x^{s-1} - x^{r-1}$ with $p < s < r$). They prove existence and multiplicity results which are global in $\lambda > 0$ (bifurcation-type result). We mention the works of Afrouzi-Brown [4], Ambrosetti-Lupo [5], Ambrosetti-Mancini [6], Papageorgiou-Rădulescu-Repovš [7], Rădulescu-Repovš [8], (semilinear equations) and by Aizicovici-Papageorgiou-Staicu [9], Dong [10], Gasiński-O'Regan-Papageorgiou [11], Iannizzotto-Papageorgiou [12], Papageorgiou-Rădulescu [13], and Takeuchi [14,15] (nonlinear equations). We also mention the works of Gasiński-Papageorgiou [16] (double phase equations) and Iannizzotto-Mosconi-Papageorgiou [17] (fractional equations). All the aforementioned works deal with superdiffusive problems.

The study of anisotropic logistic equations is lagging behind. There are no works in this direction. Only Papageorgiou-Rădulescu-Tang [18] considered logistic equations driven by the $p(z)$-Laplacian and having a Robin boundary condition. They consider the superdiffusive case (that is, the parametric term $\lambda x^{\tau(z)-1}$ with $p(z) < \tau(z)$ for all $z \in \overline{\Omega}$). From the isotropic literature, we know that for such problems, we have a multiplicity of positive solutions. The authors in [18] show that the same is true for $p(z)$-logisitic equations. They prove a multiplicity result which is global in the parameter $\lambda > 0$ (a bifurcation-type theorem), see Theorem 22 in [18]. In [18], the equidiffusive case is studied only in the context of isotropic equations (that is, p is constant).

So, to the best of our knowledge, there are no earlier works on anisotropic equidiffusive logistic equations. Our work fills in the void in the literature. From the isotropic theory (see Kamin-Véron [19], p-Laplace equations), we know that equidiffusive problems exhibit uniqueness properties.

Here, we deal with an equidiffusive equation and for such problems we have uniqueness of solutions. Indeed, here we prove a global existence and uniqueness result. More precisely, if $\widehat{\lambda}_1(q) > 0$ denotes the principal eigenvalue of $(-\Delta_q, W_0^{1,q}(\Omega))$, we show that (P_λ) has a positive solution if and only if $\lambda > \widehat{\lambda}_1(q)$, and moreover, this solution is unique if the quotient function $x \longmapsto \frac{f(z,x)}{x^{q-1}}$ is strictly increasing on $(0, +\infty)$. Otherwise, we can show the existence of a minimal (smallest) positive solution u_λ^*. We also establish the monotonicity properties of the map $\lambda \longmapsto u_\lambda^*$ and determine the asymptotic behaviour of u_λ^* as $\lambda \to \widehat{\lambda}_1(q)^+$.

Finally, we explain why in our problem the smaller exponent q is constant. It is an interesting open problem whether the result remains true if this exponent is also variable.

2. Mathematical Background—Hypotheses

The study of anisotropic equations uses variable Lebesgue and Sobolev spaces. The complete theory of these spaces can be found in the book of Diening-Harjulehto-Hästö-Ružička [20].

Let $E_1 = \{r \in C(\overline{\Omega}) : 1 < r(z) \text{ for all } z \in \overline{\Omega}\}$. Given $r \in E$, we set

$$r_- = \min_{\overline{\Omega}} r, \quad r_+ = \max_{\overline{\Omega}} r.$$

By $L^0(\Omega)$, we denote the space of all measurable functions $u \colon \Omega \longrightarrow \mathbb{R}$. As usual, we identify two such functions which differ only on a Lebesgue-null set. Given $r \in E$, the variable Lebesgue space $L^{r(z)}(\Omega)$ is defined by

$$L^{r(z)}(\Omega) = \{u \in L^0(\Omega) : \varrho_r(u) = \int_\Omega |u|^{r(z)}\, dz < \infty\}.$$

We call ϱ_r the modular function corresponding to the exponent r. We equip this space with the so called "Luxemburg norm" defined by

$$\|u\|_{r(z)} = \inf\{\lambda > 0 : \varrho_r(\frac{u}{\lambda}) \leqslant 1\}.$$

With this norm, $L^{r(z)}(\Omega)$ becomes a separable and reflexive Banach space. In fact, it is uniformly convex since $x \longmapsto |x|^{r(z)}$ is a uniformly convex function. If $r' \in E_1$ is the conjugate variable exponent to $r \in E_1$ (that is, $r'(z) = \frac{r(z)}{r(z)-1}$ for all $z \in \overline{\Omega}$), then we have $L^{r(z)}(\Omega)^* = L^{r'(z)}(\Omega)$, and the following Hölder-type inequality holds

$$\int_\Omega |uv|\, dz \leqslant \left(\frac{1}{r_-} + \frac{1}{r'_-}\right)\|u\|_{r(z)}\|v\|_{r'(z)} \quad \forall u \in L^{r(z)}(\Omega),\ v \in L^{r'(z)}(\Omega).$$

If $r_1, r_2 \in E_1$ and $r_1(z) \leqslant r_2(z)$ for all $z \in \overline{\Omega}$, then $L^{r_2(z)}(\Omega) \subseteq L^{r_1(z)}(\Omega)$ continuously.

Using the variable Lebesgue spaces, we can define the corresponding variable Sobolev spaces. So, let $r \in E_1$. Then, the variable Sobolev space $W^{1,r(z)}(\Omega)$ is defined by

$$W^{1,r(z)}(\Omega) = \{u \in L^{r(z)}(\Omega) : |Du| \in L^{r(z)}(\Omega)\}.$$

By Du, we denote the weak gradient of u. We equip $W^{1,r(z)}(\Omega)$ with the following norm

$$\|u\|_{1,r(z)} = \|u\|_{r(z)} + \|Du\|_{r(z)} \quad \forall u \in W^{1,r(z)}(\Omega),$$

with $\|Du\|_{r(z)} = \||Du|\|_{r(z)}$. Furthermore, if $r \in E_1 \cap C^{0,1}(\overline{\Omega})$ (that is, r is Lipschitz continuous on $\overline{\Omega}$), then we define

$$W_0^{1,r(z)}(\Omega) = \overline{C_c^\infty}^{\|\cdot\|_{1,r(z)}}.$$

Both spaces $W^{1,r(z)}(\Omega)$ and $W_0^{1,r(z)}(\Omega)$ are separable and reflexive Banach spaces (in fact uniformly convex). On $W_0^{1,r(z)}(\Omega)$, the Poincaré inequality holds. Namely, there exists $\hat{c} > 0$ such that

$$\|u\|_{r(z)} \leqslant \hat{c}\|Du\|_{r(z)} \quad \forall u \in W_0^{1,r(z)}(\Omega).$$

So, on $W_0^{1,r(z)}(\Omega)$, we consider the following equivalent norm

$$\|u\| = \|Du\|_{r(z)} \quad \forall u \in W_0^{1,r(z)}(\Omega).$$

Let $r^*(z)$ be the variable critical Sobolev exponent defined by

$$r^*(z) = \begin{cases} \frac{Nr(z)}{N-r(z)} & \text{if} \quad r(z) < N \\ +\infty & \text{if} \quad N \leqslant r(z) \end{cases}$$

for all $z \in \overline{\Omega}$. We have the following extension to variable spaces of the classical Sobolev embedding theorem (see [20] (p. 266)).

Proposition 1. ***(a)*** *If $r \in E_1 \cap C^{0,1}(\overline{\Omega})$, $s \in E_1$ with $s_+ < N$ and $s(z) \leqslant r^*(z)$ for all $z \in \overline{\Omega}$, then $W_0^{1,r(z)}(\Omega) \subseteq L^{s(z)}(\Omega)$ continuously.*

(b) *If $r \in E_1 \cap C^{0,1}(\overline{\Omega})$, $s \in E_1$ with $s_+ < N$ and $s(z) < r^*(z)$ for all $z \in \overline{\Omega}$, then $W_0^{1,r(z)}(\Omega) \subseteq L^{s(z)}(\Omega)$ compactly.*

There is a close relation between the modular function

$$\varrho_r(Du) = \int_\Omega |Du|^{r(z)}\, dz$$

and the norm $\|u\| = \|Du\|_{r(z)}$ for all $u \in W_0^{1,r(z)}(\Omega)$ (see [20] (p. 73)).

Proposition 2. *If $r \in E_1$, then*
(a) $\|u\| = \lambda \Longleftrightarrow \varrho_r(\frac{Du}{\lambda}) = 1$ *for all* $u \in W_0^{1,r(z)}(\Omega) \setminus \{0\}$.
(b) $\|u\| < 1$ *(resp.* $= 1, > 1$*)* $\Longleftrightarrow \varrho_r(Du) < 1$ *(resp.* $= 1, > 1$*).*
(c) $\|u\| < 1 \Longrightarrow \|u\|^{r_+} \leqslant \varrho_r(Du) \leqslant \|u\|^{r_-}$.
(d) $\|u\| > 1 \Longrightarrow \|u\|^{r_-} \leqslant \varrho_r(Du) \leqslant \|u\|^{r_+}$.
(e) $\|u\| \to 0$ *(resp.* $\to \infty$*)* $\Longleftrightarrow \varrho_r(Du) \to 0$ *(resp.* $\to \infty$*).*

Given $r \in E_1 \cap C^{1,0}(\overline{\Omega})$, we have

$$W_0^{1,r(z)}(\Omega)^* = W^{-1,r'(z)}(\Omega).$$

Let $A_r\colon W_0^{1,r(z)}(\Omega) \to W^{-1,r'(z)}(\Omega)$ be the nonlinear operator defined by

$$\langle A_r(u), h\rangle = \int_\Omega |Du|^{r(z)-2}(Du, Dh)_{\mathbb{R}^N}\, dz \quad \forall u, h \in W_0^{1,r(z)}(\Omega).$$

This operator has the following properties (see Gasiński-Papageorgiou [21] (Proposition 2.5)).

Proposition 3. *The operator $A_r\colon W_0^{1,r(z)}(\Omega) \to W^{-1,r'(z)}(\Omega)$ is bounded (maps bounded sets to bounded sets), continuous, strictly monotone (thus maximal monotone too) and of type $(S)_+$, that is, "$u_n \xrightarrow{w} u$ in $W_0^{1,r(z)}(\Omega)$ and $\limsup_{n\to+\infty}\langle A_r(u_n), u_n - u\rangle \leqslant 0$, imply that $u_n \longrightarrow u$ in $W_0^{1,r(z)}(\Omega)$".*

In addition to the variable Lebesgue and Sobolev spaces, the anisotropic regularity theory of Fan [22] will lead us to the space

$$C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) :\ u|_{\partial\Omega} = 0\}.$$

This is an ordered Banach space with positive (order) cone

$$C_+ = \{u \in C_0^1(\overline{\Omega}) :\ u \geqslant 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\mathrm{int}C_+ = \{u \in C_+ :\ u(z) > 0 \text{ for all } z \in \Omega,\ \frac{\partial u}{\partial n}|_{\partial\Omega} < 0\},$$

with $\frac{\partial u}{\partial n} = (Du, n)_{\mathbb{R}^N}$, where n is the outward unit normal on $\partial\Omega$.

Let $u \in L^0(\Omega)$, then we define

$$u^+(z) = \max\{u(z), 0\}, \quad u^-(z) = \min\{-u(z), 0\} \quad \forall z \in \Omega.$$

Evidently, $u^\pm \in L^0(\Omega)$, $u = u^+ - u^-$, and $|u| = u^+ + u^-$. If $u \in W_0^{1,r(z)}(\Omega)$, then $u^\pm \in W_0^{1,r(z)}(\Omega)$. Given $u, v \in L^0(\Omega)$, we write $u \prec v$ if for any compact set $K \subseteq \Omega$, we have

$$0 < c_K \leqslant v(z) - u(z) \quad \text{for a.a. } z \in K.$$

Note that if $u, v \in C(\Omega)$ and $u(z) < v(z)$ for all $z \in \Omega$, then $u \prec v$.

Consider the following nonlinear eigenvalue problem

$$\begin{cases} -\Delta_q u(z) = \widehat{\lambda}_1 |u(z)|^{q-2} u(z) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \ u > 0, \ 1 < q < \infty. \end{cases} \tag{1}$$

We know that (1) admits smallest eigenvalue $\lambda_1(q) > 0$ which is isolated and simple. It has the following variational characterization:

$$0 < \widehat{\lambda}_1(q) = \inf\left\{ \frac{\|Du\|_q^q}{\|u\|_q^q} : u \in W_0^{1,q}(\Omega), \ u \neq 0 \right\}. \tag{2}$$

The infimum in (2) is realized on the corresponding one dimensional eigenspace, the elements of which have a fixed sign. By $\widehat{u}_1(q)$, we denote the positive, L^p-normalized (that is, $\|\widehat{u}_1(q)\|_q = 1$) eigenfunction corresponding to $\widehat{\lambda}_1(q) > 0$. The isotropic nonlinear regularity theory of Lieberman [23] implies that $\widehat{u}_1(q) \in C_+ \setminus \{0\}$. Finally, from the nonlinear maximum principle (see, for example, Gasiński-Papageorgiou [24] (p. 736)), we have that $\widehat{u}_1(q) \in \mathrm{int}C_+$.

Our hypotheses on the data of (P_λ) are the following:

$\underline{H_0}$: $p \in C^{0,1}(\overline{\Omega})$ and $1 < q < p_- \leqslant p_+ < N$.

$\underline{H_1}$: $f \colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function such that

(i) $0 \leqslant f(z,x) \leqslant a(z)(1 + x^{r(z)-1})$ for a.a. $z \in \Omega$, all $x \geqslant 0$, with $a \in L^\infty(\Omega)$, $r \in C(\overline{\Omega})$ and $p(z) < r(z) < p^*(z)$ for all $z \in \overline{\Omega}$;

(ii) $\lim\limits_{x \to +\infty} \frac{f(z,x)}{x^{p(z)-1}} = +\infty$ uniformly for a.a. $z \in \Omega$;

(iii) $\lim\limits_{x \to 0^+} \frac{f(z,x)}{x^{q-1}} = 0$ uniformly for a.a. $z \in \Omega$;

(iv) for every $\varrho > 0$, there exists $\widehat{\xi}_\varrho > 0$ such that for a.a. $z \in \Omega$, the map $x \longmapsto \widehat{\xi}_\varrho x^{p(z)-1} - f(z,x)$ is nondecreasing on $[0,\varrho]$;

(v) for a.a. $z \in \Omega$, $x \longmapsto \frac{f(z,x)}{x^{q-1}}$ is strictly increasing on $(0,+\infty)$.

Remark 1. Since we want to find positive solutions and the above conditions concern the positive semiaxis $\mathbb{R}_+ = [0,+\infty)$, without any loss of generality we may assume that $f(z,x) = 0$ for a.a. $z \in \Omega$, all $x \leqslant 0$. Hypothesis $H_1(iii)$ implies that $f(z,0) = 0$ for a.a. $z \in \Omega$. In the isotropic case (that is, p is constant), if $f(z,x) = f(x) = x^{r-1}$ for all $x \geqslant 0$, with $p < r < p^*$, then we have the standard equidiffusive logistic (p,q) equation. More generally, consider the function

$$f(z,x) = \begin{cases} (x^+)^{p(z)-1} - (x^+)^{r(z)-1} & \text{if} \quad x \leqslant 1, \\ x^{p(z)-1} \ln x & \text{if} \quad x > 1, \end{cases}$$

with $r \in C(\overline{\Omega})$, $p(z) < r(z)$ for all $z \in \overline{\Omega}$. Then, $f(z,x)$ satisfies hypotheses H_1.

In what follows, $V \colon W_0^{1,p(z)}(\Omega) \longrightarrow W^{-1,p'}(\Omega)$ is the nonlinear operator defined by

$$V(u) = A_p(u) + A_q(u) \quad \forall u \in W_0^{1,p(z)}(\Omega).$$

On account of Proposition 3, we know that V is bounded, continuous, strictly monotone (thus maximal monotone too) and of type $(S)_+$.

3. Positive Solutions

We start with a nonexistence result.

Proposition 4. *If hypotheses H_0, H_1 hold and $0 < \lambda \leqslant \widehat{\lambda}_1(q)$, then problem (P_λ) has no positive solution.*

Proof. Arguing by contradiction, suppose that the parameter $\lambda \in (0, \widehat{\lambda}_1(q)]$ is admissible. Then, we can find $u \in W_0^{1,p(z)}(\Omega) \setminus \{0\}$, $u \geqslant 0$ such that

$$\langle V(u), h \rangle = \lambda \int_\Omega u^{q-1} h \, dz - \int_\Omega f(z,u) h \, dz \quad \forall h \in W^{1,p(z)}(\Omega). \tag{3}$$

In (3), we use the test function $h = u \in W_0^{1,p}(\Omega)$. Then,

$$\varrho_p(Du) + \|Du\|_q^q \leqslant \lambda \|u\|_q^q$$

(since $f \geqslant 0$), so

$$(1 - \frac{\lambda}{\widehat{\lambda}_1(q)}) \|Du\|_q^q < 0$$

(see (2) and note that $\varrho_p(Du) < 0$), a contradiction. □

Next, we prove existence and uniqueness of positive solutions for $\lambda > \widehat{\lambda}_1(q)$.

Proposition 5. *If hypotheses H_0, H_1 hold and $\lambda > \widehat{\lambda}_1(q)$, then problem (P_λ) has a unique positive solution $u_\lambda \in \mathrm{int}C_+$.*

Proof. Let $F(z,x) = \int_0^x f(z,s)\, ds$ and consider the C^1-functional $\varphi_\lambda \colon W_0^{1,p(z)}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\varphi_\lambda(u) = \int_\Omega \frac{1}{p(z)} |Du|^{p(z)} \, dz + \frac{1}{q} \|Du\|_q^q - \frac{\lambda}{q} \|u^+\|_q^q + \int_\Omega F(z,u^+), dz$$

for all $u \in W^{1,p(z)}(\Omega)$.

Since $F \geqslant 0$ and $q < p_-$, we see that φ_λ is coercive. Furthermore, using Proposition 1, we see that φ_λ is sequentially weakly lower semicontinuous. Therefore, by the Weierstrass–Tonelli theorem, we can find $u_\lambda \in W_0^{1,p(z)}(\Omega)$ such that

$$\varphi_\lambda(u_\lambda) = \inf_{u \in W_0^{1,p(z)}(\Omega)} \varphi_\lambda(u). \tag{4}$$

On account of hypothesis $H_1(iii)$, given $\varepsilon > 0$, we can find $\delta > 0$ such that

$$F(z,x) \leqslant \frac{\varepsilon}{q} x^q \quad \text{for a.a. } z \in \Omega, \text{ all } 0 \leqslant x \leqslant \delta. \tag{5}$$

Recall that $\widehat{u}_1 = \widehat{u}_1(q) \in \mathrm{int}C_+$. So, for $t \in (0,1)$ small, we have

$$0 \leqslant t\widehat{u}_1(z) \leqslant \delta \quad \forall z \in \overline{\Omega}. \tag{6}$$

Using (5) and (6), we see that

$$\varphi_\lambda(t\widehat{u}_1) \leqslant \frac{t^{p_-}}{p_-} \varrho_p(D\widehat{u}_1) + \frac{t^q}{q} (\widehat{\lambda}_1(q) + \varepsilon - \lambda)$$

(see (2) and recall that $\|\widehat{u}_1\|_q = 1$).

Choosing $\varepsilon \in (0, \lambda - \widehat{\lambda}_1(q))$, we obtain

$$\varphi_\lambda(t\widehat{u}_1) \leqslant c_1 t^{p_-} - c_2 t^q,$$

for some $c_1, c_2 > 0$. Since $q < p_-$ (see hypotheses H_0), choosing $t \in (0,1)$ even smaller if necessary, we have

$$\varphi_\lambda(t\widehat{u}_1) < 0,$$

so

$$\varphi_\lambda(u_\lambda) < 0 = \varphi_\lambda(0)$$

(see (4)) and thus $u_\lambda \neq 0$.

From (4), we have

$$\langle \varphi'_\lambda(u_\lambda), h\rangle = 0 \quad \forall h \in W_0^{1,p(z)}(\Omega),$$

so

$$\langle V(u_\lambda), h\rangle = \int_\Omega (\lambda (u_\lambda^+)^{q-1} - f(z, u_\lambda^+))h\,dz \quad \forall h \in W_0^{1,p(z)}(\Omega). \tag{7}$$

In (7), we use the test function $h = -u_\lambda^- \in W_0^{1,p(z)}(\Omega)$. Then,

$$\varrho_p(Du_\lambda^-) + \|Du_\lambda^-\|_q^q = 0,$$

so $u_\lambda \geqslant 0$, $u_\lambda \neq 0$.

From Fan-Zhao [25], we know that $u_\lambda \in L^\infty(\Omega)$. Then, the anisotropic regularity theory of Fan [22] (extension of the isotropic theory of Lieberman [23]) implies that $u_\lambda \in C_+ \setminus \{0\}$. Finally, the anisotropic maximum principle (see Zhang [26] (Theorem 1.2)) and Papageorgiou-Rădulescu-Zhang [27] (Proposition A.2)) implies that $u_\lambda \in \operatorname{int} C_+$.

Next, we show that this positive solution is unique. To this end, we introduce the integral functional $j\colon L^1(\Omega) \longrightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$j(u) = \begin{cases} \int_\Omega \frac{1}{p(z)} |Du^{\frac{1}{q}}|^{p(z)}\,dz + \frac{1}{q}\|Du^{\frac{1}{q}}\|_q^q & \text{if } u \geqslant 0,\ u^{\frac{1}{q}} \in W_0^{1,p(z)}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

From Takáč-Giacomoni [28] (Theorem 2.2), we know that j is convex. Let $\operatorname{dom} j = \{u \in L^1(\Omega) : j(u) < \infty\}$ (the effective domain of j). If v_λ is another positive solution of (P_λ), then again we have $v_\lambda \in \operatorname{int} C_+$. Using Proposition 4.1.22 of Papageorgiou-Rădulescu-Repovš [29] (p. 274), we have

$$\frac{v_\lambda}{u_\lambda} \in L^\infty \quad \text{and} \quad \frac{u_\lambda}{v_\lambda} \in L^\infty. \tag{8}$$

Let $h = u_\lambda^q - v_\lambda^q \in C_0^1(\overline{\Omega})$. From (8), it follows that for $t \in (0,1)$ small, we have

$$u_\lambda^q + th \in \operatorname{dom} j \quad \text{and} \quad v_\lambda^q + th \in \operatorname{dom} j.$$

Then, exploiting the convexity of j, we can compute the directional derivatives of j at u_λ^q and v_λ^q in the direction h. Using Theorem 2.5 of Takáč-Giacomoni [28], we have

$$j'(u_\lambda^q)(h) = \frac{1}{q}\int_\Omega \frac{-\Delta_{p(z)}u_\lambda - \Delta_q u_\lambda}{u_\lambda^{q-1}} h\,dz = \frac{1}{q}\int_\Omega (\lambda - \frac{f(z,u_\lambda)}{u_\lambda^{q-1}})h\,dz$$

and

$$j'(v_\lambda^q)(h) = \frac{1}{q}\int_\Omega \frac{-\Delta_{p(z)}v_\lambda - \Delta_q v_\lambda}{v_\lambda^{q-1}} h\,dz = \frac{1}{q}\int_\Omega (\lambda - \frac{f(z,v_\lambda)}{v_\lambda^{q-1}})h\,dz.$$

The convexity of j implies the monotonicity of the directional derivative. Therefore, we have

$$0 \leqslant \int_\Omega (\frac{f(z,v_\lambda)}{v_\lambda^{q-1}} - \frac{f(z,u_\lambda)}{u_\lambda^{q-1}})(u_\lambda^q - v_\lambda^q)\,dz \leqslant 0,$$

so $u_\lambda = v_\lambda$ (see hypothesis $H_1(v)$).

This proves the uniqueness of the positive solution of (P_λ) for all $\lambda > \widehat{\lambda}_1(q)$. $\square$

4. Extremal Positive Solutions

If we drop hypothesis $H_1(v)$ (the strict monotonicity of the quotient map $x \longmapsto \frac{f(z,x)}{x^{q-1}}$ on $(0,+\infty)$), then we cannot guarantee the uniqueness of the positive solution of (P_λ). In this case, we can show the existence of the smallest positive solution (minimal positive solution).

So, now our hypotheses on $f(z,x)$ are the following:

$\underline{H_1'}$ $f\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function satisfying hypotheses $H_1(i)$–(iv).

These hypotheses imply that given $\varepsilon \in (0,\lambda)$, we can find $c_3 = c_3(\varepsilon) > 0$ such that

$$0 \leqslant f(z,x) \leqslant \varepsilon x^{q-1} + c_3 x^{r(z)-1} \quad \text{for a.a. } z \in \Omega, \text{ all } x \geqslant 0. \tag{9}$$

The growth restriction on $f(z,\cdot)$ leads to the following auxiliary Dirichlet problem

$$(Q_\lambda) \qquad \begin{cases} -\Delta_{p(z)}u(z) - \Delta_q u(z) = (\lambda-\varepsilon)u(z)^{q-1} - c_3 u(z)^{r(z)-1} & \text{in } \Omega, \\ u|_{\partial\Omega} = 0,\ u > 0. \end{cases}$$

Since $\varepsilon \in (0,\lambda)$ and $q < p_- \leqslant p(z) < r(z)$, using Proposition 5, we have the following existence and uniqueness result for problem (Q_λ).

Proposition 6. *If hypotheses H_0 hold and $\lambda > \widehat{\lambda}_1(q)$, then problem (Q_λ) has a unique solution $\overline{u}_\lambda \in \mathrm{int}C_+$.*

Let S_λ denote the set of positive solutions of problem (P_λ). We already know that

$$\lambda > \widehat{\lambda}_1(q) \implies \varnothing \neq S_\lambda \subseteq C_+.$$

The unique solution $\overline{u}_\lambda \in \mathrm{int}C_+$ of (Q_λ) provides a lower bound for the elements of S_λ.

Proposition 7. *If hypotheses H_0, H_1' hold and $\lambda > \widehat{\lambda}_1(q)$, then $\overline{u}_\lambda \leqslant u$ for all $u \in S_\lambda$.*

Proof. Let $u_0 \in S_\lambda \subseteq \mathrm{int}C_+$. We introduce the Carathéodory function $k_\lambda(z,x)$ defined by

$$k_\lambda(z,x) = \begin{cases} (\lambda-\varepsilon)(x^+)^{q-1} - c_3(x^+)^{r(z)-1} & \text{if } \ x \leqslant u_0(z), \\ (\lambda-\varepsilon)u_0(z)^{q-1} - c_3 u_0(z)^{r(z)-1} & \text{if } \ x > u_0(z). \end{cases} \tag{10}$$

We set $K_\lambda(z,x) = \int_0^x k_\lambda(z,s)\,ds$ and consider the C^1-functional $\psi_\lambda\colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\psi_\lambda(u) = \int_\Omega \frac{1}{p(z)}|Du|^{p(z)}\,dz + \frac{1}{q}\|Du\|_q^q - \int_\Omega K_\lambda(z,u)\,dz \quad \forall u \in W_0^{1,p(z)}(\Omega).$$

From (10), we see that ψ_λ is coercive. Furthermore, using Proposition 1, we see that ψ_λ is sequentially weakly lower semicontinuous. So, we can find $\widetilde{u}_\lambda \in W_0^{1,p(z)}(\Omega)$ such that

$$\psi_\lambda(\widetilde{u}_\lambda) = \inf_{u \in W_0^{1,p(z)}(\Omega)} \psi_\lambda(u). \tag{11}$$

We know that $u_0 \in \mathrm{int}C_+$. Using Proposition 4.1.22 of Papageorgiou-Rădulescu-Repovš [29] (p. 274), we can find $t \in (0,1)$ small so that

$$0 \leqslant t\widehat{u}_1 \leqslant u_0$$

(recall that $\widehat{u}_1 = \widehat{u}_1(q) \in \mathrm{int}C_+$). Then, from (10) and since $q < p_-$, we see that

$$\psi_\lambda(t\widehat{u}_1) < 0,$$

so

$$\psi_\lambda(\widetilde{u}_\lambda) < 0 = \psi_\lambda(0)$$

(see (11)) and thus $\widetilde{u}_\lambda \neq 0$.

From (11), we have

$$\langle \psi'_\lambda(\widetilde{u}_\lambda), h\rangle = 0 \quad \forall h \in W_0^{1,p}(\Omega),$$

so

$$\langle V(\widetilde{u}_\lambda), h\rangle = \int_\Omega k_\lambda(z, \widetilde{u}_\lambda) h\, dz \quad \forall h \in W^{1,p(z)}(\Omega). \tag{12}$$

In (12), we use $h = -\widetilde{u}_\lambda^- \in W_0^{1,p(z)}(\Omega)$ and obtain

$$\varrho_p(D\widetilde{u}_\lambda^-) \leqslant 0,$$

so $\widetilde{u}_\lambda \geqslant 0$, $\widetilde{u}_\lambda \neq 0$ (see Proposition 2).

Next, in (12) we use $h = (\widetilde{u}_\lambda - u_0)^+ \in W_0^{1,p(z)}(\Omega)$. Then,

$$\begin{aligned} & \langle V(\widetilde{u}_\lambda), (\widetilde{u}_\lambda - u_0)^+\rangle \\ = \ & \int_\Omega \left((\lambda - \varepsilon)u_0^{q-1} - c_3 u_0^{r(z)-1}\right)(\widetilde{u}_\lambda - u_0)^+\, dz \\ \leqslant \ & \int_\Omega \left(\lambda u_0^{q-1} - f(z, u_0)\right)(\widetilde{u}_\lambda - u_0)^+\, dz \\ = \ & \langle V(u_0), (\widetilde{u}_\lambda - u_0)^+\rangle \end{aligned}$$

(see (10), (18) and since $u_0 \in S_\lambda$), so $\widetilde{u}_\lambda \leqslant u_0$ (see Proposition 3).

Therefore, we have $0 \leqslant \widetilde{u}_\lambda \leqslant u_0$, $\widetilde{u}_\lambda \neq 0$. This fact, together with (10), (12), and Proposition 6, implies that

$$\widetilde{u}_\lambda = \overline{u}_\lambda,$$

so $\overline{u}_\lambda \leqslant u$ for all $u \in S_\lambda$. $\square$

Using this lower bound, we can show the existence of a smallest (minimal) positive solution.

Proposition 8. *If hypotheses H_0, H_1' hold and $\lambda > \widehat{\lambda}_1(q)$, then problem (P_λ) has a smallest positive solution $u_\lambda^* \in \mathrm{int}C_+$; that is, $u_\lambda^* \leqslant u$ for all $u \in S_\lambda$.*

Proof. The set S_λ is downwardly directed (that is, if $u_1, u_2 \in S_\lambda$, then there exists $u \in S_\lambda$ such that $u \leqslant u_1$, $u \leqslant u_2$; see Filippakis-Papageorgiou [30]). Using Theorem 5.109 of Hu-Papageorgiou [31] (p. 308), we can find a decreasing sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq S_\lambda$ such that

$$\inf S_\lambda = \inf_{n\in\mathbb{N}} u_n.$$

We have

$$\langle V(u_n), h\rangle = \int_\Omega \left(\lambda u_n^{q-1} - f(z, u_n)\right) h\, dz \quad \forall h \in W_0^{1,p(z)}(\Omega),\ n \in \mathbb{N}, \tag{13}$$

$$\overline{u}_\lambda \leqslant u_n \leqslant u_1 \quad \forall n \in \mathbb{N}. \tag{14}$$

In (13), we choose the test function $h = u_n \in W_0^{1,p(z)}(\Omega)$. Using (14) and hypothesis $H_1'(i)$, we see that the sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p(z)}(\Omega)$ is bounded.

So, we may assume that

$$u_n \xrightarrow{w} u_\lambda^* \quad \text{in } W_0^{1,p(z)}(\Omega), \quad u_n \longrightarrow u_\lambda^* \quad \text{in } L^{r(z)}(\Omega), \tag{15}$$
$$\overline{u}_\lambda \leqslant u_\lambda^* \tag{16}$$

(see (14)).

In (13), we use $h = u_n - u_\lambda^* \in W^{1,p(z)}(\Omega)$, pass to the limit as $n \to +\infty$ and use (15). Then,

$$\lim_{n\to+\infty} \langle V(u_n), u_n - u_\lambda^* \rangle = 0,$$

so

$$u_n \longrightarrow u_\lambda^* \quad \text{in } W_0^{1,p(z)}(\Omega)$$

and thus

$$\langle V(u_\lambda^*), h \rangle = \int_\Omega \big(\lambda (u_\lambda^*)^{q-1} - f(z, u_\lambda^*)\big) h \, dz \quad \forall h \in W_0^{1,p(z)}(\Omega)$$

(see (13)), so $u_\lambda^* \in S_\lambda$, $u_\lambda^* = \inf S_\lambda$. $\square$

We examine the monotonicity of the minimal solution map $\lambda \longmapsto u_\lambda^*$ and determine its asymptotic behaviour as $\lambda \to \widehat{\lambda}_1(q)^+$.

Proposition 9. *If hypotheses H_0, H_1' hold, then*
(a) *the map $\lambda \longmapsto u_\lambda^*$ is strictly increasing on $(\widehat{\lambda}_1(q), +\infty))$; that is,*

$$\widehat{\lambda}_1(q) < \mu < \lambda \implies u_\lambda^* - u_\mu^* \in \mathrm{int} C_+;$$

(b) *if $\lambda \to \widehat{\lambda}_1(q)^+$, then $u_\lambda^* \to 0$ in $C_0^1(\overline{\Omega})$.*

Proof. **(a)** Let $\widehat{\lambda}_1(q) < \mu < \lambda$. First, we show that

$$\overline{u}_\mu \leqslant u_\mu^* \leqslant u_\lambda^*. \tag{17}$$

The inequality $\overline{u}_\mu \leqslant u_\mu^*$ follows from Proposition 7. Next, we show the inequality $u_\mu^* \leqslant u_\lambda^*$ in (17). Note that

$$-\Delta_{p(z)} u_\lambda^* - \Delta_q u_\lambda^* = \lambda (u_\lambda^*)^{q-1} - f(z, u_\lambda^*) \geqslant \mu (u_\lambda^*)^{q-1} - f(z, u_\lambda^*) \quad \text{in } \Omega. \tag{18}$$

We introduce the Carathéodory function $\beta_\mu \colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\beta_\mu(z, x) = \begin{cases} \mu (x^+)^{q-1} - f(z, x^+) & \text{if } \ x \leqslant u_\lambda^*(z), \\ \mu u_\lambda^*(z)^{q-1} - f(z, u_\lambda^*(z)) & \text{if } \ x > u_\lambda^*(z). \end{cases} \tag{19}$$

We set $B_\mu(z, x) = \int_0^x \beta_\mu(z, s)\, ds$ and consider the C^1-functional $\widehat{\psi}_\mu \colon W_0^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\psi}_\mu(u) = \int_\Omega \frac{1}{p(z)} |Du|^{p(z)}\, dz + \frac{1}{q} \|Du\|_q^q - \int_\Omega B_\mu(z, u)\, dz \quad \forall u \in W_0^{1,p(z)}(\Omega).$$

As in the proof of Proposition (7), using (19) and the Weierstrass–Tonelli theorem, we can find $u_\mu \in W_0^{1,p(z)}(\Omega)$ such that

$$\widehat{\psi}_\mu(u_\mu) = \inf_{u \in W_0^{1,p(z)}(\Omega)} \widehat{\psi}_\mu(u), \quad u_\mu \leqslant u_\lambda^*. \tag{20}$$

Since u_μ is a critical point of $\widehat{\psi}_\mu$ (see (20)), from (19) and (20), we see that $u_\mu \in S_\mu$ and so $u_\mu^* \leqslant u_\mu \leqslant u_\mu^*$. This proves (17).

Let $\varrho = \|u_\lambda^*\|_\infty$ and let $\widehat{\xi}_\varrho > 0$ be as postulated by hypothesis $H_1'(iv)$. We have

$$\begin{aligned} & -\Delta_{p(z)} u_\mu^* - \Delta_q u_\mu^* + \widehat{\xi}_\varrho (u_\mu^*)^{p(z)-1} \\ = \ & \mu (u_\mu^*)^{q-1} - f(z, u_\mu^*) + \widehat{\xi}_\varrho (u_\mu^*)^{p(z)-1} \\ \leqslant \ & \mu (u_\lambda^*)^{q-1} - f(z, u_\lambda^*) + \widehat{\xi}_\varrho (u_\lambda^*)^{p(z)-1} \\ \leqslant \ & -\Delta_{p(z)} u_\lambda^* - \Delta_q u_\lambda^* + \widehat{\xi}_\varrho (u_\lambda^*)^{p(z)-1} \end{aligned} \tag{21}$$

(see (17) and (18)). Since $u_\mu^* \in \mathrm{int}C_+$ and $\mu < \lambda$, we see that $0 \prec (\lambda - \mu) u_\mu^*$. So, from (21) and Proposition 2.3 of Papageorgiou-Winkert [32], we infer that

$$u_\lambda^* - u_\mu^* \in \mathrm{int}C_+,$$

therefore the map $\lambda \longmapsto u_\lambda^*$ is strictly increasing on $(\widehat{\lambda}_1(q), +\infty)$.
(b) Let $\lambda > \widehat{\lambda}_1(q)$. We have

$$\langle V(u_\lambda^*), h \rangle = \int_\Omega (\lambda (u_\lambda^*)^{q-1} - f(z, u_\lambda^*)) h \, dz \quad \forall h \in W_0^{1,p(z)}(\Omega).$$

Using the test function $h = u_\lambda^* \in W_0^{1,p(z)}(\Omega)$, we obtain

$$\varrho_p(Du_\lambda^*) + \|Du_\lambda^*\|_q^q \leqslant \lambda \|u_\lambda^*\|_q^q,$$

so

$$\varrho_p(Du_\lambda^*) \leqslant (\lambda - \widehat{\lambda}_1(q)) \|u_\lambda^*\|_q^q$$

(see 2), so

$$u_\lambda^* \to 0 \quad \text{in } W_0^{1,p(z)}(\Omega) \quad \text{as } \lambda \to \widehat{\lambda}_1(q)^+. \tag{22}$$

Note that for $\lambda \in (\widehat{\lambda}_1(q), \vartheta]$, we have $u_\lambda^* \leqslant u_\vartheta^* \in \mathrm{int}C_+$ and so the anisotropic regularity theory of Fan [22] implies that there exist $\alpha \in (0,1)$ and $c_4 > 0$ such that

$$u_\lambda^* \in C_0^{1,\alpha}(\overline{\Omega}) \quad \text{and} \quad \|u_\lambda^*\|_{C_0^{1,\alpha}(\overline{\Omega})} \leqslant c_4 \quad \forall \lambda \in (\widehat{\lambda}_1(q), \vartheta].$$

Then, the compactness of the embedding $C_0^{1,\alpha}(\overline{\Omega}) \subseteq C_0^1(\overline{\Omega})$ and (22) imply that

$$u_\lambda^* \to 0 \quad \text{in } C_0^1(\overline{\Omega}) \quad \text{as } \lambda \to \widehat{\lambda}_1(q)^+.$$

□

5. Main Theorem—Conclusions

Summarizing our findings in this paper, we can state the following theorem concerning problem (P_λ).

Theorem 1. ***(a)*** *If hypotheses H_0, H_1 hold and $\lambda > \widehat{\lambda}_1(q)$, then problem (P_λ) has a unique positive solution $u_\lambda \in \mathrm{int}C_+$.*
(b) *If hypotheses H_0, H_1' hold and $\lambda > \widehat{\lambda}_1(q)$, then problem (P_λ) has the smallest positive solution $u_\lambda^* \in \mathrm{int}C_+$, the map $\lambda \longmapsto u_\lambda^*$ is strictly increasing and*

$$u_\lambda^* \to 0 \quad \textit{in } C_0^1(\overline{\Omega}) \quad \textit{as } \lambda \to \widehat{\lambda}_1(q)^+.$$

Conclusions

In this paper, we studied anisotropic logistic equations of the equidiffusive type. Apparently, this is the first work of this kind in the literature. For equations driven by the $(p(z), q)$-Laplacian, we show that we can have the uniqueness of the positive solution and more generally we show the existence of a minimal positive solution u_λ^* and determine the properties of the map $\lambda \longmapsto u_\lambda^*$.

If the second exponent is variable too, then we encounter serious difficulties and it is not clear to us how we can overcome them. First, the difficulty is that the spectral properties of $(-\Delta_{q(z)}, W_0^{1,q(z)}(\Omega))$ are more complicated due to the nonhomogeneity of the operator. We need restrictive monotonicity conditions on q (see Fan-Zhang-Zao [33]). The second and more serious difficulty is that the anisotropic Diaz-Saa inequality of Takáč-Giacomoni [28], does not work since $q \neq q_-$. So, to prove the uniqueness and existence of minimal solutions, we need to come up with a new approach. We do not know if this is possible.

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