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Metric Dimension of Circulant Graphs with 5 Consecutive Generators

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Abstract: The problem of finding the metric dimension of circulant graphs with t generators $1, 2, \ldots, t$ (and their inverses) has been extensively studied. The problem is solved for $t = 2, 3, 4$, and some exact values and bounds are known also for $t = 5$. We solve all the open cases for $t = 5$.

Keywords: metric dimension; resolving set; circulant graph

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1. Introduction

The metric dimension is an invariant that has wide applications, for example, in pharmaceutical chemistry [1], Sonar and coast guard Loran [2], and robot navigation [3]. Applications of the metric dimension to the problem of pattern recognition and image processing can be found in [4]. The concept of metric dimension was introduced by Slater [2] in 1975.

For a graph G with set of vertices $V(G)$, the distance $d(u, v)$ between two vertices $u, v \in V(G)$ is the number of edges in a shortest path between u and v. The vertices u and v are resolved by a vertex w if $d(u, w) \neq d(v, w)$. For an ordered set $W = \{w_1, w_2, \ldots, w_z\}$, the representation of distances of v with respect to W is the ordered z-tuple

$$r(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_z)).$$

The set $W \subset V(G)$ is a resolving set of G if all the vertices of G have distinct representations (if every pair of vertices of G is resolved by a vertex of W). The number of vertices in a smallest resolving set is the metric dimension $\dim(G)$.

Circulant graphs have been extensively studied because they are particularly symmetric, and it is easy to set up such networks and check their properties. For $t \geq 2$ and $n \geq 2t + 1$, the circulant graph $C_n(1, 2, \ldots, t)$ has vertices $v_0, v_1, v_2, \ldots, v_{n-1}$ and edges $v_i v_{i+1}, v_i v_{i+2}, \ldots, v_i v_{i+t}$, where $i = 0, 1, 2, \ldots, n-1$, and the subscripts are taken modulo n. The integers $\pm 1, \pm 2, \ldots, \pm t$ are the generators of $C_n(1, 2, \ldots, t)$.

Let us suppose that $n \geq 2t + 2$ since $C_n(1, 2, \ldots, t)$ is a complete graph for $n = 2t + 1$. We present the known results on $\dim(C_n(1, 2, \ldots, t))$ for small t (and $n \geq 2t + 2$). By [5,6], we have

$$\dim(C_n(1,2)) = \begin{cases} 3 & \text{if } n \equiv 0, 2, 3 \pmod 4, \\ 4 & \text{if } n \equiv 1 \pmod 4. \end{cases}$$

By [5,7],

$$\dim(C_n(1,2,3)) = \begin{cases} 4 & \text{if } n \equiv 0,2,3,4,5 \pmod{6}, \\ 5 & \text{if } n \equiv 1 \pmod{6}. \end{cases}$$

For $n \notin \{11, 19\}$, by [8],

$$\dim(C_n(1,2,3,4)) = \begin{cases} 4 & \text{if } n \equiv 2 \pmod{8}, \\ 5 & \text{if } n \equiv 0,1,3,4 \pmod{8}, \\ 6 & \text{if } n \equiv 5,6,7 \pmod{8}, \end{cases}$$

and $\dim(C_{11}(1,2,3,4)) = \dim(C_{19}(1,2,3,4)) = 4$.

The metric dimension of $C_n(1,2,3,4)$ was studied also in [9], the graphs $C_n(1,3)$ were investigated in [10], $C_n(2,3)$ in [11], $C_n(1,4)$ in [12], $C_n(1, \frac{n}{2})$ where n is even in [13], $C_n(1,2,4)$ in [14], and $C_n(1,2,5)$ in [15].

In [16], it was shown that $\dim(C_n(1,2,\ldots,t)) \geq \lceil \frac{2t}{3} \rceil + 2$ for $t \geq 6$, and by [17], for each $t \geq 6$ there exists an n such that the bound is sharp. However, this bound does not hold for $t = 5$. Therefore, we are very interested in the case $t = 5$, and we solve all the open cases for $\dim(C_n(1,2,3,4,5))$.

Let us denote $C_n(1,2,3,4,5)$ by G_n. Since $n \geq 2t + 2$, we have $n \geq 12$. From Theorem 2.5 given in [18], we have $\dim(G_n) \geq 6$ for $n \equiv 0,1,7,8,9 \pmod{10}$. This result was extended by Chau and Gosselin [19], who proved in their Theorem 2.7 that $\dim(G_n) \geq 6$ also for $n \equiv 2,6 \pmod{10}$. By Theorem 2.13 from [19], $\dim(G_n) \geq 6$ for $n \equiv 3 \pmod{10}$, where $n \geq 23$. By Lemma 8 and Theorem 10 given in [16], $\dim(G_n) \geq 6$ for $n \equiv 4,5 \pmod{10}$. Thus, by [16,18,19], for $n \geq 12$ where $n \neq 13$,

$$\dim(G_n) \geq 6. \tag{1}$$

From Table 3 presented in [19], we have $\dim(G_{13}) = 5$.

By Grigorious et al. [20] (see Proposition 1.2 in [19]), we have

$$\dim(G_n) \leq 6 \text{ if } n \equiv 2,3,4,5,6,7 \pmod{10}.$$

By Theorem 2.9 presented in [19], $\dim(G_n) \leq 6$ for $n \equiv 8 \pmod{10}$. Thus, by [16,19,20], for $n \geq 12$ where $n \neq 13$,

$$\dim(G_n) = 6 \text{ if } n \equiv 2,3,4,5,6,7,8 \pmod{10}, \text{ and } \dim(G_{13}) = 5. \tag{2}$$

In this paper, we study $\dim(G_n)$ for $n \equiv 0,1,9 \pmod{10}$.

2. Results

First, we prove an upper bound on $\dim(G_n)$ for $n \equiv 9 \pmod{10}$.

Theorem 1. *Let $n = 10k + 9$ where $k \geq 2$. Then,* $\dim(G_n) \leq 6$.

Proof. We show that $W = \{v_0, v_3, v_7, v_{5k}, v_{5k+4}, v_{5k+7}\}$ is a resolving set of G_n. Representations of distances of all the vertices in $V(G_n) \setminus W$ with respect to W are given in Table 1.

Table 1. Representations of vertices in $V(G_n) \setminus W$ with respect to W.

Representation	v_0	v_3	v_7	v_{5k}	v_{5k+4}	v_{5k+7}
v_1	1	1	2	k	$k+1$	$k+1$
v_2	1	1	1	k	$k+1$	$k+1$
v_4	1	1	1	k	k	$k+1$
v_5	1	1	1	$k-1$	k	$k+1$
v_6	2	1	1	$k-1$	k	$k+1$
v_{5i+1} $(2 \leq i \leq k-1)$	$i+1$	i	$i-1$	$k-i$	$k-i+1$	$k-i+2$
v_{5i+2} $(2 \leq i \leq k-1)$	$i+1$	i	$i-1$	$k-i$	$k-i+1$	$k-i+1$
v_{5i+3} $(1 \leq i \leq k-1)$	$i+1$	i	i	$k-i$	$k-i+1$	$k-i+1$
v_{5i+4} $(1 \leq i \leq k-1)$	$i+1$	$i+1$	i	$k-i$	$k-i$	$k-i+1$
v_{5i+5} $(1 \leq i \leq k-2)$	$i+1$	$i+1$	i	$k-i-1$	$k-i$	$k-i+1$
v_{5k+1}	$k+1$	k	$k-1$	1	1	2
v_{5k+2}	$k+1$	k	$k-1$	1	1	1
v_{5k+3}	$k+1$	k	k	1	1	1
v_{5k+5}	$k+1$	$k+1$	k	1	1	1
v_{5k+6}	$k+1$	$k+1$	k	2	1	1
v_{5k+8}	$k+1$	$k+1$	$k+1$	2	1	1
v_{5k+9}	k	$k+1$	$k+1$	2	1	1
v_{5k+10}	k	$k+1$	$k+1$	2	2	1
$v_{10k-5i+6}$ $(1 \leq i \leq k-1)$	$i+1$	$i+2$	$i+2$	$k-i+2$	$k-i+1$	$k-i$
$v_{10k-5i+7}$ $(1 \leq i \leq k-1)$	$i+1$	$i+1$	$i+2$	$k-i+2$	$k-i+1$	$k-i$
$v_{10k-5i+8}$ $(1 \leq i \leq k-1)$	$i+1$	$i+1$	$i+2$	$k-i+2$	$k-i+1$	$k-i+1$
$v_{10k-5i+9}$ $(1 \leq i \leq k-1)$	i	$i+1$	$i+2$	$k-i+2$	$k-i+1$	$k-i+1$
$v_{10k-5i+10}$ $(1 \leq i \leq k-1)$	i	$i+1$	$i+2$	$k-i+2$	$k-i+2$	$k-i+1$
v_{10k+6}	1	2	2	$k+1$	$k+1$	k
v_{10k+7}	1	1	2	$k+1$	$k+1$	k
v_{10k+8}	1	1	2	$k+1$	$k+1$	$k+1$

Since any two vertices have different representations, $\dim(G_n) \leq 6$. □

Let v_x be a vertex from a resolving set W. Then, v_x resolves v_{x+5} and v_{x+6}, but it does not resolve any pair from $v_{x+1}, v_{x+2}, \ldots, v_{x+5}$. Analogously, v_x resolves v_{x+10} and v_{x+11}, but it does not resolve any pair from $v_{x+6}, v_{x+7}, \ldots, v_{x+10}$. If v_y and v_{y+1} are resolved by v_x and $v_x \notin \{v_y, v_{y+1}\}$, then we say that v_x creates a border between v_y and v_{y+1}. Observe that borders caused by v_x split the vertices of G_n into sequences of non-resolved vertices (by v_x). The only exception is v_x itself since it does not create borders between v_{x-1} and v_x and between v_x and v_{x+1} (recall that we require $v_x \notin \{v_y, v_{y+1}\}$ in the definition of borders). The reason is that v_x does not distinguish v_{x-1} and v_{x+1}. Of course, v_x does not distinguish also v_{x-6} from v_{x+6} though in the sequence $v_{x-6}, v_{x-5}, \ldots, v_{x+6}$ there are two borders caused by v_x. But our aim is to construct a lower bound, so we need a necessary condition, not a sufficient.

So take all vertices of W and create all borders caused by these vertices. Then, these borders split $V(G_n)$ around the cycle $C_n = (v_0, v_1, \ldots, v_{n-1})$ into many small sequences, which we call states. And the necessary condition for W to be a resolving set is that every state contains at most one vertex which is not in W. We use this condition in the proofs of the next theorems.

We need to distinguish the distance in G_n from the distance in a subgraph C_n $(= (v_0, v_1, \ldots, v_{n-1}))$ of G_n. Therefore, by the index distance, we call the distance between the vertices of G_n in C_n. For example, v_x and v_{x+5} (the indices are always taken modulo n) have distance 1 in G_n, but their index distance is 5.

We start with a lower bound on $\dim(G_n)$ for $n \equiv 1 \pmod{10}$.

Theorem 2. *Let $n = 10d + 1$, where $d \geq 2$. Then,* $\dim(G_n) \geq 8$.

Proof. By way of contradiction, suppose that $V(G_n)$ contains a subset W containing 7 vertices which are resolving. Observe that G_n has diameter d and for every vertex v_x, the vertices at distance d from v_x are the 10 consecutive vertices (consecutive by index distance) $v_{x+n'-4}, v_{x+n'-3}, \ldots, v_{x+n'}, v_{x-n'}, v_{x-n'+1}, \ldots, v_{x-n'+4}$, where $n' = \frac{n-1}{2}$.

Let $v_x \in W$. Since W must distinguish $v_{x-n'}$ from $v_{x+n'}$, we have $v_{x-n'} \in W$ or $v_{x+n'} \in W$ or there is a vertex $v_y \in W$ at index distance $5k$ from v_x for some $k \in \mathbb{N}$. But since both $v_{x-n'}$ and $v_{x+n'}$ have index distance $n' = \frac{n-1}{2} = 5d$ from v_x, we conclude that W must contain a vertex at index distance $5k$ from v_x.

So for every $v_x \in W$, there is $v_y \in W$ such that the index distance between v_x and v_y is a multiple of 5. Obviously, such a role as that which v_y plays for v_x is played also by v_x for v_y. And since $|W| = 7$, there must be a vertex in W, say v_a, such that two vertices from W, say v_b and v_c, have an index distance that is a multiple of 5 from v_a.

Not only this, but let $L = \{v_{a-1}, v_{a-2}, \ldots, v_{a-n'}\}$ and $R = \{v_{a+1}, v_{a+2}, \ldots, v_{a+n'}\}$. Then, L, R, $\{v_a\}$ is a partition of $V(G_n)$. If both v_b and v_c are in R (or if they are both in L), then all mutual index distances between pairs of vertices of $\{v_a, v_b, v_c\}$ are multiples of 5. Hence, by symmetry, we may assume that $v_b \in L$, $v_c \in R$, and we may also assume that $a = 0$.

Now, we split L and R, each into five sets according to the index distance from v_0 ($= v_a$). Let $0 \le i \le 4$. By L^i (by R^i), we denote vertices of L (of R) whose index distance from v_0 is congruent to $i \pmod 5$. (Observe that the index distance cannot be greater than n'.) Hence, $v_b \in L^0$ and $v_c \in R^0$.

As mentioned above, for every $v_z \in W \setminus \{v_a, v_b, v_c\}$, to distinguish $v_{z-n'}$ from $v_{z+n'}$, there must be a vertex in W which is at index distance $5k$ from v_z. For instance, if $v_z \in L^0$, then to distinguish $v_{z-n'}$ from $v_{z+n'}$, W contains a vertex, say v_y, such that either $v_y \in L^0$ (if v_y is in L), or $v_y \in R^0$ (when $v_y \in R$ and the shortest index distance $v_z - v_y$ path contains v_a), or $v_y \in R^1$ (when $v_y \in R$ and the shortest index distance $v_z - v_y$ path does not contain v_a). In Table 2, we have all 10 cases according to v_z being in $L^0, L^1, \ldots, R^4$. In each case, the vertex v_y is from the union of the three sets.

Table 2. The 10 cases according to v_z being in $L^0, L^1, \ldots, R^4$.

L^0: L^0, R^0, R^1	R^0: R^0, L^0, L^1
L^1: L^1, R^4, R^0	R^1: R^1, L^4, L^0
L^2: L^2, R^3, R^4	R^2: R^2, L^3, L^4
L^3: L^3, R^2, R^3	R^3: R^3, L^2, L^3
L^4: L^4, R^1, R^2	R^4: R^4, L^1, L^2

Let $i \le 4$. Suppose that we have borders caused by $7 - i$ vertices of W and the borders caused by the remaining i vertices of W are not considered yet. Moreover, suppose that there is a sequence of $i + 1$ vertices, say $v_x, v_{x+1}, \ldots, v_{x+i}$, such that none of the $7 - i$ already considered vertices is in $\{v_x, v_{x+1}, \ldots, v_{x+i}\}$ and the already considered vertices do not create borders between any consecutive pair from $\{v_x, v_{x+1}, \ldots, v_{x+i}\}$. Then, $v_{x+i-n'-5}, v_{x+i-n'-4}, \ldots, v_{x-n'+4}$ cannot be among the remaining vertices of W. The reason is that each of the remaining i vertices of W causes, at most, one border in the sequence $v_x, v_{x+1}, \ldots, v_{x+i}$ and if it is in $\{v_{x+i-n'-5}, v_{x+i-n'-4}, \ldots, v_{x-n'+4}\} = \{v_{x+i+n'-4}, v_{x+i+n'-3}, \ldots, v_{x+n'+5}\}$, then it causes no border in the sequence since every member of the sequence is at distance d from all the vertices from $\{v_{x+i-n'-5}, v_{x+i-n'-4}, \ldots, v_{x-n'+4}\}$.

Denote 11 vertices which are at distance at most 1 from v_a by α, and denote 10 vertices which are at distance d from v_a by β. Then, both α and β form sequences of consecutive vertices. We consider the situation in α and β. In Figure 1, we describe α and β as they occur in the cycle $C_n(1)$ (the other vertices are not depicted). Circles represent vertices of G_n, and those whose position is already known, such as v_a, are depicted instead of circles. Vertical lines are borders caused by v_a, and in the space between vertices, we indicate which vertices may form the border. In fact, such vertices cause borders if and only if they are not in α or β, and therefore, situations when W contains vertices of α or β should be considered separately.

$$| \circ \substack{L^1\\R^4} \circ \substack{L^2\\R^3} \circ \substack{L^3\\R^2} \circ \substack{L^4\\R^1} \circ \substack{L^0\\R^0} \circ \substack{L^1\\R^4} \circ \substack{L^2\\R^3} \circ \substack{L^3\\R^2} \circ \substack{L^4\\R^1} \circ |$$

$$| \circ \substack{L^0\\R^1} \circ \substack{L^4\\R^2} \circ \substack{L^3\\R^3} \circ \substack{L^2\\R^4} \circ \substack{L^1\\R^0} v_a \substack{L^0\\R^1} \circ \substack{L^4\\R^2} \circ \substack{L^3\\R^3} \circ \substack{L^2\\R^4} \circ \substack{L^1\\R^0} \circ |$$

Figure 1. Initial distribution.

Regarding the position of v_b and v_c in β, by symmetry, we distinguish three main cases.

Case 1. $v_b, v_c \in \beta$.

See Figure 2, where we specify also the borders in α caused by v_b and v_c.

$$| \circ \substack{L^1\\R^4} \circ \substack{L^2\\R^3} \circ \substack{L^3\\R^2} \circ \substack{L^4\\R^1} v_b \substack{L^0\\R^0} v_c \substack{L^1\\R^4} \circ \substack{L^2\\R^3} \circ \substack{L^3\\R^2} \circ \substack{L^4\\R^1} \circ |$$

$$| \circ \substack{v_b\\|} \circ \substack{L^4\\R^2} \circ \substack{L^3\\R^3} \circ \substack{L^2\\R^4} \circ \substack{L^1\\R^0} v_a \substack{L^0\\R^1} \circ \substack{L^4\\R^2} \circ \substack{L^3\\R^3} \circ \substack{L^2\\R^4} \circ \substack{v_c\\|} \circ |$$

Figure 2. Case 1.

The vertices v_{-1} and v_1 must be distinguished, so there must be a border on the left-hand side of v_a (or $v_{a-1} \in W$) or on the right-hand side of v_a (or $v_{a+1} \in W$). By symmetry, we may assume that this border is on the right-hand side of v_a. So for the next vertex $v_e \in W$, we have either $e = a+1$, or $v_e \in R^1$ but $v_e \notin \alpha$, or $v_e \in L^0$. We consider these subcases separately.

Subcase 1.1. $e = a+1$.

So, $v_e = v_{a+1} = v_1 \in R^1$. Now, the positions of four vertices of W are determined, see Figure 3. It remains to be determined the positions of the last three vertices, say, v_f, v_g and v_h.

$$| \circ \substack{L^1\\R^4} \circ \substack{L^2\\R^3} \circ \substack{L^3\\R^2} \circ \substack{L^4\\R^1} v_b \substack{L^0\\R^0} v_c \substack{L^1\\R^4} \circ \substack{L^2\\R^3} \circ \substack{L^3\\R^2} \circ \substack{v_e\\|} \circ |$$

$$| \circ \substack{v_b\\|} \circ \substack{L^4\\R^2} \circ \substack{L^3\\R^3} \circ \substack{L^2\\R^4} \circ \substack{L^1\\R^0} v_a \substack{L^0\\R^1} v_e \substack{L^4\\R^2} \circ \substack{L^3\\R^3} \circ \substack{L^2\\R^4} \circ \substack{v_c\\|} \circ |$$

Figure 3. Subcase 1.1.

At the moment, there is a sequence of four consecutive vertices in $L \cap \alpha$ without a border, namely, $v_{-4}, v_{-3}, v_{-2}, v_{-1}$, so $R \cap \beta \cap W = \{v_c\}$ since it remains to be determined the three vertices of W. Also, there is a sequence of four consecutive vertices in $L \cap \beta$ without a border, namely $v_{-n'+1}, v_{-n'+2}, v_{-n'+3}, v_{-n'+4}$, so $R \cap \alpha \cap W = \{v_e\}$. Thus, there is a border between v_2 and v_3, and also a border between v_3 and v_4. This yields four possibilities.

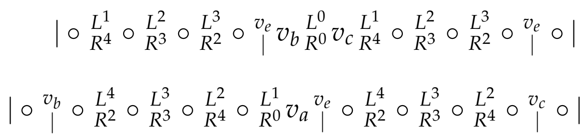
(a) $v_f \in L^3$ and $v_g \in L^2$. Since $v_{-n'+1}$ must be distinguished from $v_{n'-1}$, we have $v_h \in L^4 \cup R^1 \cup L^0 \cup R^0 \cup L^1 \cup R^4$ or $v_h = v_{-n'+1} \in L^4$. In any case, by Table 2, W does not contain a vertex at index distance $5k$ from v_f.

(b) $v_f \in L^3$ and $v_g \in R^4$. By Table 2, $v_h \in L^3 \cup R^2 \cup R^3$ since W must contain a vertex at index distance $5k$ from v_f. At the same time, $v_h \in R^4 \cup L^1 \cup L^2$ since W must contain a vertex at index distance $5k$ from v_g, a contradiction.

(c) $v_f \in R^3$ and $v_g \in L^2$. Since $v_{n'-2}$ must be distinguished from $v_{n'-3}$ and $v_{n'-2}, v_{n'-3} \notin W$, we have $v_h \in L^3 \cup R^2$. On the other hand, $v_{n'+1}$ must be distinguished from $v_{n'-1}$ and so $v_h \in L^4 \cup R^1 \cup L^0 \cup R^0 \cup L^1 \cup R^4$, a contradiction.

(d) $v_f \in R^3$ and $v_g \in R^4$. Since $v_{n'-2}$ must be distinguished from $v_{n'-3}$, we have $v_h \in L^3 \cup R^2$. However, by Table 2, W does not contain a vertex at index distance $5k$ from v_g.

Subcase 1.2. $v_e \in R^1$ but $v_e \notin \alpha$ (see Figure 4).



$v_e \in R^1$

Figure 4. Subcase 1.2.

At the moment, there are sequences of four consecutive vertices without a border in $L \cap \alpha$ $(v_{-4}, v_{-3}, v_{-2}, v_{-1})$, in $R \cap \alpha$ (v_1, v_2, v_3, v_4), and in $L \cap \beta$ $(v_{-n'+1}, v_{-n'+2}, v_{-n'+3}, v_{-n'+4})$. Therefore, $R \cap \beta \cap W = \{v_c\}$, $L \cap \beta \cap W = \{v_b\}$ and $R \cap \alpha \cap W = \emptyset$. Hence, there is a border between v_2 and v_3 and also a border between v_3 and v_4. Analogously as in Subcase 1.1, we consider four possibilities.

(a) $v_f \in L^3$ and $v_g \in L^2$. Since $v_{-n'+4}$ must be distinguished from $v_{-n'+3}$ and $L \cap \beta \cap W = \{v_b\}$, we have $v_h \in L^1 \cup R^4$. In any case, by Table 2, W does not contain a vertex at index distance $5k$ from v_f.

(b) $v_f \in L^3$ and $v_g \in R^4$. This case can be solved analogously as in Subcase 1.1.

(c) $v_f \in R^3$ and $v_g \in L^2$. Since $v_{n'-2}$ must be distinguished from $v_{n'-3}$ and $v_{n'-2}, v_{n'-3} \notin W$, we have $v_h \in L^3 \cup R^2$. On the other hand, v_1 must be distinguished from v_2 and $v_1, v_2 \notin W$, and so $v_h \in L^4 \cup R^2$. Consequently, $v_h \in R^2$, and by Table 2, W does not contain a vertex at index distance $5k$ from v_h.

(d) $v_f \in R^3$ and $v_g \in R^4$. This case can be solved analogously as in Subcase 1.1.

Subcase 1.3. $v_e \in L^0$.

This situation is presented in Figure 5. Here, $v_e \in \alpha$ if $v_e = v_{-5}$, or $v_e \notin \alpha$. In any case, $v_e \notin \beta$.

$v_e \in L^0$

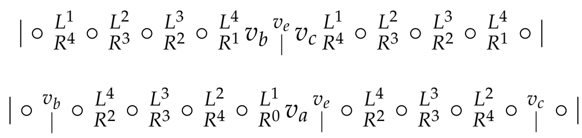


Figure 5. Subcase 1.3.

In this subcase, we have four consecutive vertices without a border in all $L \cap \alpha$, $R \cap \alpha$, $L \cap \beta$ and $R \cap \beta$, and so $\alpha \cap W = \{v_a\}$ and $\beta \cap W = \{v_b, v_c\}$. Hence, there is a border between v_2 and v_3, and also a border between v_3 and v_4. This gives the same possibilities as in the previous two subcases, and they can be solved analogously as in Subcase 1.2.

Case 2. $v_b \in \beta$ but $v_c \notin \beta$ (see Figure 6).

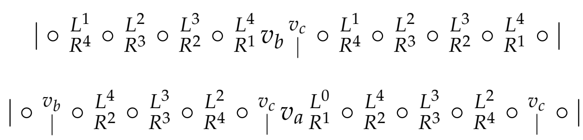


Figure 6. Case 2.

The positions of three vertices of W are determined, and it remains to be determined the last four vertices. But at the moment, there is a sequence of five consecutive vertices without a border in $R \cap \beta$, namely $v_{n'-4}, v_{n'-3}, v_{n'-2}, v_{n'-1}, v_{n'}$, and so $L \cap \alpha = \emptyset$. Thus, we distinguish eight possibilities.

(a) $v_e \in L^4$, $v_f \in L^3$ and $v_g \in L^2$. By Table 2, $v_h \in L^4 \cup R^1 \cup R^2$ since W must contain a vertex at index distance $5k$ from v_e. At the same time, $v_h \in L^2 \cup R^3 \cup R^4$ since W must contain a vertex at index distance $5k$ from v_g, a contradiction.

(b) $v_e \in L^4$, $v_f \in L^3$ and $v_g \in R^4$. By Table 2, $v_h \in L^4 \cup R^1 \cup R^2$ since W must contain a vertex at index distance $5k$ from v_e. At the same time, $v_h \in R^4 \cup L^1 \cup L^2$ since W must contain a vertex at index distance $5k$ from v_g, a contradiction.

(c) $v_e \in L^4$, $v_f \in R^3$ and $v_g \in L^2$. Since $v_{n'}$ must be distinguished from $v_{n'-1}$, we have $v_h \in L^1 \cup R^4$ or $v_h = v_{n'}$ or $v_h = n_{n'-1}$. Summing up, $v_h \in L^1 \cup R^4 \cup R^0$. At the same time, $v_h \in L^4 \cup R^1 \cup R^2$ since W must contain a vertex at index distance $5k$ from v_e, a contradiction.

(d) $v_e \in L^4$, $v_f \in R^3$ and $v_g \in R^4$. By Table 2, $v_h \in L^4 \cup R^1 \cup R^2$ since W must contain a vertex at index distance $5k$ from v_e. At the same time, $v_h \in R^4 \cup L^1 \cup L^2$ since W must contain a vertex at index distance $5k$ from v_g, a contradiction.

(e) $v_e \in R^2$, $v_f \in L^3$ and $v_g \in L^2$. To distinguish $v_{n'}$ from $v_{n'-1}$, we have $v_h \in L^1 \cup R^4$ or $v_h = v_{n'}$ or $v_h = v_{n'-1}$. But the case $v_h = v_{n'}$ was already considered in Case 1. Hence, $v_h \in L^1 \cup R^4$. At the same time, W must contain a vertex at index distance $5k$ from v_g, and so $v_h \in R^4$. To distinguish $v_{n'-3}$ from $v_{n'-4}$, W must contain a vertex in $L^4 \cup R^1$ or $v_h = v_{n'-3}$ or $v_h = v_{n'-4}$. Consequently, $v_e = v_{n'-3}$. However, then, v_e does not create the border between v_{-4} and v_{-3}. Since W does not contain a vertex in L^4 and $v_{-4}, v_{-3} \notin W$, the vertices v_{-4} and v_{-3} are not resolved, a contradiction.

(f) $v_e \in R^2$, $v_f \in L^3$ and $v_g \in R^4$. First, suppose that $v_e \neq v_{n'-3}$. To distinguish $v_{n'-3}$ from $v_{n'-4}$, we have $v_h \in L^4 \cup R^1$ or $v_h = v_{n'-3}$ or $v_h = v_{n=-4}$. Hence, $v_h \in L^4 \cup R^1 \cup R^2$. At the same time, $v_h \in R^4 \cup L^1 \cup L^2$ since W must contain a vertex at index distance $5k$ from v_g, a contradiction. So $v_e = v_{n'-3}$. But then, v_e does not create a border between v_{-4} and v_{-3} since $v_{-4}, v_{-3} \notin W$, $v_h \in L^4 \cup R^2$. But then, W does not contain a vertex at index distance $5k$ from v_g, a contradiction.

(g) $v_e \in R^2$, $v_f \in R^3$ and $v_g \in L^2$. To distinguish $v_{n'}$ from $v_{n'-1}$, we must have $v_h \in L^1 \cup R^4$ or $v_h = v_{n'}$ or $v_h = v_{n'-1}$. Summing up, $v_h \in L^1 \cup R^4 \cup R^0$. At the same time $v_h \in R^2 \cup L^3 \cup L^4$ since W must contain a vertex at index distance $5k$ from v_e, a contradiction.

(h) $v_e \in R^2$, $v_f \in R^3$ and $v_g \in R^4$. By Table 2, $v_h \in R^2 \cup L^3 \cup L^4$ since W must contain a vertex at index distance $5k$ from v_e. At the same time $v_h \in R^4 \cup L^1 \cup L^2$ since W must contain a vertex at index distance $5k$ from v_g, a contradiction.

Case 3. $v_b, v_c \notin \beta$ (see Figure 7).

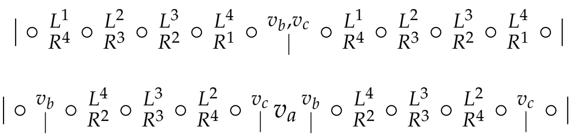


Figure 7. Case 3.

Analogously as in Case 2, at the moment there are sequences of five consecutive vertices without a border in $L \cap \beta$ and $R \cap \beta$, and so $R \cap \alpha = \emptyset$ and $L \cap \alpha = \emptyset$. Thus, we distinguish eight possibilities analogously as in Case 2. And we can use the proofs from Case 2 since in Case 3, we have more restrictions (so it is even easier to obtain a contradiction). □

Finally, we present a lower bound on $\dim(G_n)$ for $n \equiv 0 \pmod{10}$.

Theorem 3. *Let $n = 10d$, where $d \geq 2$. Then, $\dim(G_n) \geq 7$.*

Proof. We proceed analogously as in the proof of Theorem 2. Let $n' = \frac{n}{2}$. Observe that the index distance is at most n' in G_n. Obviously, G_n has diameter d and for every vertex v_x. There are nine consecutive vertices at distance d from v_x, namely, $v_{x+n'-4}, v_{x+n'-3}, \ldots, v_{x+n'+4}$.

By way of contradiction, suppose that $V(G_n)$ contains a subset W of six vertices, which is resolving. Let $v_a \in W$. Denote $L = \{v_{a-1}, v_{a-2}, \ldots, v_{a-n'}\}$ and $R = \{v_{a+1}, v_{a+2}, \ldots, v_{a+n'}\}$. Hence, $L \cap R = \{v_{a+n'}\}$. Analogously as in the proof of Theorem 2, we split L and R each into five sets according to the index distance from v_a. Let $0 \leq i \leq 4$. By L^i (by R^i), we denote vertices of L (of R) whose index distance from v_a is congruent to $i \pmod 5$. (Recall that the index distance does not exceed n'.) To simplify the notation, we assume that $a = 0$.

Let $i \leq 4$. Suppose that we have borders caused by $6 - i$ vertices of W and the borders caused by the remaining i vertices of W are not considered yet. Moreover, suppose that there is a sequence of $i + 1$ vertices, say $v_x, v_{x+1}, \ldots, v_{x+i}$ such that none of the $6 - i$ already considered vertices is in $\{v_x, v_{x+1}, \ldots, v_{x+i}\}$ and the already considered vertices do not form borders between any consecutive pair from $\{v_x, v_{x+1}, \ldots, v_{x+i}\}$. Then, $v_{x+n'+i-4}, v_{x+n'+i-3}, \ldots, v_{x+n'+4}$ cannot be among the remaining vertices of W. The reason is that each of the remaining i vertices of W causes at most one border in the sequence $v_x, v_{x+1}, \ldots, v_{x+i}$ and if it is in $\{v_{x+n'+i-4}, v_{x+n'+i-3}, \ldots, v_{x+n'+4}\}$, then it causes no border in the sequence since every member of the sequence is at distance d from all the vertices from $\{v_{x+n'+i-4}, v_{x+n'+i-3}, \ldots, v_{x+n'+4}\}$.

Denote 11 vertices which are at distance at most 1 from v_a by α, and denote 9 vertices which are at distance d from v_a by β. Then, both α and β form sequences of vertices, where neighboring vertices have index distance 1. We consider mainly the situation in α and β. In Figure 8, we describe α and β analogously as in the proof of Theorem 2. Circles represent vertices of G_n, and those whose position is already known, such as v_a, are depicted instead of circles. Vertical borders are borders caused by v_a, and in the space between vertices, we indicate which vertices may form the border. Again, such vertices cause borders if and only if they are not in α or β, and therefore situations when W contains vertices of α or β should be considered separately.

$$\Big| \circ \substack{L^1\\R^3} \circ \substack{L^2\\R^2} \circ \substack{L^3\\R^1} \circ \substack{L^4\\R^0} \circ \substack{L^0\\R^4} \circ \substack{L^1\\R^3} \circ \substack{L^2\\R^2} \circ \substack{L^3\\R^1} \circ \Big|$$

$$\Big| \circ \substack{L^0\\R^1} \circ \substack{L^4\\R^2} \circ \substack{L^3\\R^3} \circ \substack{L^2\\R^4} \circ \substack{L^1\\R^0} v_a \substack{L^0\\R^1} \circ \substack{L^4\\R^2} \circ \substack{L^3\\R^3} \circ \substack{L^2\\R^4} \circ \substack{L^1\\R^0} \circ \Big|$$

Figure 8. Initial distribution.

Before we start with cases, we exclude the possibility that two consecutive vertices are in W. So by way of contradiction, suppose that $v_a, v_{a+1} \in W$, see Figure 9.

$$\Big| \circ \substack{L^1\\R^3} \circ \substack{L^2\\R^2} \circ \substack{L^3\\R^1} \circ \substack{L^4\\R^0} \circ \substack{L^0\\R^4} \circ \substack{L^1\\R^3} \circ \substack{L^2\\R^2} \circ \substack{v_b\\|} \circ \Big|$$

$$\Big| \circ \substack{v_b\\|} \circ \substack{L^4\\R^2} \circ \substack{L^3\\R^3} \circ \substack{L^2\\R^4} \circ \substack{L^1\\R^0} v_a \substack{L^0\\R^1} v_b \substack{L^4\\R^2} \circ \substack{L^3\\R^3} \circ \substack{L^2\\R^4} \circ \substack{L^1\\R^0} \circ \Big|$$

Figure 9. $v_a, v_{a+1} \in W$.

This yields a sequence of six consecutive vertices $v_{n'-1}, v_{n'}, \ldots, v_{n'+4}$ without a border caused by v_a or v_b. However, any vertex from $W \setminus \{v_a, v_b\}$ creates at most one border in this sequence, and so W cannot distinguish all the vertices in this sequence, a contradiction.

To distinguish the three vertices $v_{n'-1}, v_{n'}, v_{n'+1}$, we need two vertices of W, say, v_b and v_c. In the following cases, we distinguish their position with respect to β.

Case 1. $v_b, v_c \notin \beta$. (By symmetry, we consider three subcases.)

Subcase 1.1. $v_b \in L^4$ and $v_c \in R^4$ (see Figure 10).

$$\Big| \circ \substack{L^1\\R^3} \circ \substack{L^2\\R^2} \circ \substack{L^3\\R^1} \circ \substack{v_b\\|} \circ \substack{v_c\\|} \circ \substack{L^1\\R^3} \circ \substack{L^2\\R^2} \circ \substack{L^3\\R^1} \circ \Big|$$

$$\Big| \circ \substack{L^0\\R^1} \circ \substack{v_b\\|} \circ \substack{L^3\\R^3} \circ \substack{v_c\\|} \circ \substack{L^1\\R^0} v_a \substack{L^0\\R^1} \circ \substack{v_b\\|} \circ \substack{L^3\\R^3} \circ \substack{v_c\\|} \circ \substack{L^1\\R^0} \circ \Big|$$

Figure 10. Subcase 1.1.

At the moment, there is a sequence of four vertices without a border in $L \cap \beta$, namely, $v_{n'+1}, v_{n'+2}, v_{n'+3}, v_{n'+4}$. Thus, $v_{n'+1+n'+3-4} = v_0, v_1, \ldots, v_{n'+1+n'+4} = v_5$ are not among the vertices of $W \setminus \{v_a, v_b, v_c\}$, and so $R \cap \alpha \cap (W \setminus \{v_c\}) = \emptyset$. Analogously, $v_{n'-4}, v_{n'-3}, v_{n'-4}$, $v_{n'-1}$ form a sequence of 4 consecutive vertices without a border in $R \cap \beta$, and so $L \cap \alpha \cap (W \setminus \{v_b\}) = \emptyset$.

Observe that $v_b \notin \{v_{-3}, v_{-2}\}$ even if $v_b \in \alpha$. Hence, to distinguish v_{-3} from v_{-2}, we need $v_e \in L^3 \cup R^3$. By now, this subcase is symmetric, so we may assume that $v_e \in L^3$. Recall that as mentioned above, $v_e \notin \alpha$. Now, we consider four possibilities with respect to $\{v_b, v_c\} \cap \alpha$.

(a) $v_b, v_c \notin \alpha$. Recall that $v_f, v_g \notin \alpha$. Since v_{-5} must be distinguished from v_{-4}, we have $v_f \in L^0 \cup R^1$, and since v_4 must be distinguished from v_5, we have $v_g \in L^1 \cup R^0$. However, also $v_{n'-3}$ must be distinguished from $v_{n'-2}$, and so W must contain a vertex in $L^2 \cup R^2 \cup \{v_{n'-3}, v_{n'-2}\}$, that is a vertex in $L^2 \cup R^2 \cup R^3$, a contradiction.

(b) $v_b \in \alpha$ and $v_c \notin \alpha$. In this case $v_b = v_{-4}$. To distinguish v_{-5} from v_{-3} we have $v_f \in L^0 \cup L^4 \cup R^1 \cup R^2$, and to distinguish v_4 from v_5 we have $v_g \in L^1 \cup R^0$. However, to distinguish $v_{n'-2}$ from $v_{n'-3}$ we conclude that $v_f \in R^2$, and to distinguish $v_{n'+3}$ from $v_{n'+4}$, we conclude that $v_g \in L^1$.

So we have $v_b = v_{-4}$, $v_c \in R^4$, $v_e \in L^3$, $v_f \in R^2$ and $v_g \in L^1$. This choice works well in $\alpha \cup \beta$, see Figure 11. So denote $\alpha_a = \alpha$, $\beta_a = \beta$, $L_a = L$, $R_a = R$, $L_a^i = L^i$ and $R_a^i = R^i$, $0 \le i \le 4$, and analogously construct α_x, β_x, L_x, R_x, L_x^i and R_x^i where $x \in \{b, c, e, f, g\}$.

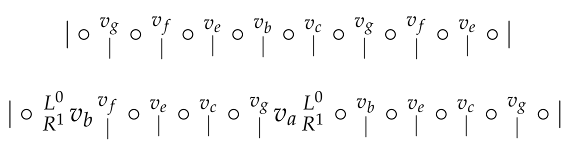


Figure 11. Subcase 1.1.

Consider v_g. We have $v_a \in R_g^1$, $v_b \in R_g^2$, $v_c \in L_g^0 \cup R_g^0$, $v_e \in L_g^2 \cup R_g^3$ and $v_f \in L_g^2 \cup R_g^3$. Observe that W does not contain a vertex in $L_g^4 \cup R_g^4$. Thus, the unique vertex v_c must distinguish three consecutive vertices $v_{g+n'-1}, v_{g+n'}, v_{g+n'+1}$ in β_g (see Figure 8), a contradiction.

(c) $v_b \notin \alpha$ and $v_c \in \alpha$. In this case, $v_c = v_4$. To distinguish v_{-5} from v_{-4}, we have $v_f \in L^0 \cup R^1$, and to distinguish v_3 from v_5, we have $v_g \in L^1 \cup L^2 \cup R^0 \cup R^4$. However, to distinguish $v_{n'-3}$ from $v_{n'-2}$ we conclude that $v_g \in L^2$, and to distinguish $v_{n'-2}$ from $v_{n'-1}$, we need a vertex of W in $L^1 \cup R^3$ or $v_{n'-1} \in R^4$ (different from v_c) or $v_{n'-2} \in R^3$, a contradiction.

(d) $v_b, v_c \in \alpha$, that is $v_b = v_{-4}$ and $v_c = v_4$. To distinguish v_{-5} from v_{-3}, we have $v_f \in L^0 \cup L^4 \cup R^1 \cup R^2$, and to distinguish v_3 from v_5, we have $v_g \in L^1 \cup L^2 \cup R^0 \cup R^4$. However, to distinguish $v_{n'-2}$ from $v_{n'-1}$, the set W must contain a vertex of $L^1 \cup R^3$ or $v_{n'-2} \in R^3$ or $v_{n'-1} \in R^4$ (different from v_c). Thus, $v_g = v_{n'-1}$ or $v_g \in L^1$. And to distinguish $v_{n'-3}$ from $v_{n'-2}$, the set W must contain a vertex in $L^2 \cup R^2$ or $v_{n'-3} \in R^3$ or $v_{n'-2} \in R^2$, and so $v_f \in R^2$. Now, if $v_g = v_{n'-1}$, then $v_{n'+3}$ is not distinguished from $v_{n'+4}$, so we conclude that $v_g \in L^1$. This yields a situation which was already solved in case (b) above.

Subcase 1.2. $v_b \in L^0$ and $v_c \in R^0$ (see Figure 12).

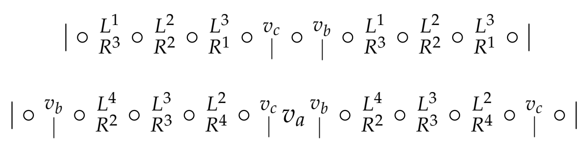


Figure 12. Subcase 1.2.

In each $L \cap \alpha$, $R \cap \alpha$, $L \cap \beta$ and $R \cap \beta$, there are four consecutive vertices which are not resolved by v_a, v_b, v_c, even if $v_b \in \alpha$ or $v_c \in \alpha$. Since the sequence v_1, v_2, v_2, v_4 yields $\{v_{n'}, v_{n'+1}, \ldots, v_{n'+5}\} \cap (W \setminus \{v_a, v_b, c_c\}) = \emptyset$, and the sequence $v_{n'+1}, v_{n'+2}, v_{n'+3}, v_{n'+4}$ yields $\{v_0, v_1, \ldots, v_5\} \cap (W \setminus \{v_a, v_b, c_c\}) = \emptyset$, we have $\{v_e, v_f, v_g\} \cap (\alpha \cup \beta) = \emptyset$.

Hence, to distinguish $v_{n'+3}$ from $v_{n'+4}$, we have $v_e \in L^1 \cup R^3$; to distinguish $v_{n'+2}$ from v_{n+3}, we have $v_f \in L^2 \cup R^2$; and to distinguish $v_{n'+1}$ from $v_{n'+2}$, we have $v_g \in L^3 \cup R^1$.

But also, v_{-4} must be distinguished from v_{-3}, and so $v_f \in R^2$. And since v_{-2} must be distinguished from v_{-1}, W must contain a vertex in $L^2 \cup R^4$, a contradiction.

Subcase 1.3. $v_b \in L^0$ and $v_c \in L^4$, see Figure 13 (the case $v_b \in R^0$ and $v_c \in R^4$ is analogous, by symmetry).

$$| \circ \substack{L^1\\R^3} \circ \substack{L^2\\R^2} \circ \substack{L^3\\R^1} \circ \substack{v_c\\|} \circ \substack{v_b\\|} \circ \substack{L^1\\R^3} \circ \substack{L^2\\R^2} \circ \substack{L^3\\R^1} \circ |$$

$$| \circ \substack{v_b\\|} \circ \substack{v_c\\|} \circ \substack{L^3\\R^3} \circ \substack{L^2\\R^4} \circ \substack{L^1\\R^0} v_a \substack{v_b\\|} \circ \substack{v_c\\|} \circ \substack{L^3\\R^3} \circ \substack{L^2\\R^4} \circ \substack{L^1\\R^0} \circ |$$

Figure 13. Subcase 1.3.

Recall that $v_b, v_c \notin \beta$. So at the moment, the vertices $v_{n'+1}, v_{n'+2}, v_{n'+3}, v_{n'+4}$ are not distinguished, and so $\{v_0, v_1 \ldots, v_5\} \cap \{v_e, v_f, v_g\} = \emptyset$. That is, $v_e, v_f, v_g \notin R \cap \alpha$. Analogously v_2, v_3, v_4, v_5 are not distinguished since $v_b, v_c \in L$, and so $v_e, v_f, v_g \notin \{v_{n'+1}, v_{n'+2}, \ldots, v_{n'+6}\}$.

Hence, to distinguish v_2 from v_3, we have $v_e \in L^3 \cup R^3$; to distinguish v_3 from v_4, we have $v_f \in L^2 \cup R^4$; and to distinguish v_4 from v_5, we have $v_g \in L^1 \cup R^0$. On the other hand, also $v_{n'+1}$ must be distinguished from $v_{n'+2}$, and so $v_e \in L^3$. Since $v_{n'+2}$ must be distinguished from $v_{n'+3}$, we have $v_f \in L^2$. And since $v_{n'+3}$ must be distinguished from $v_{n'+4}$, we have $v_g \in L^1$.

So we have $v_b \in L^0$, $v_c \in L^4$, $v_e \in L^3$, $v_f \in L^2$ and $v_g \in L^1$. Observe that if $x \in \{b, c, e, f, g\}$, then $v_{x+n'-1}, v_{x+n'}, v_{x+n'+1} \notin W$. Hence, relabeling a with x, we remain in Case 1, and in four out of these five possible relabelings, not all remaining vertices of W are in L_x and also they are not all in R_x. So, each of the four relabelings reduces this case to one of the previous ones.

Case 2. $v_b = v_{n'+1}$ and $v_c = v_{n'-1}$ (see Figure 14).

$$| \circ \substack{L^1\\R^3} \circ \substack{L^2\\R^2} \circ \substack{L^3\\R^1} v_b \substack{L^4\\R^0} \circ \substack{L^0\\R^4} v_c \substack{L^1\\R^3} \circ \substack{L^2\\R^2} \circ \substack{L^3\\R^1} \circ |$$

$$| \circ \substack{L^0\\R^1} \circ \substack{v_b\\|} \circ \substack{L^3\\R^3} \circ \substack{L^2\\R^4} \circ \substack{L^1\\R^0} v_a \substack{L^0\\R^1} \circ \substack{L^4\\R^2} \circ \substack{L^3\\R^3} \circ \substack{v_c\\|} \circ \substack{L^1\\R^0} \circ |$$

Figure 14. Case 2.

First, we focus on α. To distinguish v_{-1} from v_1 we have $v_e \in L^0 \cup Ł^1 \cup R^0 \cup R^1$. Recall that consecutive vertices cannot be in W. So, to distinguish v_{-2} from v_{-1} we have $v_f \in L^2 \cup R^4$, which covers also the case $v_f = v_{-2}$. And to distinguish v_1 from v_2, we have $v_g \in L^4 \cup R^2$, which covers also the case $v_g = v_2$. But we need to distinguish also v_{-3} from v_{-2}, for which we need a vertex of $L^3 \cup R^3 \cup \{v_{-3}, v_{-2}\}$. Consequently, $v_f = v_{-2} \in L^2$. But to distinguish v_{-3} from v_{-1} we need a vertex in $L^2 \cup L^3 \cup R^3 \cup R^4$ other than v_f, a contradiction.

Case 3. $v_b = v_{n'+1}$ and $v_c \notin \beta$.

Thus, $v_c \in L^0 \cup R^4$. We distinguish two subcases.

Subcase 3.1. $v_c \in L^0$ (see Figure 15).

$$| \circ \substack{L^1\\R^3} \circ \substack{L^2\\R^2} \circ \substack{L^3\\R^1} v_b \substack{L^4\\R^0} \circ \substack{v_c\\|} \circ \substack{L^1\\R^3} \circ \substack{L^2\\R^2} \circ \substack{L^3\\R^1} \circ |$$

$$| \circ \substack{v_c\\|} \circ \substack{v_b\\|} \circ \substack{L^3\\R^3} \circ \substack{L^2\\R^4} \circ \substack{L^1\\R^0} v_a \substack{v_c\\|} \circ \substack{L^4\\R^2} \circ \substack{L^3\\R^3} \circ \substack{L^2\\R^4} \circ \substack{L^1\\R^0} \circ |$$

Figure 15. Subcase 3.1.

At the moment, we have a sequence of 5 consecutive vertices $v_1, v_2, \ldots, v_5$ without a border. The vertices in this sequence should be distinguished by three remaining vertices of W, which is impossible.

Subcase 3.2. $v_c \in R^4$, but $v_c \neq v_{n'-1}$ (see Figure 16).

$$| \circ \substack{L^1\\R^3} \circ \substack{L^2\\R^2} \circ \substack{L^3\\R^1} v_b \substack{L^4\\R^0} \circ \substack{v_c\\|} \circ \substack{L^1\\R^3} \circ \substack{L^2\\R^2} \circ \substack{L^3\\R^1} \circ |$$

$$| \circ \substack{L^0\\R^1} \circ \substack{v_b\\|} \circ \substack{L^3\\R^3} \circ \substack{v_c\\|} \circ \substack{L^1\\R^0} v_a \substack{L^0\\R^1} \circ \substack{L^4\\R^2} \circ \substack{L^3\\R^3} \circ \substack{v_c\\|} \circ \substack{L^1\\R^0} \circ |$$

Figure 16. Subcase 3.2.

At the moment, there is a sequence of four vertices without a border $v_{n'-4}, v_{n'-3}, v_{n'-2}, v_{n'-1}$, and so $v_e, v_f, v_g \notin \{v_{-5}, v_{-4}, \ldots, v_0\}$. To distinguish v_{-5} from v_{-4}, we have $v_e \in L^0 \cup R^1$, and to distinguish v_{-3} from v_{-2}, we have $v_f \in L^3 \cup R^3$. Finally, to distinguish v_1 from v_2, we have $v_g \in L^4 \cup R^2$ which covers also the case $v_g = v_2$ (recall that $v_1 \notin W$ since W does not contain consecutive vertices).

Now, we consider β. To distinguish $v_{n'-2}$ from $v_{n'-1}$, W must contain a vertex of $L^1 \cup R^3$, which covers also $v_{n'-2}$ (recall that the case $v_{n'-1} \in W$ was already considered in Case 2), and so $v_f \in R^3$. To distinguish $v_{n'+2}$ from $v_{n'+3}$, W must contain a vertex of $L^2 \cup R^2$ (since $v_{n'+2} \notin W$), and so $v_g \in R^2$.

Now, if $v_e \in R^1$, then $v_c, v_e, v_f, v_g \in R$. So considering α_e and β_e instead of α_a and β_a reduces the problem to Case 1 (recall that $v_e \neq v_1$). Hence, $v_e \in L^0$.

So $v_c \in R^4$, $v_e \in L^0$, $v_f \in R^3$ and $v_g \in R^2$. But to distinguish $v_{n'}$ from $v_{n'+2}$, W must contain a vertex in $L^3 \cup L^4 \cup R^0 \cup R^1$ other than v_b, a contradiction.

Thus, it remains to consider the last case, namely, when $v_{n'} \in W$. But since all the other cases were solved already, we may assume that $v_e = v_{c+n'}$ and $v_g = v_{f+n'}$.

Case 4. $v_b = v_{n'}$, $v_e = v_{c+n'}$ and $v_g = v_{f+n'}$ (see Figure 17).

$$| \circ \substack{L^1\\R^3} \circ \substack{L^2\\R^2} \circ \substack{L^3\\R^1} \circ \substack{L^4\\R^0} v_b \substack{L^0\\R^4} \circ \substack{L^1\\R^3} \circ \substack{L^2\\R^2} \circ \substack{L^3\\R^1} \circ |$$

$$| \circ \substack{v_b\\|} \circ \substack{L^4\\R^2} \circ \substack{L^3\\R^3} \circ \substack{L^2\\R^4} \circ \substack{L^1\\R^0} v_a \substack{L^0\\R^1} \circ \substack{L^4\\R^2} \circ \substack{L^3\\R^3} \circ \substack{L^2\\R^4} \circ \substack{v_b\\|} \circ |$$

Figure 17. Case 4.

Obviously, two vertices from v_c, v_e, v_f, v_g are in L, and two are in R. We assume that $v_c, v_f \in L$ and $v_e, v_g \in R$. To distinguish v_{-2} from v_{-1}, W must contain a vertex in $L^2 \cup R^4$, which covers also the case v_{-2}. So we distinguish 2 subcases.

Subcase 4.1. $v_c \in L^2$.

Then, $v_e \in R^3$; see Figure 18.

$$| \circ \substack{v_e\\|} \circ \substack{v_c\\|} \circ \substack{L^3\\R^1} \circ \substack{L^4\\R^0} v_b \substack{L^0\\R^4} \circ \substack{v_e\\|} \circ \substack{v_c\\|} \circ \substack{L^3\\R^1} \circ |$$

$$| \circ \substack{v_b\\|} \circ \substack{L^4\\R^2} \circ \substack{v_e\\|} \circ \substack{v_c\\|} \circ \substack{L^1\\R^0} v_a \substack{L^0\\R^1} \circ \substack{L^4\\R^2} \circ \substack{v_e\\|} \circ \substack{v_c\\|} \circ \substack{v_b\\|} \circ |$$

Figure 18. Subcase 4.1.

To distinguish v_1 from v_2, W must contain a vertex in $L^4 \cup R^2$, and so either $v_f \in L^4$ and $v_g \in R^1$, or $v_g \in R^2$ and $v_f \in L^3$. Since in the later case v_{-1} and v_1 are not distinguished, we conclude that $v_f \in L^4$ and $v_g \in R^1$.

Now, consider α_c and β_c. We have $v_a \in R_c^2$, $v_b \in L_c^3$, and either $v_f \in L_c^2$ and $v_g \in R_c^3$ or $v_f \in R_c^3$ and $v_g \in L_c^2$. However, both cases were already excluded in the previous paragraph.

Subcase 4.2. $v_e \in R^4$.

Then, $v_c \in L^1$; see Figure 19.

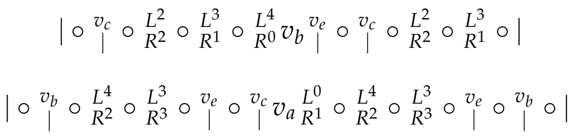


Figure 19. Subcase 4.2.

To distinguish v_1 from v_2, W must contain a vertex in $L^4 \cup R^2$, and so either $v_f \in L^4$ and $v_g \in R^1$, or $v_g \in R^2$ and $v_f \in L^3$. Since in the former case $v_{n'+2}$ and $v_{n'+3}$ are not distinguished, we conclude that $v_f \in L^3$ and $v_g \in R^2$.

Now consider α_f and β_f. We have $v_a \in R^3_f$, $v_b \in L^2_f$, and either $v_c \in R^2_f$ and $v_e \in L^3_f$, or $v_c \in L^3_f$ and $v_e \in R^2_f$. However, both cases were already excluded, which completes the proof. □

3. Conclusions

From our Theorem 1, we have $\dim(G_n) \leq 6$ for $n \equiv 9 \pmod{10}$ where $n \geq 29$. By (1), we have $\dim(G_n) \geq 6$, and thus,

$$\dim(G_n) = 6 \text{ if } n \equiv 9 \pmod{10} \text{ where } n \geq 29. \tag{3}$$

By Theorems 2.18 and 2.19 from [19], $\dim(G_n) \leq 8$ for $n \equiv 0, 1 \pmod{10}$. By Theorem 2.17 given in [19], $\dim(G_n) \geq 7$ for $n \equiv 1 \pmod{10}$. We improved the lower bound in our Theorem 2 by proving that $\dim(G_n) \geq 8$ for $n \equiv 1 \pmod{10}$. Thus,

$$\dim(G_n) = 8 \text{ if } n \equiv 1 \pmod{10}. \tag{4}$$

From our Theorem 3, we obtain $\dim(G_n) \geq 7$ for $n \equiv 0 \pmod{10}$. By [21], for the same values of n, we have $\dim(G_n) \leq 7$, and thus

$$\dim(G_n) = 7 \text{ if } n \equiv 0 \pmod{10}. \tag{5}$$

Hence, by (2), (3), (4) and (5), for $n \geq 12$ where $n \notin \{13, 19\}$,

$$\dim(G_n) = \begin{cases} 6 & \text{if } n \equiv 2,3,4,5,6,7,8,9 \pmod{10}, \\ 7 & \text{if } n \equiv 0 \pmod{10}, \\ 8 & \text{if } n \equiv 1 \pmod{10}. \end{cases}$$

Note that from Table 3 given in [19], we have $\dim(G_{13}) = 5$ and $\dim(G_{19}) = 7$.

Thus, the problem of finding the metric dimension of $C_n(1, 2, \ldots, t)$ is now completely solved for $t \leq 5$ and any n. We suggest continuing to try to solve the problem completely for $t > 5$.

Problem 1. *Find the metric dimension of $C_n(1, 2, \ldots, t)$ for $t \geq 6$ and any n.*

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