

On the Structure of $SO(3)$: Trace and Canonical Decompositions

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Abstract: This paper is devoted to some selected topics of the theory of special orthogonal group $SO(3)$. First, we discuss the trace of orthogonal matrices and its relation to the characteristic polynomial; on this basis, the partition of $SO(3)$ formed by conjugation classes is described by trace mapping. Second, we show that every special orthogonal matrix can be expressed as the product of three elementary special orthogonal matrices. Explicit formulas for the decomposition as needed for applications in differential geometry and physics as symmetry transformations are given.

Keywords: special orthogonal group; orbit stabilizer; trace; $SO(3)$ matrix decomposition

MSC: 15B10; 50E40; 70H33; 83C40

1. Introduction

Our objective in this paper is a well-known classical subject: the special orthogonal group $SO(3)$. We search for innovations needed for new applications of $SO(3)$ as a transformation group on topological and smooth manifolds.

As a set, $SO(3)$ carries several mathematical structures. Its basic *algebraic structure* (the *group structure*) is induced by the Euclidean scalar product on $\mathbf{R}^3$. The canonical *topological structure* of $SO(3)$ is induced by the Euclidean topology of $\mathbf{R}^9$. Clearly, $SO(3)$ also has *smooth manifold* and *Lie group structures*. Indeed, these structures are all compatible. We wish to revisit and summarize properties of these structures in a coherent way suitable for various applications.

Standard and modern topics and problems related to applications of $SO(3)$ do not need a broad introduction. $SO(3)$ -symmetry appears in elementary analysis, geometry and invariance theory. As a fundamental concept, $SO(3)$ appears in classical mechanics (e.g., the Kepler problem) and in general relativity and field theory (e.g., gravitational fields with spherical symmetry, the Schwarzschild solution of the Einstein equations and the Birkhoff theorem). In variational geometry, $SO(3)$ invariance implies conservation laws for solutions of the Euler-Lagrange equations. Geometric and topological problems include a classification of manifolds endowed with a $SO(3)$ group action.

Indeed, some of these problems still need a more exact formulation of basic theorems and their proofs. Exact assertions are also needed for new applications. In general, we are interested in two questions: (a) the decomposition theorems of $SO(3)$ in terms of its subgroups isomorphic to the special orthogonal group $SO(2)$; and (b) orbit structure of continuous actions of $SO(3)$ on topological spaces.

We wish to consider in this paper the underlying *algebraic* part of the theory. In the proofs, only the *group structure* of $SO(3)$ is used. In particular, we do not need an interpretation of $SO(3)$ as a rotation group. It should be pointed out that in accordance with motivations included in work by Krupka and Brajerčík [1], basic algebraic theory will be extended to the topological and smooth structures of $SO(3)$ in subsequent articles.



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Section 2 is devoted to standard general definitions and the notation as needed in proofs (e.g., Alperin, Bell [2] and Kurosh [3]), and a version of the orbit stabilizer theorem is presented (Bui, [4]).

In Section 3, $SO(3)$ is studied. We prefer a direct, straightforward approach based on its algebraic and geometric origin. As usual, special orthogonal matrices are introduced by the *orthogonality conditions* and the *determinant condition*. Sometimes, we also apply an equivalent definition based on the *cross-product* of vectors in the Euclidean vector space $\mathbf{R}^3$.

Within this framework, innovations in this article include:

- the trace properties of special orthogonal matrices (relationship of the trace and the characteristic polynomial, explicit form of the partition of $SO(3)$ associated to the trace mapping, simple description of stable points),
- a canonical decomposition formula for any special orthogonal matrix in terms of *elementary* special orthogonal matrices canonically identified with elements of $SO(2)$.

The text and the notation also provide a coherent, self-contained basis for the discussion of the topological and smooth structures of $SO(3)$. Because of the lack of space, we will discuss these concepts elsewhere.

Everywhere in this paper $\mathbf{R}$ denotes the field of real numbers and $\mathbf{R}^n$ is the Euclidean vector space of dimension n endowed with the Euclidean scalar product.

2. Group Actions: Orbit Stabilizer Theorem

2.1. Groups, Subgroups, Coset Partitions

In this section, G is a group. The group multiplication is denoted as a mapping $G \times G \ni (g_1, g_2) \rightarrow g_1 \cdot g_2 \in G$. e_G is the *identity element* of G , and g^{-1} is the *inverse* of g .

For any two subsets H and K of a group G we define the *group product* of H and K to be the subset $H \cdot K = \{g \in G \mid g = h \cdot k, h \in H, k \in K\}$ of G . If H is a one-point set, $H = \{h\}$, we write $h \cdot K = \{g \in G \mid g = h \cdot k, h \in H, k \in K\}$ instead of $\{h\} \cdot K$.

Consider a special case when H and K are *subgroups* of G .

Lemma 1. *The group product $H \cdot K$ of subgroups H and K of a group G is a subgroup of G if and only if*

$$H \cdot K = K \cdot H. \quad (1)$$

Proof. Only sufficiency of condition (1) needs proof., which is straightforward. $\square$

For any *subgroup* H of G and any $g \in G$ the set gH is called the *left coset* of H in G *generated by* g . Since H contains the identity element, g always belongs to gH .

Lemma 2. *Let G be a group and let H be a subgroup of G .*

- Two left cosets g_1H, g_2H coincide if and only if $g_1^{-1}g_2 \in H$.*
- Given a left coset gH , then for any two elements $g_1, g_2 \in gH$,*

$$gH = g_1H = g_2H. \quad (2)$$

- Left cosets form a partition of the group G .*

Proof. (a) Suppose that $g_1H = g_2H$. Then $g_1h_1 = g_2h_2$ for some $h_1, h_2 \in H$ hence $g_1^{-1}g_2 = h_1h_2^{-1} \in H$. Conversely if $g_1^{-1}g_2 = h \in H$ for some $h \in H$, then for any $g \in g_1H$ formula $g = g_1h = g_2 \in g_2H$ yields $g_1H = g_2H$.

(b) Let gH be a left coset, let g_1, g_2 be two points belonging to gH . Then $g_1 = gh_1$ and $g_2 = gh_2$ for some $h_1, h_2 \in H$. Hence $g = g_1h_1^{-1} = g_2h_2^{-1}$ and $g_1^{-1}g_2 = h_1^{-1}h_2 \in H$. Then by (a), $g_1H = g_2H$. Replacing g_1 by g we also have $gH = g_2H$ proving (2).

(c) Formula (2) implies that a left coset is generated by *any* of its elements. Thus, two left cosets are either identical or disjoint. $\square$

The partition of G formed by left cosets gH is the *left partition* of G by the subgroup H and is denoted by G/H . The corresponding quotient projection is the mapping $G \ni g \rightarrow gH \in G/H$.

Similar construction arises when we use the *right cosets* generated by g $Hg = \{g' \in G | g' = hg, h \in H\}$ instead of gH . The *canonical* (one-to-one) *correspondence* between right and left cosets is given by

$$Hg \ni k \rightarrow gkg^{-1} \in gH. \quad (3)$$

Lemma 3. Let G be a group and let H be a subgroup of G .

(a) Two right cosets Hg_1, Hg_2 coincide if and only if $g_1g_2^{-1} \in H$.

(b) Given a right coset Hg , then for any two elements $g_1, g_2 \in Hg$,

$$Hg = Hg_1 = Hg_2.$$

(c) Right cosets form a partition of the group G .

Proof. See Lemma 2. $\square$

The partition of G formed by right cosets Hg is the *right partition* of G by H . The corresponding quotient projection is the mapping $X \ni x \rightarrow Hx \in G/H$. Formula (3) shows that the right partition differs from the left partition G/H by a canonical bijection.

2.2. Group Actions

Let G be a group and let X be a set. Recall that a *right action* of G on H is a mapping $X \times G \ni (x, g) \rightarrow x \cdot g \in X$ such that $x \cdot e_G = x$ for all $x \in X$ and $x \cdot (g_1g_2) = (x \cdot g_1)g_2$ for all $g_1, g_2 \in G$ and all $x \in X$. A set X endowed with a right action of G is called a *right G -set*.

A mapping $\Phi: X_1 \rightarrow X_2$ of a right G -set X_1 into a right G -set X_2 is said to be *G -equivariant*, if $\Phi(x \cdot g) = \Phi(x) \cdot g$ for all $x \in X_1$ and $g \in G$.

A right action $(g, x) \rightarrow x \cdot g$ of G on X is *transitive*, if for any two points $x_1, x_2 \in X$ there exists $g \in G$ such that $x_2 = x_1 \cdot g$. A right action $(g, x) \rightarrow x \cdot g$ is *free*, if for any $x \in X$ equation $x \cdot g = x$ has a unique solution $g = e_G$; equivalently we also say that G acts on X *without fixed points*.

The *G -orbit* of a point x in a right G -set X , or just an *orbit* of x , is the set $xG = \{x' \in X | x' = x \cdot g, g \in G\}$. Clearly, a G -orbit of x is a right G -set and the restriction of the right action of G to xG is *transitive*.

The *stabilizer* of a point $x \in X$ is the subgroup $G_x = \{g \in G | x = x \cdot g\}$ of G . It is easily seen that $G_{xg} = g^{-1}G_xg$ for all g ; in particular, the stabilizers of the points of X along a G -orbit are canonically isomorphic.

G -orbits can be considered as equivalence classes of the equivalence relation on X " $x_1 \sim x_2$ if there exists a point $g \in G$ such that $x_2 = x_1 \cdot g$ ". The corresponding quotient set denoted by X/G consists of different orbits and is called *the orbit set*. The *quotient projection* assigns to $x \in X$ the G -orbit xG .

Set for every point $x \in X$ and every $g \in G$

$$\Theta_x(g) = x \cdot g.$$

This formula defines the *G -orbit mapping* at the point x , $\Theta_x: G \rightarrow X$. Restricting the range of Θ_x we get a surjective mapping of the group G onto the G -orbit xG . Since $x \cdot (g_1g_2) = (x \cdot g_1) \cdot g_2$ for any $g_1, g_2 \in G$, Θ_x satisfies

$$\Theta_x(g_1g_2) = \Theta_x(g_1) \cdot g_2 = \Theta_{g_1 \cdot x}(g_2).$$

In particular, Θ_x is *G -equivariant*.

A *left action* of G on X is a mapping $G \times X \ni (g, x) \rightarrow g \cdot x \in X$ such that $e_G \cdot x = x$ for all $x \in X$ and $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$ for all $g_1, g_2 \in G$ and all $x \in X$. A set X endowed with a left action of G is called a *left G -set*.

The concepts like a *stabilizer*, a *G -equivariant mapping* of left G -sets, a *transitive* and a *free* left action, the *orbit* and the *orbit set* of a right action extend in an obvious way to the left actions. Note, however, that $G_{gx} = gG_x g^{-1}$.

2.3. Orbit Stabilizer Theorem

Let X be a left G -set, let H be a subgroup of G . Setting for any left coset gH and any $g' \in G$

$$g' \cdot gH = (g'g)H$$

We get a *left action* $(g', gH) \rightarrow (g'g)H$ of G on the set of left cosets G/H . With this action G/H becomes a left G -set. For any $x \in X$ we have the diagram

$$\begin{array}{ccc} G & \xrightarrow{\Theta_x} & Gx \\ \downarrow & & \\ G/H & & \end{array} \quad (4)$$

The following is a version of the *Orbit stabilizer theorem* for the G -orbit mapping Θ_x (cf. e.g., Hong Thien An Bui [4], Theorem 8.1).

Theorem 1. *Let X be a left G -set. For any point $x \in X$ there exists a mapping $\theta_x : G/G_x \rightarrow Gx$ such that the diagram*

$$\begin{array}{ccc} G & \xrightarrow{\Theta_x} & Gx \\ \downarrow & \nearrow \theta_x & \\ G/G_x & & \end{array} \quad (5)$$

commutes. θ_x is unique and is a G -equivariant isomorphism of G/G_x and the G -orbit Gx .

Proof. Fix $x \in X$ and consider the G -orbit mapping $\Theta_x(g) = g \cdot x$ and the stabilizer G_x of x . We know that two left cosets $g_1 G_x, g_2 G_x$ are equal if and only if $g_1^{-1} g_2 \in G_x$. Consider a class $[g] \in G/G_x$ and choose $g_1, g_2 \in [g]$. Then $g_2^{-1} g_1 \in G_x$ hence $g_2^{-1} g_1 \cdot x = x$ and $g_1 \cdot x = g_2 \cdot x$ that is $\Theta_x(g_1) = \Theta_x(g_2)$. Consequently, the point $\Theta_x(g)$ depends on the class $[g]$ only. Thus, formula

$$\theta_x([g]) = \theta_x(gG_x) = \Theta_x(g)$$

defines a mapping $\theta_x : G/G_x \rightarrow X$. This proves the existence of θ_x completing the diagram (4) to the commutative diagram (5). The uniqueness of θ_x is obvious: If $\theta_x[g] = x$ then $\Theta_x(g) = x$ and if $\tilde{\theta}_x[g] = \tilde{x}$ then $\Theta_x(g) = \tilde{x}$; then however, $x = \tilde{x}$ and $\theta_x = \tilde{\theta}_x$.

If $[h] \in G/G_x$ and $g \in G$, then

$$\theta_x(g \cdot [h]) = \theta_x([g \cdot h]) = \Theta_x(gh) = g \cdot (h \cdot x) = g \cdot \Theta_x(h) = g \cdot \theta_x([h])$$

proving that θ_x is G -equivariant.

We show that θ_x is bijective. To prove injectivity, suppose that $\theta_x([g]) = \theta_x([h])$ that is, $g \cdot x = h \cdot x$, then $h^{-1}g \cdot x = x$ hence, $h^{-1}g \in G_x$; this implies that the left cosets hG_x and gG_x are equal, thus, θ_x is injective. To prove surjectivity, since we already have the commutative diagram (5), it is sufficient to verify that the orbit mapping Θ_x is surjective, but this is obviously true since $\Theta_x(G) = G \cdot x$ is the G -orbit. $\square$

3. Special Orthogonal Group SO(3)

3.1. SO(2)

A (2×2) -matrix with real entries

$$\tau = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix} \quad (6)$$

is said to be *special orthogonal* if its row vectors $(\tau_{11}, \tau_{12}), (\tau_{21}, \tau_{22})$ satisfy the *orthogonality conditions*

$$\tau_{11}^2 + \tau_{12}^2 = 1, \tau_{21}^2 + \tau_{22}^2 = 1, \tau_{11}\tau_{21} + \tau_{12}\tau_{22} = 0 \quad (7)$$

and the *determinant condition*

$$\begin{vmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{vmatrix} = 1. \quad (8)$$

The set of special orthogonal matrices (6) together with the matrix multiplication is a group called the *special orthogonal group* (in two dimensions) and is denoted by SO(2).

Clearly, orthogonality conditions state that the rows of a special orthogonal matrix constitute an *orthonormal basis* of the vector space $\mathbf{R}^2$.

Let ${}^t\tau$ denote the *transpose* of the matrix τ . Identity

$$\tau \cdot {}^t\tau = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix} \cdot \begin{pmatrix} \tau_{11} & \tau_{21} \\ \tau_{12} & \tau_{22} \end{pmatrix} = \begin{pmatrix} \tau_{11}^2 + \tau_{12}^2 & \tau_{11}\tau_{21} + \tau_{12}\tau_{22} \\ \tau_{11}\tau_{21} + \tau_{22}\tau_{12} & \tau_{21}^2 + \tau_{22}^2 \end{pmatrix}$$

together with orthogonality and determinant conditions (7) and (8) show that τ is special orthogonal if and only if

$$\tau^{-1} = {}^t\tau.$$

The following is an immediate consequence of the definition.

Lemma 4. *The following conditions are equivalent:*

- (a) τ is special orthogonal.
- (b) τ is of the form

$$\tau = \begin{pmatrix} \tau_{11} & -\tau_{21} \\ \tau_{21} & \tau_{11} \end{pmatrix},$$

where

$$\tau_{11}^2 + \tau_{21}^2 = 1.$$

- (c) τ has exactly one of the following two expressions

$$\begin{pmatrix} \tau_{11} & -\sqrt{1-\tau_{11}^2} \\ \sqrt{1-\tau_{11}^2} & \tau_{11} \end{pmatrix}, \begin{pmatrix} \tau_{11} & \sqrt{1-\tau_{11}^2} \\ -\sqrt{1-\tau_{11}^2} & \tau_{11} \end{pmatrix},$$

where τ_{11} is a number such that $-1 \leq \tau_{11} \leq 1$.

Remark 1. The mapping

$$\text{SO}(2) \ni \begin{pmatrix} \tau_{11} & -\tau_{21} \\ \tau_{21} & \tau_{11} \end{pmatrix} \rightarrow (\tau_{11}, \tau_{21}) \in S^1$$

is a bijection, the *canonical bijection* of SO(2) and the circle $S^1 \subset \mathbf{R}^2$.

3.2. $SO(3)$

A (3×3) -matrix τ with real entries

$$\tau = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix} \quad (9)$$

is said to be *special orthogonal*, if its entries τ_{ij} satisfy the *orthogonality conditions*

$$\begin{aligned} \tau_{11}^2 + \tau_{12}^2 + \tau_{13}^2 &= 1, \\ \tau_{21}^2 + \tau_{22}^2 + \tau_{23}^2 &= 1, \\ \tau_{31}^2 + \tau_{32}^2 + \tau_{33}^2 &= 1, \\ \tau_{11}\tau_{21} + \tau_{12}\tau_{22} + \tau_{13}\tau_{23} &= 0, \\ \tau_{11}\tau_{31} + \tau_{12}\tau_{32} + \tau_{13}\tau_{33} &= 0, \\ \tau_{21}\tau_{31} + \tau_{22}\tau_{32} + \tau_{23}\tau_{33} &= 0 \end{aligned} \quad (10)$$

and the *determinant condition*

$$\begin{vmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{vmatrix} = 1. \quad (11)$$

The set of special orthogonal matrices of the form (9) endowed with the matrix multiplication is called the *special orthogonal group* (in three dimensions) and is denoted by $SO(3)$.

The orthogonality conditions show that the rows of a special orthogonal matrix constitute an *orthonormal basis* of the vector space $\mathbf{R}^3$.

First examples of special orthogonal matrices are

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Let ${}^t\tau$ be the *transpose* of the matrix τ . Identity

$$\begin{aligned} & \tau \cdot {}^t\tau \\ &= \begin{pmatrix} \tau_{11}^2 + \tau_{12}^2 + \tau_{13}^2 & \tau_{11}\tau_{21} + \tau_{12}\tau_{22} + \tau_{13}\tau_{23} & \tau_{11}\tau_{31} + \tau_{12}\tau_{32} + \tau_{13}\tau_{33} \\ \tau_{11}\tau_{21} + \tau_{12}\tau_{22} + \tau_{13}\tau_{23} & \tau_{21}^2 + \tau_{22}^2 + \tau_{23}^2 & \tau_{21}\tau_{31} + \tau_{22}\tau_{32} + \tau_{23}\tau_{33} \\ \tau_{11}\tau_{31} + \tau_{12}\tau_{32} + \tau_{13}\tau_{33} & \tau_{21}\tau_{31} + \tau_{22}\tau_{32} + \tau_{23}\tau_{33} & \tau_{31}^2 + \tau_{32}^2 + \tau_{33}^2 \end{pmatrix} \end{aligned}$$

shows that τ is special orthogonal if and only if τ is invertible and

$$\tau^{-1} = {}^t\tau.$$

In the following two remarks we give an example of a *finite* subgroup of the special orthogonal group $SO(3)$ and an example of a subgroup homomorphic with the special orthogonal group $SO(2)$.

Remark 2 (Cyclic permutation matrices). Recall that a *permutation matrix* is a square matrix with real entries that has exactly one entry 1 in every column and every row and all

other entries are 0. In $SO(3)$ there are exactly three permutation matrices, namely the *cyclic permutation matrices*

$$\nu_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \nu_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \nu_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

These cyclic matrices form a subgroup of $SO(3)$. Each permutation matrix defines a *cyclic permutation* of the rows of the matrices τ (6):

$$\nu_1 \cdot \tau = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix}, \nu_2 \cdot \tau = \begin{pmatrix} \tau_{31} & \tau_{32} & \tau_{33} \\ \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \end{pmatrix}, \nu_3 \cdot \tau = \begin{pmatrix} \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \\ \tau_{11} & \tau_{12} & \tau_{13} \end{pmatrix}.$$

If for example

$$\tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \tau_{22} & \tau_{23} \\ 0 & \tau_{32} & \tau_{33} \end{pmatrix},$$

then

$$\nu_1 \cdot \tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \tau_{22} & \tau_{23} \\ 0 & \tau_{32} & \tau_{33} \end{pmatrix}, \nu_2 \cdot \tau = \begin{pmatrix} 0 & \tau_{32} & \tau_{33} \\ 1 & 0 & 0 \\ 0 & \tau_{22} & \tau_{23} \end{pmatrix}, \nu_3 \cdot \tau = \begin{pmatrix} 0 & \tau_{22} & \tau_{23} \\ 0 & \tau_{32} & \tau_{33} \\ 1 & 0 & 0 \end{pmatrix}.$$

Remark 3. A matrix τ of the form

$$\tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \tau_{11} & \tau_{12} \\ 0 & \tau_{21} & \tau_{22} \end{pmatrix}$$

is special orthogonal if and only if its submatrix

$$\begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix} \quad (12)$$

is special orthogonal. Special orthogonal matrices of the form (12) constitute a subgroup of $SO(3)$ isomorphic with $SO(2)$.

3.3. Cross-Product

Recall that the *cross-product* of two vectors (y_1, y_2, y_3) and (z_1, z_2, z_3) in $\mathbf{R}^3$ is defined to be the vector

$$\chi((y_1, y_2, y_3), (z_1, z_2, z_3)) = \left(\begin{vmatrix} y_2 & y_3 \\ z_2 & z_3 \end{vmatrix}, -\begin{vmatrix} y_1 & y_3 \\ z_1 & z_3 \end{vmatrix}, \begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix} \right).$$

The matrix

$$\begin{pmatrix} \begin{vmatrix} y_2 & y_3 \\ z_2 & z_3 \end{vmatrix} & y_1 & z_1 \\ -\begin{vmatrix} y_1 & y_3 \\ z_1 & z_3 \end{vmatrix} & y_2 & z_2 \\ \begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix} & y_3 & z_3 \end{pmatrix} \quad (13)$$

is called the *cross-product matrix* of (y_1, y_2, y_3) and (z_1, z_2, z_3) .

Note that χ assigns to the pair of column vectors $(y_1, y_2, y_3) = (0, 1, 0)$, $(z_1, z_2, z_3) = (0, 0, 1)$ the column vector $(1, 0, 0)$; in this case the cross-product matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

represents the canonical *counterclockwise* orthonormal frame of $\mathbf{R}^3$.

Lemma 5. *The cross-product matrix of two vectors (y_1, y_2, y_3) , (z_1, z_2, z_3) forming an orthonormal system is special orthogonal.*

Proof 1. Suppose that the vectors (y_1, y_2, y_3) , (z_1, z_2, z_3) form an orthonormal system, that is,

$$\begin{aligned} y_1^2 + y_2^2 + y_3^2 &= 1, \\ y_1 z_1 + y_2 z_2 + y_3 z_3 &= 0, \\ z_1^2 + z_2^2 + z_3^2 &= 1. \end{aligned}$$

Then

$$\begin{vmatrix} y_2 & y_3 \\ z_2 & z_3 \end{vmatrix}^2 + \begin{vmatrix} y_1 & y_3 \\ z_1 & z_3 \end{vmatrix}^2 + \begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix}^2 = y_1^2 + y_2^2 + y_3^2 = 1,$$

proving that the first column in (1) is of length 1. Identities

$$\begin{aligned} y_1 \begin{vmatrix} y_2 & y_3 \\ z_2 & z_3 \end{vmatrix} - y_2 \begin{vmatrix} y_1 & y_3 \\ z_1 & z_3 \end{vmatrix} + y_3 \begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix} &= 0, \\ z_1 \begin{vmatrix} y_2 & y_3 \\ z_2 & z_3 \end{vmatrix} - z_2 \begin{vmatrix} y_1 & y_3 \\ z_1 & z_3 \end{vmatrix} + z_3 \begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix} &= 0 \end{aligned}$$

then shows that the matrix (13) satisfies the orthogonality conditions. The determinant condition can be verified by a direct calculation. $\square$

We show that the cross-product is *surjective*.

Lemma 6. *For any unit vector (x_1, x_2, x_3) in $\mathbf{R}^3$ there exist unit vectors (y_1, y_2, y_3) and (z_1, z_2, z_3) such that the matrix*

$$\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}$$

is special orthogonal.

Proof. Lemma 6 just says that every unit vector can be completed to an orthonormal basis. $\square$

The cross-product is unique way how to construct special orthogonal matrices:

Lemma 7. *Suppose that we have two special orthogonal matrices of the form*

$$\tau = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix}, \quad \kappa = \begin{pmatrix} \kappa_{11} & \kappa_{12} & \kappa_{13} \\ \kappa_{21} & \kappa_{22} & \kappa_{23} \\ \kappa_{31} & \kappa_{32} & \kappa_{33} \end{pmatrix}. \quad (14)$$

Then

$$\tau_{11} = \kappa_{11}, \quad \tau_{21} = \kappa_{21}, \quad \tau_{31} = \kappa_{31}. \quad (15)$$

Proof. By hypothesis matrices (14) satisfy the orthogonality conditions and the determinant condition; thus,

$$\begin{aligned} \tau_{11}^2 + \tau_{12}^2 + \tau_{13}^2 &= 1, & \kappa_{11}^2 + \tau_{12}^2 + \tau_{13}^2 &= 1, \\ \tau_{21}^2 + \tau_{22}^2 + \tau_{23}^2 &= 1, & \kappa_{21}^2 + \tau_{22}^2 + \tau_{23}^2 &= 1, \\ \tau_{31}^2 + \tau_{32}^2 + \tau_{33}^2 &= 1, & \kappa_{31}^2 + \tau_{32}^2 + \tau_{33}^2 &= 1, \\ \tau_{11}\tau_{21} + \tau_{12}\tau_{22} + \tau_{13}\tau_{23} &= 0, & \kappa_{11}\kappa_{21} + \tau_{12}\tau_{22} + \tau_{13}\tau_{23} &= 0, \\ \tau_{11}\tau_{31} + \tau_{12}\tau_{32} + \tau_{13}\tau_{33} &= 0, & \kappa_{11}\kappa_{31} + \tau_{12}\tau_{32} + \tau_{13}\tau_{33} &= 0, \\ \tau_{21}\tau_{31} + \tau_{22}\tau_{32} + \tau_{23}\tau_{33} &= 0, & \kappa_{21}\kappa_{31} + \tau_{22}\tau_{32} + \tau_{23}\tau_{33} &= 0. \end{aligned} \quad (16)$$

and

$$\begin{vmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{vmatrix} = 1, \quad \begin{vmatrix} \kappa_{11} & \tau_{12} & \tau_{13} \\ \kappa_{21} & \tau_{22} & \tau_{23} \\ \kappa_{31} & \tau_{32} & \tau_{33} \end{vmatrix} = 1. \quad (17)$$

We claim that these equations force the identity $(\tau_{11}, \tau_{21}, \tau_{31}) = (\kappa_{11}, \kappa_{21}, \kappa_{31})$ (15).

Suppose the opposite. Then we have two vectors $(\tau_{11}, \tau_{21}, \tau_{31})$, $(\kappa_{11}, \kappa_{21}, \kappa_{31})$, satisfying (16) and (17) such that $(\tau_{11}, \tau_{21}, \tau_{31}) \neq (\kappa_{11}, \kappa_{21}, \kappa_{31})$. From (16) we conclude that $\tau_{11}^2 = \kappa_{11}^2$, $\tau_{21}^2 = \kappa_{21}^2$, $\tau_{31}^2 = \kappa_{31}^2$, $\tau_{11}\tau_{21} = \kappa_{11}\kappa_{21}$, $\tau_{11}\tau_{31} = \kappa_{11}\kappa_{31}$, $\tau_{21}\tau_{31} = \kappa_{21}\kappa_{31}$, or, equivalently,

$$\begin{aligned} (\tau_{11} - \kappa_{11})(\tau_{11} + \kappa_{11}) &= 0, & (\tau_{21} - \kappa_{21})(\tau_{21} + \kappa_{21}) &= 0, \\ (\tau_{31} - \kappa_{31})(\tau_{31} + \kappa_{31}) &= 0 \end{aligned} \quad (18)$$

and

$$\tau_{11}\tau_{21} = \kappa_{11}\kappa_{21}, \quad \tau_{11}\tau_{31} = \kappa_{11}\kappa_{31}, \quad \tau_{21}\tau_{31} = \kappa_{21}\kappa_{31}. \quad (19)$$

If $\tau_{11} - \kappa_{11} \neq 0$, then the first equation yields $\tau_{11} = -\kappa_{11} \neq 0$. Then Equations (18) and (19) transforms to

$$\begin{aligned} (\tau_{21} - \kappa_{21})(\tau_{21} + \kappa_{21}) &= 0, & (\tau_{31} - \kappa_{31})(\tau_{31} + \kappa_{31}) &= 0, \\ \tau_{21} + \kappa_{21} &= 0, & \tau_{31} + \kappa_{31} &= 0, \\ \tau_{21}\tau_{31} &= \kappa_{21}\kappa_{31} \end{aligned}$$

hence

$$\tau_{21} + \kappa_{21} = 0, \quad \tau_{31} + \kappa_{31} = 0. \quad (20)$$

Then, Equations (19) and (20) contradict the determinant conditions (17). Consequently $\tau_{11} - \kappa_{11} = 0$ as required.

The same contradictions arise when we suppose that $\tau_{21} - \kappa_{21} \neq 0$ or $\tau_{31} - \kappa_{31} \neq 0$. $\square$

Theorem 2. Every special orthogonal matrix τ is a cross-product matrix of two vectors forming an orthonormal system,

$$\tau = \begin{pmatrix} \begin{vmatrix} \tau_{22} & \tau_{32} \\ \tau_{23} & \tau_{33} \end{vmatrix} & \tau_{12} & \tau_{13} \\ -\begin{vmatrix} \tau_{12} & \tau_{32} \\ \tau_{13} & \tau_{33} \end{vmatrix} & \tau_{22} & \tau_{23} \\ \begin{vmatrix} \tau_{12} & \tau_{22} \\ \tau_{13} & \tau_{23} \end{vmatrix} & \tau_{32} & \tau_{33} \end{pmatrix}.$$

Proof. Let τ be any special orthogonal matrix,

$$\tau = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix},$$

and let κ be the cross-product matrix of the column vectors $(\tau_{12} \ \tau_{22} \ \tau_{32})$ and $(\tau_{13} \ \tau_{23} \ \tau_{33})$,

$$\kappa = \begin{pmatrix} \begin{vmatrix} \tau_{22} & \tau_{32} \\ \tau_{23} & \tau_{33} \end{vmatrix} & \tau_{12} & \tau_{13} \\ -\begin{vmatrix} \tau_{12} & \tau_{32} \\ \tau_{13} & \tau_{33} \end{vmatrix} & \tau_{22} & \tau_{23} \\ \begin{vmatrix} \tau_{12} & \tau_{22} \\ \tau_{13} & \tau_{23} \end{vmatrix} & \tau_{32} & \tau_{33} \end{pmatrix}.$$

Then, by Lemma 7, $\tau = \kappa$ as desired. $\square$

Clearly, Theorem 2 holds *mutatis mutandis* whenever the first column in a special orthogonal matrix is replaced by any of its columns and similarly for the rows. Consequently, we obtain the following useful identities for the entries of special orthogonal matrices.

Theorem 3. Let τ be a matrix expressed as

$$\tau = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix}, \quad (21)$$

and suppose

$$\det \tau = 1. \quad (22)$$

The following two conditions are equivalent:

- (a) The entries of τ satisfy the conditions (2), Section 3.2.
- (b) The entries of τ satisfy

$$\begin{aligned} \tau_{11} &= \begin{vmatrix} \tau_{22} & \tau_{32} \\ \tau_{23} & \tau_{33} \end{vmatrix}, \tau_{21} = -\begin{vmatrix} \tau_{12} & \tau_{32} \\ \tau_{13} & \tau_{33} \end{vmatrix}, \tau_{31} = \begin{vmatrix} \tau_{12} & \tau_{22} \\ \tau_{13} & \tau_{23} \end{vmatrix}, \\ \tau_{12} &= \begin{vmatrix} \tau_{23} & \tau_{33} \\ \tau_{21} & \tau_{31} \end{vmatrix}, \tau_{22} = -\begin{vmatrix} \tau_{13} & \tau_{33} \\ \tau_{11} & \tau_{31} \end{vmatrix}, \tau_{32} = \begin{vmatrix} \tau_{13} & \tau_{23} \\ \tau_{11} & \tau_{21} \end{vmatrix}, \\ \tau_{13} &= \begin{vmatrix} \tau_{21} & \tau_{31} \\ \tau_{22} & \tau_{32} \end{vmatrix}, \tau_{23} = -\begin{vmatrix} \tau_{11} & \tau_{31} \\ \tau_{12} & \tau_{32} \end{vmatrix}, \tau_{33} = \begin{vmatrix} \tau_{11} & \tau_{21} \\ \tau_{12} & \tau_{22} \end{vmatrix}. \end{aligned} \quad (23)$$

Proof. 1. Since cyclic permutations of columns do not change special orthogonality, we can write

$$\begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix} = \begin{pmatrix} \begin{vmatrix} \tau_{22} & \tau_{32} \\ \tau_{23} & \tau_{33} \end{vmatrix} & \tau_{12} & \tau_{13} \\ -\begin{vmatrix} \tau_{12} & \tau_{32} \\ \tau_{13} & \tau_{33} \end{vmatrix} & \tau_{22} & \tau_{23} \\ \begin{vmatrix} \tau_{12} & \tau_{22} \\ \tau_{13} & \tau_{23} \end{vmatrix} & \tau_{32} & \tau_{33} \end{pmatrix},$$

$$\begin{pmatrix} \tau_{13} & \tau_{11} & \tau_{12} \\ \tau_{23} & \tau_{21} & \tau_{22} \\ \tau_{33} & \tau_{31} & \tau_{32} \end{pmatrix} = \begin{pmatrix} \begin{vmatrix} \tau_{21} & \tau_{31} \\ \tau_{22} & \tau_{32} \end{vmatrix} & \tau_{11} & \tau_{12} \\ -\begin{vmatrix} \tau_{11} & \tau_{31} \\ \tau_{12} & \tau_{32} \end{vmatrix} & \tau_{21} & \tau_{22} \\ \begin{vmatrix} \tau_{11} & \tau_{21} \\ \tau_{12} & \tau_{22} \end{vmatrix} & \tau_{31} & \tau_{32} \end{pmatrix},$$

$$\begin{pmatrix} \tau_{12} & \tau_{13} & \tau_{11} \\ \tau_{22} & \tau_{23} & \tau_{21} \\ \tau_{32} & \tau_{33} & \tau_{31} \end{pmatrix} = \begin{pmatrix} \begin{vmatrix} \tau_{23} & \tau_{33} \\ \tau_{21} & \tau_{31} \end{vmatrix} & \tau_{13} & \tau_{11} \\ -\begin{vmatrix} \tau_{13} & \tau_{33} \\ \tau_{11} & \tau_{31} \end{vmatrix} & \tau_{23} & \tau_{21} \\ \begin{vmatrix} \tau_{13} & \tau_{23} \\ \tau_{11} & \tau_{21} \end{vmatrix} & \tau_{33} & \tau_{31} \end{pmatrix}$$

proving (23).

2. Suppose that we have a matrix τ whose entries satisfy (10) and (11). Then by a direct calculation

$$\begin{aligned} \tau_{11}^2 + \tau_{12}^2 + \tau_{13}^2 &= \tau_{11} \begin{vmatrix} \tau_{22} & \tau_{32} \\ \tau_{23} & \tau_{33} \end{vmatrix} + \tau_{12} \begin{vmatrix} \tau_{23} & \tau_{33} \\ \tau_{21} & \tau_{31} \end{vmatrix} + \tau_{13} \begin{vmatrix} \tau_{21} & \tau_{31} \\ \tau_{22} & \tau_{32} \end{vmatrix} \\ &= \tau_{11} \begin{vmatrix} \tau_{22} & \tau_{32} \\ \tau_{23} & \tau_{33} \end{vmatrix} - \tau_{12} \begin{vmatrix} \tau_{33} & \tau_{23} \\ \tau_{31} & \tau_{21} \end{vmatrix} + \tau_{13} \begin{vmatrix} \tau_{21} & \tau_{31} \\ \tau_{22} & \tau_{32} \end{vmatrix} = \begin{vmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{vmatrix} = 1, \end{aligned}$$

$$\begin{aligned} \tau_{21}^2 + \tau_{22}^2 + \tau_{23}^2 &= -\tau_{21} \begin{vmatrix} \tau_{12} & \tau_{32} \\ \tau_{13} & \tau_{33} \end{vmatrix} - \tau_{22} \begin{vmatrix} \tau_{13} & \tau_{33} \\ \tau_{11} & \tau_{31} \end{vmatrix} - \tau_{23} \begin{vmatrix} \tau_{11} & \tau_{31} \\ \tau_{12} & \tau_{32} \end{vmatrix} \\ &= -\tau_{21} \begin{vmatrix} \tau_{12} & \tau_{13} \\ \tau_{32} & \tau_{33} \end{vmatrix} + \tau_{22} \begin{vmatrix} \tau_{11} & \tau_{13} \\ \tau_{31} & \tau_{33} \end{vmatrix} - \tau_{23} \begin{vmatrix} \tau_{11} & \tau_{12} \\ \tau_{31} & \tau_{32} \end{vmatrix} = \begin{vmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{vmatrix} = 1, \end{aligned}$$

$$\begin{aligned} \tau_{31}^2 + \tau_{32}^2 + \tau_{33}^2 &= \tau_{31} \begin{vmatrix} \tau_{12} & \tau_{22} \\ \tau_{13} & \tau_{23} \end{vmatrix} + \tau_{32} \begin{vmatrix} \tau_{13} & \tau_{23} \\ \tau_{11} & \tau_{21} \end{vmatrix} + \tau_{33} \begin{vmatrix} \tau_{11} & \tau_{21} \\ \tau_{12} & \tau_{22} \end{vmatrix} \\ &= -\begin{vmatrix} \tau_{31} & \tau_{32} & \tau_{33} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{11} & \tau_{12} & \tau_{13} \end{vmatrix} = \begin{vmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{vmatrix} = 1. \end{aligned}$$

Similar calculations yield

$$\begin{aligned} \tau_{11}\tau_{21} + \tau_{12}\tau_{22} + \tau_{13}\tau_{23} &= -\tau_{11} \begin{vmatrix} \tau_{12} & \tau_{32} \\ \tau_{13} & \tau_{33} \end{vmatrix} - \tau_{12} \begin{vmatrix} \tau_{13} & \tau_{33} \\ \tau_{11} & \tau_{31} \end{vmatrix} - \tau_{13} \begin{vmatrix} \tau_{11} & \tau_{31} \\ \tau_{12} & \tau_{32} \end{vmatrix} \\ &= -\tau_{11} \begin{vmatrix} \tau_{12} & \tau_{32} \\ \tau_{13} & \tau_{33} \end{vmatrix} + \tau_{12} \begin{vmatrix} \tau_{33} & \tau_{13} \\ \tau_{31} & \tau_{11} \end{vmatrix} - \tau_{13} \begin{vmatrix} \tau_{11} & \tau_{31} \\ \tau_{12} & \tau_{32} \end{vmatrix} = -\begin{vmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{vmatrix} = 0, \end{aligned}$$

$$\begin{aligned} \tau_{11}\tau_{31} + \tau_{12}\tau_{32} + \tau_{13}\tau_{33} &= \tau_{11} \begin{vmatrix} \tau_{12} & \tau_{22} \\ \tau_{13} & \tau_{23} \end{vmatrix} + \tau_{12} \begin{vmatrix} \tau_{13} & \tau_{23} \\ \tau_{11} & \tau_{21} \end{vmatrix} + \tau_{13} \begin{vmatrix} \tau_{11} & \tau_{21} \\ \tau_{12} & \tau_{22} \end{vmatrix} \\ &= \tau_{11} \begin{vmatrix} \tau_{12} & \tau_{13} \\ \tau_{22} & \tau_{23} \end{vmatrix} - \tau_{12} \begin{vmatrix} \tau_{11} & \tau_{21} \\ \tau_{13} & \tau_{23} \end{vmatrix} + \tau_{13} \begin{vmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{vmatrix} = \begin{vmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \end{vmatrix} = 0, \end{aligned}$$

$$\begin{aligned}\tau_{21}\tau_{31} + \tau_{22}\tau_{32} + \tau_{23}\tau_{33} &= \tau_{21}\begin{vmatrix} \tau_{12} & \tau_{22} \\ \tau_{13} & \tau_{23} \end{vmatrix} + \tau_{22}\begin{vmatrix} \tau_{13} & \tau_{23} \\ \tau_{11} & \tau_{21} \end{vmatrix} + \tau_{23}\begin{vmatrix} \tau_{11} & \tau_{21} \\ \tau_{12} & \tau_{22} \end{vmatrix} \\ &= \tau_{21}\begin{vmatrix} \tau_{12} & \tau_{13} \\ \tau_{22} & \tau_{23} \end{vmatrix} - \tau_{22}\begin{vmatrix} \tau_{11} & \tau_{13} \\ \tau_{21} & \tau_{23} \end{vmatrix} + \tau_{23}\begin{vmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{vmatrix} = \begin{vmatrix} \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \end{vmatrix} = 0.\end{aligned}$$

□

3.4. Traces of Special Orthogonal Matrices

Recall that the *trace* of a matrix

$$\tau = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix}$$

is the number

$$\text{tr } \tau = \tau_{11} + \tau_{22} + \tau_{33}.$$

The real-valued function $\tau \rightarrow \text{tr } \tau$ is the *trace function*.

Set for any two (3×3) -matrices κ and τ

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} = \begin{pmatrix} \kappa_{11} & \kappa_{12} & \kappa_{13} \\ \kappa_{21} & \kappa_{22} & \kappa_{23} \\ \kappa_{31} & \kappa_{32} & \kappa_{33} \end{pmatrix} \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix}.$$

Calculating the trace of this matrix product we easily obtain

$$\begin{aligned}\sigma_{11} + \sigma_{22} + \sigma_{33} &= (\kappa_{11} + \kappa_{22} + \kappa_{33})(\tau_{11} + \tau_{22} + \tau_{33}) \\ &\quad - \begin{vmatrix} \kappa_{11} & \tau_{21} \\ \kappa_{12} & \tau_{22} \end{vmatrix} - \begin{vmatrix} \kappa_{11} & \tau_{12} \\ \kappa_{21} & \tau_{22} \end{vmatrix} - \begin{vmatrix} \kappa_{22} & \tau_{32} \\ \kappa_{23} & \tau_{33} \end{vmatrix} - \begin{vmatrix} \kappa_{22} & \tau_{23} \\ \kappa_{32} & \tau_{33} \end{vmatrix} - \begin{vmatrix} \kappa_{33} & \tau_{31} \\ \kappa_{13} & \tau_{11} \end{vmatrix} - \begin{vmatrix} \kappa_{33} & \tau_{13} \\ \kappa_{31} & \tau_{11} \end{vmatrix} \\ &\quad - \kappa_{11}(\tau_{33} - \tau_{22}) - \kappa_{22}(\tau_{11} - \tau_{33}) - \kappa_{33}(\tau_{22} - \tau_{11}).\end{aligned}\tag{24}$$

In particular,

$$\text{tr}(\kappa \cdot \tau) = \text{tr}(\tau \cdot \kappa).\tag{25}$$

Consequently, for any nonsingular matrix ν the trace of the conjugate matrix $\nu^{-1}\tau\nu$ is

$$\text{tr}(\nu^{-1}\tau\nu) = \text{tr}(\tau\nu \cdot \nu^{-1}) = \text{tr } \tau.\tag{26}$$

This formula says that the trace function is constant on the conjugacy classes of square (3×3) -matrices τ .

Properties (25) and (26) hold for any $(n \times n)$ -matrices κ and τ where n is a positive integer.

If τ is *special orthogonal*, we have the identities (23)

$$\tau_{11} = \begin{vmatrix} \tau_{22} & \tau_{32} \\ \tau_{23} & \tau_{33} \end{vmatrix}, \quad \tau_{22} = -\begin{vmatrix} \tau_{13} & \tau_{33} \\ \tau_{11} & \tau_{31} \end{vmatrix}, \quad \tau_{33} = \begin{vmatrix} \tau_{11} & \tau_{21} \\ \tau_{12} & \tau_{22} \end{vmatrix}$$

showing that the trace $\text{tr } \tau$ can also be calculated from the formula

$$\text{tr } \tau = \tau_{22}\tau_{33} - \tau_{23}\tau_{32} - \tau_{13}\tau_{31} + \tau_{11}\tau_{33} + \tau_{11}\tau_{22} - \tau_{12}\tau_{21}.\tag{27}$$

Lemma 8. For any special orthogonal matrix $\kappa \in \text{SO}(3)$,

$$\text{tr}(\kappa \cdot \kappa) = \text{tr } \kappa \cdot (\text{tr } \kappa - 2).\tag{28}$$

Proof. For $\tau = \kappa$ Formula (24) yields

$$\begin{aligned}
 & \sigma_{11} + \sigma_{22} + \sigma_{33} \\
 &= \kappa_{11}\kappa_{11} + \kappa_{12}\kappa_{21} + \kappa_{13}\kappa_{31} + \kappa_{21}\kappa_{12} + \kappa_{22}\kappa_{22} + \kappa_{23}\kappa_{32} + \kappa_{31}\kappa_{13} + \kappa_{32}\kappa_{23} + \kappa_{33}\kappa_{33} \\
 &= (\kappa_{11} + \kappa_{22} + \kappa_{33})(\kappa_{11} + \kappa_{22} + \kappa_{33}) \\
 &- \begin{vmatrix} \kappa_{11} & \kappa_{21} \\ \kappa_{12} & \kappa_{22} \end{vmatrix} - \begin{vmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{vmatrix} - \begin{vmatrix} \kappa_{22} & \kappa_{32} \\ \kappa_{23} & \kappa_{33} \end{vmatrix} - \begin{vmatrix} \kappa_{22} & \kappa_{23} \\ \kappa_{32} & \kappa_{33} \end{vmatrix} - \begin{vmatrix} \kappa_{33} & \kappa_{31} \\ \kappa_{13} & \kappa_{11} \end{vmatrix} - \begin{vmatrix} \kappa_{33} & \kappa_{13} \\ \kappa_{31} & \kappa_{11} \end{vmatrix} \\
 &= (\kappa_{11} + \kappa_{22} + \kappa_{33})^2 - 2 \begin{vmatrix} \kappa_{11} & \kappa_{21} \\ \kappa_{12} & \kappa_{22} \end{vmatrix} - 2 \begin{vmatrix} \kappa_{22} & \kappa_{32} \\ \kappa_{23} & \kappa_{33} \end{vmatrix} - 2 \begin{vmatrix} \kappa_{33} & \kappa_{31} \\ \kappa_{13} & \kappa_{11} \end{vmatrix}.
 \end{aligned}$$

Substituting for the determinants from Formula (23), we have

$$\begin{aligned}
 & \sigma_{11} + \sigma_{22} + \sigma_{33} \\
 &= (\kappa_{11} + \kappa_{22} + \kappa_{33})^2 - 2(\kappa_{11} + \kappa_{22} + \kappa_{33}) \\
 &= (\kappa_{11} + \kappa_{22} + \kappa_{33})(\kappa_{11} + \kappa_{22} + \kappa_{33} - 2)
 \end{aligned}$$

as required. $\square$

Lemma 9. (a) The trace $\text{tr } \kappa$ of any special orthogonal matrix $\kappa \in \text{SO}(3)$ satisfies

$$-1 \leq \text{tr } \kappa \leq 3.$$

The endpoints -1 and 3 of the interval $[-1, 3]$ are achieved by the special orthogonal matrices

$$\begin{aligned}
 J &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, J_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, J_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
 I &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
 \end{aligned}$$

(b) The trace function $\text{SO}(3) \ni \tau \rightarrow \text{tr } \tau \in [-1, 3] \subset \mathbf{R}$ is surjective.

Proof. (a) Since the product $\kappa \cdot \kappa$ is special orthogonal, then $\text{tr}(\kappa \cdot \kappa) \leq 3$ and Formula (28) implies

$$\text{tr } \kappa \cdot (\text{tr } \kappa - 2) \leq 3.$$

Since $(\text{tr } \kappa)^2 - 2\text{tr } \kappa - 3 = (\text{tr } \kappa - 3)(\text{tr } \kappa + 1)$ then equivalently

$$(\text{tr } \kappa - 3)(\text{tr } \kappa + 1) \leq 0.$$

The case $\text{tr } \kappa - 3 > 0$ has no solution. The same is true if $\text{tr } \kappa + 1 < 0$. The case $\text{tr } \kappa - 3 \leq 0$, $\text{tr } \kappa + 1 \geq 0$ is solved by any κ such that $-1 \leq \text{tr } \kappa \leq 3$.

(b) Given any number $c \in [-1, 3]$ we can always find a special orthogonal matrix τ such that $\text{tr}(\tau) = c$. Indeed, set

$$\tau_c = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{c-1}{2} & -\frac{1}{2}\sqrt{(3-c)(1+c)} \\ 0 & \frac{1}{2}\sqrt{(3-c)(1+c)} & \frac{c-1}{2} \end{pmatrix}.$$

Obviously, this matrix is well defined because by assumption always $(3 - c)(1 + c) \geq 0$. Clearly τ_c satisfies the orthogonality conditions,

$$\frac{(c-1)^2}{4} + \frac{(3-c)(1+c)}{4} = \frac{c^2 - 2c + 1 + 3 + 2c - c^2}{4} = 1,$$

$$\frac{c-1}{2} \cdot \frac{1}{2} \sqrt{(3-c)(1+c)} - \frac{1}{2} \sqrt{(3-c)(1+c)} \frac{c-1}{2} = 0$$

and the determinant condition

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & \frac{c-1}{2} & -\frac{1}{2} \sqrt{(3-c)(1+c)} \\ 0 & \frac{1}{2} \sqrt{(3-c)(1+c)} & \frac{c-1}{2} \end{vmatrix} = \frac{c^2 - 2c + 1 + 3 + 2c - c^2}{4} = 1.$$

Thus, $\tau \in \text{SO}(3)$. The trace of τ is

$$\text{tr} \tau = 1 + \frac{c-1}{2} + \frac{c-1}{2} = 1 + c - 1 = c$$

as required. $\square$

Lemma 10. A special orthogonal matrix κ is symmetric if and only if $\text{tr} \kappa = -1$ or $\text{tr} \kappa = 3$.

Proof. If κ is symmetric, then $\kappa = {}^t \kappa$ hence by (28)

$$\text{tr}(\kappa \cdot \kappa) = \text{tr}(\kappa \cdot \kappa^{-1}) = \text{tr} I = (\text{tr} \kappa)^2 - 2\text{tr} \kappa = 3$$

hence $(\text{tr} \kappa)^2 - 2\text{tr} \kappa - 3 = (\text{tr} \kappa + 1)(\text{tr} \kappa - 3) = 0$. Then necessarily $\text{tr} \kappa = -1$ or $\text{tr} \kappa = 3$.

The inverse follows from Lemma 9, (a). $\square$

3.5. Characteristic Polynomial

In this section we show that the characteristic polynomial of a special orthogonal matrix $\tau \in \text{SO}(3)$

$$\det(\tau - \lambda I) = \begin{vmatrix} \tau_{11} - \lambda & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} - \lambda & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} - \lambda \end{vmatrix} \quad (29)$$

is completely determined by the trace $\text{tr} \tau$ of τ . A partition of special orthogonal matrices is constructed on this basis, which can be considered as a classification of special orthogonal matrices by the trace mapping.

Lemma 11. Any matrix $\tau \in \text{SO}(3)$ satisfies the identity

$$\begin{vmatrix} \tau_{11} - 1 & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} - 1 & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} - 1 \end{vmatrix} = 0. \quad (30)$$

Proof. Determinant (30) can be calculated as follows:

$$\begin{aligned} \det(\tau - I) &= \det \tau \cdot \det(\tau^{-1} - I) = \det {}^t \tau \cdot \det(\tau - I) = \det({}^t \tau \cdot (\tau - I)) \\ &= \det({}^t \tau \cdot \tau - {}^t \tau) = \det(\tau^{-1} \cdot \tau - {}^t \tau) = \det(I - {}^t \tau). \end{aligned}$$

But

$$\begin{aligned} \det(I - {}^t\tau) &= \begin{vmatrix} 1 - \tau_{11} & -\tau_{21} & -\tau_{31} \\ -\tau_{12} & 1 - \tau_{22} & -\tau_{32} \\ -\tau_{13} & -\tau_{23} & 1 - \tau_{33} \end{vmatrix} \\ &= \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} \begin{vmatrix} \tau_{11} - 1 & \tau_{21} & \tau_{31} \\ \tau_{12} & \tau_{22} - 1 & \tau_{32} \\ \tau_{13} & \tau_{23} & \tau_{33} - 1 \end{vmatrix} \\ &= \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} \cdot \begin{vmatrix} \tau_{11} - 1 & \tau_{21} & \tau_{31} \\ \tau_{12} & \tau_{22} - 1 & \tau_{32} \\ \tau_{13} & \tau_{23} & \tau_{33} - 1 \end{vmatrix} = -\det(\tau - I) \end{aligned}$$

hence $\det(\tau - I) = 0$. $\square$

Lemma 11 says that the number $\lambda = 1$ is a *characteristic root* of the matrix τ . To find the other characteristic roots we express the characteristic polynomial (29) explicitly. Standard calculations yield

$$\begin{aligned} &\begin{vmatrix} \tau_{11} - \lambda & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} - \lambda & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} - \lambda \end{vmatrix} = -\lambda^3 + \lambda^2(\tau_{11} + \tau_{22} + \tau_{33}) \\ &\quad + \lambda(-\tau_{11}\tau_{33} + \tau_{13}\tau_{31} - \tau_{11}\tau_{22} + \tau_{12}\tau_{21} - \tau_{22}\tau_{33} + \tau_{32}\tau_{23}) \\ &\quad + \tau_{11}(\tau_{22}\tau_{33} - \tau_{32}\tau_{23}) - \tau_{12}(\tau_{21}\tau_{33} - \tau_{31}\tau_{23}) + \tau_{13}(\tau_{21}\tau_{32} - \tau_{31}\tau_{22}). \end{aligned}$$

But, the last row in this expression is $\det \tau$ and the coefficient at λ is $-\text{tr } \tau$ (27); thus

$$\det(\tau - \lambda I) = -\lambda^3 + \lambda^2 \text{tr } \tau - \lambda \text{tr } \tau + 1.$$

On the other hand, according to Lemma 11, the characteristic polynomial is also expressible as

$$\begin{aligned} \det(\tau - \lambda I) &= (\lambda - 1)(P\lambda^2 + Q\lambda + R) \\ &= P\lambda^3 + \lambda^2(Q - P) + \lambda(R - Q) - R. \end{aligned}$$

Comparing the coefficients we find $P = -1$, $Q - P = \text{tr } \tau$, $R - Q = -\text{tr } \tau$ and $-R = 1$ hence $P = -1$, $R = -1$ and $1 + Q = \text{tr } \tau$ hence

$$\det(\tau - \lambda I) = (\lambda - 1)(-\lambda^2 + (\text{tr } \tau - 1)\lambda - 1).$$

The corresponding *characteristic equation*

$$(\lambda - 1)(-\lambda^2 + (\text{tr } \tau - 1)\lambda - 1) = 0$$

includes the trace $\text{tr } \tau$ as a parameter.

Theorem 4. (a) The image of the trace mapping $\text{tr} : \text{SO}(3) \rightarrow \mathbf{R}$ is the closed interval $[-1, 3] \subset \mathbf{R}$.
(b) The family of subsets of $\text{SO}(3)$ labelled by $c \in [-1, 3]$,

$$\{\tau \in \text{SO}(3) | \text{tr } \tau = c, c \in [-1, 3]\},$$

is a partition of $\text{SO}(3)$ formed by conjugacy classes of special orthogonal matrices

$$\tau_c = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{c-1}{2} & -\frac{1}{2}\sqrt{(3-c)(1+c)} \\ 0 & \frac{1}{2}\sqrt{(3-c)(1+c)} & \frac{c-1}{2} \end{pmatrix} \quad (31)$$

Proof. (a) See Lemma 9.

(b) We shall determine the characteristic roots solving the quadratic equation

$$-\lambda^2 + (\operatorname{tr} \tau - 1)\lambda - 1 = 0. \quad (32)$$

Since in this equation τ is a special orthogonal matrix, it is always supposed that $\operatorname{tr} \tau \in [-1, 3]$ (Lemma 6).

Solutions λ to Equation (32) are classified by its *discriminant*

$$D = (\operatorname{tr} \tau - 1)^2 - 4 = (\operatorname{tr} \tau - 3)(\operatorname{tr} \tau + 1). \quad (33)$$

Consider separately the following three complementary cases: (b1) $D > 0$, (b2) $D = 0$, and (b3) $D < 0$.

(b1) $D > 0$. Then either $\operatorname{tr} \tau > 3$ and $\operatorname{tr} \tau > -1$, or $\operatorname{tr} \tau < 3$ and $\operatorname{tr} \tau < -1$; there are no special orthogonal matrices satisfying these inequalities.

(b2) $D = 0$. In this case either $\operatorname{tr} \tau = 3$ or $\operatorname{tr} \tau = -1$; in both cases $\operatorname{tr} \tau \in [-1, 3]$.

If $\operatorname{tr} \tau = 3$, then τ has a double characteristic root

$$\lambda = -\frac{1 - \operatorname{tr} \tau}{2} = -\frac{1 - 3}{2} = 1.$$

In this case τ belongs to special orthogonal matrices of the form

$$\tau = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix}$$

satisfying an additional condition

$$\tau_{11} + \tau_{22} + \tau_{33} = 3.$$

There is a unique matrix $I \in \operatorname{SO}(3)$ whose trace is maximal (equal to 3), namely

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

If $\operatorname{tr} \tau = -1$, then we have a double characteristic root

$$\lambda = -\frac{1 - \operatorname{tr} \tau}{2} = -\frac{1 + 1}{2} = -1.$$

In this case τ belongs to special orthogonal matrices of the form

$$\tau = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix}$$

satisfying

$$\tau_{11} + \tau_{22} + \tau_{33} = -1.$$

This condition is obviously satisfied by the special orthogonal matrix

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and by all special orthogonal matrices belonging to the *conjugacy class* of J , $\{\tau \in \text{SO}(3) \mid \tau = \kappa J \kappa^{-1}, \kappa \in \text{SO}(3)\}$. Two examples of such matrices are

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(b3) $D < 0$. Then since by (33) $(\text{tr } \tau - 3)(\text{tr } \tau + 1) < 0$, we have either $\text{tr } \tau > 3$, $\text{tr } \tau < -1$ or $\text{tr } \tau < 3$, $\text{tr } \tau > -1$. Thus, condition $D < 0$ implies

$$-1 < \text{tr } \tau < 3$$

and the characteristic polynomial of τ has two complex conjugate roots

$$\lambda_{1,2} = -\frac{1 - \text{tr } \tau}{2} \pm i \frac{\sqrt{(-\text{tr } \tau + 3)(\text{tr } \tau + 1)}}{2}. \quad (34)$$

A matrix $\tau = \tau_c$ whose trace is $D < 0$ can be constructed for any $-1 < c < 3$ as in the proof of Lemma 6, (b). We have

$$\tau_c = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{c-1}{2} & -\frac{1}{2}\sqrt{(3-c)(1+c)} \\ 0 & \frac{1}{2}\sqrt{(3-c)(1+c)} & \frac{c-1}{2} \end{pmatrix}. \quad (35)$$

The set of all matrices τ with characteristic roots (6) is now given as the conjugacy class $\{\tau \in \text{SO}(3) \mid \tau = \kappa \tau_c \kappa^{-1}, \kappa \in \text{SO}(3)\}$.

Note that Formula (7) is also valid for $c = -1$ and $c = 3$. $\square$

Remark 4. The matrix τ_c in Theorem 4 can be replaced by one of the following two matrices

$$\begin{pmatrix} \frac{c-1}{2} & 0 & \frac{1}{2}\sqrt{(3-c)(1+c)} \\ 0 & 1 & 0 \\ -\frac{1}{2}\sqrt{(3-c)(1+c)} & 0 & \frac{c-1}{2} \end{pmatrix}, \begin{pmatrix} \frac{c-1}{2} & -\frac{1}{2}\sqrt{(3-c)(1+c)} & 0 \\ \frac{1}{2}\sqrt{(3-c)(1+c)} & \frac{c-1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (36)$$

Indeed, these matrices are obtained by cyclic permutations of the rows and columns in τ_c which preserve both orthogonality and determinant conditions.

3.6. Stable Points

Since every $\tau \in \text{SO}(3)$ has an eigenvalue equal to 1, equation for stable points $u = (u_1, u_2, u_3)$ of τ in $\mathbf{R}^3$,

$$\tau \cdot u = u, \quad (37)$$

has always a nontrivial solution. Solutions of this equation determine the stable points by explicit formulas.

If τ and κ are conjugate, that is, $\kappa = \nu \tau \nu^{-1}$, then $\tau \cdot u = \tau \nu^{-1} \nu \cdot u = u$ hence

$$\nu \tau \nu^{-1} \cdot \nu u = \nu u. \quad (38)$$

Thus, a stable point u of τ already determines a stable point νu of $\nu \tau \nu^{-1}$. It is therefore sufficient to find solutions to Equation (37) for special orthogonal matrices $\tau = \tau_c$ (31), where $c \in [-1, 3]$,

$$\tau_c = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{c-1}{2} & -\frac{1}{2}\sqrt{(3-c)(1+c)} \\ 0 & \frac{1}{2}\sqrt{(3-c)(1+c)} & \frac{c-1}{2} \end{pmatrix}.$$

In particular,

$$\tau_{(-1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \tau_{(3)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In the following theorem we write

$$v = \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{pmatrix}.$$

Theorem 5. (a) If $c = 3$ then any vector of length 1

$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

is a stable point of τ_3 .

(b) Let $c \in [-1, 3)$. Then for every $v \in \text{SO}(3)$ there is a unique stable eigenvector u of length 1 of the matrix $v\tau_c v^{-1}$. This eigenvector is given by

$$u = \begin{pmatrix} v_{11} \\ v_{21} \\ v_{31} \end{pmatrix}.$$

Proof. (a) Obvious.

(b) Let $c \in [-1, 3)$. For every τ_c the eigenvector w of τ_c is necessarily of the form

$$w = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Now, we apply Formula (2). $\square$

Remark 5. The matrix τ_c in Theorem 5 can be replaced by one of the matrices (36). These replacements will cause some obvious changes in parts (b) and (c) of Theorem 5.

3.7. Canonical Decomposition

In this section the following decomposability problem is considered. Given a special orthogonal matrix $\tau \in \text{SO}(3)$ expressed as

$$\tau = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix},$$

we ask whether there exist three elementary special orthogonal matrices

$$\tau_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_1 & -s_1 \\ 0 & s_1 & c_1 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} c_2 & 0 & s_2 \\ 0 & 1 & 0 \\ -s_2 & 0 & c_2 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} c_3 & -s_3 & 0 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (39)$$

such that

$$\tau = \tau_3 \tau_2 \tau_1. \quad (40)$$

Expression (40) is called the *canonical decomposition* of τ .

Note that the submatrices of the matrices (39)

$$\begin{pmatrix} c_1 & -s_1 \\ s_1 & c_1 \end{pmatrix}, \begin{pmatrix} c_2 & s_2 \\ -s_2 & c_2 \end{pmatrix}, \begin{pmatrix} c_3 & -s_3 \\ s_3 & c_3 \end{pmatrix}$$

are necessarily special orthogonal; thus, by Lemma 4, for constructing decomposition (40) we always suppose that

$$c_1^2 + s_1^2 = 1, c_2^2 + s_2^2 = 1, c_3^2 + s_3^2 = 1. \quad (41)$$

Since the matrix product $\tau_3 \tau_2 \tau_1$ has an expression

$$\begin{pmatrix} c_3 & -s_3 & 0 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_2 & 0 & s_2 \\ 0 & 1 & 0 \\ -s_2 & 0 & c_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_1 & -s_1 \\ 0 & s_1 & c_1 \end{pmatrix} \\ = \begin{pmatrix} c_3 c_2 & c_3 s_2 s_1 - s_3 c_1 & c_3 s_2 c_1 + s_3 s_1 \\ s_3 c_2 & s_3 s_2 s_1 + c_3 c_1 & s_3 s_2 c_1 - c_3 s_1 \\ -s_2 & c_2 s_1 & c_2 c_1 \end{pmatrix},$$

condition (40) induces the *decomposability equation* for the unknowns $c_1, c_2, c_3, s_1, s_2, s_3$

$$\begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix} = \begin{pmatrix} c_3 c_2 & c_3 s_2 s_1 - s_3 c_1 & c_3 s_2 c_1 + s_3 s_1 \\ s_3 c_2 & s_3 s_2 s_1 + c_3 c_1 & s_3 s_2 c_1 - c_3 s_1 \\ -s_2 & c_2 s_1 & c_2 c_1 \end{pmatrix}. \quad (42)$$

The constraints are the orthogonality condition (10) and the determinant condition (11) satisfied by the matrix on the left-hand side and the orthogonality conditions and conditions (41) satisfied by the right-hand side.

The following compatibility lemma is a direct consequence of the group structure of $\text{SO}(3)$. A simple direct proof can also be given.

Lemma 12. *The matrix*

$$\begin{pmatrix} c_3 c_2 & c_3 s_2 s_1 - s_3 c_1 & c_3 s_2 c_1 + s_3 s_1 \\ s_3 c_2 & s_3 s_2 s_1 + c_3 c_1 & s_3 s_2 c_1 - c_3 s_1 \\ -s_2 & c_2 s_1 & c_2 c_1 \end{pmatrix} \quad (43)$$

whose entries satisfy conditions (3), is special orthogonal.

Proof. Orthogonality and determinant conditions (10) and (11) can be verified by straightforward calculation. $\square$

Remark 6. Clearly, if (43) is special orthogonal, then replacing (c_1, s_1) by any of the pairs $(-c_1, s_1)$, $(c_1, -s_1)$, $(-c_1, -s_1)$ we also get a special orthogonal matrix; and analogously for (c_2, s_2) and (c_3, s_3) .

Now we are in a position to find solutions to the decomposability Equation (42).

Theorem 6. (a) *For every $\tau \in \text{SO}(3)$ there exist elementary special orthogonal matrices $\tau_1, \tau_2, \tau_3 \in \text{SO}(3)$ such that*

$$\tau = \tau_3 \tau_2 \tau_1.$$

(b) If $\tau_{31} = 1$, then

$$\tau_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_1 & -s_1 \\ 0 & s_1 & c_1 \end{pmatrix}, \tau_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\tau_3 = \begin{pmatrix} -s_1\tau_{12} + c_1\tau_{22} & c_1\tau_{12} + s_1\tau_{22} & 0 \\ -c_1\tau_{12} - s_1\tau_{22} & -s_1\tau_{12} + c_1\tau_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where τ_1 is arbitrary.

If $\tau_{31} = -1$, then

$$\tau_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_1 & -s_1 \\ 0 & s_1 & c_1 \end{pmatrix}, \tau_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

$$\tau_3 = \begin{pmatrix} s_1\tau_{12} + c_1\tau_{22} & c_1\tau_{12} - s_1\tau_{22} & 0 \\ -c_1\tau_{12} + s_1\tau_{22} & s_1\tau_{12} + c_1\tau_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (44)$$

where τ_1 is arbitrary.

If $-1 < \tau_{31} < 1$, then τ admits exactly 2 decompositions

$$\tau_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\tau_{33}}{\sqrt{1-\tau_{31}^2}} & -\frac{\tau_{32}}{\sqrt{1-\tau_{31}^2}} \\ 0 & \frac{\tau_{32}}{\sqrt{1-\tau_{31}^2}} & \frac{\tau_{33}}{\sqrt{1-\tau_{31}^2}} \end{pmatrix}, \tau_2 = \begin{pmatrix} \sqrt{1-\tau_{31}^2} & 0 & -\tau_{31} \\ 0 & 1 & 0 \\ \tau_{31} & 0 & \sqrt{1-\tau_{31}^2} \end{pmatrix},$$

$$\tau_3 = \begin{pmatrix} \frac{\tau_{11}}{\sqrt{1-\tau_{31}^2}} & -\frac{\tau_{21}}{\sqrt{1-\tau_{31}^2}} & 0 \\ \frac{\tau_{21}}{\sqrt{1-\tau_{31}^2}} & \frac{\tau_{11}}{\sqrt{1-\tau_{31}^2}} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$\tau_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{\tau_{33}}{\sqrt{1-\tau_{31}^2}} & \frac{\tau_{32}}{\sqrt{1-\tau_{31}^2}} \\ 0 & -\frac{\tau_{32}}{\sqrt{1-\tau_{31}^2}} & -\frac{\tau_{33}}{\sqrt{1-\tau_{31}^2}} \end{pmatrix}, \tau_2 = \begin{pmatrix} -\sqrt{1-\tau_{31}^2} & 0 & -\tau_{31} \\ 0 & 1 & 0 \\ \tau_{31} & 0 & -\sqrt{1-\tau_{31}^2} \end{pmatrix},$$

$$\tau_3 = \begin{pmatrix} -\frac{\tau_{11}}{\sqrt{1-\tau_{31}^2}} & \frac{\tau_{21}}{\sqrt{1-\tau_{31}^2}} & 0 \\ -\frac{\tau_{21}}{\sqrt{1-\tau_{31}^2}} & -\frac{\tau_{11}}{\sqrt{1-\tau_{31}^2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Proof. It is sufficient to prove assertion (b). We consider the following three complementary cases (b1) $\tau_{31} = 1$, (b2) $\tau_{31} = -1$ and (b3) $-1 < \tau_{31} < 1$ separately.

(b1) $\tau_{31} = 1$. Orthogonality conditions (2), Section 3.2, yield

$$\begin{aligned} \tau_{11}^2 + \tau_{12}^2 + \tau_{13}^2 &= 1, \\ \tau_{21}^2 + \tau_{22}^2 + \tau_{23}^2 &= 1, \\ \tau_{32}^2 + \tau_{33}^2 &= 0, \\ \tau_{11}\tau_{21} + \tau_{12}\tau_{22} + \tau_{13}\tau_{23} &= 0, \\ \tau_{11} + \tau_{12}\tau_{32} + \tau_{13}\tau_{33} &= 0, \\ \tau_{21} + \tau_{22}\tau_{32} + \tau_{23}\tau_{33} &= 0. \end{aligned}$$

Clearly, we have analogous conditions for the transposed matrix ${}^t\tau$. Then $\tau_{32} = 0$, $\tau_{33} = 0$, $\tau_{21} = 0$, $\tau_{11} = 0$ and the remaining components of τ satisfy

$$\tau_{12}^2 + \tau_{13}^2 = 1, \tau_{22}^2 + \tau_{23}^2 = 1, \tau_{12}\tau_{22} + \tau_{13}\tau_{23} = 0.$$

In particular, the matrix

$$\begin{pmatrix} \tau_{12} & \tau_{13} \\ \tau_{22} & \tau_{23} \end{pmatrix}$$

is special orthogonal and by Lemma 4

$$\begin{pmatrix} \tau_{12} & \tau_{13} \\ \tau_{22} & \tau_{23} \end{pmatrix} = \begin{pmatrix} \tau_{12} & -\tau_{22} \\ \tau_{22} & \tau_{12} \end{pmatrix},$$

where $\tau_{12}^2 + \tau_{22}^2 = 1$. Thus, our assumption $\tau_{31} = 1$ restricts the matrix τ on the left-hand side of decomposition equation (44) to the matrices of the form

$$\tau = \begin{pmatrix} 0 & \tau_{12} & -\tau_{22} \\ 0 & \tau_{22} & \tau_{12} \\ 1 & 0 & 0 \end{pmatrix}, \quad (45)$$

where

$$\tau_{12}^2 + \tau_{22}^2 = 1.$$

For these matrices, the decomposability equation becomes

$$\begin{pmatrix} 0 & \tau_{12} & -\tau_{22} \\ 0 & \tau_{22} & \tau_{12} \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} c_3c_2 & c_3s_2s_1 - s_3c_1 & c_3s_2c_1 + s_3s_1 \\ s_3c_2 & s_3s_2s_1 + c_3c_1 & s_3s_2c_1 - c_3s_1 \\ -s_2 & c_2s_1 & c_2c_1 \end{pmatrix},$$

where it is understood that $\tau_{12}^2 + \tau_{22}^2 = 1$. Solving this system in the set where $c_1^2 + s_1^2 = 1$, $c_2^2 + s_2^2 = 1$, $c_3^2 + s_3^2 = 1$, we have

$$s_2 = -1, c_2 = 0$$

and $\tau_{12} = -c_3s_1 - s_3c_1$, $\tau_{22} = -s_3s_1 + c_3c_1$. In a matrix form

$$\begin{pmatrix} -s_1 & -c_1 \\ c_1 & -s_1 \end{pmatrix} \begin{pmatrix} c_3 \\ s_3 \end{pmatrix} = \begin{pmatrix} \tau_{12} \\ \tau_{22} \end{pmatrix}.$$

Then by orthogonality

$$\begin{pmatrix} -s_1 & c_1 \\ -c_1 & -s_1 \end{pmatrix} \begin{pmatrix} -s_1 & -c_1 \\ c_1 & -s_1 \end{pmatrix} \begin{pmatrix} c_3 \\ s_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_3 \\ s_3 \end{pmatrix} = \begin{pmatrix} c_3 \\ s_3 \end{pmatrix},$$

hence

$$\begin{pmatrix} c_3 \\ s_3 \end{pmatrix} = \begin{pmatrix} -s_1 & c_1 \\ -c_1 & -s_1 \end{pmatrix} \begin{pmatrix} \tau_{12} \\ \tau_{22} \end{pmatrix} = \begin{pmatrix} -s_1\tau_{12} + c_1\tau_{22} \\ -c_1\tau_{12} - s_1\tau_{22} \end{pmatrix},$$

that is, $c_3 = -s_1\tau_{12} + c_1\tau_{22}$ and $s_3 = -c_1\tau_{12} - s_1\tau_{22}$. We conclude that in the matrix τ (45) has a decomposition $\tau = \tau_3\tau_2\tau_1$, where

$$\begin{aligned}\tau_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_1 & -s_1 \\ 0 & s_1 & c_1 \end{pmatrix}, \\ \tau_2 &= \begin{pmatrix} c_2 & 0 & s_2 \\ 0 & 1 & 0 \\ -s_2 & 0 & c_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\ \tau_3 &= \begin{pmatrix} c_3 & -s_3 & 0 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -s_1\tau_{12} + c_1\tau_{22} & -c_1\tau_{12} - s_1\tau_{22} & 0 \\ c_1\tau_{12} + s_1\tau_{22} & -s_1\tau_{12} + c_1\tau_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}.\end{aligned}$$

A direct verification shows that τ_3 is special orthogonal:

$$\begin{aligned}c_1^2\tau_{12}^2 + 2c_1\tau_{12}s_1\tau_{22} + s_1^2\tau_{22}^2 + s_1^2\tau_{12}^2 - 2s_1\tau_{12}c_1\tau_{22} + c_1^2\tau_{22}^2 \\ = (c_1^2 + s_1^2)\tau_{12}^2 + (c_1^2 + s_1^2)\tau_{22}^2 = \tau_{12}^2 + \tau_{22}^2 = 1.\end{aligned}$$

(b2) $\tau_{31} = -1$. In this case orthogonality conditions (10) yield

$$\begin{aligned}\tau_{11}^2 + \tau_{12}^2 + \tau_{13}^2 &= 1, \\ \tau_{21}^2 + \tau_{22}^2 + \tau_{23}^2 &= 1, \\ \tau_{32}^2 + \tau_{33}^2 &= 0, \\ \tau_{11}\tau_{21} + \tau_{12}\tau_{22} + \tau_{13}\tau_{23} &= 0, \\ -\tau_{11} + \tau_{12}\tau_{32} + \tau_{13}\tau_{33} &= 0, \\ -\tau_{21} + \tau_{22}\tau_{32} + \tau_{23}\tau_{33} &= 0.\end{aligned}$$

Then $\tau_{32} = 0$, $\tau_{33} = 0$, $\tau_{21} = 0$, $\tau_{11} = 0$, thus

$$\tau = \begin{pmatrix} 0 & \tau_{12} & \tau_{13} \\ 0 & \tau_{22} & \tau_{23} \\ -1 & 0 & 0 \end{pmatrix}. \quad (46)$$

The remaining components of τ satisfy

$$\tau_{12}^2 + \tau_{13}^2 = 1, \quad \tau_{22}^2 + \tau_{23}^2 = 1, \quad \tau_{12}\tau_{22} + \tau_{13}\tau_{23} = 0,$$

which implies that the upper right corner (2×2) -submatrix is orthogonal. The determinant condition (23) requires

$$\begin{aligned}\begin{vmatrix} 0 & \tau_{12} & \tau_{13} \\ 0 & \tau_{22} & \tau_{23} \\ -1 & 0 & 0 \end{vmatrix} &= -\tau_{12}\begin{vmatrix} 0 & \tau_{23} \\ -1 & 0 \end{vmatrix} + \tau_{13}\begin{vmatrix} 0 & \tau_{22} \\ -1 & 0 \end{vmatrix} \\ &= -\tau_{12}\tau_{23} + \tau_{13}\tau_{22} = -\begin{vmatrix} \tau_{12} & \tau_{13} \\ \tau_{22} & \tau_{23} \end{vmatrix} = 1\end{aligned}$$

that is,

$$\begin{vmatrix} \tau_{13} & \tau_{12} \\ \tau_{23} & \tau_{22} \end{vmatrix} = 1.$$

Consequently, the matrix

$$\begin{pmatrix} \tau_{13} & \tau_{12} \\ \tau_{23} & \tau_{22} \end{pmatrix}$$

is special orthogonal hence by Lemma 4

$$\begin{pmatrix} \tau_{13} & \tau_{12} \\ \tau_{23} & \tau_{22} \end{pmatrix} = \begin{pmatrix} \tau_{13} & -\tau_{23} \\ \tau_{23} & \tau_{13} \end{pmatrix}$$

and $\tau_{13}^2 + \tau_{23}^2 = 1$. Substitution $\tau_{12} = -\tau_{23}$ and $\tau_{22} = \tau_{13}$ in (46) yields

$$\tau = \begin{pmatrix} 0 & \tau_{12} & \tau_{22} \\ 0 & \tau_{22} & -\tau_{12} \\ -1 & 0 & 0 \end{pmatrix}. \quad (47)$$

Summarizing, we can say that our assumption (b2) $\tau_{31} = -1$ restricts the special orthogonal matrices τ on the left-hand side of decomposition Equation (42) to matrices of the form (47), where

$$\tau_{12}^2 + \tau_{22}^2 = 1.$$

For these matrices, the decomposability Equation (44) reads

$$\begin{pmatrix} 0 & \tau_{12} & \tau_{22} \\ 0 & \tau_{22} & -\tau_{12} \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} c_3 c_2 & c_3 s_2 s_1 - s_3 c_1 & c_3 s_2 c_1 + s_3 s_1 \\ s_3 c_2 & s_3 s_2 s_1 + c_3 c_1 & s_3 s_2 c_1 - c_3 s_1 \\ -s_2 & c_2 s_1 & c_2 c_1 \end{pmatrix},$$

where it is understood that

$$\tau_{12}^2 + \tau_{22}^2 = 1.$$

Solving this system in the set where $c_1^2 + s_1^2 = 1$, $c_2^2 + s_2^2 = 1$, $c_3^2 + s_3^2 = 1$, we have

$$s_2 = 1, c_2 = 0$$

and $\tau_{12} = c_3 s_1 - s_3 c_1$, $\tau_{22} = c_3 c_1 + s_3 s_1$. In a matrix form

$$\begin{pmatrix} s_1 & -c_1 \\ c_1 & s_1 \end{pmatrix} \begin{pmatrix} c_3 \\ s_3 \end{pmatrix} = \begin{pmatrix} \tau_{12} \\ \tau_{22} \end{pmatrix}.$$

Then by orthogonality

$$\begin{pmatrix} s_1 & c_1 \\ -c_1 & s_1 \end{pmatrix} \begin{pmatrix} s_1 & -c_1 \\ c_1 & s_1 \end{pmatrix} \begin{pmatrix} c_3 \\ s_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_3 \\ s_3 \end{pmatrix} = \begin{pmatrix} c_3 \\ s_3 \end{pmatrix},$$

hence

$$\begin{pmatrix} c_3 \\ s_3 \end{pmatrix} = \begin{pmatrix} s_1 & c_1 \\ -c_1 & s_1 \end{pmatrix} \begin{pmatrix} \tau_{12} \\ \tau_{22} \end{pmatrix} = \begin{pmatrix} s_1 \tau_{12} + c_1 \tau_{22} \\ -c_1 \tau_{12} + s_1 \tau_{22} \end{pmatrix},$$

that is, $c_3 = s_1 \tau_{12} + c_1 \tau_{22}$ and $s_3 = -c_1 \tau_{12} + s_1 \tau_{22}$. We conclude that in this case $\tau_{31} = -1$ the matrix τ (47) has a decomposition $\tau = \tau_3 \tau_2 \tau_1$, where

$$\tau_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_1 & -s_1 \\ 0 & s_1 & c_1 \end{pmatrix},$$

$$\tau_2 = \begin{pmatrix} c_2 & 0 & s_2 \\ 0 & 1 & 0 \\ -s_2 & 0 & c_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

$$\tau_3 = \begin{pmatrix} c_3 & -s_3 & 0 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} s_1 \tau_{12} + c_1 \tau_{22} & c_1 \tau_{12} - s_1 \tau_{22} & 0 \\ -c_1 \tau_{12} + s_1 \tau_{22} & s_1 \tau_{12} + c_1 \tau_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

A direct verification shows that τ_3 is special orthogonal:

$$\begin{aligned} s_1^2 \tau_{12}^2 + 2s_1 \tau_{12} c_1 \tau_{22} + c_1^2 \tau_{22}^2 + c_1^2 \tau_{12}^2 - 2c_1 \tau_{12} s_1 \tau_{22} + s_1^2 \tau_{22}^2 \\ = s_1^2 \tau_{12}^2 + c_1^2 \tau_{22}^2 + c_1^2 \tau_{12}^2 + s_1^2 \tau_{22}^2 = 1. \end{aligned}$$

(b3) $-1 < \tau_{31} < 1$. Equation (44) now implies

$$s_2 = -\tau_{31}$$

and by (41) c_2 is one of the following two non-zero numbers

$$c_2 = \pm \sqrt{1 - s_2^2} = \pm \sqrt{1 - \tau_{31}^2}.$$

Then again, from (41)

$$c_3 = \frac{\tau_{11}}{c_2}, s_3 = \frac{\tau_{21}}{c_2}, s_1 = \frac{\tau_{32}}{c_2}, c_1 = \frac{\tau_{33}}{c_2}.$$

Thus, we get two solutions:

$$\begin{aligned} & (c_1, s_1, c_2, s_2, c_3, s_3) \\ & = \left(\frac{\tau_{33}}{\sqrt{1 - \tau_{31}^2}}, \frac{\tau_{32}}{\sqrt{1 - \tau_{31}^2}}, \sqrt{1 - \tau_{31}^2}, -\tau_{31}, \frac{\tau_{11}}{\sqrt{1 - \tau_{31}^2}}, \frac{\tau_{21}}{\sqrt{1 - \tau_{31}^2}} \right) \end{aligned}$$

and

$$\begin{aligned} & (c_1, s_1, c_2, s_2, c_3, s_3) \\ & = \left(-\frac{\tau_{33}}{\sqrt{1 - \tau_{31}^2}}, -\frac{\tau_{32}}{\sqrt{1 - \tau_{31}^2}}, -\sqrt{1 - \tau_{31}^2}, -\tau_{31}, -\frac{\tau_{11}}{\sqrt{1 - \tau_{31}^2}}, -\frac{\tau_{21}}{\sqrt{1 - \tau_{31}^2}} \right). \end{aligned}$$

The corresponding τ_1, τ_2, τ_3 are given by

$$\tau_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_1 & -s_1 \\ 0 & s_1 & c_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\tau_{33}}{\sqrt{1 - \tau_{31}^2}} & -\frac{\tau_{32}}{\sqrt{1 - \tau_{31}^2}} \\ 0 & \frac{\tau_{32}}{\sqrt{1 - \tau_{31}^2}} & \frac{\tau_{33}}{\sqrt{1 - \tau_{31}^2}} \end{pmatrix},$$

$$\tau_2 = \begin{pmatrix} c_2 & 0 & s_2 \\ 0 & 1 & 0 \\ -s_2 & 0 & c_2 \end{pmatrix} = \begin{pmatrix} \sqrt{1 - \tau_{31}^2} & 0 & -\tau_{31} \\ 0 & 1 & 0 \\ \tau_{31} & 0 & \sqrt{1 - \tau_{31}^2} \end{pmatrix},$$

$$\tau_3 = \begin{pmatrix} c_3 & -s_3 & 0 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\tau_{11}}{\sqrt{1 - \tau_{31}^2}} & -\frac{\tau_{21}}{\sqrt{1 - \tau_{31}^2}} & 0 \\ \frac{\tau_{21}}{\sqrt{1 - \tau_{31}^2}} & \frac{\tau_{11}}{\sqrt{1 - \tau_{31}^2}} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

or

$$\tau_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_1 & -s_1 \\ 0 & s_1 & c_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{\tau_{33}}{\sqrt{1 - \tau_{31}^2}} & \frac{\tau_{32}}{\sqrt{1 - \tau_{31}^2}} \\ 0 & -\frac{\tau_{32}}{\sqrt{1 - \tau_{31}^2}} & -\frac{\tau_{33}}{\sqrt{1 - \tau_{31}^2}} \end{pmatrix},$$

$$\tau_2 = \begin{pmatrix} c_2 & 0 & s_2 \\ 0 & 1 & 0 \\ -s_2 & 0 & c_2 \end{pmatrix} = \begin{pmatrix} -\sqrt{1-\tau_{31}^2} & 0 & -\tau_{31} \\ 0 & 1 & 0 \\ \tau_{31} & 0 & -\sqrt{1-\tau_{31}^2} \end{pmatrix},$$

$$\tau_3 = \begin{pmatrix} c_3 & -s_3 & 0 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{\tau_{11}}{\sqrt{1-\tau_{31}^2}} & \frac{\tau_{21}}{\sqrt{1-\tau_{31}^2}} & 0 \\ -\frac{\tau_{21}}{\sqrt{1-\tau_{31}^2}} & -\frac{\tau_{11}}{\sqrt{1-\tau_{31}^2}} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

respectively. This concludes the proof of (b3). $\square$

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