Article

A Blow-Up Criterion for the Density-Dependent Incompressible Magnetohydrodynamic System with Zero Viscosity

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Abstract: In this paper, we provide a blow-up criterion for the density-dependent incompressible magnetohydrodynamic system with zero viscosity. The proof uses the L^p-method and the Kato–Ponce inequalities in the harmonic analysis. The novelty of our work lies in the fact that we deal with the case in which the resistivity η is positive.

Keywords: magnetohydrodynamic system; incompressible; blow-up criterion

MSC: 35Q35; 76D03

1. Introduction

Magnetohydrodynamics (MHD) is concerned with the study of applications between magnetic fields and fluid conductors of electricity. The application of magnetohydrodynamics covers a very wide range of physical objects, from liquid metals to cosmic plasmas.

We consider the following 3D density-dependent incompressible magnetohydrodynamic system:

$$\partial_t \rho + u \cdot \nabla \rho = 0, \tag{1}$$

$$\rho \partial_t u + \rho(u \cdot \nabla)u + \nabla \pi = \operatorname{rot} b \times b, \tag{2}$$

$$\partial_t b + u \cdot \nabla b - b \cdot \nabla u = \eta \Delta b, \tag{3}$$

$$\operatorname{div} u = 0, \quad \operatorname{div} b = 0 \quad \text{in } \mathbb{R}^3 \times (0, \infty), \tag{4}$$

$$\lim_{|x| \to \infty} (\rho, u, b) = (1, 0, 0), \tag{5}$$

$$(\rho, u, b)(\cdot, 0) = (\rho_0, u_0, b_0) \quad \text{in } \mathbb{R}^3. \tag{6}$$

The unknowns are the fluid velocity field $u = u(x,t)$, the pressure $\pi = \pi(x,t)$, the density $\rho = \rho(x,t)$, and the magnetic field $b = b(x,t)$. $\eta > 0$ is the resistivity coefficient. The term $\operatorname{rot} b \times b$ in (2) is the Lorentz force with low regularity, and thus it is the difficult term.

For the case of $b = 0$, there are many studies. Beirão da Veiga and Valli [1,2] and Valli and Zajaczkowski [3] proved the unique solvability, local in time, in some supercritical Sobolev spaces and Hölder spaces in bounded domains. It is worth pointing out that, in 1995, Berselli [4] discussed the standard ideal flow. Danchin [5] and Danchin and Fanelli [6] (see also [7,8]) proved the unique solvability, local in time, in some critical Besov spaces. Recently, Bae et al [9] showed a regularity criterion:

$$\nabla u \in L^1(0, T; L^\infty(\mathbb{R}^3)). \tag{7}$$

This refined the previous blow-up criteria [5–7]:

$$\omega := \operatorname{rot} u \in L^1(0, T; \dot{B}_{2,1}^{\frac{d}{2}}(\mathbb{R}^d)), \tag{8}$$

$$\nabla u \in L^1(0, T; L^\infty) \quad \text{and} \quad \nabla \pi \in L^1(0, T; B_{\infty,r}^{s-1}), s \geq 1, 1 \leq r \leq \infty. \tag{9}$$

In [10], the authors proved the local well-posedness of smooth solutions in Sobolev spaces. The aim of this article is to prove (7) for the system (1)–(6). We will prove the following.

Theorem 1. *Let $0 < \inf \rho_0 \le \rho_0 \le C, \nabla \rho_0 \in H^2, u_0, b_0 \in H^3$ with $\mathrm{div}\, u_0 = \mathrm{div}\, b_0 = 0$ in $\mathbb{R}^3$. Let (ρ, u, b) be the unique solution to the problem (1)–(6). If (7) holds true with some $0 < T < \infty$, then the solution (ρ, u, b) can be extended beyond $T > 0$.*

Remark 1. *In [8], Zhou, Fan and Xin showed the same blow-up criterion (8), which is refined by (7) for the ideal MHD system.*

Remark 2. *When $\eta = 0$, we are unable to show a similar result.*

In the following proofs, we will use the bilinear commutator and product estimates due to Kato–Ponce [11]:

$$\|\Lambda^s(fg) - f\Lambda^s g\|_{L^p} \le C(\|\nabla f\|_{L^{p_1}}\|\Lambda^{s-1}g\|_{L^{q_1}} + \|g\|_{L^{p_2}}\|\Lambda^s f\|_{L^{q_2}}), \tag{10}$$

$$\|\Lambda^s(fg)\|_{L^p} \le C(\|f\|_{L^{p_1}}\|\Lambda^s g\|_{L^{q_1}} + \|\Lambda^s f\|_{L^{p_2}}\|g\|_{L^{q_2}}), \tag{11}$$

with $s > 0, \Lambda := (-\Delta)^{\frac{1}{2}}$ and $\dfrac{1}{p} = \dfrac{1}{p_1} + \dfrac{1}{q_1} = \dfrac{1}{p_2} + \dfrac{1}{q_2}$.

2. Proof of Theorem 1

We only need to prove a priori estimates.

First, thanks to the maximum principle, it is easy to see that

$$\frac{1}{C} \le \rho \le C. \tag{12}$$

We will use the identity

$$b \cdot \nabla b + b \times \mathrm{rot}\, b = \frac{1}{2}\nabla |b|^2. \tag{13}$$

Testing (2) by u, using (1), (4) and (13), we find that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho |u|^2 \mathrm{d}x = \int (b \cdot \nabla)b \cdot u \mathrm{d}x. \tag{14}$$

Testing (3) by b and using (4), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |b|^2 \mathrm{d}x + \eta \int |\nabla b|^2 \mathrm{d}x = \int (b \cdot \nabla)u \cdot b \mathrm{d}x. \tag{15}$$

Summing up (14) and (15), we have the well-known energy identity

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int (\rho |u|^2 + |b|^2)\mathrm{d}x + \eta \int |\nabla b|^2 \mathrm{d}x = 0,$$

and hence

$$\int (|u|^2 + |b|^2)\mathrm{d}x + \int_0^T \int |\nabla b|^2 \mathrm{d}x \mathrm{d}t \le C. \tag{16}$$

It is easy to deduce that

$$\|\nabla \rho\|_{L^\infty} \le \|\nabla \rho_0\|_{L^\infty} \exp\left(\int_0^t \|\nabla u(s)\|_{L^\infty}\mathrm{d}s\right) \le C. \tag{17}$$

Testing (3) by $|b|^{q-2}b$ $(2 < q < \infty)$ and using (4), we derive

$$\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\|b\|_{L^q}^q + \eta\int|b|^{q-2}|\nabla b|^2\mathrm{d}x + \eta\int\frac{1}{2}\nabla|b|^2 \cdot \nabla|b|^{q-2}\mathrm{d}x$$

$$= \int b \cdot \nabla u \cdot |b|^{q-2}b\mathrm{d}x \le \|\nabla u\|_{L^\infty}\|b\|_{L^q}^q,$$

and therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\|b\|_{L^q} \le \|\nabla u\|_{L^\infty}\|b\|_{L^q},$$

which gives

$$\|b\|_{L^q} \le \|b_0\|_{L^q}\exp\left(\int_0^t\|\nabla u(s)\|_{L^\infty}\mathrm{d}s\right). \tag{18}$$

Taking $q \to \infty$, one has

$$\|b\|_{L^\infty} \le C. \tag{19}$$

(2) can be rewritten as

$$\partial_t u + u \cdot \nabla u + \frac{1}{\rho}\nabla\pi = \frac{1}{\rho}\mathrm{rot}\, b \times b. \tag{20}$$

Testing (20) by $\nabla\pi$ and using (4), it follows that

$$\int\frac{1}{\rho}|\nabla\pi|^2\mathrm{d}x = -\int u \cdot \nabla u \cdot \nabla\pi\mathrm{d}x + \int\left(\frac{1}{\rho}\mathrm{rot}\, b \times b\right)\cdot\nabla\pi\mathrm{d}x$$

$$\le C(\|u \cdot \nabla u\|_{L^2} + \|\mathrm{rot}\, b \times b\|_{L^2})\|\nabla\pi\|_{L^2}$$

$$\le C(\|u\|_{L^2}\|\nabla u\|_{L^\infty} + \|b\|_{L^\infty}\|\nabla b\|_{L^2})\|\nabla\pi\|_{L^2},$$

whence

$$\|\nabla\pi\|_{L^2} \le C\|\nabla u\|_{L^\infty} + C\|\nabla b\|_{L^2}. \tag{21}$$

Taking rot to (20) and denoting the vorticity $\omega := \mathrm{rot}\, u$, we obtain

$$\partial_t\omega + u \cdot \nabla\omega = \omega \cdot \nabla u - \nabla\frac{1}{\rho}\times\nabla\pi + \mathrm{rot}\left(\frac{\mathrm{rot}\, b}{\rho}\times b\right). \tag{22}$$

Testing (22) by ω and using (4), (17) and (19), we compute

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\omega|^2\mathrm{d}x$$

$$= \int\omega \cdot \nabla u \cdot \omega\mathrm{d}x - \int\left(\nabla\frac{1}{\rho}\times\nabla\pi\right)\omega\mathrm{d}x + \int\mathrm{rot}\left(\frac{\mathrm{rot}\, b}{\rho}\times b\right)\cdot\omega\mathrm{d}x$$

$$\le \|\nabla u\|_{L^\infty}\int|\omega|^2\mathrm{d}x + \left\|\nabla\frac{1}{\rho}\right\|\|\nabla\pi\|_{L^2}\|\omega\|_{L^2}$$

$$+ C\|\nabla\rho\|_{L^\infty}\|b\|_{L^\infty}\|\nabla b\|_{L^2}\|\omega\|_{L^2} + C\|b\|_{L^\infty}\|\Delta b\|_{L^2}\|\omega\|_{L^2} + C\|\nabla b\|_{L^4}^2\|\omega\|_{L^2}$$

$$\le C\|\nabla u\|_{L^\infty}\int|\omega|^2\mathrm{d}x + C\|\nabla\pi\|_{L^2}\|\omega\|_{L^2} + C\|\nabla b\|_{L^2}\|\omega\|_{L^2}$$

$$+ \frac{\eta}{4}\|\Delta b\|_{L^2}^2 + C\|\omega\|_{L^2}^2. \tag{23}$$

Here, we have used the Gagliardo–Nirenberg inequality

$$\|\nabla b\|_{L^4}^2 \le C\|b\|_{L^\infty}\|\Delta b\|_{L^2}. \tag{24}$$

On the other hand, testing (3) by $-\Delta b$ and using (4) and (19), we achieve

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}&\int|\nabla b|^2\mathrm{d}x + \eta\int|\Delta b|^2\mathrm{d}x\\
&= \sum_j\int(u\cdot\nabla)b\partial_j^2 b\,\mathrm{d}x - \int(b\cdot\nabla)u\cdot\Delta b\,\mathrm{d}x\\
&= -\sum_j\int\partial_j u\cdot\nabla b\cdot\partial_j b\,\mathrm{d}x - \int b\cdot\nabla u\cdot\Delta b\,\mathrm{d}x\\
&\leq C\|\nabla u\|_{L^\infty}\|\nabla b\|_{L^2}^2 + \|b\|_{L^\infty}\|\nabla u\|_{L^2}\|\Delta b\|_{L^2}\\
&\leq \frac{\eta}{4}\|\Delta b\|_{L^2}^2 + C\|\nabla u\|_{L^\infty}\|\nabla b\|_{L^2}^2 + C\|\omega\|_{L^2}^2.
\end{aligned}
\tag{25}
$$

Here, we have used the inequality

$$
\|\nabla u\|_{L^r} \leq C\|\omega\|_{L^r} \text{ with } 1 < r < \infty.
\tag{26}
$$

Summing up (23) and (25) and using (21) and the Gronwall inequality, we reach

$$
\|\omega\|_{L^2} + \|\nabla b\|_{L^2} \leq C.
\tag{27}
$$

Taking div to (20) and using (4), we observe that

$$
-\Delta\pi = f := \rho\,\mathrm{div}\,(u\cdot\nabla u) + \rho\nabla\frac{1}{\rho}\cdot\nabla\pi - \rho\,\mathrm{div}\left(\frac{\mathrm{rot}\,b}{\rho}\times b\right),
\tag{28}
$$

from which, with (17), (19), (21) and (27), we have

$$
\begin{aligned}
\|\Delta\pi\|_{L^4} &\leq \|f\|_{L^4} \leq C\|\nabla u\|_{L^\infty}\|\nabla u\|_{L^4} + C\|\nabla\rho\|_{L^\infty}\|\nabla\pi\|_{L^4}\\
&\quad + C\|\nabla\rho\|_{L^\infty}\|b\|_{L^\infty}\|\mathrm{rot}\,b\|_{L^4} + C\|\Delta b\|_{L^4}\|b\|_{L^\infty} + C\|\nabla b\|_{L^8}^2\\
&\leq C\|\nabla u\|_{L^\infty}\|\omega\|_{L^4} + C\|\nabla\pi\|_{L^4} + C\|\mathrm{rot}\,b\|_{L^4} + C\|\Delta b\|_{L^4}\\
&\leq C\|\nabla u\|_{L^\infty}\|\omega\|_{L^4} + C\|\nabla\pi\|_{L^2}^{\frac{4}{7}}\|\Delta\pi\|_{L^4}^{\frac{3}{7}} + C\|\mathrm{rot}\,b\|_{L^4} + C\|\Delta b\|_{L^4}\\
&\leq \frac{1}{2}\|\Delta\pi\|_{L^4} + C\|\nabla u\|_{L^\infty}\|\omega\|_{L^4} + C\|\nabla u\|_{L^\infty} + C + C\|\mathrm{rot}\,b\|_{L^4} + C\|\Delta b\|_{L^4},
\end{aligned}
$$

which yields

$$
\|\Delta\pi\|_{L^4} \leq C\|\nabla u\|_{L^\infty}\|\omega\|_{L^4} + C\|\nabla u\|_{L^\infty} + C + C\|\mathrm{rot}\,b\|_{L^4} + C|\Delta b\|_{L^4}.
\tag{29}
$$

Here, we have used the Gagliardo–Nirenberg inequalities

$$
\begin{aligned}
\|\nabla b\|_{L^8}^2 &\leq C\|b\|_{L^\infty}\|\Delta b\|_{L^4},\\
\|\nabla\pi\|_{L^4} &\leq C\|\nabla\pi\|_{L^2}^{\frac{4}{7}}\|\Delta\pi\|_{L^4}^{\frac{3}{7}}.
\end{aligned}
\tag{30}
\tag{31}
$$

Testing (22) by $|\omega|^2\omega$ and using (4), (17), (19), (29), (30) and (31), we have

$$
\begin{aligned}
\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_{L^4}^4 &\leq C\|\nabla u\|_{L^\infty}\|\omega\|_{L^4}^4 + C\left\|\nabla\frac{1}{\rho}\right\|_{L^\infty}\|\nabla\pi\|_{L^4}\|\omega\|_{L^4}^3\\
&\quad + C(\|b\|_{L^\infty}\|\Delta b\|_{L^4} + \|\nabla b\|_{L^8}^2)\|\omega\|_{L^4}^3 + C\left\|\nabla\frac{1}{\rho}\right\|_{L^\infty}\|b\|_{L^\infty}\|\mathrm{rot}\,b\|_{L^4}\|\omega\|_{L^4}^3\\
&\leq C\|\nabla u\|_{L^\infty}\|\omega\|_{L^4}^4 + C\|\nabla\pi\|_{L^4}\|\omega\|_{L^4}^3 + C(\|\Delta b\|_{L^4} + \|\mathrm{rot}\,b\|_{L^4})\|\omega\|_{L^4}^3,
\end{aligned}
$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_{L^4}^2 \leq C\|\nabla u\|_{L^\infty}\|\omega\|_{L^4}^2 + C(\|\nabla u\|_{L^\infty} + 1 + \|\mathrm{rot}\,b\|_{L^4} + \|\Delta b\|_{L^4})\|\omega\|_{L^4}$$

$$\leq C\|\nabla u\|_{L^\infty}\|\omega\|_{L^4}^2 + C(\|\nabla u\|_{L^\infty} + 1 + \|\mathrm{rot}\,b\|_{L^4})\|\omega\|_{L^4} + C\|\omega\|_{L^4}^2 + C\|\Delta b\|_{L^4}^2,$$

and thus

$$\begin{aligned}
\|\omega\|_{L^4}^2 &\leq \|\omega_0\|_{L^4}^2 + C\int_0^t [(\|\nabla u\|_{L^\infty} + 1)\|\omega\|_{L^4}^2 \\
&\quad + (\|\nabla u\|_{L^\infty} + 1 + \|\mathrm{rot}\,b\|_{L^4})\|\omega\|_{L^4}]\mathrm{d}s + \int_0^t \|\Delta b\|_{L^4}^2 \mathrm{d}s.
\end{aligned} \tag{32}$$

On the other hand, using the $L^2(0, T; W^{2,4})$-theory of the heat equation, it follows that

$$\begin{aligned}
\int_0^t \|\Delta b\|_{L^4}^2 \mathrm{d}s &\leq C + C\int_0^t \|u \cdot \nabla b - b \cdot \nabla u\|_{L^4}^2 \mathrm{d}s \\
&\leq C + C\int_0^t (\|u\|_{L^\infty}^2 \|\nabla b\|_{L^4}^2 + \|b\|_{L^\infty}^2 \|\nabla u\|_{L^4}^2)\mathrm{d}s \\
&\leq C + C\int_0^t (\|u\|_{L^6}^{\frac{4}{3}} \|\nabla u\|_{L^\infty}^{\frac{2}{3}} \|\mathrm{rot}\,b\|_{L^4}^2 + \|\omega\|_{L^4}^2)\mathrm{d}s \\
&\leq C + C\int_0^t (\|\nabla u\|_{L^\infty}^{\frac{2}{3}} \|\mathrm{rot}\,b\|_{L^4}^2 + \|\omega\|_{L^4}^2)\mathrm{d}s.
\end{aligned} \tag{33}$$

Here, we have used the inequality

$$\|\nabla b\|_{L^p} \leq C\|\mathrm{rot}\,b\|_{L^p} \quad \text{with } 1 < p < \infty. \tag{34}$$

Inserting (33) into (32), we have

$$\begin{aligned}
\|\omega\|_{L^4}^2 &\leq C + C\int_0^t (\|\nabla u\|_{L^\infty} +)\|\omega\|_{L^4}^2 \mathrm{d}s + C\int_0^t (\|\nabla u\|_{L^\infty} + \|\mathrm{rot}\,b\|_{L^4})\|\omega\|_{L^4} \mathrm{d}s \\
&\quad + C\int_0^t \|\nabla u\|_{L^\infty}^{\frac{2}{3}} \|\mathrm{rot}\,b\|_{L^4}^2 \mathrm{d}s.
\end{aligned} \tag{35}$$

Taking rot to (3) and denoting the current $J := \mathrm{rot}\,b$, we infer that

$$\partial_t J - \eta \Delta J + \mathrm{rot}\,(u \cdot \nabla b - b \cdot \nabla u) = 0. \tag{36}$$

Testing (36) by $|J|^2 J$, using (17), (19) and (34), we derive

$$\begin{aligned}
&\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\|J\|_{L^4}^4 + \eta \int |J|^2 |\nabla J|^2 \mathrm{d}x + \eta \int \frac{1}{2}\nabla|J|^2 \cdot \nabla|J|^2 \mathrm{d}x \\
&= \int (b \cdot \nabla u - u \cdot \nabla b)\mathrm{rot}\,(|J|^2 J)\mathrm{d}x \\
&= \int (b \cdot \nabla u - u \cdot \nabla b)(|J|^2 \mathrm{rot}\,J + \nabla|J|^2 \times J)\mathrm{d}x \\
&\leq \|b \cdot \nabla u - u \cdot \nabla b\|_{L^4}\|J\|_{L^4}\||J| \cdot |\nabla J|\|_{L^2} \\
&\leq C(\|b\|_{L^\infty}\|\nabla u\|_{L^4} + \|u\|_{L^\infty}\|\nabla b\|_{L^4})\|J\|_{L^4}\||J| \cdot |\nabla J|\|_{L^2} \\
&\leq \frac{\eta}{2}\int |J|^2 |\nabla J|^2 \mathrm{d}x + C(\|\omega\|_{L^4}^2 + \|u\|_{L^\infty}^2\|J\|_{L^4}^2)\|J\|_{L^4}^2,
\end{aligned}$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\|J\|_{L^4}^2 \leq C\|\omega\|_{L^4}^2 + C\|\nabla u\|_{L^\infty}^{\frac{2}{3}}\|J\|_{L^4}^2.$$

Integrating the above inequality, one has

$$\|\operatorname{rot} b\|_{L^4}^2 \leq C + C \int_0^t (\|\omega\|_{L^4}^2 + \|\nabla u\|_{L^\infty}^{\frac{2}{3}} \|\operatorname{rot} b\|_{L^4}^2) \, ds. \tag{37}$$

Summing up (35) and (37), using the Gronwall inequality, we arrive at

$$\|\nabla u\|_{L^4} + \|\nabla b\|_{L^4} + \int_0^T \|\Delta b\|_{L^4}^2 \, dt \leq C, \tag{38}$$

$$\|u\|_{L^\infty} + \int_0^T \|\nabla b\|_{L^\infty}^2 \, dt \leq C. \tag{39}$$

Applying Λ^3 to (1), testing by $\Lambda^3 \rho$ and using (4) and (10), we obtain

$$\frac{1}{2} \frac{d}{dt} \int (\Lambda^3 \rho)^2 dx = - \int (\Lambda^3 (u \cdot \nabla \rho) - u \cdot \nabla \Lambda^3 \rho) \Lambda^3 \rho \, dx$$
$$\leq C(\|\nabla u\|_{L^\infty} \|\Lambda^3 \rho\|_{L^2} + \|\nabla \rho\|_{L^\infty} \|\Lambda^3 u\|_{L^2}) \|\Lambda^3 \rho\|_{L^2}. \tag{40}$$

Applying Λ^3 to (2), testing by $\Lambda^3 u$ and using (1) and (4), we have

$$\frac{1}{2} \frac{d}{dt} \int \rho |\Lambda^3 u|^2 dx = \int (\Lambda^3 (b \cdot \nabla b) - b \cdot \nabla \Lambda^3 b) \Lambda^3 u \, dx + \int b \cdot \nabla \Lambda^3 b \cdot \Lambda^3 u \, dx$$
$$- \int (\Lambda^3 (\rho \partial_t u) - \rho \Lambda^3 \partial_t u) \Lambda^3 u \, dx - \int (\Lambda^3 (\rho u \cdot \nabla u) - \rho u \cdot \nabla \Lambda^3 u) \Lambda^3 u \, dx$$
$$=: I_1 + I_2 + I_3 + I_4. \tag{41}$$

Applying Λ^3 to (3), testing by $\Lambda^3 b$ and using (4), we have

$$\frac{1}{2} \frac{d}{dt} \int |\Lambda^3 b|^2 dx + \eta \int |\Lambda^4 b|^2 dx$$
$$= \int (\Lambda^3 (b \cdot \nabla u) - b \cdot \nabla \Lambda^3 u) \Lambda^3 b \, dx + \int b \cdot \Lambda^3 u \cdot \Lambda^3 b \, dx$$
$$- \int (\Lambda^3 (u \cdot \nabla b) - u \cdot \nabla \Lambda^3 b) \Lambda^3 b \, dx =: I_5 + I_6 + I_7. \tag{42}$$

Summing up (41) and (42) and noting that $I_2 + I_6 = 0$, we have

$$\frac{1}{2} \frac{d}{dt} \int (\rho |\Lambda^3 u|^2 + |\Lambda^3 b|^2) dx + \eta \int |\Lambda^4 b|^2 dx = I_1 + I_3 + I_4 + I_5 + I_7. \tag{43}$$

Using (10) and (11), we bound I_1, I_4, I_5 and I_7 as follows.

$$I_1 \leq C \|\nabla b\|_{L^\infty} (\|\Lambda^3 b\|_{L^2}^2 + \|\Lambda^3 u\|_{L^2}^2);$$
$$I_4 \leq C(\|\nabla (\rho u)\|_{L^\infty} \|\Lambda^3 u\|_{L^2} + \|\nabla u\|_{L^\infty} \|\Lambda^3 (\rho u)\|_{L^2}) \|\Lambda^3 u\|_{L^2}$$
$$\leq C(\|\nabla u\|_{L^\infty} + 1) \|\Lambda^3 u\|_{L^2}^2 + C \|\nabla u\|_{L^\infty} (\|\Lambda^3 u\|_{L^2} + \|u\|_{L^\infty} \|\Lambda^3 \rho\|_{L^2}) \|\Lambda^3 u\|_{L^2}$$
$$\leq C(\|\nabla u\|_{L^\infty} + 1) \|\Lambda^3 u\|_{L^2}^2 + C \|\nabla u\|_{L^\infty} \|\Lambda^3 \rho\|_{L^2}^2;$$
$$I_5 + I_7 \leq C(\|\nabla b\|_{L^\infty} \|\Lambda^3 u\|_{L^2} + \|\nabla u\|_{L^\infty} \|\Lambda^3 b\|_{L^2}) \|\Lambda^3 b\|_{L^2}.$$

To bound I_3, we proceed as follows.

$$
\begin{aligned}
I_3 &\leq C(\|\partial_t u\|_{L^\infty}\|\Lambda^3\rho\|_{L^2} + \|\nabla\rho\|_{L^\infty}\|\Lambda^2\partial_t u\|_{L^2})\|\Lambda^3 u\|_{L^2}\\
&\leq C\left\|u\cdot\nabla u + \frac{1}{\rho}\nabla\pi - \frac{\mathrm{rot}\,b}{\rho}\times b\right\|_{L^\infty}\|\Lambda^3\rho\|_{L^2}\|\Lambda^3 u\|_{L^2}\\
&\quad + C\left\|\Delta\left(u\cdot\nabla u + \frac{1}{\rho}\nabla\pi - \frac{\mathrm{rot}\,b}{\rho}\times b\right)\right\|_{L^2}\|\Lambda^3 u\|_{L^2}\\
&\leq C(\|\nabla u\|_{L^\infty} + \|\nabla\pi\|_{L^\infty} + \|\nabla b\|_{L^\infty})\|\Lambda^3\rho\|_{L^2}\|\Lambda^3 u\|_{L^2}\\
&\quad + C\left(\|u\|_{L^\infty}\|\Lambda^3 u\|_{L^2} + \|\Lambda^2\nabla\pi\|_{L^2} + \|\nabla\pi\|_{L^3}\left\|\Lambda^2\frac{1}{\rho}\right\|_{L^6}\right.\\
&\quad \left. + \left\|\Delta\frac{1}{\rho}\right\|_{L^6}\|\mathrm{rot}\,b\|_{L^3}\|b\|_{L^\infty} + \|b\|_{L^\infty}\|\Lambda^3 b\|_{L^2}\right)\|\Lambda^3 u\|_{L^2}\\
&\leq C(\|\nabla u\|_{L^\infty} + \|\nabla\pi\|_{L^\infty} + \|\nabla b\|_{L^\infty})\|\Lambda^3\rho\|_{L^2}\|\Lambda^3 u\|_{L^2}\\
&\quad + C(\|\Lambda^3 u\|_{L^2} + \|\Lambda^2\nabla\pi\|_{L^2} + \|\nabla\pi\|_{L^3}\|\Delta\rho\|_{L^6} + \|\Delta\rho\|_{L^6} + \|\Lambda^3 b\|_{L^2})\|\Lambda^3 u\|_{L^2}\\
&\leq C(\|\nabla u\|_{L^\infty} + \|\nabla\pi\|_{L^\infty} + \|\nabla b\|_{L^\infty})\|\Lambda^3\rho\|_{L^2}\|\Lambda^3 u\|_{L^2}\\
&\quad + C(\|\Lambda^3 u\|_{L^2} + \|\nabla f\|_{L^2} + \|\nabla\pi\|_{L^3}\|\Lambda^3\rho\|_{L^2} + \|\Lambda^3\rho\|_{L^2} + \|\Lambda^3 b\|_{L^2})\|\Lambda^3 u\|_{L^2}. \quad (44)
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
\|\nabla\pi\|_{L^\infty} &\leq C\|\nabla\pi\|_{L^2} + C\|\Delta\pi\|_{L^4}\\
&\leq C\|\nabla u\|_{L^\infty} + C + C\|\Delta b\|_{L^4};
\end{aligned} \quad (45)
$$

$$
\begin{aligned}
\|\nabla\Delta\pi\|_{L^2} &= \|\nabla f\|_{L^2}\\
&\leq C\|\nabla\rho\|_{L^\infty}\|\nabla u\|_{L^4}^2 + C\|u\|_{L^\infty}\|\Lambda^3 u\|_{L^2} + C\|\nabla\rho\|_{L^\infty}\|\nabla^2\pi\|_{L^2}\\
&\quad + C\|\Delta\rho\|_{L^6}\|\nabla\pi\|_{L^3} + C\|\nabla\rho\|_{L^\infty}(\|\Delta b\|_{L^2}\|b\|_{L^\infty} + \|\nabla b\|_{L^4}^2 + \|\nabla\rho\|_{L^\infty}\|\nabla b\|_{L^2})\\
&\quad + C\|b\|_{L^\infty}\|\Lambda^3 b\|_{L^2} + C\left\|\Delta\frac{1}{\rho}\right\|_{L^6}\|b\|_{L^\infty}\|\nabla b\|_{L^3}\\
&\leq C + C\|\Lambda^3 u\|_{L^2} + C\|\nabla^2\pi\|_{L^2} + C\|\Lambda^3\rho\|_{L^2}\|\nabla\pi\|_{L^3} + C\|\Lambda^3 b\|_{L^2} + C\|\Lambda^3\rho\|_{L^2}\\
&\leq \frac{1}{2}\|\nabla f\|_{L^2} + C\|\nabla\pi\|_{L^2} + C + C\|\Lambda^3 u\|_{L^2}\\
&\quad + C\|\Lambda^3\rho\|_{L^2} + C\|\Lambda^3 b\|_{L^2} + C\|\Lambda^3\rho\|_{L^2}\|\nabla\pi\|_{L^3},
\end{aligned}
$$

which gives

$$
\begin{aligned}
\|\nabla f\|_{L^2} &\leq C + C\|\nabla u\|_{L^\infty} + C\|\Lambda^3\rho\|_{L^2} + C\|\Lambda^3 u\|_{L^2}\\
&\quad + C\|\Lambda^3 b\|_{L^2} + C\|\Lambda^3\rho\|_{L^2}(\|\nabla\pi\|_{L^2} + \|\Delta\pi\|_{L^4})\\
&\leq C + C\|\nabla u\|_{L^\infty} + C\|\Lambda^3(\rho,u,b)\|_{L^2} + C\|\Lambda^3\rho\|_{L^2}(\|\nabla u\|_{L^\infty} + 1 + \|\Delta b\|_{L^4}).
\end{aligned}
$$

Inserting the above estimates into (44), we obtain

$$
\begin{aligned}
I_3 &\leq C(\|\nabla u\|_{L^\infty} + 1 + \|\nabla b\|_{L^\infty} + \|\Delta b\|_{L^4})(\|\Lambda^3\rho\|_{L^2}^2 + \|\Lambda^3 u\|_{L^2}^2) + C\|\Lambda^3(\rho,u,b)\|_{L^2}^2\\
&\leq C(\|\nabla u\|_{L^\infty} + 1 + \|\nabla b\|_{L^\infty} + \|\Delta b\|_{L^4})\|\Lambda^3(\rho,u,b)\|_{L^2}^2.
\end{aligned}
$$

Inserting the above estimates of I_1, I_3, I_4, I_5 and I_7 into (43), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int(\rho|\Lambda^3 u|^2 + |\Lambda^3 b|^2)\mathrm{d}x + \eta\int|\Lambda^4 b|^2\mathrm{d}x\\
&\leq C(\|\nabla u\|_{L^\infty} + 1 + \|\nabla b\|_{L^\infty} + \|\Delta b\|_{L^4})\|\Lambda^3(\rho,u,b)\|_{L^2}^2. \quad (46)
\end{aligned}
$$

Summing up (40) and (46) and using the Gronwall inequality, we conclude that

$$
\|\Lambda^3(\rho,u,b)\|_{L^2} + \int_0^T\int|\Lambda^4 b|^2\mathrm{d}x\mathrm{d}t \leq C.
$$

This completes the proof. □

3. Conclusions

In this paper, we prove a refined blow-up criterion for the inhomogeneous incompressible MHD system with zero viscosity, which is important and can be used for the simulation of MHD. For $\rho = 1$ and $\eta = 0$, Caflisch et al. [12] showed the following regularity criterion:

$$\text{rot}\, u, \text{rot}\, b \in L^1(0, T; L^\infty). \tag{47}$$

Since the problem is very challenging, we are unable to present further developments.

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