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*Article*

# Some Oscillatory Criteria for Second-Order Emden–Fowler Neutral Delay Differential Equations

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**Abstract:** In this paper, by using the Riccati transformation and integral inequality technique, we establish several oscillation criteria for second-order Emden–Fowler neutral delay differential equations under the canonical case and non-canonical case, respectively. Compared with some recent results reported in the literature, we extend the range of the neutral coefficient. Therefore, our results generalize to some of the results presented in the literature. Furthermore, several examples are provided to illustrate our conclusions.

**Keywords:** delay differential equations; second order; neutral equation; Emden–Fowler type; oscillation

**MSC:** 34C10; 34K11

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## 1. Introduction

In this paper, we consider the oscillation of the following second-order Emden–Fowler neutral delay differential equation

$$(a(t)|z'(t)|^{\alpha-1}z'(t))' + q(t)|x(\sigma(t))|^{\beta-1}x(\sigma(t)) = 0, \tag{1}$$

where $t \geq t_0$, $z(t) = x(t) + p(t)x(\tau(t))$, $a(t), \tau(t), \sigma(t) \in C^1([t_0,\infty),(0,\infty))$, $p(t), q(t) \in C([t_0,\infty),[0,\infty))$, $\alpha > 0$, and $\beta > 0$. Additionally, the following two assumptions are satisfied:

(H1). $a'(t) \geq 0, \tau'(t) > 0, \sigma'(t) > 0, \sigma(t) \leq t,\ \tau(t) \leq t,\ \lim\limits_{t\to\infty} \tau(t) = \lim\limits_{t\to\infty} \sigma(t) = \infty;$
(H2). $p(t) > 1$.

If

$$\int_{t_0}^{\infty} \frac{1}{a^{1/\alpha}(t)} dt = \infty, \tag{2}$$

then we say that Equation (1) satisfies the canonical case.

If

$$\int_{t_0}^{\infty} \frac{1}{a^{1/\alpha}(t)} dt < \infty, \tag{3}$$

then we say that Equation (1) satisfies the non-canonical case.

In this paper, we investigate the oscillation of Equation (1) when it satisfies (2) and (3), respectively.

We only consider the nontrivial solution of (1), which satisfies $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq t_0$.

**Definition 1.** *A nontrivial solution of (1) is oscillatory if it has an arbitrarily large zero point on the interval $I = [t_0,\infty)$. Otherwise, it is nonoscillatory.*

**Definition 2.** *Equation (1) is oscillatory if all its solutions are oscillatory.*

The Emden–Fowler equation is in honor of astrophysicist Jacob Robert Emden (1862–1940) and astronomer Sir Ralph Howard Fowler. This equation was established by Fowler to model some phenomena in fluid mechanics [1]. With the development of science, this equation has many applications to model various physical phenomena, such as in the study of astrophysics, gas dynamics, fluid physics, and nuclear physics [2–5]. Wong [5] established the oscillation criteria of the following second-order super-linear equation

$$x''(t) + q(t)|x(t)|^{\beta} sgnx(t) = 0.$$

Since then, many researchers have found that delay and oscillation effects are often formulated with the help of external sources and/or nonlinear diffusion, perturbing the natural evolution of related systems; see, e.g., [6–8]. Therefore, the oscillatory properties for Emden–Fowler delay differential equations have attracted the attention of many researchers. We refer the reader to the papers [9–30].

In [11,17–20,26], the authors considered the oscillation of the following second-order half-linear equation:

$$(a(t)(z'(t))^{\alpha})' + q(t)x^{\alpha}(\sigma(t)) = 0,\ t \geq t_0, \tag{4}$$

where $\alpha \in Q^*$, $Q^*$ represents the set of all the ratios of odd positive integers. When the neutral coefficient $p(t)$ satisfied $0 \leq p(t) < 1$, Agarwal et al. [11] studied the oscillation of Equation (4) under the non-canonical case. Grace et al. [17] and Jadlovská [20] considered the oscillation of Equation (4) under the canonical case. When the neutral coefficient $p(t)$ satisfied $p(t) \equiv p_0 > 1$, Hassan et al. [19] studied the oscillation of Equation (4) under the non-canonical case. In [26], based on condition (2), Moaaz et al. obtained several oscillation criteria for (4) under the condition $0 \leq p(t) \leq p_0 < \infty$ ($p_0$ is a constant). In [18], based on condition (3), Hindi et al. provided several oscillation criteria for (4) under the condition $0 \leq p(t) \leq p_0 < \infty$. These results expanded the range of the neutral coefficient $p(t)$ in [11,17,20].

Abdelnaser et al. [10] studied the oscillation of the following second-order Emden–Fowler equation under the canonical case

$$(a(t)(z'(t))^{\alpha})' + \sum_{i=1}^{l} q_i(t)x^{\beta}(\sigma_i(t)) = 0, \tag{5}$$

where $\alpha, \beta \in Q^*$.

When $l = 1$, Equation (5) becomes the following second-order Emden–Fowler-type equation:

$$(a(t)(z'(t))^{\alpha})' + q(t)x^{\beta}(\sigma(t)) = 0. \tag{6}$$

Under the non-canonical case and $0 \leq p(t) < 1$, Agarwal et al. [9] provided some oscillation criteria for Equation (6) when $\beta \geq \alpha$, $\beta < \alpha$ and $\beta < \alpha = 1$ are satisfied, respectively. By introducing some new comparison theorems, Baculíková et al. [13] established several new results. They transformed the study of second-order neutral differential equations into the research on first-order delay differential equations and extended the range of $p(t)$ from $0 \leq p(t) < 1$ to $0 \leq p(t) \leq p_0 < \infty$. Based on the assumptions that $0 < \beta \leq 1$ and the neutral coefficient $p(t)$ satisfies $0 \leq p(t) \leq p_0 < \infty$, $p(t) \equiv p_0 \neq 1$ and $p(t) > 1$, respectively, Li et al. [22] provided some oscillation criteria for Equation (6) under the canonical case and non-canonical case, respectively.

In [21,23,27–29], the scholars obtained some oscillation criteria for Equation (1). When $\alpha = 1$, then (1) becomes the following equation:

$$(a(t)z'(t))' + q(t)|x(\sigma(t))|^{\beta-1}x(\sigma(t)) = 0. \tag{7}$$

Under the non-canonical case, $\beta \geq 1$ and $0 \leq p(t) < 1$, Li et al. [21] obtained some oscillation criteria for (7). In [23], Li et al. provided some oscillation criteria of (7) under the canonical case. They extended the range of $p(t)$ from $0 \leq p(t) < 1$ to $0 \leq p(t) \leq p_0 < \infty$, with the conditions $\beta > 1$, $\tau'(t) \geq \tau_0 > 0$, and $\sigma \circ \tau = \tau \circ \sigma$.

Under the canonical case and non-canonical case, respectively, and $0 \leq p(t) < 1$, Wu et al. [27,28] and Zeng et al. [29] provided some different oscillation criteria for Equation (1) by different methods.

In this article, we study the oscillation of Equation (1). When the neutral coefficient $p(t)$ satisfies $p(t) > 1$ and $\alpha = \beta = 1$, compared with the results of Baculíková et al. [13], Li et al. [22,23], and Moaaz et al. [26], we establish a new oscillation criterion of Equation (1) without the condition $\tau \circ \sigma = \sigma \circ \tau$. For the same Equation (1), compared with the above results of [21,27–29], we extend the range of neutral coefficient $p(t)$ from $0 \leq p(t) < 1$ to $p(t) > 1$ (see also [31,32]). The main difficulty is, under the non-canonical case, when $p(t) > 1$ and $z'(t) < 0$ hold, the inequality

$$x(t) > (1 - p(t))z(t)$$

is not valid. Moreover, we extend the ranges of $\alpha$ and $\beta$. Compared with the research of [9,29], we do not need to discuss $\alpha \geq \beta$ or $\alpha \leq \beta$ separately because we provide a unified form of the oscillation criteria for Equation (1). Therefore, our results extend the works of previous researchers. At the end of this article, we provide some examples to verify our criteria.

## 2. Main Results

For simplicity, we introduce the following temporary notation:

$$P(t) := \frac{1}{p(t)}\left(1 - \frac{(\tau^{-1}(t))^{\frac{1}{l}}}{p(\tau^{-1}(t))t^{\frac{1}{l}}}\right), \quad P^*(t) := \frac{1}{p(t)}\left(1 - \frac{1}{p(\tau^{-1}(t))}\right),$$

where $l \in (0,1)$, $p(\tau^{-1}(t)) > (\frac{\tau^{-1}(t)}{t})^{\frac{1}{l}}$.

Before starting to present our main results, we first introduce the following useful lemmas.

**Lemma 1** ([30])**.** *Let $g(u) = Eu - Fu^{\frac{\beta+1}{\beta}}$; where E and F are positive constants, $\beta$ is a quotient of odd natural numbers. Then, g attains its maximum value on $\Re^+$ at $u^* = (\frac{\beta a}{(\beta+1)b})^{\beta}$ and*

$$\max_{u \in \Re^+} g = g(u^*) = \frac{\beta^{\beta}}{(\beta+1)^{\beta+1}} \frac{E^{\beta+1}}{F^{\beta}}.$$

**Lemma 2** ([33,34])**.** *If a function $f(t)$ satisfies $f^{(i)}(t) > 0$, $i = 1, 2, \cdots, k$, and $f^{(k+1)}(t) \leq 0$, then, for every $l \in (0,1)$, $\frac{f(t)}{f'(t)} \geq \frac{lt}{k}$.*

First, we consider the oscillation criteria for Equation (1), which satisfy the canonical case; that is, (2) holds.

### *2.1. Equation (1) Satisfies Condition (2)*

**Theorem 1.** *Assume that (2) and*

$$\int_{t_0}^{\infty} q(s)(P(\tau^{-1}(\sigma(s))))^{\beta} ds = \infty \tag{8}$$

*hold; then, (1) is oscillatory.*

**Proof.** Conversely, suppose that (1) has a nonoscillatory solution $x(t)$. Without loss of generality, we may assume that $x(t)$ is eventually positive. That is, there exists $t_1 \geq t_0$, such that $x(t) > 0$ , $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \geq t_1$. A similar approach applies to the case that $x(t)$ is an eventually negative solution. According to (1), we obtain

$$(a(t)|z'(t)|^{\alpha-1}z'(t))' = -q(t)x^{\beta}(\sigma(t)) < 0, \quad t \geq t_1. \tag{9}$$

Therefore, $a(t)|z'(t)|^{\alpha-1}z'(t)$ is decreasing. Thus, we know that $z'(t) < 0$ or $z'(t) > 0$ for $t \geq t_1$.

Case 1. $z'(t) < 0$ for $t \geq t_1$. By means of the fact that $a(t)|z'(t)|^{\alpha-1}z'(t)$ is decreasing, we have

$$a(t)|z'(t)|^{\alpha-1}z'(t) \leq a(t_2)|z'(t_2)|^{\alpha-1}z'(t_2), \quad t \geq t_2 \geq t_1. \tag{10}$$

Dividing both sides of (10) by $a(t)$, integrating from $t_2$ to $t$ and using (2), we obtain

$$z(t) \leq z(t_2) - a^{\frac{1}{\alpha}}(t_2)|z'(t_2)| \int_{t_2}^{t} a^{-\frac{1}{\alpha}}(s)ds \to -\infty \quad as \quad t \to \infty;$$

this contradicts the fact that $z(t) > 0$.

Case 2. $z'(t) > 0$ for $t \geq t_1$. According to (1), $a(t) > 0$ and $a'(t) \geq 0$; we know that $z''(t) < 0$. Thus, by Lemma 2, we have $\frac{z(t)}{z'(t)} \geq lt$, where $0 < l < 1$. Then, we have that $\frac{z(t)}{t^{\frac{1}{l}}}$ is nonincreasing. By $\tau(t) \leq t$ and $\tau'(t) \geq 0$, we obtain $(\tau^{-1}(t))' \geq 0$; thus, $\tau^{-1}(t) \leq \tau^{-1}(\tau^{-1}(t))$, and then

$$\frac{z(\tau^{-1}(t))}{(\tau^{-1}(t))^{\frac{1}{l}}} \geq \frac{z(\tau^{-1}(\tau^{-1}(t)))}{(\tau^{-1}(\tau^{-1}(t)))^{\frac{1}{l}}}. \tag{11}$$

By the definition of $z(t)$, we have

$$\begin{aligned} x(t) &= \frac{1}{p(\tau^{-1}(t))}(z(\tau^{-1}(t)) - x(\tau^{-1}(t))) \\ &= \frac{z(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{1}{p(\tau^{-1}(t))}\left(\frac{z(\tau^{-1}(\tau^{-1}(t)))}{p(\tau^{-1}(\tau^{-1}(t)))} - \frac{x(\tau^{-1}(\tau^{-1}(t)))}{p(\tau^{-1}(\tau^{-1}(t)))}\right) \\ &\geq \frac{z(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{1}{p(\tau^{-1}(t))}\frac{z(\tau^{-1}(\tau^{-1}(t)))}{p(\tau^{-1}(\tau^{-1}(t)))}. \end{aligned} \tag{12}$$

By (11) and (12), we arrive at

$$x(t) \geq z(\tau^{-1}(t))P(\tau^{-1}(t)),\ t \geq t_3 \geq t_2. \tag{13}$$

From (1), we have

$$(a(t)(z'(t))^{\alpha})' \leq -q(t)z^{\beta}(\tau^{-1}(\sigma(t)))P^{\beta}(\tau^{-1}(\sigma(t))). \tag{14}$$

Now, we introduce a Riccati substitution

$$u(t) := \frac{a(t)(z'(t))^{\alpha}}{z^{\beta}(\tau^{-1}(\sigma(t)))},\ t \geq t_3. \tag{15}$$

Then, $u(t) > 0$ on $[t_3, \infty)$, and we have

$$\begin{aligned} u'(t) &= \frac{(a(t)(z'(t))^{\alpha})'}{z^{\beta}(\tau^{-1}(\sigma(t)))} - \frac{\beta a(t)(z'(t))^{\alpha}z'(\tau^{-1}(\sigma(t)))(\tau^{-1})'(v)|_{v=\sigma(t)}\sigma'(t)}{z^{\beta+1}(\tau^{-1}(\sigma(t)))} \\ &\leq -q(t)P^{\beta}(\tau^{-1}(\sigma(t))) - \frac{\beta a(t)(z'(t))^{\alpha}z'(\tau^{-1}(\sigma(t)))(\tau^{-1})'(v)|_{v=\sigma(t)}\sigma'(t)}{z^{\beta+1}(\tau^{-1}(\sigma(t)))}. \end{aligned} \tag{16}$$

By $\beta > 0$, $a(t) > 0$, $(\tau^{-1})'(t) \geq 0$, $\sigma'(t) > 0$, $z(t) > 0$, and $z'(t) > 0$, we obtain

$$u'(t) + q(t)P^{\beta}(\tau^{-1}(\sigma(t))) \leq 0, \ t \geq t_3. \tag{17}$$

Integrating both sides of (17) from $t_3$ to $t$ and using (8), we obtain

$$u(t) \leq u(t_3) - \int_{t_3}^{t} q(s)P^{\beta}(\tau^{-1}(\sigma(s)))ds \to -\infty, \ as \ t \to \infty,$$

which contradicts the positivity of $u(t)$. Therefore, the assumption does not hold. □

Obviously, Theorem 1 is a generalization of [27] (Theorem 2.1).

If condition (2) is satisfied and condition (8) is not valid, we can also provide an oscillation criterion of Equation (1). First of all, we need the following useful lemmas.

**Lemma 3.** *Assume that $x(t)$ is an eventually positive solution of (1), $u(t)$ is defined by Equation (15), and $\sigma(t) \leq \tau(t)$. Then,*

$$u'(t) \leq -q(t)P^{\beta}(\tau^{-1}(\sigma(t))) - \frac{\vartheta(\tau^{-1})'(v)|_{v=\sigma(t)}\sigma'(t)Q}{a^{\frac{1}{\vartheta}}(\phi(t))}u^{\frac{\vartheta+1}{\vartheta}}(t), \tag{18}$$

*where $\vartheta = \min\{\alpha, \beta\}$ and*

$$Q := \begin{cases} 1, & \alpha = \beta \\ const > 0, & \alpha \neq \beta \end{cases}, \quad \phi(t) := \begin{cases} t, & \alpha > \beta \\ \tau^{-1}(\sigma(t)), & \alpha \leq \beta \end{cases}.$$

**Proof.** Continuing the proof of Case 2 of Theorem 1, we obtain (16). By $\sigma(t) \leq \tau(t)$ and $z''(t) < 0$, we have

$$\begin{aligned} u'(t) &\leq -q(t)P^{\beta}(\tau^{-1}(\sigma(t))) - \frac{\beta a(t)(z'(t))^{\alpha}z'(\tau^{-1}(\sigma(t)))(\tau^{-1})'(v)|_{v=\sigma(t)}\sigma'(t)}{z(\tau^{-1}(\sigma(t)))^{\beta+1}} \\ &\leq -q(t)P^{\beta}(\tau^{-1}(\sigma(t))) - \frac{\beta a(t)(z'(t))^{\alpha+1}(\tau^{-1})'(v)|_{v=\sigma(t)}\sigma'(t)}{z(\tau^{-1}(\sigma(t)))^{\beta+1}}. \end{aligned} \tag{19}$$

If $\beta \geq \alpha$, by the fact that $z(\tau^{-1}(\sigma(t)))$ is increasing, then there exist constant $Q_1 > 0$ and $t_4 \geq t_3$ such that $[z(\tau^{-1}(\sigma(t)))]^{\frac{\beta-\alpha}{\alpha}} \geq Q_1$ for $t \geq t_4$. Thus, according to (19), $a'(t) \geq 0$, $a(t) > 0$, and $\sigma(t) \leq \tau(t)$, we obtain

$$\begin{aligned} u'(t) &\leq -q(t)P^{\beta}(\tau^{-1}(\sigma(t))) - \frac{\beta(\tau^{-1})'(v)|_{v=\sigma(t)}\sigma'(t)[z(\tau^{-1}(\sigma(t)))]^{\frac{\beta-\alpha}{\alpha}}}{a^{\frac{1}{\alpha}}(\tau^{-1}(\sigma(t)))}u^{\frac{\alpha+1}{\alpha}}(t) \\ &\leq -q(t)P^{\beta}(\tau^{-1}(\sigma(t))) - \frac{\alpha(\tau^{-1})'(v)|_{v=\sigma(t)}\sigma'(t)Q_1}{a^{\frac{1}{\alpha}}(\tau^{-1}(\sigma(t)))}u^{\frac{\alpha+1}{\alpha}}(t). \end{aligned} \tag{20}$$

Obviously, if $\alpha = \beta$, $Q_1 = 1$.

Next, if $\alpha > \beta$, by $z''(t) < 0$, then $z'(t)$ is decreasing and $[z'(t)]^{\frac{\beta-\alpha}{\beta}}$ is increasing. Then, there exist constant $Q_2 > 0$ and $t_5 \geq t_4$ such that $[z'(t)]^{\frac{\beta-\alpha}{\beta}} \geq Q_2$ for $t \geq t_5$. Hence, by (19), it has

$$\begin{aligned} u'(t) &\leq -q(t)P^{\beta}(\tau^{-1}(\sigma(t))) - \frac{\beta(\tau^{-1})'(v)|_{v=\sigma(t)}\sigma'(t)[z'(t)]^{\frac{\beta-\alpha}{\beta}}}{a^{\frac{1}{\beta}}(t)}u^{\frac{\beta+1}{\beta}}(t) \\ &\leq -q(t)P^{\beta}(\tau^{-1}(\sigma(t))) - \frac{\beta(\tau^{-1})'(v)|_{v=\sigma(t)}\sigma'(t)Q_2}{a^{\frac{1}{\beta}}(t)}u^{\frac{\beta+1}{\beta}}(t). \end{aligned} \tag{21}$$

Combining (20) and (21), we obtain that inequality (18) holds. □

In order to continue the analysis, we need the following notation:

$$M(t) := \int_t^\infty q(s)P^\beta(\tau^{-1}(\sigma(s)))ds,\ A(t) := \frac{\vartheta(\tau^{-1})'(v)|_{v=\sigma(t)}\sigma'(t)Q}{a^{\frac{1}{\vartheta}}(\phi(t))}. \tag{22}$$

Define a sequence of functions $\{\omega_n\}_{n0}^\infty$ by

$$\omega_0 = M(t),\ t \geq t_0,$$

and

$$\omega_n(t) = \int_t^\infty A(s)\omega_{n-1}^{\frac{\vartheta+1}{\vartheta}}(s)ds + \omega_0(t),\ t \geq t_0,\ n = 1,2,3,\ldots. \tag{23}$$

By induction, it is thus established that $\omega_n(t) \leq \omega_{n+1}(t),\ t \geq t_0,\ n = 1,2,3,\ldots$.

**Lemma 4.** *Assume that $x(t)$ is an eventually positive solution of (1). Then, we obtain $\omega_n(t) \leq u(t)$, where $u(t)$ and $\omega_n(t)$ are defined by (15) and (23), respectively. Moreover, there exists a function $\omega(t) \in C([T,\infty),(0,\infty))$, such that $\lim_{t\to\infty} \omega_n(t) = \omega(t)$ for $t \geq T \geq t_0$ and*

$$\omega(t) = \int_t^\infty A(s)\omega^{\frac{\vartheta+1}{\vartheta}}(s)ds + \omega_0(t), \quad t \geq T. \tag{24}$$

**Proof.** Proceeding as in the proof of Lemma 3, we have

$$u'(t) \leq -q(t)P^\beta(\tau^{-1}(\sigma(t))) - A(t)u^{\frac{\vartheta+1}{\vartheta}}(t). \tag{25}$$

Thus, $u(t)$ is decreasing. Then, integrating both sides of (25) from $t$ to $t'$, we obtain

$$u(t') - u(t) + \int_t^{t'} q(s)P^\beta(\tau^{-1}(\sigma(s)))ds + \int_t^{t'} A(s)u^{\frac{\vartheta+1}{\vartheta}}(s)ds \leq 0. \tag{26}$$

Then, it is not difficult to know that

$$u(t') - u(t) + \int_t^{t'} A(s)u^{\frac{\vartheta+1}{\vartheta}}(s)ds \leq 0. \tag{27}$$

Thus, we claim that

$$\int_t^\infty A(s)u^{\frac{\vartheta+1}{\vartheta}}(s)ds < \infty, \quad t \geq T. \tag{28}$$

Otherwise, by (27), $u(t') \leq u(t) - \int_t^{t'} A(s)u^{\frac{\vartheta+1}{\vartheta}}(s)ds \to -\infty$ as $t' \to \infty$, which contradicts the positivity of $u(t)$. By $u(t) > 0$ and the fact that $u(t)$ is decreasing, from (28), we obtain $\lim_{t\to\infty} u(t) = 0$. Thus, from (26), we have

$$u(t) \geq M(t) + \int_t^\infty A(s)u^{\frac{\vartheta+1}{\vartheta}}(s)ds = \omega_0(t) + \int_t^\infty A(s)u^{\frac{\vartheta+1}{\vartheta}}(s)ds; \tag{29}$$

that is,

$$u(t) \geq M(t) = \omega_0(t).$$

Moreover, by induction, we obtain that $u(t) \geq \omega_n(t)$ for $t \geq t_0$, $n = 1,2,3,\ldots$. Thus, since the sequence $\{\omega_n(t)\}_{n=0}^\infty$ is monotone increasing and bounded above, it converges to $\omega(t)$. Using Lebesgue's monotone convergence theorem in (23), we obtain that (24) holds. □

**Theorem 2.** *Suppose (2) is valid and (8) is not satisfied. Assume that*

$$\liminf_{t\to\infty} \frac{1}{M(t)} \int_t^\infty A(s) M^{\frac{\vartheta+1}{\vartheta}}(s) ds = \infty \tag{30}$$

*holds, where $\vartheta$, $M(t)$, and $A(t)$ are defined by (18) and (22). Then, (1) is oscillatory.*

**Proof.** Suppose that $x(t)$ is an eventually positive solution of (1). Then, proceeding as in the proof of Lemmas 3 and 4, we obtain (29) and

$$\frac{u(t)}{M(t)} \geq 1 + \frac{1}{M(t)} \int_t^\infty A(s) M^{\frac{\vartheta+1}{\vartheta}}(s) \left( \frac{u(s)}{M(s)} \right)^{\frac{\vartheta+1}{\vartheta}} ds, \quad t \geq T. \tag{31}$$

From (30), we obtain that the following inequality is satisfied

$$\liminf_{t\to\infty} \frac{1}{M(t)} \int_t^\infty A(s) M^{\frac{\vartheta+1}{\vartheta}}(s) ds > \frac{\vartheta}{(\vartheta+1)^{\frac{\vartheta+1}{\vartheta}}}. \tag{32}$$

Then, by (32), we know that there exists a constant $C$ such that

$$\liminf_{t\to\infty} \frac{1}{M(t)} \int_t^\infty A(s) M^{\frac{\vartheta+1}{\vartheta}}(s) ds > C > \frac{\vartheta}{(\vartheta+1)^{\frac{\vartheta+1}{\vartheta}}}.$$

Thus, there exists $T_0$ sufficiently large enough such that

$$\inf_{t\geq T_0} \frac{1}{M(t)} \int_t^\infty A(s) M^{\frac{\vartheta+1}{\vartheta}}(s) ds > C > \frac{\vartheta}{(\vartheta+1)^{\frac{\vartheta+1}{\vartheta}}}. \tag{33}$$

Let $\lambda = \inf_{t\geq T_0} \frac{u(t)}{M(t)}$. Then from (31), we have

$$\lambda \geq 1 + \lambda^{\frac{\vartheta+1}{\vartheta}} C. \tag{34}$$

According to Lemma 1 and (33), we obtain

$$\begin{aligned} \lambda - C\lambda^{\frac{\vartheta+1}{\vartheta}} &\leq \frac{\vartheta^\vartheta}{(\vartheta+1)^{\vartheta+1}} \frac{1}{C^\vartheta} \\ &< 1. \end{aligned}$$

Thus, we have

$$\lambda < 1 + C\lambda^{\frac{\vartheta+1}{\vartheta}},$$

which contradicts inequality (34). Therefore, the assumption does not hold. Thus, Equation (1) is oscillatory. □

**Remark 1.** *The conclusion of Theorem 2 remains valid if we replace (32) with (30).*

Finally, we provide an oscillation criterion for Equation (1) that satisfies the non-canonical case; that is, (3) holds.

*2.2. Equation (1) Satisfies Condition (3)*

**Theorem 3.** *Suppose (3) holds. If there exists a function $\delta(t) \in C^1([t_0,\infty),(0,\infty))$ such that for all constants $Q > 0$, $N > 0$, the following integral formulas are satisfied*

$$\lim_{t\to\infty} \int_T^t \left[ q(s)\delta(s) P^\beta(\tau^{-1}(\sigma(s))) - \frac{(\delta'(t))^{\vartheta+1} a(\phi(s))}{(\vartheta+1)^{\vartheta+1} (\delta(s)(\tau^{-1})'(v)|_{v=\sigma(s)} \sigma'(s) Q)^\vartheta} \right] ds = \infty \tag{35}$$

*and*

$$\lim_{t\to\infty}\int_T^t\left[\kappa^\theta(s)q(s)\left(\frac{\kappa(\tau^{-1}(\tau^{-1}(\sigma(s))))}{\kappa(\sigma(s))}P^*(\tau^{-1}(\sigma(s)))\right)^\beta-\frac{\epsilon}{\kappa(s)a^{1/\alpha}(s)}\right]ds=\infty, \tag{36}$$

*where* $\vartheta=\min\{\alpha,\beta\}$, $\theta=\max\{\alpha,\beta\}$,

$$\phi(t):=\begin{cases}t, & \alpha>\beta\\ \tau^{-1}(\sigma(t)), & \alpha\le\beta\end{cases},\quad \kappa(t):=\lim_{l\to\infty}\int_t^l\frac{1}{a^{\frac{1}{\alpha}}(s)}ds,$$

$\epsilon=(\frac{\theta}{\theta+1})^{\theta+1}(\frac{\theta}{N})^\theta$ *(when* $\alpha=\beta,\ Q=1,\ N=\alpha$*), and then (1) is oscillatory.*

**Proof.** To obtain a contradiction, suppose that (1) has a nonoscillatory solution $x(t)$. Without loss of generality, we may assume that $x(t)$ is eventually positive. That is, there exists $t_1\ge t_0$, such that $x(t)>0$, $x(\tau(t))>0$, and $x(\sigma(t))>0$ for $t\ge t_1$. If $x(t)$ is an eventually negative solution, it can be proved in a similar way. According to (1), we obtain

$$(a(t)|z'(t)|^{\alpha-1}z'(t))'=-q(t)x^\beta(\sigma(t))<0,\quad t\ge t_1. \tag{37}$$

Therefore, $a(t)|z'(t)|^{\alpha-1}z'(t)$ is decreasing. Thus, we have two possible cases for $z'(t)$. That is, there exists a $t_2\ge t_1$ such that $z'(t)>0$ or $z'(t)<0$ for $t\ge t_2$.

Case 1. $z'(t)>0$ for $t\ge t_2$. In view of (10), we know that $a(t)(z'(t))^\alpha$ is decreasing. Proceeding as in the proof of Case 2 of Theorem 1, we have that (13) and (14) hold. Define a function $w(t)$ as follows

$$w(t):=\delta(t)\frac{a(t)(z'(t))^\alpha}{z^\beta(\tau^{-1}(\sigma(t)))},\quad t\ge t_3. \tag{38}$$

Then, $w(t)>0$ for $t\ge t_3$. Taking differentiation on both sides of (38), we obtain

$$\begin{aligned}w'(t)=&\ \delta'(t)\frac{a(t)(z'(t))^\alpha}{z^\beta(\tau^{-1}(\sigma(t)))}+\delta(t)\frac{(a(t)(z'(t))^\alpha)'}{z^\beta(\tau^{-1}(\sigma(t)))}\\ &-\delta(t)\frac{\beta a(t)(z'(t))^\alpha z'(\tau^{-1}(\sigma(t)))(\tau^{-1})'(v)|_{v=\sigma(t)}\sigma'(t)}{z^{\beta+1}(\tau^{-1}(\sigma(t)))}.\end{aligned} \tag{39}$$

If $\beta\ge\alpha$, in view of the fact that $z(\tau^{-1}(\sigma(t)))$ is increasing, $z^{\frac{\beta-\alpha}{\alpha}}(\tau^{-1}(\sigma(t)))$ is thus increasing, and there exist constants $Q_1$ and $t_4\ge t_3$, such that $z^{\frac{\beta-\alpha}{\alpha}}(\tau^{-1}(\sigma(t)))\ge Q_1$ for $t_4\ge t_3$. According to (39) and (14), $a'(t)\ge 0$, $a(t)>0$, and $\sigma(t)\le\tau(t)$, and we obtain

$$\begin{aligned}w'(t)\le&-q(t)\delta(t)P^\beta(\tau^{-1}(\sigma(t)))+\frac{\delta'(t)w(t)}{\delta(t)}\\ &-\frac{\beta(\tau^{-1})'(v)|_{v=\sigma(t)}\sigma'(t)z^{\frac{\beta-\alpha}{\alpha}}(\tau^{-1}(\sigma(t)))}{(a(\tau^{-1}(\sigma(t)))\delta(t))^{\frac{1}{\alpha}}}w^{\frac{\alpha+1}{\alpha}}(t)\\ \le&-q(t)\delta(t)P^\beta(\tau^{-1}(\sigma(t)))+\frac{\delta'(t)w(t)}{\delta(t)}-\frac{\alpha(\tau^{-1})'(v)|_{v=\sigma(t)}\sigma'(t)Q_1}{(a(\tau^{-1}(\sigma(t)))\delta(t))^{\frac{1}{\alpha}}}w^{\frac{\alpha+1}{\alpha}}(t).\end{aligned} \tag{40}$$

It is not difficult to know that, if $\alpha=\beta$, $Q_1=1$.

If $\alpha>\beta$, in view of $(a(t)(z'(t))^\alpha)'\le 0$ and $a'(t)\ge 0$, then $z''(t)<0$. Thus, $z'(t)$ is decreasing and $[z'(t)]^{\frac{\beta-\alpha}{\beta}}$ is increasing. Then, there exist constants $Q_2$ and $t_5\ge t_4$, such that $[z'(t)]^{\frac{\beta-\alpha}{\beta}}\ge Q_2$ for $t\ge t_5$. Thus, by (39) and (14), we obtain

$$
\begin{aligned}
w'(t) \leq & -q(t)\delta(t)P^{\beta}(\tau^{-1}(\sigma(t))) + \frac{\delta'(t)w(t)}{\delta(t)} \\
& - \frac{\beta(\tau^{-1})'(v)|_{v=\sigma(t)}\sigma'(t)[z'(t)]^{\frac{\beta-\alpha}{\beta}}}{(a(t)\delta(t))^{\frac{1}{\beta}}} w^{\frac{\beta+1}{\beta}}(t) \\
\leq & -q(t)\delta(t)P^{\beta}(\tau^{-1}(\sigma(t))) + \frac{\delta'(t)w(t)}{\delta(t)} - \frac{\beta(\tau^{-1})'(v)|_{v=\sigma(t)}\sigma'(t)Q_1}{(a(t)\delta(t))^{\frac{1}{\beta}}} w^{\frac{\beta+1}{\beta}}(t).
\end{aligned} \tag{41}
$$

Similarly, proceeding as in the proof of Lemma 3, we obtain that the following inequality holds for any $\alpha > 0$ and $\beta > 0$,

$$
w'(t) \leq -q(t)\delta(t)P^{\beta}(\tau^{-1}(\sigma(t))) + \frac{\delta'(t)w(t)}{\delta(t)} - \frac{\vartheta(\tau^{-1})'(v)|_{v=\sigma(t)}\sigma'(t)Q}{(a(\phi(t))\delta(t))^{\frac{1}{\vartheta}}} w^{\frac{\vartheta+1}{\vartheta}}(t) \tag{42}
$$

for $t \geq t_5$, where $\vartheta = \min\{\alpha, \beta\}$ and

$$
Q := \begin{cases} 1, & \alpha = \beta \\ const > 0, & \alpha \neq \beta \end{cases}, \quad \phi(t) := \begin{cases} t, & \alpha > \beta \\ \tau^{-1}(\sigma(t)), & \alpha \leq \beta \end{cases}.
$$

Let $y = w(t)$, $E(t) := \frac{\delta'(t)}{\delta(t)}$, and $F(t) := \frac{\vartheta(\tau^{-1})'(v)|_{v=\sigma(t)}\sigma'(t)Q}{(a(\phi(t))\delta(t))^{\frac{1}{\vartheta}}}$, where $E > 0$, $y \geq 0$, and $F > 0$. By (42) and Lemma 1, we have

$$
\begin{aligned}
Ey - Fy^{\frac{\vartheta+1}{\vartheta}} \leq & \frac{\vartheta^{\vartheta}}{(\vartheta+1)^{\vartheta+1}} \frac{E^{\vartheta+1}}{F^{\vartheta}} \\
= & \frac{(\delta'(t))^{\vartheta+1}a(\phi(t))}{(\vartheta+1)^{\vartheta+1}(\delta(t)(\tau^{-1})'(v)|_{v=\sigma(t)}\sigma'(t)Q)^{\vartheta}}.
\end{aligned}
$$

Then, we obtain

$$
w'(t) \leq -q(t)\delta(t)P^{\beta}(\tau^{-1}(\sigma(t))) + \frac{(\delta'(t))^{\vartheta+1}a(\phi(t))}{(\vartheta+1)^{\vartheta+1}(\delta(t)(\tau^{-1})'(v)|_{v=\sigma(t)}\sigma'(t)Q)^{\vartheta}}.
$$

Integrating both sides of the above inequality from $T$ to $t$, $T \geq t_5$, we obtain

$$
\begin{aligned}
& w(t) \leq w(T) - \\
& \int_T^t \left[ q(s)\delta(s)P^{\beta}(\tau^{-1}(\sigma(s))) - \frac{(\delta'(s))^{\vartheta+1}a(\phi(s))}{(\vartheta+1)^{\vartheta+1}(\delta(s)(\tau^{-1})'(v)|_{v=\sigma(s)}\sigma'(s)Q)^{\vartheta}} \right] ds.
\end{aligned} \tag{43}
$$

Letting $t \to \infty$ in (43) and using (35), we obtain $w(t) < -\infty$, which contradicts the fact that $w(t) > 0$.

Case 2. $z'(t) < 0$ for $t \geq t_2$. By means of (1), we have

$$
(a(t)(-z'(t))^{\alpha})' > 0, \quad t \geq t_2.
$$

Then, $a(t)(-z'(t))^{\alpha}$ is increasing and the following inequality holds

$$
z'(s) \leq \left(\frac{a(t)}{a(s)}\right)^{\frac{1}{\alpha}} z'(t), \quad s \geq t \geq t_2. \tag{44}
$$

Integrating both sides of (44) from $t$ to $l$, we obtain

$$
z(l) \leq z(t) + a^{\frac{1}{\alpha}}(t)z'(t)\int_t^l a^{-\frac{1}{\alpha}}(s)ds, \quad l \geq t \geq t_2.
$$

Letting $l \to \infty$, we obtain

$$z(t) \geq \kappa(t)a^{\frac{1}{\alpha}}(t)(-z'(t)),\ t \geq t_2. \tag{45}$$

Then, we have

$$z^{\alpha}(t) \geq \kappa^{\alpha}(t)a(t)(-z'(t))^{\alpha},\ t \geq t_2.$$

Define

$$W(t) := \frac{a(t)(-z'(t))^{\alpha}}{z^{\beta}(t)},\ t \geq t_2. \tag{46}$$

Then, $W(t) > 0$ for $t \geq t_2$.

If $\alpha \geq \beta$, then $z^{\alpha-\beta}(t)$ is decreasing and thus there exists a constant $m_1 > 0$ such that $z^{\alpha-\beta}(t) \leq m_1$ for $t \geq t_2$. Hence, we have

$$\kappa^{\alpha}(t)W(t) \leq z^{\alpha-\beta}(t) \leq m_1,\ t \geq t_2. \tag{47}$$

If $\alpha < \beta$, then $(a^{\frac{1}{\alpha}}(t)(-z'(t)))^{\alpha-\beta}$ is decreasing and thus there exists a constant $m_2 > 0$ such that

$$\kappa^{\beta}(t)W(t) \leq (a^{\frac{1}{\alpha}}(t)(-z'(t)))^{\alpha-\beta} \leq m_2,\ t \geq t_2. \tag{48}$$

Using (47) and (48), we obtain

$$0 < \kappa^{\theta}(t)W(t) \leq m,\ t \geq t_2,$$

where $\theta = \max\{\alpha, \beta\}$ and $m = \max\{m_1, m_2\}$.

According to (45), we know that $\frac{z(t)}{\kappa(t)}$ is nondecreasing for $t \geq t_2$. Thus, by $\tau'(t) > 0$, $t \leq \tau^{-1}(t) \leq \tau^{-1}(\tau^{-1}(t))$, $z'(t) < 0$, and (12), we have

$$x(t) \geq z(\tau^{-1}(\tau^{-1}(t)))P^*(\tau^{-1}(t)) \geq \frac{z(t)\kappa(\tau^{-1}(\tau^{-1}(t)))}{\kappa(t)}P^*(\tau^{-1}(t)).$$

Thus,

$$x^{\beta}(\sigma(t)) \geq z^{\beta}(t)\left(\frac{\kappa(\tau^{-1}(\tau^{-1}(\sigma(t))))}{\kappa(\sigma(t))}P^*(\tau^{-1}(\sigma(t)))\right)^{\beta}. \tag{49}$$

Using (1) and (49), we obtain

$$(a(t)(-z'(t))^{\alpha})' - q(t)\left(\frac{\kappa(\tau^{-1}(\tau^{-1}(\sigma(t))))}{\kappa(\sigma(t))}P^*(\tau^{-1}(\sigma(t)))\right)^{\beta}z^{\beta}(t) \geq 0. \tag{50}$$

Differentiating on both sides of (46), using (50), we obtain

$$W'(t) \geq q(t)\left(\frac{\kappa(\tau^{-1}(\tau^{-1}(\sigma(t))))}{\kappa(\sigma(t))}P^*(\tau^{-1}(\sigma(t)))\right)^{\beta} + \frac{\beta a(t)(-z'(t))^{\alpha+1}}{z^{\beta+1}(t)}. \tag{51}$$

If $\alpha \geq \beta$, $z^{\frac{\beta-\alpha}{\alpha}}(t)$ is increasing, and then there exists a constant $N_1 > 0$, such that $z^{\frac{\beta-\alpha}{\alpha}}(t) \geq N_1$. From (51), we have

$$\begin{aligned} W'(t) &\geq q(t)\left(\frac{\kappa(\tau^{-1}(\tau^{-1}(\sigma(t))))}{\kappa(\sigma(t))}P^*(\tau^{-1}(\sigma(t)))\right)^{\beta} + \frac{\beta}{a^{1/\alpha}(t)}z^{\frac{\beta-\alpha}{\alpha}}(t)W^{\frac{\alpha+1}{\alpha}}(t) \\ &\geq q(t)\left(\frac{\kappa(\tau^{-1}(\tau^{-1}(\sigma(t))))}{\kappa(\sigma(t))}P^*(\tau^{-1}(\sigma(t)))\right)^{\beta} + \frac{\beta N_1}{a^{1/\alpha}(t)}W^{\frac{\alpha+1}{\alpha}}(t). \end{aligned} \tag{52}$$

Obviously, if $\alpha = \beta$, $N_1 = 1$.

If $\alpha < \beta$, $(a^{1/\alpha}(t)(-z'(t)))^{\frac{\beta-\alpha}{\beta}}$ is increasing and there exists a constant $N_2 > 0$, such that $(a^{1/\alpha}(t)(-z'(t)))^{\frac{\beta-\alpha}{\beta}} > N_2$. By (51), we obtain

$$\begin{aligned}
W'(t) \geq\, & q(t)\left(\frac{\kappa(\tau^{-1}(\tau^{-1}(\sigma(t))))}{\kappa(\sigma(t))}P^*(\tau^{-1}(\sigma(t)))\right)^{\beta} \\
& + \frac{\beta}{a^{1/\alpha}(t)}(a^{1/\alpha}(t)(-z(t)))^{\frac{\beta-\alpha}{\beta}}W^{\frac{\beta+1}{\beta}}(t) \\
\geq\, & q(t)\left(\frac{\kappa(\tau^{-1}(\tau^{-1}(\sigma(t))))}{\kappa(\sigma(t))}P^*(\tau^{-1}(\sigma(t)))\right)^{\beta} + \frac{\beta N_2}{a^{1/\alpha}(t)}W^{\frac{\beta+1}{\beta}}(t). \qquad (53)
\end{aligned}$$

Thus, combining (52) and (53), we have

$$W'(t) \geq q(t)\left(\frac{\kappa(\tau^{-1}(\tau^{-1}(\sigma(t))))}{\kappa(\sigma(t))}P^*(\tau^{-1}(\sigma(t)))\right)^{\beta} + \frac{N}{a^{1/\alpha}(t)}W^{\frac{\theta+1}{\theta}}(t), \qquad (54)$$

where $\theta = \max\{\alpha, \beta\}$ and $N = \begin{cases} \alpha, & \alpha = \beta \\ K, & \alpha \neq \beta \end{cases}$.

Multiplying both sides of (54) by $\kappa^{\theta}(t)$ and integrating from $T$ to $t$, we obtain

$$\begin{aligned}
\int_T^t \kappa^{\theta}(s)q(s)\left(\frac{\kappa(\tau^{-1}(\tau^{-1}(\sigma(s))))}{\kappa(\sigma(s))}P^*(\tau^{-1}(\sigma(s)))\right)^{\beta} ds \leq & \int_T^t \kappa^{\theta-1}(s)a^{-1/\alpha}(s)[\theta W(s) \\
& - N\kappa(s)W^{\frac{\theta+1}{\theta}}(s)]ds + \kappa^{\theta}(t)W(t) - \kappa^{\theta}(T)W(T).
\end{aligned}$$

Let $y = W(s)$, $E = \theta$, and $F = N\kappa(s)$. By Lemma 1, we arrive at

$$\begin{aligned}
&\int_T^t \kappa^{\theta}(s)q(t)\left(\frac{\kappa(\tau^{-1}(\tau^{-1}(\sigma(t))))}{\kappa(\sigma(t))}P^*(\tau^{-1}(\sigma(t)))\right)^{\beta} ds \\
&\qquad \leq \int_T^t \frac{\epsilon}{\kappa(s)a^{1/\alpha}(s)}ds + \kappa^{\theta}(t)W(t) - \kappa^{\theta}(T)W(T).
\end{aligned}$$

Thus, we have

$$\int_T^t \left[\kappa^{\theta}(s)q(s)\left(\frac{\kappa(\tau^{-1}(\tau^{-1}(\sigma(s))))}{\kappa(\sigma(s))}P^*(\tau^{-1}(\sigma(s)))\right)^{\beta} - \frac{\epsilon}{\kappa(s)a^{1/\alpha}(s)}\right]ds \leq m,$$

where $\epsilon = (\frac{\theta}{\theta+1})^{\theta+1}(\frac{\theta}{N})^{\theta}$, which contradicts condition (36). The proof is complete. □

Clearly, Theorem 3 is also a generalization of ([27], Theorem 2.5).

## 3. Example

**Example 1.** *Consider the following second-order neutral differential equation*

$$(|z'(t)|^{\alpha-1}z'(t))' + q(t)|x(\frac{t}{2})|^{\beta-1}x(\frac{t}{2}) = 0, \qquad (55)$$

*where $z(t) = x(t) + 8x(\frac{t}{8})$, $a(t) \equiv 1$, $p(t) = 8$, $\alpha > 0$, $\beta > 0$, $\tau(t) = \frac{t}{8}$, $\sigma(t) = \frac{t}{2}$, and $q(t) \in U_1 = \{bt^n | n \in \mathbb{Z}, n \geq -1, b \in \Re^+\}$. It is clear that $U_1$ is a semigroup under the usual multiplication operation.*

*Letting $l = \frac{1}{2}$, then $\int_t^{\infty} q(s)(P(\tau^{-1}(\sigma(s))))^{\beta}ds = \infty$ for $\beta > 0$. Thus, it is not difficult to verify that all conditions of Theorem 1 are satisfied. Therefore, Equation (55) is oscillatory.*

**Example 2.** *Consider the following second-order neutral differential equation*

$$(t^{-2}|z'(t)|^{\alpha-1}z'(t))' + q_0 t^{-3}|x(\frac{t}{8})|^{\beta-1}x(\frac{t}{8}) = 0, \tag{56}$$

*where* $z(t) = x(t) + 8x(\frac{t}{2})$, $a(t) = t^{-2}$, $p(t) = 8$, $\alpha > 0$, $\beta > 0$, $\tau(t) = \frac{t}{2}$, $\sigma(t) = \frac{t}{8}$, $\tau^{-1}(\sigma(t)) = \frac{t}{4}$, $q(t) = q_0 t^{-3}, q_0 > 0$, *and* $l = \frac{1}{2}$. *Note that Equation (56) satisfies the canonical case.*

$$M(t) = \frac{q_0}{2}\left(\frac{1}{16}\right)^{\beta} t^{-2},$$

$$A(t) = \frac{1}{4}Q\vartheta\phi^{\frac{2}{\vartheta}}(t).$$

*For* $\alpha > \beta$, $\phi^{\frac{2}{\vartheta}}(t) = t^{\frac{2}{\vartheta}}$, *and* $\vartheta = \beta$, *we obtain*

$$\begin{aligned}
&\liminf_{t\to\infty} \frac{1}{M(t)} \int_t^\infty A(s)[M(s)]^{\frac{\beta+1}{\beta}} ds \\
&= \liminf_{t\to\infty} \frac{2}{q_0}16^\beta t^2 \int_t^\infty \frac{1}{4}Q\beta s^{\frac{2}{\beta}} \left(\frac{q_0}{2}(\frac{1}{16})^\beta s^{-2}\right)^{\frac{\beta+1}{\beta}} ds \\
&= \liminf_{t\to\infty} (\frac{q_0}{32})^{\frac{1}{\beta}} \frac{Q\beta}{4} t = \infty.
\end{aligned}$$

*For* $\alpha \le \beta$, $\phi^{\frac{2}{\vartheta}}(t) = (\frac{t}{4})^{\frac{2}{\vartheta}}$, *and* $\vartheta = \alpha$, *we obtain*

$$\begin{aligned}
&\liminf_{t\to\infty} \frac{1}{M(t)} \int_t^\infty A(s)[M(s)]^{\frac{\alpha+1}{\alpha}} ds \\
&= \liminf_{t\to\infty} \frac{2}{q_0}16^\beta t^2 \int_t^\infty \frac{1}{4}Q\alpha(\frac{1}{4})^{\frac{2}{\alpha}} s^{\frac{2}{\alpha}} \left(\frac{q_0}{2}(\frac{1}{16})^\beta s^{-2}\right)^{\frac{\alpha+1}{\alpha}} ds \\
&= \liminf_{t\to\infty} (\frac{q_0}{2})^{\frac{1}{\alpha}} (\frac{1}{16})^{\frac{\beta+1}{\alpha}} \frac{Q\alpha}{4} t = \infty.
\end{aligned}$$

*It is clear that all conditions of Theorem 2 are satisfied if* $q_0 > 0$. *Therefore, Equation (56) is oscillatory.*

**Remark 2.** *When* $\alpha \ne \beta \ne 1$, *the oscillation criteria of [13,18,19,22,23,26] cannot be applied to Equation (56) because they are different equations, and the oscillation criteria of [21,27–29] cannot be applied to Equation (56) because* $p(t) > 1$.

*When* $\alpha = \beta = 1$, *Equation (56) becomes the following special case*

$$(t^{-2}z'(t))' + q_0 t^{-3}x(t/8) = 0 \tag{57}$$

*where* $z(t) = x(t) + 8x(t/2)$, $\alpha = 1$, *and* $\beta = 1$. *According to Example 2, we know that Equation (57) satisfies the canonical case and is oscillatory if* $q_0 > 0$ *by Theorem 2.*

*Using Corollary 4 of [13] or Theorem 2.8 of [22] or Theorem 2.2 of [26] (letting* $\varphi(t) = t^2$*), it is not difficult to verify that Equation (57) is oscillatory if* $q_0 > 0$. *However, the additional condition* $\sigma \circ \tau = \tau \circ \sigma$ *is necessary in these results. In [23], Li et al. considered Equation (57) under the condition* $\beta > 1$. *Thus, the oscillation criteria of [23] cannot be applied to Equation (57). The oscillation criteria of [18,19] cannot be applied to Equation (57) because they considered the non-canonical case.*

**Example 3.** *Consider the following second-order neutral differential equation*

$$(t^8|z'(t)|^3z'(t))' + q_0 t^5|x(\frac{t}{8})|x(\frac{t}{8}) = 0, \tag{58}$$

*where* $z(t) = x(t) + 8x(\frac{t}{2})$, $a(t) = t^8$, $p(t) = 8$, $\alpha = 4$, $\beta = 2$, $\tau(t) = \frac{t}{2}$, $\sigma(t) = \frac{t}{8}$, $\tau^{-1}(\sigma(t)) = \frac{t}{4}$, $q(t) = q_0 t^5$, $q_0 > 0$.

*According to Equation (58), we know that Equation (58) satisfies the non-canonical case. Thus, letting* $\delta(t) = 1$ *and* $l = \frac{1}{2}$, *we obtain*

$$\lim_{t\to\infty}\int_T^t \left[q_0(\frac{1}{16})^2\right]s^5 ds = \infty,$$

*which implies that (35) holds. Letting* $N > 0$, *we have*

$$\lim_{t\to\infty}\int_T^t \left[\kappa^\theta(s)q(s)\left(\frac{\kappa(\tau^{-1}(\tau^{-1}(\sigma(s))))}{\kappa(\sigma(s))}P^*(\tau^{-1}(\sigma(s)))\right)^\beta - \frac{\epsilon}{\kappa(s)a^{1/\alpha}(s)}\right]ds$$
$$= \lim_{t\to\infty}\int_T^t \left[\frac{q_0 7^2}{128^2}s - \frac{4^9}{5^5 N^4}s^{-2}\right]ds = \infty,$$

*which implies that (36) holds. Thus, all conditions of Theorem 3 are satisfied. Therefore, Equation (58) is oscillatory.*

*If* $\alpha = 4$, $\beta = 5$, $q(t) = q_0 t^6$, *and other parameters remain unchanged, it is not difficult to verify that conditions (35) and (36) are satisfied. Thus, Equation (58) is oscillatory.*

*By view of* $\alpha = 4$ *and* $\beta = 2$, *the oscillation criteria of [10,13,18,19,22,23,26] cannot be applied to Equation (56) because they are different equations, and the oscillation criteria of [21,27–29] cannot be applied to Equation (56) because* $p(t) > 1$.

## 4. Conclusions

In this paper, by using the Riccati transformation and integral inequality technique, we establish several oscillation criteria for second-order Emden–Fowler neutral delay differential equations under the canonical case and non-canonical case, respectively. Compared with some recent results reported in the literature, we extend the range of the neutral coefficient. Therefore, our results generalize to some of the recent results reported in the literature. Furthermore, we provide some examples to verify our criteria. For researchers interested in this field, and as part of our future research, we would like to further investigate the oscillatory properties of the following even-order Emden–Fowler differential equation:

$$(a(t)|z^{(n-1)}(t)|^{\alpha-1}z^{(n-1)}(t))' + q(t)|x(\sigma(t))|^{\beta-1}x(\sigma(t)) = 0$$

under conditions (2) and (3), respectively, where $n$ is even.

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