Article

Parameterizations of Delaunay Surfaces from Scratch

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Abstract: Starting with the most fundamental differential-geometric principles we derive here new explicit parameterizations of the Delaunay surfaces of revolution which depend on two real parameters with fixed ranges. Besides, we have proved that these parameters have very clear geometrical meanings. The first one is responsible for the size of the surface under consideration and the second one specifies its shape. Depending on the concrete ranges of these parameters any of the Delaunay surfaces which is neither a cylinder nor the plane is classified unambiguously either as a first or a second kind Delaunay surface. According to this classification spheres are Delaunay surfaces of first kind while the catenoids are Delaunay surfaces of second kind. Geometry of both classes is established meaning that the coefficients of their fundamental forms are found in explicit form.

Keywords: Delaunay surfaces; surfaces of revolution; constant mean curvature surfaces; elliptic functions and integrals, roulettes

MSC: 53A04; 53A05; 53A10; 33E05

1. Introduction

By its very definition (cf. any book on differential geometry), the *mean curvature H* at some point of a given surface $\mathcal{S}$ in the three-dimensional Euclidean space is the arithmetic mean of the two principal curvatures (maximal and minimal ones) at that point.

In the most intuitive representation (the so-called Monge parameterization) of $\mathcal{S}$, which relies on the Cartesian coordinates (x, y, z) in the space $\mathbb{R}^3$, the surface is given as a graph defined over some domain U of the plane, i.e.,

$$\mathcal{S} = \{(x, y, z), \ z = z(x, y), \ (x, y) \in U \subset \mathbb{R}^2\}. \tag{1}$$

Then, according to the formula available in the classical differential geometry (see [1–4]), the mean curvature is expressed in the form

$$H(x, y) = \frac{r(1 + q^2) - 2pqs + t(1 + p^2)}{(1 + p^2 + q^2)^{3/2}} \tag{2}$$

in which

$$p = \frac{\partial z}{\partial x}(x, y), \qquad q = \frac{\partial z}{\partial y}(x, y)$$

$$r = \frac{\partial^2 z}{\partial^2 x}(x, y), \qquad s = \frac{\partial^2 z}{\partial x \partial y}(x, y), \qquad t = \frac{\partial^2 z}{\partial^2 y}(x, y).$$

Unfortunately, solving the equation

$$H(x, y) = c \equiv \text{const} \tag{3}$$

turns out to be a formidably complicated task!

In this situation the French astronomer and mathematician Charles-Eugène Delaunay immediately abandoned the general problem stated above and has restricted his considerations to axially-symmetric surfaces. In doing this he undertakes a genuinely geometric approach by showing that all profile curves (meridional sections) of the CMC surfaces of revolution (except for the plane) can be obtained as roulettes of the conic sections [5] (named after him *Delaunay roulettes*).

In fact, these Delaunay roulettes are nothing else, but traces of the foci of the non-degenerate conic sections when they roll (without slipping) along the axis of revolution (Figure 1).

In parallel, in a note to the same Delaunay's paper, Sturm [6] had proposed another approach of variational nature in which these surfaces are characterized as surfaces of revolution having a minimal lateral surface area for a given (fixed in advance) volume.

More recent accounts on the subject can found in the paper by Eells [7] and in the handbook by Krivoshapko-Ivanov [8] (pp. 155–157).

Let us recall that in the case of a rotational CMC surface the principle curvatures are known as a parallel κ_π and a meridional κ_μ curvatures and therefore one should solve the equation

$$H := \frac{\kappa_\pi + \kappa_\mu}{2} = c \qquad (4)$$

where $c \in \mathbb{R}$ is an arbitrary constant.

This has been done already by Delaunay in 1841, who has found also the first analytical (integral) representation of the *constant mean curvatures* (CMC) surfaces of revolution [5]. The complete list of such surfaces consists of planes, cylinders, sequences of spheres, catenoids, unduloids, and nodoids (see Figure 2), known nowadays just as *Delaunay surfaces*.

Cylinder	Sphere	Catenoid	Unduloid	Nodoid
Circle	Line Segment	Parabola	Ellipse	Hyperbola
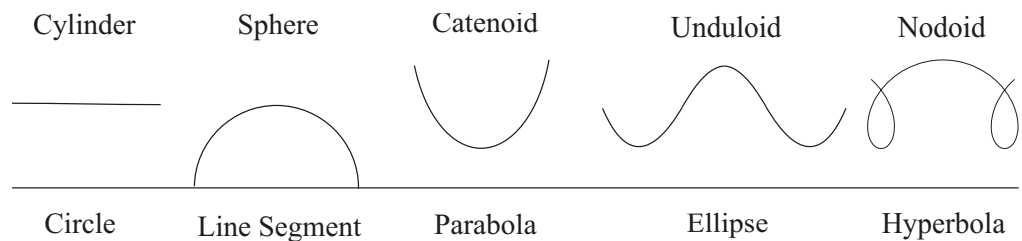
Figure 1. Delaunay roulettes (the profile curves of the Delaunay surfaces, excluding the plane) are generated as traces of the foci of the non-degenerated conic sections by rolling them along the axis of revolution (the horizontal line).
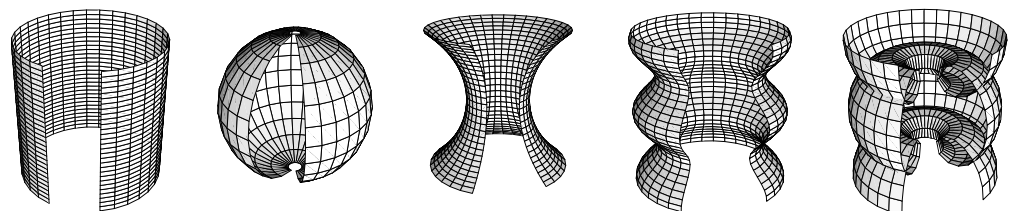
Figure 2. The open parts of the cylinder, sphere, catenoid, unduloid and nodoid shown here are obtained by revolving the profile curves in Figure 1 about symmetry axis taken this time in vertical position.

In his paper Delaunay has introduced the constant mean curvature of the surfaces in question in the form

$$H = \frac{1}{2}\left(\frac{1}{\mathcal{R}_\pi} + \frac{1}{\mathcal{R}_\mu}\right) = \frac{1}{2a} \tag{5}$$

where by definition $\mathcal{R}_\pi$ and $\mathcal{R}_\mu$ are the parallel and meridional curvature radii respectively and at the beginning the parameter a was just an arbitrary constant. Latter on this parameter has been recognized as the major semi-axis of the rolling conic.

In the variational approach respectively, a initially has played the role of the Lagrangian multiplier, but at the end it was also identified as a length of the semi-axis. In both considerations, during the processes of integrations another integration constant appears which was recognized readily as the length of the minor semi-axis of the rolling conic.

Viewed in this way, the situation is a little bit strange: the profile curve depends on two parameters, but the mean curvature of the generated rotational surface depends only on one of them!

In what follows the problem will be considered from a different viewpoint and the first step in this direction will be to rewrite the expression for the mean curvature H in another form.

This possibility is based on the obvious observation that any profile curve has a point which is situated at a maximal distance, say R from the symmetry axis. It is clear also that at such point the tangent to the curve is parallel to the symmetry axis of the surface and that its normal is parallel to the OX axis. From here, one can conclude that the radius of the parallel curvature $\mathcal{R}_\pi$ is just R and that along the circle (which will be further referred as equator) with this radius the meridional curvature $\mathcal{R}_\mu$ is also constant (due to the basic assumption (4) about the surface). The ratio of these radii, $\mathcal{R}_\pi(R)/\mathcal{R}_\mu(R)$ will be denoted further on by ν, i.e.,

$$\mathcal{R}_\pi(R) \equiv R = \nu \mathcal{R}_\mu(R). \tag{6}$$

So, we have in place of (5) the equation

$$H = \frac{1}{2}\left(\frac{1}{\mathcal{R}_\pi(R)} + \frac{1}{\mathcal{R}_\mu(R)}\right) = \frac{1}{2}\left(\frac{1}{R} + \frac{\nu}{R}\right) = \frac{\nu + 1}{2R}. \tag{7}$$

The bottom line of all above is that the problem of describing the Delaunay surfaces with mean curvature specified by (4) transforms into that of finding the axially-symmetric surfaces which mean curvature is fixed by formula (7).

Remark 1. *The above reasoning is valid mutatis mutandis for the minimal distance r as well!*

The actual organization of the paper is as follows. The main results are presented in the next Section 2 in which by relying on the first differential-geometric principles the fundamental equations behind the Delaunay surfaces are derived. Their solutions are formulated as two Propositions associated with the two different kinds of Delaunay surfaces and later on they are unified as Theorem 1 describing in explicit form all Delaunay surfaces. The above results are illustrated by various graphics which aims are to make them more clear.

As it is well known the intrinsic geometry of the surfaces in the Euclidean space is encoded into their fundamental forms which have been found explicitly in Section 3.

The Section 4 points out the importance of these surfaces for other branches of the sciences outside the mathematics. Finally, we have added an Appendix A in which we have collected all known (to the authors) explicit parameterizations of the Delaunay surfaces that can be useful for theoretical and applied purposes in the future.

2. The New Parameterizations

The starting point of our considerations will be some basic facts in the classical differential geometry which can be easily located in any of the above cited books [1–4].

Namely, let the axially-symmetric surface $\mathcal{S} \subset \mathbb{R}^3$ is generated by revolving about the "vertical" axis OZ a curve $\mathcal{C}$ lying in the XOZ-plane of the Cartesian coordinate system $OXYZ$ (Figure 3). If the *generating curve* $\mathcal{C}$ is parameterized as

$$\mathcal{C} = \{(x, z), \ z = z(x), \ x \geq 0\} \tag{8}$$

then the principal curvatures of $\mathcal{S}$ can be calculated by the well-known formulas

$$\kappa_\pi(x) = \frac{\dot{z}(x)}{x\left(1 + \dot{z}^2(x)\right)^{1/2}}, \qquad \kappa_\mu(x) = \frac{\ddot{z}(x)}{\left(1 + \dot{z}^2(x)\right)^{3/2}} \tag{9}$$

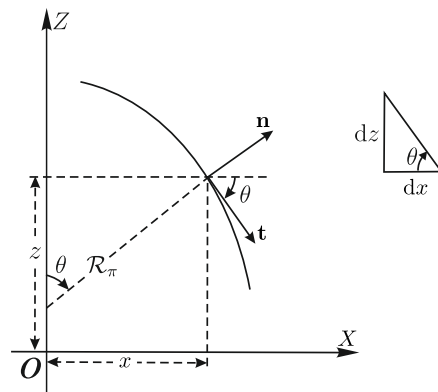where $\dot{z}(x) \equiv \mathrm{d}z/\mathrm{d}x$, etc.

Figure 3. Geometry of the generating profile curve $\mathcal{C}$.

Besides, by making use of the explicit formulas in (9) it is straightforward to check directly that the above expressions satisfy the fundamental identity

$$\kappa_\mu(x) = \frac{\mathrm{d}\left(x\kappa_\pi(x)\right)}{\mathrm{d}x}. \tag{10}$$

Combined with the defining Equation (4), the last equality produces (after integration) the solution

$$\kappa_\pi(x) = c + \frac{m}{x^2} \tag{11}$$

where $m \in \mathbb{R}$ is some integration constant. Inserting this back in (10) gives directly

$$\kappa_\mu(x) = c - \frac{m}{x^2}. \tag{12}$$

In this setting it remains only to fix the new integration constant m and for this purpose let us consider the last equality at $x \equiv R$. As it is well-known at such point of $\mathcal{S}$ we have (cf. (7))

$$c \equiv \frac{\nu + 1}{2R} \qquad \text{and} \qquad \kappa_\mu(R) = \frac{1}{\mathcal{R}_\mu(R)} = \frac{\nu}{\mathcal{R}_\pi(R)} = \nu\kappa_\pi(R) = \frac{\nu}{R} \tag{13}$$

and it is easy to conclude from (12) that

$$m \equiv \frac{1 - \nu}{2}R. \tag{14}$$

Having all this, we can write the final results for both curvatures as

$$\kappa_\pi(x) = \frac{1 + \nu}{2R} + \frac{(1 - \nu)R}{2x^2}, \qquad \kappa_\mu(x) = \frac{1 + \nu}{2R} - \frac{(1 - \nu)R}{2x^2}. \tag{15}$$

The natural question that arises: for what these formulas are useful, can be answered immediately.

First, we can solve the first equation in (9) with respect to $\dot{z}(x)$ and by doing this we obtain the differential equation

$$\frac{\mathrm{d}z(x)}{\mathrm{d}x} = \pm \frac{x\kappa_\pi}{\sqrt{1 - (x\kappa_\pi)^2}}. \tag{16}$$

Second, by inserting there the expression for $\kappa_\pi(x)$ from (15) we end up with the explicit equation

$$\frac{\mathrm{d}z(x)}{\mathrm{d}x} = \pm \frac{(\nu + 1)x^2 - (\nu - 1)R^2}{\sqrt{(R^2 - x^2)\left((\nu + 1)^2 x^2 - (\nu - 1)^2 R^2\right)}}. \tag{17}$$

Let us replace x with another independent variable, say χ, via the formula

$$x = R\chi. \tag{18}$$

Performing this change in (17) one can see that z is defined by the integral

$$z(\chi) = \pm R \int \frac{\left((\nu + 1)\chi^2 - \nu + 1\right)\mathrm{d}\chi}{\sqrt{(1 - \chi^2)\left((\nu + 1)^2 \chi^2 - (\nu - 1)^2\right)}}. \tag{19}$$

One thing that this formula make clear is that the parameters R and ν which appear in the expression (7) of the mean curvature are separated multiplicatively and that the problem of finding the CMC surfaces reduces to the evaluation of the integral in (19).

Technically, this depends on the location of the roots of the quartic polynomial under the radical sign. Two of them are known, namely, these are $\chi_{1,2} = \pm 1$ coming from the first factor in the quartic. It is easy to see that the other two roots can be easily found to be $\chi_{3,4} = \pm\sqrt{\sigma(\nu)}$, where

$$\sigma(\nu) = \left(\frac{\nu - 1}{\nu + 1}\right)^2. \tag{20}$$

Still on technical level one needs to know also if these roots are inside or outside of the interval $[-1, +1]$ on the real line. Some elementary analysis tells us that for $\nu \in (-\infty, 0)$, $\sqrt{\sigma(\nu)} > 1$ while for $\nu \in (0, +\infty)$, $\sqrt{\sigma(\nu)} < 1$.
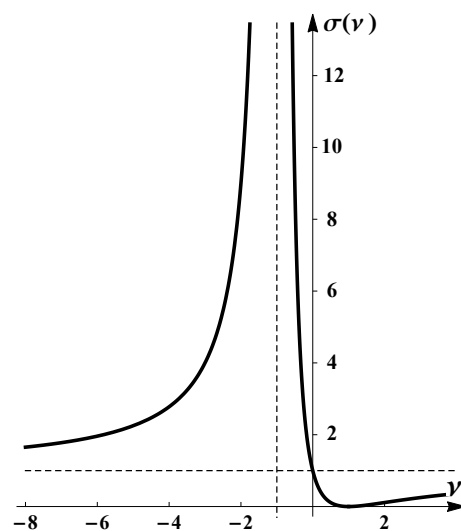
This situation is depicted by the graphics in Figure 4.

Figure 4. Graphics of σ versus ν. The dashed lines indicate the asymptotic values of the function.

We will consider first the case when $v > 0$, which implies that the integral in (19) can be cast in the form

$$z(\chi) = \pm R \int \frac{\left(\chi^2 - \frac{v-1}{v+1}\right) d\chi}{\sqrt{(1-\chi^2)(\chi^2 - \sigma(v))}}. \tag{21}$$

Taking into account that the uniformization of the integral

$$\int \frac{d\chi}{\sqrt{(1-\chi^2)(\chi^2 - \sigma(v))}} \tag{22}$$

meaning its reduction to some trivial integrals, is ensured by the Jacobian elliptic function $dn(u, k)$, where u is the uniformization variable and $k^2 = 1 - \sigma(v)$ is the so-called elliptical modulus, the result of the integration can be written in the form

$$z = \pm R \left(E\big(am(u, k(v)), k(v)\big) - \frac{v-1}{v+1} u \right), \qquad k(v) = \frac{2\sqrt{v}}{v+1}, \qquad v \in \mathbb{R}^+. \tag{23}$$

Here $am(u, k)$ is the Jacobian amplitude function and $\mathbb{R}^+$ denotes the set of all positive real numbers. More details about Jacobian elliptic function, elliptic integrals and related subjects can be found in any of the books [9,10] and the well-known Tables of elliptic integrals [11–14].

We should add also that all of the above mentioned differential geometry books [1–4] offer an implementation of the computer algebra systems for visualization of the analytical results and in this way provide a great help in increasing the reader's understanding.

Figurea 5 and 6 below illustrate an unduloid and a nodoid generated via formula (23).
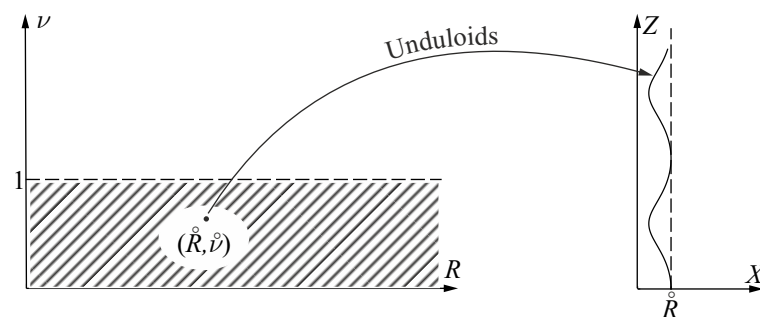
Figure 5. Any point within the striped area where $v < 1$ on the left produces (via Equation (23)) an unduloid shown on the right.
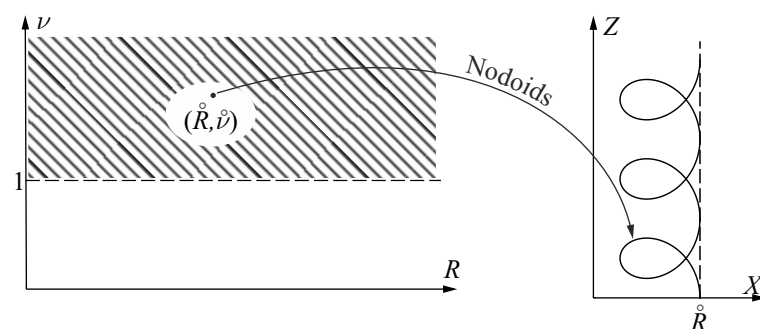
Figure 6. With any point in the striped region $v > 1$ in the left part is associated a nodoid (via Equation (23)) shown on the right side of the figure.

Remark 2. *Let us notice that in the special case when $\nu \equiv 1$ the integral in* (21) *amounts to*

$$z = \pm R \sqrt{1 - \chi^2}. \tag{24}$$

Taken together with (18) *they represent a circle $x^2 + z^2 = R^2$ of radius R playing the role of the profile curve of the sphere with the same radius.*

Our next task will be the case when $\nu < 0$, i.e., the parameter is assumed to be negative. Then, the relevant integral for z is of the form

$$z(\chi) = \pm r \int \frac{((\nu + 1)\chi^2 - \nu + 1)\mathrm{d}\chi}{\sqrt{(\chi^2 - 1)((\nu - 1)^2 - (\nu + 1)^2 \chi^2)}}, \qquad \nu \in \mathbb{R}^-. \tag{25}$$

It will be convenient to replace the negative number ν in the above integral with the positive number $\tilde{\nu}$ via the formula

$$\nu = -\tilde{\nu}, \qquad \tilde{\nu} \in \mathbb{R}^+ \tag{26}$$

Performing the same change in (25) we end up with the problem of evaluation of the integral

$$
\begin{aligned}
z(\chi) &= \pm r \int \frac{(\tilde{\nu} + 1 - (\tilde{\nu} - 1)\chi^2)\mathrm{d}\chi}{\sqrt{(\chi^2 - 1)((\tilde{\nu} + 1)^2 - (\tilde{\nu} - 1)^2 \chi^2)}} \\[2mm]
&= \pm r \int \frac{\left(1 - \frac{\tilde{\nu} - 1}{\tilde{\nu} + 1}\chi^2\right)\mathrm{d}\chi}{\sqrt{(\chi^2 - 1)\left(1 - \left(\frac{\tilde{\nu} - 1}{\tilde{\nu} + 1}\right)^2 \chi^2\right)}}, \qquad \tilde{\nu} \in \mathbb{R}^+.
\end{aligned}
\tag{27}
$$

In this case the integral that has to be uniformized is

$$\int \frac{\mathrm{d}\chi}{\sqrt{(\chi^2 - 1)\left(1 - \left(\frac{\tilde{\nu} - 1}{\tilde{\nu} + 1}\right)^2 \chi^2\right)}} \tag{28}$$
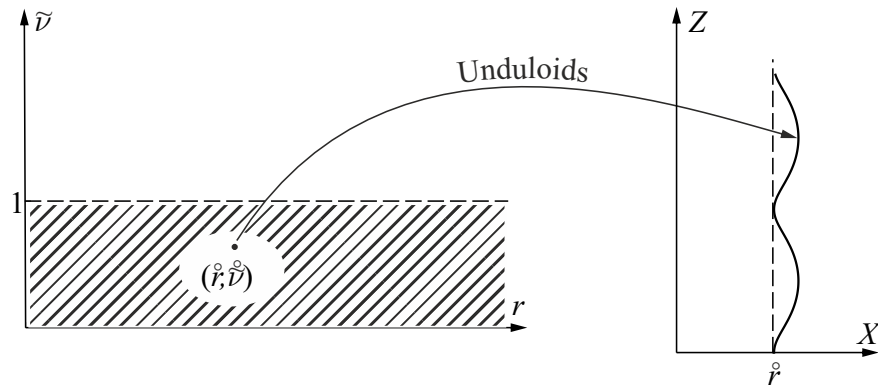
and this can be achieved introducing as uniformization variable u via the function

$$\chi = \frac{1}{\mathrm{dn}(u, k(\tilde{\nu}))}, \qquad k(\tilde{\nu}) = \sqrt{1 - \left(\frac{\tilde{\nu} - 1}{\tilde{\nu} + 1}\right)^2} = \frac{2\sqrt{\tilde{\nu}}}{\tilde{\nu} + 1}. \tag{29}$$

In these circumstances the evaluation of the integral in (27) produces at the very end the result

$$z(u) = \pm r \left(u - \frac{\tilde{\nu} + 1}{\tilde{\nu} - 1} \left(E(am(u, k(\tilde{\nu})), k(\tilde{\nu})) - k^2(\tilde{\nu}) \frac{\mathrm{sn}(u, k(\tilde{\nu}))\mathrm{cn}(u, k(\tilde{\nu}))}{\mathrm{dn}(u, k(\tilde{\nu}))} \right) \right). \tag{30}$$

An unduloid and a nodoid generated this time via the above formula are shown below in Figures 7 and 8.

Figure 7. With any point in the striped region $\nu < 1$ in the left part is associated an unduloid (via Equation (30)) shown on the right hand side of the figure.
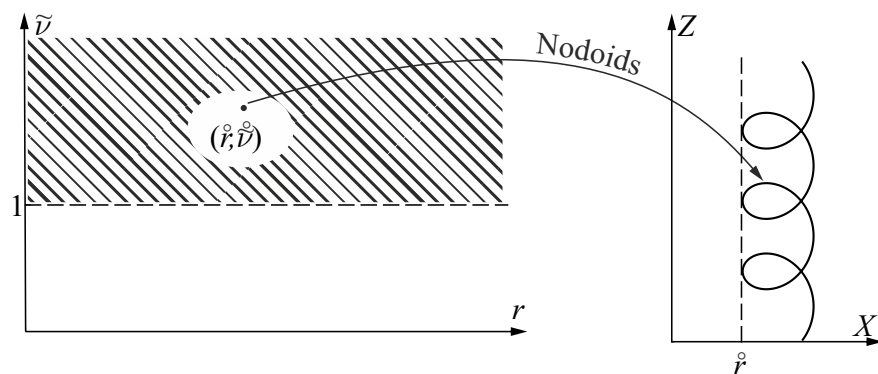
Figure 8. Any point in the striped region $\nu > 1$ on the left hand side generates a nodoid (via Equation (30)) shown on the right hand side of the figure.

Remark 3. *Let us notice that in the special case when $\tilde{\nu} \equiv 1$ the integral in (28) amounts to*

$$z = \pm r \int \frac{\mathrm{d}\chi}{\sqrt{\chi^2 - 1}} \tag{31}$$

and its uniformization is provided by $\chi = \cosh(u)$, so that we have also

$$z = r\,u \qquad \Longleftrightarrow \qquad u = \frac{z}{r}. \tag{32}$$

On the other side according to (18) we have as well

$$x = r\,\chi = r\cosh\left(\frac{z}{r}\right) \tag{33}$$

which is obviously the equation of the catenary with a focal distance r.

Actually, the results obtained above can be unified as the following propositions.

Proposition 1. *With any point within the first quadrant in the (R, ν) plane is associated a unique Delaunay surface which profile curve in XOZ plane lies inside the strip $(0, \mathring{R}) \times \mathbb{R}$. The profile curves of the line $R \times \{1\}$ are semi-circles which radii coincide with the respective $\mathring{R}$.*

Their explicit parameterizations through the parameters in the specified range are given by the formulas

$$x = R\,\mathrm{dn}\big(u, k(\nu)\big), \quad z = \pm R\left(E\big(am(u, k(\nu)), k(\nu)\big) - \frac{\nu-1}{\nu+1}u\right), \quad k(\nu) = \frac{2\sqrt{\nu}}{\nu+1} \tag{34}$$

or

$$x = R\chi, \qquad z = \pm R\sqrt{1-\chi^2}, \qquad \chi \in [0,1], \qquad \text{when} \qquad \nu \equiv 1.$$

respectively.

Proposition 2. *With any point within the first quadrant in the $(r, \tilde{\nu})$ plane is associated a unique Delaunay surface which profile curve in XOZ plane lies outside the strip $(0, \mathring{r}) \times \mathbb{R}$. The profile curves of the line $r \times \{1\}$ are catenaries which focal distances coincide with the respective $\mathring{r}$.*

Parameterizations of these surfaces associated with the positive numbers $(r, \tilde{\nu})$ are provided by the formulas

$$x = \frac{r}{\mathrm{dn}(u,k(\tilde{\nu}))}, \quad z = \pm r\left(u - \frac{\tilde{\nu}+1}{\tilde{\nu}-1}\left(E\big(am(u, k(\tilde{\nu})), k(\tilde{\nu})\big) - k^2(\tilde{\nu})\frac{\mathrm{sn}(u,k(\tilde{\nu}))\mathrm{cn}(u,k(\tilde{\nu}))}{\mathrm{dn}(u,k(\tilde{\nu}))}\right)\right) \tag{35}$$

or

$$x = r\cosh\left(\frac{z}{r}\right), \qquad z \in [-\infty, +\infty], \qquad \text{when} \qquad \tilde{\nu} \equiv 1.$$

Definition 1. *Surfaces described in Proposition 1 will be referred further on as Delaunay surfaces of first kind and those introduced in Proposition 2 as Delaunay surfaces of second kind.*

Now we are ready to formulate our results as

Theorem 1. *The analytical descriptions of all non-trivial Delaunay surfaces of the first, respectively the second kind, are given by formulas (34) and (35) which are illustrated in Figures 9 and 10.*
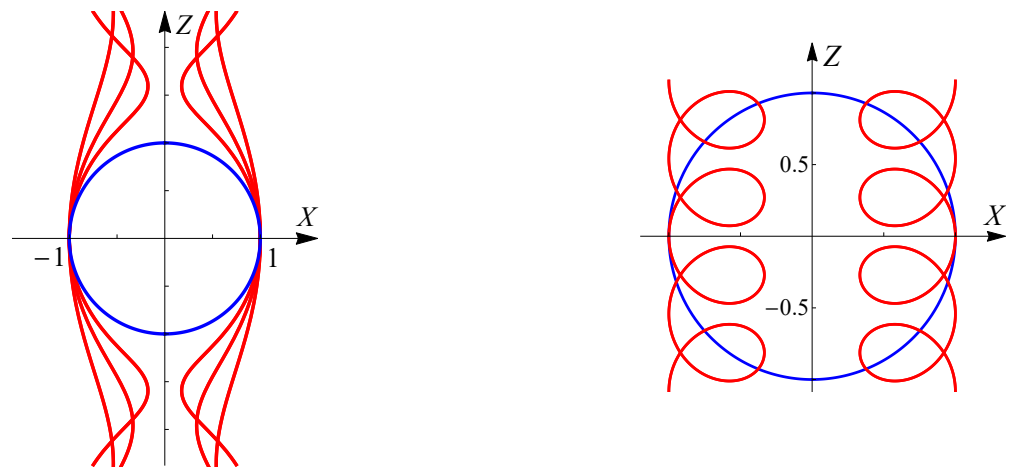

Figure 9. An ensemble of undularies $\mathcal{U}_{\nu}^{\mathrm{in}}$ (left) and a circle $\mathcal{C}_1$ along with an ensemble of nodaries $\mathcal{N}_{\nu}^{\mathrm{in}}$ and with a different scaling the same circle $\mathcal{C}_1$ (in blue) on the right.
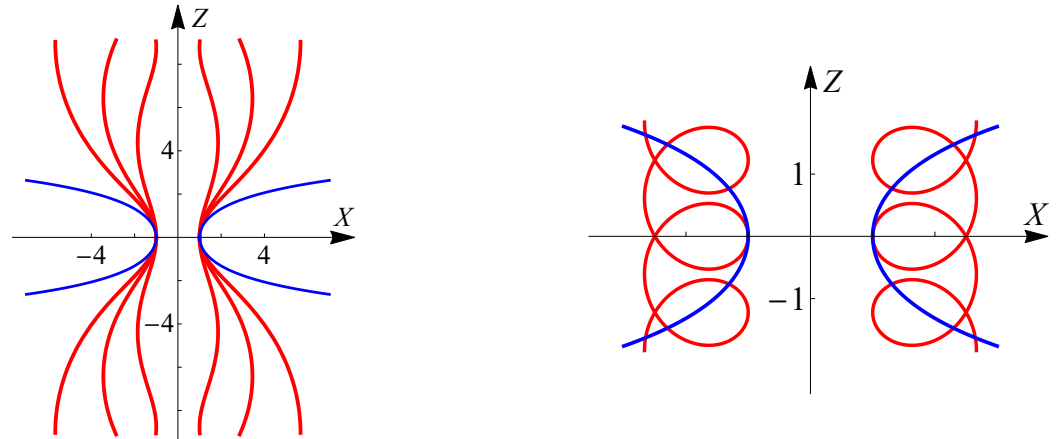
Figure 10. Portions of a few undularies $\mathcal{U}_\nu^{\text{out}}$ (**left**), along a nodary $\mathcal{N}_\nu^{\text{out}}$ (**right**) in the presence of a catenary $\tilde{\mathcal{C}}_1$ in both cases.

All results presented up to now can be illustrated schematically as shown in Figure 11.
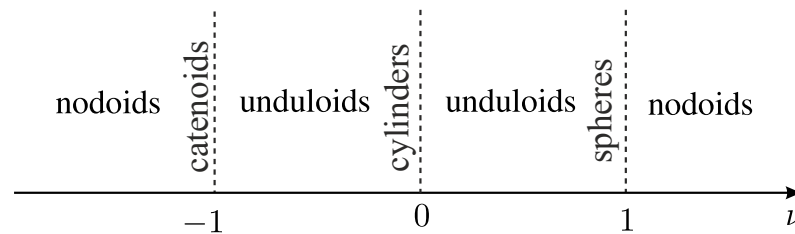
Figure 11. Distribution of the Delaunay surfaces specified by the chosen range of $\nu \in \mathbb{R}$. The normal direction to the horizontal line is reserved for the first parameter R from the suggested classification of the Delaunay surfaces.

3. Geometry of Delaunay Surfaces

One of the most fundamental theorem in the classical differential geometry [1–4] states that any surface $\mathcal{S}$

$$\mathbf{x}(u,v) = (x(u,v), y(u,v), z(u,v)), \qquad (u, v) \in \mathcal{D} \subset \mathbb{R}^2 \tag{36}$$

in the Euclidean space $\mathbb{R}^3$ is defined up to a motion there (i.e., translation and rotation) by the coefficients of its first and second fundamental forms

$$I = E\,\mathrm{d}u^2 + 2F\,\mathrm{d}u\,\mathrm{d}v + G\,\mathrm{d}v^2\,, \qquad II = \ell\,\mathrm{d}u^2 + 2m\,\mathrm{d}u\,\mathrm{d}v + n\,\mathrm{d}v^2. \tag{37}$$

These coefficients are given by the formulas

$$E = E(u,v) = \mathbf{x}_u.\,\mathbf{x}_u, \qquad F = F(u,v) = \mathbf{x}_u.\,\mathbf{x}_v, \qquad G = G(u,v) = \mathbf{x}_v.\,\mathbf{x}_v$$

$$\tag{38}$$

$$\ell = \ell(u,v) = \mathbf{x}_{uu}.\,\mathbf{U}, \qquad m = m(u,v) = \mathbf{x}_{uv}.\,\mathbf{U}, \qquad n = n(u,v) = \mathbf{x}_{vv}.\,\mathbf{U}$$

where $\mathbf{U}$ is the unit normal vector to $\mathcal{S}$

$$\mathbf{U} = \mathbf{U}(u,v) = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|}. \tag{39}$$

In our settings and after some tedious computations we end up with the following results. For the first kind Delaunay surfaces these coefficients turns out to be

$$E = \frac{4R^2\mathrm{dn}^2(u, k(v))}{(v+1)^2}, \qquad F \equiv 0, \qquad G = R^2\mathrm{dn}^2(u, k(v))$$

$$\ell = \frac{2R\left(\dfrac{v-1}{v+1} + \mathrm{dn}^2(u,k(v))\right)}{v+1}, \qquad m = 0, \qquad n = R\left(1 - \frac{2v}{v+1}\mathrm{sn}^2(u,k(v))\right). \tag{40}$$

Respectively, for the second kind Delaunay surfaces we have

$$E = \frac{4r^2}{(v+1)^2\mathrm{dn}^2(u, k(\tilde{v}))}, \qquad F \equiv 0, \qquad G = \frac{r^2}{\mathrm{dn}^2(u, k(\tilde{v}))}$$

$$\ell = \frac{2r\left(\dfrac{1-\tilde{v}}{\tilde{v}+1} - \mathrm{dn}^2(u,k(\tilde{v}))\right)}{(\tilde{v}+1)\mathrm{dn}^2(u,k(\tilde{v}))}, \qquad m = 0, \qquad n = \frac{r\left(1 - \dfrac{2\tilde{v}}{\tilde{v}+1}\mathrm{sn}^2\left(u,k(\tilde{v})\right)\right)}{\mathrm{dn}^2(u, k(\tilde{v}))}. \tag{41}$$

In the above cited books on differential geometry [1–4] the reader will find also some classical formulas which describe the two principal types of curvatures at every point of the surface. These are the so-called Gaussian curvature K, and the mean (meaning average) curvature H, the latter being of immediate interest here, i.e.,

$$K = \frac{\ell n - m^2}{EG - F^2}, \qquad H = \frac{En + G\ell - 2Fm}{2(EG - F^2)}. \tag{42}$$

It can be checked that the second formula above produces for the first kind of Delaunay surfaces the result

$$H = \frac{v+1}{2R} \tag{43}$$

while for the second kind of Delaunay surfaces one obtains

$$H = \frac{1 - \tilde{v}}{2r}. \tag{44}$$

However, despite that they seem different, one should keep in mind that according to its very definition (26), the parameter $\tilde{v}$ has just the opposite value of v, so that the above formulas produce obviously absolutely identical results (as should be).

Remark 4. *Having such complete and explicit description of the Delaunay surfaces opens the possibility to consider more subtle features of these surfaces like geodesics on them. Some preliminary results about geodesics on the unduloids can be found in [15].*

4. Conclusions

Up to now we have been interested in purely mathematical questions like how to describe Delaunay surfaces starting with the first principles in differential geometry?

Actually, using the obtained results we have been able to characterize them globally by some intrinsic entities like the coefficients of their fundamental forms or (as we shall see in Appendix A) via the intrinsic equation of their profile curves.

However, the historical development does not follow exactly the same route. The fundamental fact established by Sturm [6] about minimal property of the lateral area for a fixed volume was quite essential in the treatment of soap bubbles and liquid drops [16–19] as become evident by Plateau's [20] experiments. In fact, Plateau is also the originator who coined the names *Undulary* and *Nodary*. More recently, these surfaces have been recognized to play the main role in the explanation of the myelin shapes [21], the "pearling" states [22], the capillarity models [23] and that of cells under compression in biology [24,25].

In view of their importance in science and technology we have collected the available parameterizations in Appendix A below and we will be much obliged to the tempted readers for informing us about any other parameterization which they know but is not listed inside it.

Another our hope is that some of the talented readers of the *Mathematics* will succeed in finding the relationships between different parameters entering in these parameterizations.

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Appendix A

Here we collect the available parameterizations of the Delaunay surfaces starting with the seminal work of Delaunay on the subject. In all cases the original notation have been preserved.

- **1841** Ch. Delaunay [5] (p. 313)

$$
\begin{aligned}
x &= a\sin\varphi - a\tan\varphi\sqrt{1+\alpha-\sin^2\varphi} + \int_0^\varphi \frac{a\alpha\,\mathrm{d}\varphi}{\cos^2\varphi\sqrt{1+\alpha-\sin^2\varphi}} + \beta \\
z &= -a\cos\varphi + a\sqrt{1+\alpha-\sin^2\varphi}\,, \qquad \varphi\in\mathbb{R}.
\end{aligned}
\tag{A1}
$$

  Here $a\neq 0$, $\alpha > -1$ and β – are three arbitrary real constants. The symmetry axis of the generated surfaces is OX and their mean curvatures are $\dfrac{1}{2a}$.

- **1886** A. Lane [26] (p. 133)
  Introducing

$$
f(\alpha) = \int_0^\alpha \frac{a^2 b^2\,\mathrm{d}\alpha}{(a^2\cos^2\alpha + b^2\sin^2\alpha)^{3/2}}
\tag{A2}
$$

  in which a and b are semi-axes of the ellipse rolling on the line, he presents the respective *roulette* in the form

$$
\begin{aligned}
x &= f(\alpha) + \frac{c^2\sin\alpha\cos\alpha}{(a^2\cos^2\alpha + b^2\sin^2\alpha)^{1/2}} - \alpha\sin\alpha \\
y &= (a^2\cos^2\alpha + b^2\sin^2\alpha)^{1/2} - \alpha\cos\alpha, \qquad c^2 = a^2 - b^2.
\end{aligned}
\tag{A3}
$$

  Delaunay or Sturm are not mentioned in the paper at all.

- **1911** G. Loria [27] (p. 127)
  The symmetry axis of the surfaces is OX, a and e are real parameters and s is the natural parameter along the curve. Then

$$
y^2 = a^2\left[1 - 2e\cos\frac{s}{a} + e^2\right]
\tag{A4}
$$

  is the equation of the profile curve in which e is the eccentricity of the rolling curve. Loria gives credit about this result to the relatively obscure paper published in 1868 by Spitzer [28], in which he does not mention either Delaunay or Sturm.

However on the same page Loria presents also the *natural equation* of Delaunay curves in the form

$$\mathcal{R}(s) = a \, \frac{1 - 2\,e \cos\frac{s}{a} + e^2}{e\,(e - \cos\frac{s}{a})}. \tag{A5}$$

The natural equation is listed here for later comparison.

- **1969** M. Hady [29] (pp. 380–381)
  The profile curve with symmetry axis OZ is parameterized by the formulas

$$x = u, \qquad z = \pm \int \frac{(H_0 u^2 + C)\mathrm{d}u}{\sqrt{u^2 - (H_0 u^2 + C)^2}} \tag{A6}$$

in which H_0 and C are arbitrary real constants.

Neither Delaunay, Sturm or some of the above mentioned people were cited in the paper.

- **1980** K. Kenmotsu [30] (p. 150)
  The profile curve of CMC surfaces with symmetry axis OX is presented in the form

$$X(s; H, B) = \left(\int_0^s \frac{(1 + B \sin 2Ht)}{1 + B^2 + 2B \, \sin 2Ht)^{1/2}} \, \mathrm{d}t, \ \frac{1}{2\,|H|}\left(1 + B^2 + 2B \sin 2Hs\right)^{1/2} \right) \tag{A7}$$

in which s is the arc-length parameter, H denotes the fixed mean curvature and B is another real constant.

An updated version of this work along nice pictures can be found in the book [31].

- **2002** I. Mladenov [32] (pp. 22–24)
  The derived parameterization of the Delaunay surfaces is of the form

$$
\begin{aligned}
x[u,v] &= RE(u,k) \pm r\,F(u,k) - Rk^2 \, \frac{\sin u \cos u}{\Delta(u,k)} & u \in (-\infty, \infty) \\
y[u,v] &= \frac{r}{\Delta(u,k)} \cos v, \qquad z[u,v] = \frac{r}{\Delta(u,k)} \sin v, \quad v \in [0, 2\pi].
\end{aligned}
\tag{A8}
$$

in which R, r have the meaning used up to now, $k^2 = \frac{R^2 - r^2}{R^2}$, $\Delta(u,k) = \sqrt{1 - k^2 \sin^2 u}$ and $\pm$ signs refer to unduloids/nodoids.

- **2002** I. Mladenov [33] (p. 329)
  The symmetry axis in this case is OX, $K(k)$ and $E(k)$ denote the complete elliptic integrals of the first, respectively second kind [11–14]. The parameterization is provided by the formulas

$$
\begin{aligned}
x(u) &= a\,(1 - \varepsilon)\left[F(\tfrac{2u - \pi}{4}, k) + K(k)\right] + a\,(1 + \varepsilon)\left[E(\tfrac{2u - \pi}{4}, k) + E(k)\right] \\
y(u) &= a\sqrt{1 + \varepsilon^2 + 2\varepsilon \sin u}, \qquad k = \frac{2\sqrt{\varepsilon}}{1 + \varepsilon}, \qquad \varepsilon \in \mathbb{R}^+.
\end{aligned}
\tag{A9}
$$

The coefficients of the first and the second fundamental forms for this parameterization are found to be

$$
\begin{aligned}
E &= a^2, \qquad F = 0, \qquad G = a^2\,(1 + \varepsilon^2 + 2\,\varepsilon \sin u) \\
L &= \frac{a\,\varepsilon\,(\varepsilon + \sin u)}{1 + \varepsilon^2 + 2\,\varepsilon \sin u}, \qquad M = 0, \quad N = a\,(1 + \varepsilon \sin u)
\end{aligned}
\tag{A10}
$$

and applying the formula from (42) one immediately obtains that

$$H \equiv \frac{1}{2a}. \tag{A11}$$

- **2003** I. Mladenov and J. Oprea [15] (p. 220)
  The parameterization derived in this paper has the form

  $$
  \begin{aligned}
  x(u) &= a\,(1-\varepsilon)\,[F(\mathrm{sn}(u,k),k)+K(k)] + a\,(1+\varepsilon)\,[E(\mathrm{sn}(u,k),k)+E(k)] \\
  y(u) &= a(1+\varepsilon)\,\mathrm{dn}(u,k), \qquad \varepsilon = \sqrt{1-\tfrac{b^2}{a^2}}, \qquad k = \tfrac{2\sqrt{\varepsilon}}{1+\varepsilon}
  \end{aligned}
  \tag{A12}
  $$

  while the coefficients of the first and the second fundamental forms were found to be

  $$
  \begin{aligned}
  E &= 4a^2\,\mathrm{dn}^2(u,k), \qquad F = 0, \qquad G = a^2(1+\varepsilon)^2\,\mathrm{dn}^2(u,k) \\
  L &= \tfrac{4\,a\,\varepsilon}{1+\varepsilon}\left(1 - \tfrac{2}{1+\varepsilon}\,\mathrm{sn}^2(u,k)\right), \quad M = 0, \quad N = a\,(1+\varepsilon)\left(1 - \tfrac{2\varepsilon}{1+\varepsilon}\,\mathrm{sn}^2(u,k)\right).
  \end{aligned}
  \tag{A13}
  $$

- **2004** I. Mladenov [34] (pp. 164–166)
  This paper presents the derivation of the conformal parameterizations for *unduloids*

  $$
  \begin{aligned}
  x(u) &= \mathrm{dn}(u,k) + k\mathrm{cn}(u,k), \qquad z(u) = E(\mathrm{am}(u,k),k) + k\mathrm{sn}(u,k) \\
  k^2 &= 1-m^2, \qquad m^2 < 1
  \end{aligned}
  \tag{A14}
  $$

  and respectively, *nodoids*

  $$
  \begin{aligned}
  x(u) &= \mathrm{dn}(u/k,k) + k\mathrm{cn}(u/k,k), \qquad k^2 = \tfrac{1}{1+m^2}, \qquad m^2 > 0 \\
  z(u) &= E(\mathrm{am}(u/k,k),k) + k\mathrm{sn}(u/k,k) - \tfrac{1-k^2}{k}u.
  \end{aligned}
  \tag{A15}
  $$

  An alternative approach for finding such parameterizations can be seen in [35].
- **2008** I. Mladenov, M. Hadzhilazova, P. Djondjorov and V. Vassilev [36] (p. 278)
  The profile curve which symmetry axis is taken to be OZ, is parameterized via the natural parameter s in the form $(r(s),z(s))$. Relying on the intrinsic equation of the Delaunay curves

  $$
  \dot{\kappa} = 2(\lambda-\kappa)\sqrt{\frac{\lambda-\kappa}{\mu} - (2\lambda-\kappa)^2}
  \tag{A16}
  $$

  where $\kappa(s)$ is the curvature of the profile curve which explicit form is found to be

  $$
  \kappa(s) = \lambda\,\frac{1 - 4\lambda\mu + \sqrt{1-4\lambda\mu}\,\sin(2\lambda s)}{1 - 2\lambda\mu + \sqrt{1-4\lambda\mu}\,\sin(2\lambda s)}
  \tag{A17}
  $$

  and this leads to

  $$
  \begin{aligned}
  r(s) &= \frac{\sqrt{1-2\lambda\mu+\sqrt{1-4\lambda\mu}\,\sin(2\lambda s)}}{\sqrt{2}\lambda} \\
  z(s) &= \tfrac{\mu}{m(\lambda,\mu)}\,F(\lambda s - \tfrac{\pi}{4},k) + \tfrac{m(\lambda,\mu)}{\lambda}\,E(\lambda s - \tfrac{\pi}{4},k).
  \end{aligned}
  \tag{A18}
  $$

  Here λ and μ are real parameters with specific ranges and the other ingredients are given by the expressions

  $$
  m(\lambda,\mu) = \frac{\sqrt{1-2\lambda\mu+\sqrt{1-4\lambda\mu}}}{\sqrt{2}}, \qquad k = \sqrt{\frac{2\sqrt{1-4\lambda\mu}}{1-2\lambda\mu+\sqrt{1-4\lambda\mu}}}.
  \tag{A19}
  $$

- **2014** E. Bendito, M. Bowick and A. Medina [37] (pp. 33–35)
  If the symmetry axis is OZ, roulettes of the ellipses were found to be described by the formulas

$$
\begin{aligned}
x(t) &= \frac{b(a - c\cos(t))}{\sqrt{a^2 - c^2\cos^2(t)}}, \qquad c = \sqrt{a^2 - b^2}, \qquad a > b \\
z(t) &= \int_{t_0}^{t} \sqrt{a^2 - c^2\cos^2(u)}\,du - \frac{c\sin(t)(a - c\cos(t))}{\sqrt{a^2 - c^2\cos^2(t)}}
\end{aligned}
\tag{A20}
$$

  while that ones of hyperbolas are given by

$$
\begin{aligned}
x(t) &= \frac{b(c\cosh(t) - a)}{\sqrt{c^2\cosh^2(t) - a^2}}, \qquad c = \sqrt{a^2 + b^2} \\
z(t) &= \int_{t_0}^{t} \sqrt{c^2\cosh^2(u) - a^2}\,du - \frac{c\sinh(t)(c\cosh(t) - a)}{\sqrt{c^2\cosh^2(t) - a^2}}.
\end{aligned}
\tag{A21}
$$

- **2014** B. Athukorallage, T. Paragoda and M. Toda [38] (pp. 6–10)
  The parameterization of *undulary* with symmetry axis OX is cast into the form

$$
\begin{aligned}
x(t) &= \frac{b^2}{a} \int_0^t \frac{du}{(1 + e\cos u)\sqrt{1 - e^2\cos^2 u}} \\
y(t) &= b\sqrt{\frac{1 - e\cos u}{1 + e\cos u}}, \qquad e = \frac{\sqrt{a^2 - b^2}}{a}
\end{aligned}
\tag{A22}
$$

  and that one corresponding to *undulary* is given by the formulas

$$
\begin{aligned}
x(t) &= \frac{b^2}{a} \int_0^t \frac{\cos u\,du}{(e + \cos u)\sqrt{e^2 - \cos^2 u}} \\
y(t) &= b\sqrt{\frac{e - \cos u}{e + \cos u}}.
\end{aligned}
\tag{A23}
$$

- **2014** V. Pulov, M. Hadzhilazova and I. Mladenov [39] (p. 222)
  In this paper the reader will find the parameterization of the most interesting cases of Delaunay surfaces, namely - unduloids and nodoids, in terms of the complete set of the Weierstrassian elliptic functions, i.e., Weierstrass p-function $\wp(u) \equiv \wp(u; g_2, g_3)$, $\sigma(u) \equiv \sigma(u; g_2, g_3)$ and $\zeta(u) \equiv \zeta(u; g_2, g_3)$. Assuming that the OX-axis coincides with the axis of revolution and the generating curves of the surfaces lie in the XOZ-plane, the parameterization of the undularies (upper signs) and nodaries (lower signs) have the form

$$
\begin{aligned}
x^{\pm}(u) &= 2c^2(\mp 2a^2 + b^2 \pm c^2)J_1(u + C_1^{\pm}) \\
&\quad \mp c^2(\mp 2a^2 + b^2 \pm c^2)^2 J_2(u + C_1^{\pm}) - (b^2 \pm c^2)u + C_2^{\pm} \\
z^{\pm}(u) &= \frac{c\left(6\wp(u + C_1^{\pm}) + 10a^2 \mp 5b^2 - 3c^2\right)}{6\wp(u + C_1^{\pm}) - 2a^2 \pm b^2 + 3c^2}
\end{aligned}
\tag{A24}
$$

  where

$$
\begin{aligned}
J_1(u) &= \frac{1}{\wp'(\mathring{u})}\left(2\zeta(\mathring{u})u + \ln\frac{\sigma(u - \mathring{u})}{\sigma(u + \mathring{u})}\right) \\
J_2(u) &= -\frac{1}{\wp'^2(\mathring{u})}\left(\wp''(\mathring{u})J_1(u) + 2\wp(\mathring{u})u + \zeta(u - \mathring{u}) + \zeta(u + \mathring{u})\right).
\end{aligned}
$$

  The above functions are built with the so-called invariants of the Weierstrassian functions given by the expressions

$$
g_2 \equiv g_2^{\pm} = \frac{4}{3}(a^4 \mp a^2 b^2 + b^4), \qquad g_3 \equiv g_3^{\pm} = -\frac{4}{27}(2a^6 \mp 3a^4 b^2 - 3a^2 b^4 \pm 2b^6)
$$

in which $C_1^{\pm}$ and $C_2^{\pm}$ are arbitrary real constants. Here $\mathring{u}$ denotes the argument of $\wp(\cdot)$ which produces the value $\frac{1}{6}(2a^2 \mp b^2 - 3c^2)$ for $a > |b|$, $b \in \mathbb{R}$, $c \equiv c^{\pm} = a + \sqrt{a^2 \mp b^2}$, and $\wp'(\mathring{u}) \equiv \mathrm{d}\wp(u)/\mathrm{d}u_{|u=\mathring{u}}$, etc. The mean curvature of the generated unduloids, respectively nodoids, is $H = \pm \dfrac{1}{2a}$.

- **2017** M. Koiso and B. Palmer [40] (p. 14)

  Relying on variational arguments the authors have concluded that the surface of revolution with symmetry axis *OZ* will have CMC if its coordinates are given by the formulas

  $$r = \frac{\cos\theta \pm \sqrt{\cos^2\theta + H\,F}}{-2\,H}, \qquad z = \frac{1}{-2\,H}\int_{\theta_0}^{\theta}\left(1 \pm \frac{\cos\hat{\theta}}{\sqrt{\cos^2\hat{\theta} + H\,F}}\right)\cos\hat{\theta}\,\mathrm{d}\hat{\theta} \quad \text{(A25)}$$

  in which *H* denotes the mean curvature and *F* is the so-called *flux parameter* of the surface under consideration.

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