

Article

Reliability Assessment of Bridge Structure Using Bilal Distribution

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Abstract: Reliability assessments are pivotal in evaluating system quality and have found extensive application in manufacturing. This research delves into a system comprising five components, one of which is a bridge network. The components are presumed to follow a Bilal lifetime distribution with a failure rate that changes over time. Four distinct methods are employed to enhance the components within the system. This study involves the computation of δ -fractiles and reliability equivalence factors (REFs). Additionally, a numerical case study is provided to elucidate the theoretical findings.

Keywords: Bilal distribution; bridge network structure; reliability factors of equivalence; enhancing methods

MSC: 37M99

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1. Introduction

Reliability is prominent across diverse domains, with manufacturing being a notable application area. Consequently, numerous research endeavors have explored the reliability aspects of various systems and their associated characteristics. These systems can include elements with either constant or fluctuating failure rates.

For instance, Alghamdi and Percy [1] investigated various techniques for enhancing a series–parallel system consisting of components that are independent and identically distributed (i.i.d) and follow a Weibull distribution. Baaqeel et al. [2] enhanced the components of a bridge system which were modeled using the unit half-logistic geometric distribution, utilizing a range of different methods. Ezzati and Rasouli [3] explored compounded series and parallel systems by applying the linear exponential distribution. Recently, Jaramillo-Vacio et al. [4] applied reliability concepts in the design of electrical power transition transmission systems. Luo et al. [5] explored time-dependent reliability theory as a means to improve the performance of structural elements by decreasing their rates of failure. Moreover, Mahmood et al. [6] clarified the characteristics and use cases of the extended-cosine generalized family of distributions in the context of reliability modeling. Mohamed et al. [7] proposed the use of the type II half-logistic Weibull distribution for a reliability analysis in bladder cancer. Mustafa et al. [8] analyzed a system consisting of five components, one of which was integrated into a bridge network configuration, and enhanced its performance through the utilization of the gamma distribution. They also explored a series–parallel system with a linear exponential distribution and employed various improvement techniques [9]. Additionally, Mustafa [10] examined the reliability equivalence factor, investigating a series configuration in which component failure rates adhered to both Weibull and linearly increasing distributions. Mustafa and El Faheem [11,12] expanded the scope of the reliability equivalence method by integrating both mean and survival reliability equivalence metrics. Papaioannou and Straub [13] utilized the concept of a

sensitivity analysis based on reliability gradients. Furthermore, Peiravi et al. [14] improved system performance by implementing an innovative K-mixed redundancy approach that integrates both active and standby redundancy strategies. A fresh reliability framework encompassing technical and financial efficiency, operations, planning, and carbon footprint considerations in modern power systems was introduced by Peyghami et al. [15]. Conversely, Sarhan [16] investigated a system featuring a bridge network component consisting of five items, all exhibiting constant failure rates. In another study, Sarhan and Mustafa [17] delved into a reliability evaluation framework (REF) for a set of “ n ” identically and independently distributed (i.i.d) components, each with constant failure rates interconnected in a series configuration; for more details, see [18–22]. In the realm of energy efficiency optimization, Xia et al. [23] utilized the concept of series–parallel systems. Additionally, Xia and Zhang [24] investigated the reliability equivalent factor for a parallel system comprising n independent and identical components using the gamma distribution and improved its performance. Xu and Saleh [25] incorporated machine learning algorithms into safety applications and the field of reliability engineering, highlighting the interdisciplinary nature of reliability studies.

In this study, we focus on analyzing a bridge network system (depicted in Figure 1) in which the components 1, 2, 3, 4, and 5 are following a Bilal distribution. It is assumed that all components within the system undergo improvement using the following methods:

1. A reduction method;
2. A duplication method which includes
 - A hot duplication method;
 - Cold duplication with an imperfect switch method;
 - Cold duplication with a perfect switch method.

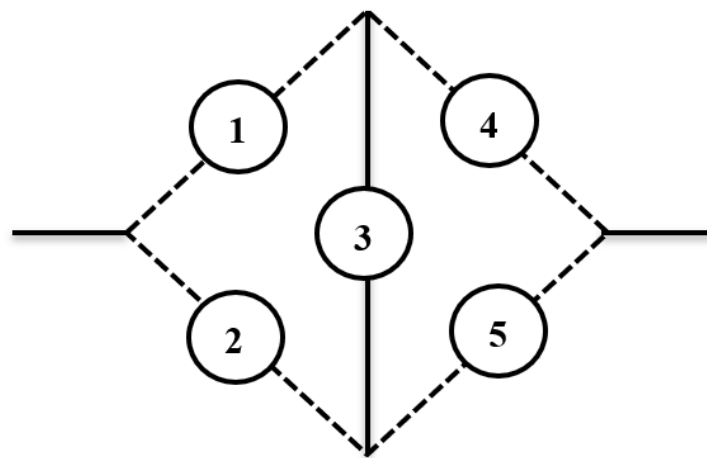


Figure 1. The bridge structure system.

The paper is organized as follows:

- In Section 2, we introduce key functions that are fundamental to the Bilal (B) distribution.
- In Section 3, presentations of reliability and mean-time-to-failure (MTTF) functions are found for the bridge network system.
- In Section 4, four distinct methods for improving the system are outlined.
- Sections 5 and 6 are dedicated to obtaining REFs and δ -fractiles for both the original and enhanced systems.
- Finally, Section 7 provides an analysis through numerical applications.

2. Fundamental Functions for the Bilal Distribution

In (2013), Abd-Elrahman [26] introduced the Bilal (B) distribution, which belongs to the category of new renewal failure rates that surpass the average. Additionally, the paper offered a thorough mathematical analysis. Abd-Elrahman also demonstrated the

existence and uniqueness of the maximum likelihood estimate and established that the moment estimates for the parameter θ can be expressed in a straightforward closed form. Its efficiency compared to the minimum variance unbiased estimate of θ stands at 99.9165%.

The cumulative function (the probability that a system fails before a given time t) and probability density function (the rate at which failure events occur per unit time, providing insights into the frequency or intensity of failures) of the Bilal distribution are provided as follows:

$$F(t) = 1 - e^{-\frac{2t}{\theta}} \left(3 - 2e^{-\frac{t}{\theta}} \right), \quad t \geq 0, \theta > 0, \quad (1)$$

$$f(t) = \frac{6}{\theta} e^{-\frac{2t}{\theta}} \left(1 - e^{-\frac{t}{\theta}} \right), \quad t \geq 0, \theta > 0. \quad (2)$$

Additionally, the reliability function (the probability of a system surviving beyond a specified duration without experiencing failure) and hazard function (the likelihood of failure occurring at different points in time, accounting for the cumulative effect of past survival) for the Bilal distribution are presented as follows:

$$R(t) = e^{-\frac{2t}{\theta}} \left(3 - 2e^{-\frac{t}{\theta}} \right), \quad t \geq 0, \theta > 0, \quad (3)$$

$$H(t) = \frac{6(1 - e^{-\frac{t}{\theta}})}{\theta(3 - 2e^{-\frac{t}{\theta}})}, \quad t \geq 0, \theta > 0. \quad (4)$$

The Bilal distribution is unimodal; where the mode, median, and mean are, respectively, at $\theta \ln(1.5)$, $\theta \ln(2)$, and $5\theta/6$. This means that they are in order. Also, it is a member of the new-better-than-used failure rate class.

3. The Bridge Structure's Reliability Function

The bridge structure comprises five identical components forming a bridge network (see Figure 1). All these components are assumed to follow the Bilal distribution with the parameter θ ($B(\theta)$). Consequently, the reliability of each component, denoted as $r_i(t)$, where $i = 1, \dots, 5$, is expressed as follows:

$$r_i(t) = e^{-\frac{2t}{\theta}} \left(3 - 2e^{-\frac{t}{\theta}} \right). \quad (5)$$

The technique of minimal paths (Figure 2) can be employed to determine the reliability of the bridge structure, where τ_i , $i = 1, \dots, 5$ is the minimal tie set.

Let us assume that $M = \{1, \dots, 5\}$ represents the set of all components, and M_i' stands for the set of all components except component i . By applying the minimal paths approach, we can express the system's reliability $R(t)$ as follows:

$$\begin{aligned} R(t) &= \sum_{i=1}^4 P(\tau_i) - \sum_{i=1}^3 \sum_{j=i+1}^4 P(\tau_i \cap \tau_j) + \sum_{i=1}^2 \sum_{j=i+1}^3 \sum_{k=j+1}^4 P(\tau_i \cap \tau_j \cap \tau_k) \\ &\quad - P\left(\bigcap_{i=1}^4 \tau_i\right) = r_1(t)r_4(t) + r_2(t)r_5(t) + r_1(t)r_3(t)r_5(t) + \\ &\quad r_2(t)r_3(t)r_4(t) - \sum_{i=1}^5 \prod_{j \in M_i'} r_i(t) + 2 \prod_{j \in M} r_j(t). \end{aligned} \quad (6)$$

Using Equation (5), Equation (6) becomes

$$R(t) = e^{-\frac{10t}{\theta}} \left(2 - 3e^{\frac{t}{\theta}} \right)^2 \left(2e^{\frac{6t}{\theta}} + 6e^{\frac{5t}{\theta}} - 49e^{\frac{4t}{\theta}} + 114e^{\frac{3t}{\theta}} - 128e^{\frac{2t}{\theta}} + 72e^{\frac{t}{\theta}} - 16 \right) \quad (7)$$

In addition, the MTTF can be denoted as

$$MTTF(t) = \int_0^{\infty} R(t)dt. \quad (8)$$

Mathematica's numerical methods are employed for computing the MTTF.

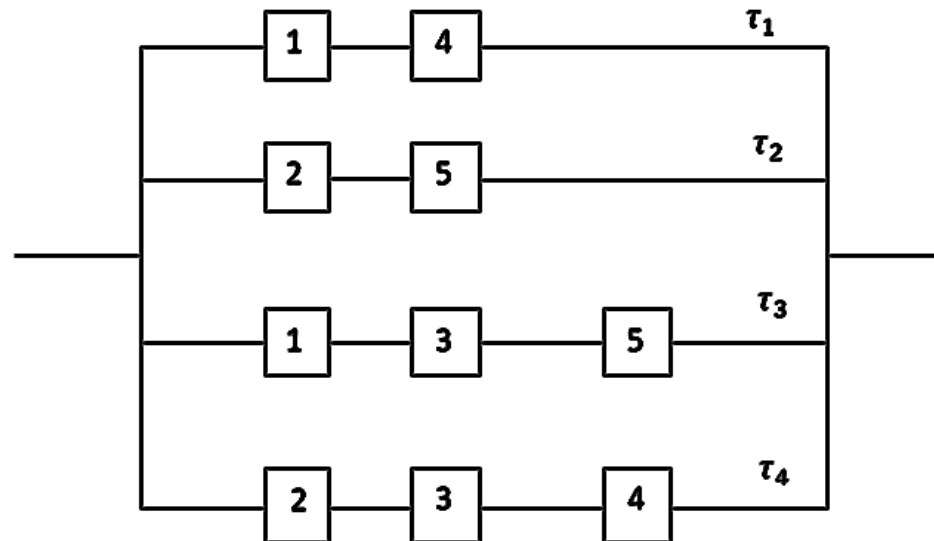


Figure 2. The use of the minimal paths technique for a bridge system.

4. Various Performance-Enhancing Techniques

In this context, improvements in the bridge system's performance are achieved through enhancements applied to certain components using four different methods. One of these methods involves reducing by a factor of η , where $0 < \eta < 1$, while the others utilize duplication techniques, as elaborated in the following subsections.

4.1. Enhancing via the Reduction Method

Let us consider set A , including the components enhanced using this technique. Denote $H_i^\eta(t)$ as the hazard function of component $i = 1, \dots, 5$, and denote $r_i^\eta(t)$ as the reliability function of that component. Consequently, we have

$$H_i^\eta(t) = \frac{6\eta(1 - e^{-\frac{t}{\theta}})}{\theta(3 - 2e^{-\frac{t}{\theta}})}, \quad t \geq 0, \theta > 0, \quad (9)$$

$$r_i^\eta(t) = e^{-\int_0^t H_i^\eta(u)du} = \left(3e^{\frac{t}{\theta}} - 2\right)^\eta e^{-\frac{3\eta t}{\theta}}. \quad (10)$$

Assume that R_A^η represents the reliability function of the improved system resulting from this technique, specifically when enhancing the components within set A . Let us also assume $r_\eta = r_i^\eta(t)$, defined in Equation (10). Then, R_A^η can be defined as follows:

- $A \in G_1 = \{\{3\}\}$

$$R_{A,G_1}^\eta(t) = e^{-\frac{12t}{\theta}} \left(2 - 3e^{\frac{t}{\theta}}\right) \left(8r_\eta - 4 + e^{\frac{t}{\theta}} \left(12 - 24r_\eta + 8e^{\frac{2t}{\theta}} r_\eta - 12e^{\frac{3t}{\theta}} r_\eta + 2e^{\frac{5t}{\theta}} (1 + r_\eta) + 9e^{\frac{t}{\theta}} (2r_\eta - 1)\right)\right), \quad (11)$$

- $A \in G_2 = \{\{1\}, \{2\}, \{4\}, \{5\}\}$

$$R_{A,G_2}^{\eta}(t) = e^{-\frac{12t}{\theta}} \left(3e^{\frac{t}{\theta}} - 2 \right) \left(e^{\frac{5t}{\theta}} (9 - 27r_{\eta}) - 16(r_{\eta} - 1) + 72e^{\frac{t}{\theta}}(r_{\eta} - 1) - 108e^{\frac{2t}{\theta}}(r_{\eta} - 1) + e^{\frac{9t}{\theta}}r_{\eta} - 2e^{\frac{6t}{\theta}}(1 + r_{\eta}) + 3e^{\frac{7t}{\theta}}(1 + r_{\eta}) + 12e^{\frac{4t}{\theta}}(3r_{\eta} - 1) + e^{\frac{3t}{\theta}}(42r_{\eta} - 50) \right), \quad (12)$$

$$\bullet \quad A \in G_3 = \{\{1, 3\}, \{2, 3\}, \{3, 4\}, \{3, 5\}\}$$

$$R_{A,G_3}^{\eta}(t) = e^{-\frac{9t}{\theta}} \left(3e^{\frac{t}{\theta}} - 2 \right) \left(8r_{\eta}(r_{\eta} - 1) - 24r_{\eta}(r_{\eta} - 1) + 18e^{\frac{2t}{\theta}}r_{\eta}(r_{\eta} - 1) + e^{\frac{6t}{\theta}}r_{\eta}(r_{\eta} + 1) + e^{\frac{4t}{\theta}}(3 + 3r_{\eta} - 9r_{\eta}^2) + e^{\frac{3t}{\theta}}(-2 - 2r_{\eta} + 6r_{\eta}^2) \right), \quad (13)$$

$$\bullet \quad A \in G_4 = \{\{1, 2\}, \{4, 5\}\}$$

$$R_{A,G_4}^{\eta}(t) = e^{-\frac{9t}{\theta}} r_{\eta} \left(3e^{\frac{t}{\theta}} - 2 \right) \left(2e^{\frac{6t}{\theta}} + e^{\frac{4t}{\theta}}(6 - 9r_{\eta}) + 8(r_{\eta} - 1) - 24e^{\frac{t}{\theta}}(r_{\eta} - 1) + 18e^{\frac{2t}{\theta}}(r_{\eta} - 1) + e^{\frac{3t}{\theta}}(6r_{\eta} - 4) \right), \quad (14)$$

$$\bullet \quad A \in G_5 = \{\{1, 5\}, \{2, 4\}\}$$

$$R_{A,G_5}^{\eta}(t) = e^{-\frac{9t}{\theta}} \left(2 - 3e^{\frac{t}{\theta}} \right) \left(-4 - 8r_{\eta}(r_{\eta} - 1) - 4e^{\frac{3t}{\theta}}r_{\eta}^2 + 6e^{\frac{4t}{\theta}}r_{\eta}^2 + e^{\frac{6t}{\theta}}r_{\eta}(r_{\eta} - 2)(1 + r_{\eta}) + 12e^{\frac{t}{\theta}}r_{\eta}(1 + 2(r_{\eta} - 1)) - 9e^{\frac{2t}{\theta}}r_{\eta}(1 + 2(r_{\eta} - 1)) \right), \quad (15)$$

$$\bullet \quad A \in G_6 = \{\{1, 4\}, \{2, 5\}\}$$

$$R_{A,G_6}^{\eta}(t) = 16e^{-\frac{9t}{\theta}}r_{\eta}(r_{\eta} - 1) + 72e^{-\frac{8t}{\theta}}r_{\eta}(r_{\eta} - 1) - 108e^{-\frac{7t}{\theta}}r_{\eta}(r_{\eta} - 1) + r_{\eta}^2 + 12e^{-\frac{5t}{\theta}}(r_{\eta} - 1)(1 + 3r_{\eta}) - 9e^{-\frac{4t}{\theta}}(r_{\eta} - 1)(1 + 3r_{\eta}) + 2e^{-\frac{6t}{\theta}}(r_{\eta} - 1)(21r_{\eta} - 2), \quad (16)$$

Furthermore, the MTTF can be determined by

$$MTTF_{A,G_i}^{\eta}(t) = \int_0^{\infty} R_{A,G_i}^{\eta}(t) dt, \quad i = 1, \dots, 6. \quad (17)$$

4.2. Duplication Techniques

In these techniques, the system is enhanced by duplicating certain components using various duplication techniques, as outlined below.

4.2.1. Hot Duplication Enhancing Technique

In this technique, the system is enhanced by duplicating specific components in a hot case, as illustrated in Figure 3.

Let $r_i^h(t)$ be the reliability of component i , which is enhanced due to this technique. Then,

$$r_i^h(t) = e^{-\frac{10t}{\theta}} \left(2 - 3e^{-\frac{t}{\theta}} \right)^2 \left(-16 + 72e^{\frac{t}{\theta}} - 128e^{\frac{2t}{\theta}} + 114e^{\frac{3t}{\theta}} + 6e^{\frac{5t}{\theta}} + 2e^{\frac{6t}{\theta}} \right). \quad (18)$$

Assume that R_B^h is the reliability function of the system enhanced due to this technique when enhancing the components belonging to B and $r_h = r_i^h(t)$ given in Equation (18); then, R_B^h is

$$\bullet \quad B \in G_1 = \{\{3\}\}$$

$$R_{B,G_1}^h(t) = e^{-\frac{12t}{\theta}} \left(2 - 3e^{\frac{t}{\theta}} \right) \left(8r_h - 4 + e^{\frac{t}{\theta}} \left(12 - 24r_h + 8e^{\frac{2t}{\theta}}r_h - 12e^{\frac{3t}{\theta}}r_h + 2e^{\frac{5t}{\theta}}(1 + r_h) + 9e^{\frac{t}{\theta}}(2r_h - 1) \right) \right), \quad (19)$$

- $B \in G_2 = \{\{1\}, \{2\}, \{4\}, \{5\}\}$

$$R_{B,G_2}^h(t) = e^{-\frac{12t}{\theta}} \left(3e^{\frac{t}{\theta}} - 2 \right) \left(e^{\frac{5t}{\theta}} (9 - 27r_h) - 16(r_h - 1) + 72e^{\frac{t}{\theta}}(r_h - 1) - 108e^{\frac{2t}{\theta}}(r_h - 1) + e^{\frac{9t}{\theta}}r_h - 2e^{\frac{6t}{\theta}}(1 + r_h) + 3e^{\frac{7t}{\theta}}(1 + r_h) + 12e^{\frac{4t}{\theta}}(3r_h - 1) + e^{\frac{3t}{\theta}}(42r_h - 50) \right), \quad (20)$$

- $B \in G_3 = \{\{1, 3\}, \{2, 3\}, \{3, 4\}, \{3, 5\}\}$

$$R_{B,G_3}^h(t) = e^{-\frac{9t}{\theta}} \left(3e^{\frac{t}{\theta}} - 2 \right) \left(8r_h(r_h - 1) - 24r_h(r_h - 1) + 18e^{\frac{2t}{\theta}}r_h(r_h - 1) + e^{\frac{6t}{\theta}}r_h(r_h + 1) + e^{\frac{4t}{\theta}}(3 + 3r_h - 9r_h^2) + e^{\frac{3t}{\theta}}(-2 - 2r_h + 6r_h^2) \right), \quad (21)$$

- $B \in G_4 = \{\{1, 2\}, \{4, 5\}\}$

$$R_{B,G_4}^h(t) = e^{-\frac{9t}{\theta}} r_h \left(3e^{\frac{t}{\theta}} - 2 \right) \left(2e^{\frac{6t}{\theta}} + e^{\frac{4t}{\theta}}(6 - 9r_h) + 8(r_h - 1) - 24e^{\frac{t}{\theta}}(r_h - 1) + 18e^{\frac{2t}{\theta}}(r_h - 1) + e^{\frac{3t}{\theta}}(6r_h - 4) \right), \quad (22)$$

- $B \in G_5 = \{\{1, 5\}, \{2, 4\}\}$

$$R_{B,G_5}^h(t) = e^{-\frac{9t}{\theta}} \left(2 - 3e^{\frac{t}{\theta}} \right) \left(-4 - 8r_h(r_h - 1) - 4e^{\frac{3t}{\theta}}r_h^2 + 6e^{\frac{4t}{\theta}}r_h^2 + e^{\frac{6t}{\theta}}r_h(r_h - 2)(1 + r_h) + 12e^{\frac{t}{\theta}}r_h(1 + 2(r_h - 1)) - 9e^{\frac{2t}{\theta}}r_h(1 + 2(r_h - 1)) \right), \quad (23)$$

- $B \in G_6 = \{\{1, 4\}, \{2, 5\}\}$

$$R_{B,G_6}^h(t) = 16e^{-\frac{9t}{\theta}}r_h(r_h - 1) + 72e^{-\frac{8t}{\theta}}r_h(r_h - 1) - 108e^{-\frac{7t}{\theta}}r_h(r_h - 1) + r_h^2 + 12e^{-\frac{5t}{\theta}}(r_h - 1)(1 + 3r_h) - 9e^{-\frac{4t}{\theta}}(r_h - 1)(1 + 3r_h) + 2e^{-\frac{6t}{\theta}}(r_h - 1)(21r_h - 2), \quad (24)$$

Now, the MTTF can be obtained as

$$MTTF_{B,G_i}^h(t) = \int_0^\infty R_{B,G_i}^h(t) dt, \quad i = 1, \dots, 6. \quad (25)$$

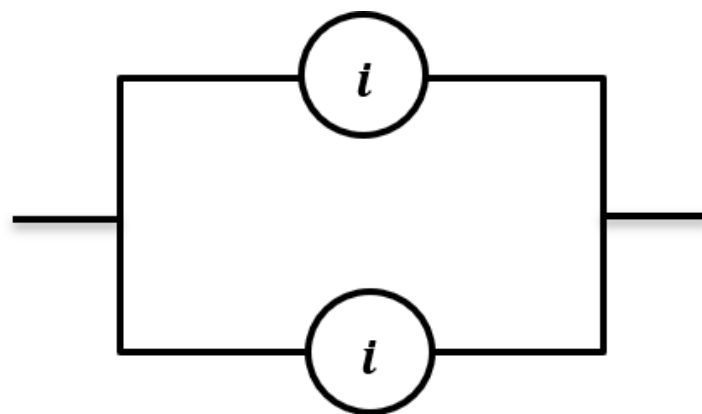


Figure 3. Hot duplication case of component i .

4.2.2. Cold Duplication with an Imperfect Switch Enhancing Technique

This method involves improving the system by duplicating certain components in a cold case while utilizing an imperfect switch, as depicted in Figure 4. In this scenario, the switch is assumed to be imperfect, with a nonzero probability of failure when switching.

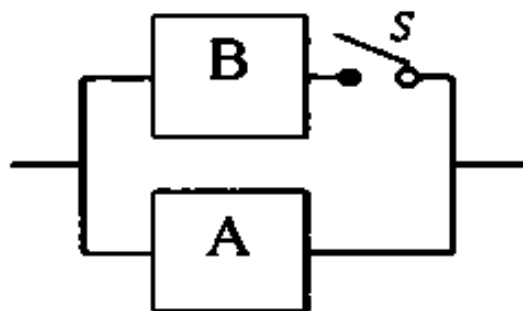


Figure 4. A cold duplication case with an imperfect switch.

Consider $r_i^{im}(t)$ the reliability of component i enhanced using this method and $r(x)$ reliability of the switch. It is assumed that the switch follows the Bilal distribution with the parameter a . Then, we have

$$r_i^{im}(t) = r(t) + \int_0^t f(x)r(x)r(t-x)dx = e^{-\frac{2t}{\theta}} \left(3 - 2e^{-\frac{t}{\theta}} \right) + \frac{a}{\theta} e^{-\frac{3t}{\theta}} \times \left(e^{t(\frac{1}{\theta} - \frac{2}{a})} \left(-27 + \frac{36\theta}{2\theta - a} \right) + \frac{12e^{t(\frac{1}{\theta} - \frac{3}{a})}(\theta - a)}{3\theta - a} + \frac{3ae^{\frac{t}{\theta}}(19\theta + 5a)}{(2\theta + a)(3\theta + a)} + \frac{2a(-19\theta + 5a)}{(2\theta + a)(6\theta^2 - 5a\theta + a^2)} + e^{-\frac{2t}{a}} \left(-18 + \frac{54\theta}{2\theta + a} \right) + e^{-\frac{3t}{a}} \left(8 - \frac{36\theta}{3\theta + a} \right) \right). \quad (26)$$

Now, Assume that R_B^{im} represents the reliability function of the improved system resulting from this method, specifically when enhancing the components within set B . Also let $r_{im} = r_i^{im}(t)$, given in Equation (26). Then, R_B^{im} can be defined as follows:

- $B \in G_1 = \{\{3\}\}$

$$R_{B,G_1}^{im}(t) = e^{-\frac{12t}{\theta}} \left(2 - 3e^{\frac{t}{\theta}} \right) \left(8r_{im} - 4 + e^{\frac{t}{\theta}} \left(12 - 24r_{im} + 8e^{\frac{2t}{\theta}}r_{im} - 12e^{\frac{3t}{\theta}}r_{im} + 2e^{\frac{5t}{\theta}}(1 + r_{im}) + 9e^{\frac{t}{\theta}}(2r_{im} - 1) \right) \right), \quad (27)$$

- $B \in G_2 = \{\{1\}, \{2\}, \{4\}, \{5\}\}$

$$R_{B,G_2}^{im}(t) = e^{-\frac{12t}{\theta}} \left(3e^{\frac{t}{\theta}} - 2 \right) \left(e^{\frac{5t}{\theta}}(9 - 27r_{im}) - 16(r_{im} - 1) + 72e^{\frac{t}{\theta}}(r_{im} - 1) - 108e^{\frac{2t}{\theta}}(r_{im} - 1) + e^{\frac{9t}{\theta}}r_{im} - 2e^{\frac{6t}{\theta}}(1 + r_{im}) + 3e^{\frac{7t}{\theta}}(1 + r_{im}) + 12e^{\frac{4t}{\theta}}(3r_{im} - 1) + e^{\frac{3t}{\theta}}(42r_{im} - 50) \right), \quad (28)$$

- $B \in G_3 = \{\{1, 3\}, \{2, 3\}, \{3, 4\}, \{3, 5\}\}$

$$R_{B,G_3}^{im}(t) = e^{-\frac{9t}{\theta}} \left(3e^{\frac{t}{\theta}} - 2 \right) \left(8r_{im}(r_{im} - 1) - 24r_{im}(r_{im} - 1) + 18e^{\frac{2t}{\theta}}r_{im}(r_{im} - 1) + e^{\frac{6t}{\theta}}r_{im}(r_{im} + 1) + e^{\frac{4t}{\theta}}(3 + 3r_{im} - 9r_{im}^2) + e^{\frac{3t}{\theta}}(-2 - 2r_{im} + 6r_{im}^2) \right), \quad (29)$$

- $B \in G_4 = \{\{1, 2\}, \{4, 5\}\}$

$$R_{B,G_4}^{im}(t) = e^{-\frac{9t}{\theta}} r_{im} \left(3e^{\frac{t}{\theta}} - 2 \right) \left(2e^{\frac{6t}{\theta}} + e^{\frac{4t}{\theta}}(6 - 9r_{im}) + 8(r_{im} - 1) - 24e^{\frac{t}{\theta}}(r_{im} - 1) + 18e^{\frac{2t}{\theta}}(r_{im} - 1) + e^{\frac{3t}{\theta}}(6r_{im} - 4) \right), \quad (30)$$

- $B \in G_5 = \{\{1, 5\}, \{2, 4\}\}$

$$R_{B,G_5}^{im}(t) = e^{\frac{-9t}{\theta}} \left(2 - 3e^{\frac{t}{\theta}} \right) \left(-4 - 8r_{im}(r_{im} - 1) - 4e^{\frac{3t}{\theta}} r_{im}^2 + 6e^{\frac{4t}{\theta}} r_{im}^2 + e^{\frac{6t}{\theta}} r_{im}(r_{im} - 2)(1 + r_{im}) + 12e^{\frac{t}{\theta}} r_{im}(1 + 2(r_{im} - 1)) - 9e^{\frac{2t}{\theta}} r_{im}(1 + 2(r_{im} - 1)) \right), \quad (31)$$

- $B \in G_6 = \{\{1, 4\}, \{2, 5\}\}$

$$R_{B,G_6}^{im}(t) = 16e^{\frac{-9t}{\theta}} r_{im}(r_{im} - 1) + 72e^{\frac{-8t}{\theta}} r_{im}(r_{im} - 1) - 108e^{\frac{-7t}{\theta}} r_{im}(r_{im} - 1) + r_{im}^2 + 12e^{\frac{-5t}{\theta}} (r_{im} - 1)(1 + 3r_{im}) - 9e^{\frac{-4t}{\theta}} (r_{im} - 1)(1 + 3r_{im}) + 2e^{\frac{-6t}{\theta}} (r_{im} - 1)(21r_{im} - 2), \quad (32)$$

The MTTF can be determined as

$$MTTF_{B,G_i}^{im}(t) = \int_0^\infty R_{B,G_i}^{im}(t) dt, \quad i = 1, \dots, 6. \quad (33)$$

4.2.3. Cold Duplication with a Perfect Switch Enhancing Technique

Here, the system is enhanced by duplicating certain components in a cold case, using a perfect switch, as depicted in Figure 5. In this scenario, the switch is assumed to be flawless, with zero probability of failure when switching.

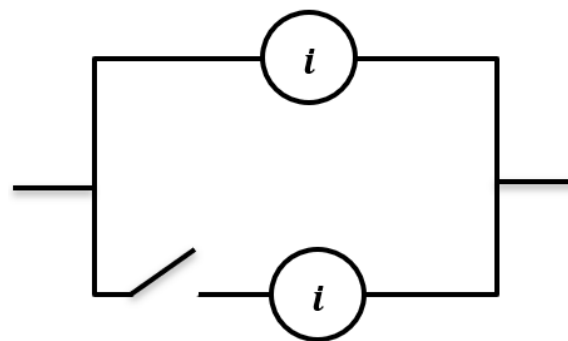


Figure 5. The cold duplication of component i with a perfect switch.

Let us assume that r_i^c represents the reliability function of the enhanced system resulting from this method, particularly when enhancing the components within set B . Then, r_i^c can be defined as follows:

$$\begin{aligned} r_i^c(t) &= r(t) + \int_0^t f(x)r(t-x)dx \\ &= e^{\frac{-2t}{\theta}} \left(3 - 2e^{\frac{-t}{\theta}} \right) + \frac{6}{\theta} e^{\frac{-3t}{\theta}} \left(5\theta + 2t + e^{\frac{t}{\theta}} (3t - 5\theta) \right). \end{aligned} \quad (34)$$

Now, let us Suppose that R_B^c represents the reliability function of the improved system resulting from this method, specifically when enhancing the components within set B . Also suppose $r_c = r_i^c(t)$, given in Equation (34). Then, R_B^c can be defined as follows:

- $B \in G_1 = \{\{3\}\}$

$$R_{B,G_1}^c(t) = e^{\frac{-12t}{\theta}} \left(2 - 3e^{\frac{t}{\theta}} \right) \left(8r_c - 4 + e^{\frac{t}{\theta}} \left(12 - 24r_c + 8e^{\frac{2t}{\theta}} r_c - 12e^{\frac{3t}{\theta}} r_c + 2e^{\frac{5t}{\theta}} (1 + r_c) + 9e^{\frac{t}{\theta}} (2r_c - 1) \right) \right), \quad (35)$$

- $B \in G_2 = \{\{1\}, \{2\}, \{4\}, \{5\}\}$

$$R_{B,G_2}^c(t) = e^{-\frac{12t}{\theta}} \left(3e^{\frac{t}{\theta}} - 2 \right) \left(e^{\frac{5t}{\theta}} (9 - 27r_c) - 16(r_c - 1) + 72e^{\frac{t}{\theta}} (r_c - 1) - 108e^{\frac{2t}{\theta}} (r_c - 1) + e^{\frac{9t}{\theta}} r_c - 2e^{\frac{6t}{\theta}} (1 + r_c) + 3e^{\frac{7t}{\theta}} (1 + r_c) + 12e^{\frac{4t}{\theta}} (3r_c - 1) + e^{\frac{3t}{\theta}} (42r_c - 50) \right), \quad (36)$$

$$\bullet \quad B \in G_3 = \{\{1, 3\}, \{2, 3\}, \{3, 4\}, \{3, 5\}\}$$

$$R_{B,G_3}^c(t) = e^{-\frac{9t}{\theta}} \left(3e^{\frac{t}{\theta}} - 2 \right) \left(8r_c(r_c - 1) - 24r_c(r_c - 1) + 18e^{\frac{2t}{\theta}} r_c(r_c - 1) + e^{\frac{6t}{\theta}} r_c(r_c + 1) + e^{\frac{4t}{\theta}} (3 + 3r_c - 9r_c^2) + e^{\frac{3t}{\theta}} (-2 - 2r_c + 6r_c^2) \right), \quad (37)$$

$$\bullet \quad B \in G_4 = \{\{1, 2\}, \{4, 5\}\}$$

$$R_{B,G_4}^c(t) = e^{-\frac{9t}{\theta}} r_c \left(3e^{\frac{t}{\theta}} - 2 \right) \left(2e^{\frac{6t}{\theta}} + e^{\frac{4t}{\theta}} (6 - 9r_c) + 8(r_c - 1) - 24e^{\frac{t}{\theta}} (r_c - 1) + 18e^{\frac{2t}{\theta}} (r_c - 1) + e^{\frac{3t}{\theta}} (6r_c - 4) \right), \quad (38)$$

$$\bullet \quad B \in G_5 = \{\{1, 5\}, \{2, 4\}\}$$

$$R_{B,G_5}^c(t) = e^{-\frac{9t}{\theta}} \left(2 - 3e^{\frac{t}{\theta}} \right) \left(-4 - 8r_c(r_c - 1) - 4e^{\frac{3t}{\theta}} r_c^2 + 6e^{\frac{4t}{\theta}} r_c^2 + e^{\frac{6t}{\theta}} r_c(r_c - 2)(1 + r_c) + 12e^{\frac{t}{\theta}} r_c(1 + 2(r_c - 1)) - 9e^{\frac{2t}{\theta}} r_c(1 + 2(r_c - 1)) \right), \quad (39)$$

$$\bullet \quad B \in G_6 = \{\{1, 4\}, \{2, 5\}\}$$

$$R_{B,G_6}^c(t) = 16e^{-\frac{9t}{\theta}} r_c(r_c - 1) + 72e^{-\frac{8t}{\theta}} r_c(r_c - 1) - 108e^{-\frac{7t}{\theta}} r_c(r_c - 1) + r_c^2 + 12e^{-\frac{5t}{\theta}} (r_c - 1)(1 + 3r_c) - 9e^{-\frac{4t}{\theta}} (r_c - 1)(1 + 3r_c) + 2e^{-\frac{6t}{\theta}} (r_c - 1)(21r_c - 2), \quad (40)$$

The MTTF can be determined as

$$MTTF_{B,G_i}^c(t) = \int_0^\infty R_{B,G_i}^c(t) dt, \quad i = 1, \dots, 6. \quad (41)$$

5. The Equivalence Factors of Reliability

The factors denoted $\eta^E(\alpha)$, where $E = h, im, c$ represent the adjustments required to match the failure rate of the main system with that of the improved system, which is achieved through hot and cold duplication methods using either a perfect or imperfect switch. The values of $\eta^E(\epsilon)$ can be determined by setting the failure rates of the original and enhanced systems equal to each other, which involves solving the following systems of equations:

$$R_{A_i}^\eta(t) = \epsilon, \quad R_{B_i}^E(t) = \epsilon, \quad E = h, im, c, \quad i = 1, \dots, 6, \quad (42)$$

where $R_{A_i}^\eta(t)$, $R_{B_i}^h(t)$, $R_{B_i}^{im}(t)$, and $R_{B_i}^c(t)$ are denoted in Equations (11)–(16), (19)–(24), (27)–(32), and (35)–(40) respectively.

Solving the aforementioned systems of equations requires the application of numerical techniques.

6. The Fractiles of δ

The fractiles of δ for the main system are determined by finding a solution for the following equation in terms of the variable L :

$$\int_L^\infty f(t) dt = \delta,$$

alternatively, stated differently,

$$R(L) = e^{-\frac{2L}{\theta}} \left(3 - 2e^{-\frac{L}{\theta}} \right) = \delta. \quad (43)$$

The equation mentioned earlier does not have an analytical solution in terms of L , which means that we need to determine the value of L using numerical techniques. Similarly, the quantiles of δ for the systems enhanced through duplication methods can be calculated by determining solutions for the following equations concerning the variable L :

$$R_{B_i}^E(L) = \delta, \quad E = h, im, c, \quad (44)$$

where $R_{B_i}^E(L)$ is given in Equations (11)–(16), (19)–(24), (27)–(32), and (35)–(40). The fractiles of δ for the improved systems resulting from duplication methods can also be acquired through numerical methods.

7. Numerical Application

This section presents a practical numerical example to demonstrate the theoretical findings. In this example, we calculate the REFs for a bridge system structure based on these assumptions.

- All components are assumed to be independent and follow a Bilal distribution.
- The scale parameter for the primary component is $\theta = 2.5$, and for the imperfect switch, it is $a = 2.4$.
- Within the reduction method, the failure rates of components that are part of the set A , where A is a subset of G_i for $i = 1, \dots, 6$, are reduced by a common factor denoted as η .
- Components belonging to set B , which is a subset of G_i for $i = 1, \dots, 6$, are enhanced through hot and cold duplication methods to improve the bridge's performance.

Table 1 provides the MTTF values for both the main and enhanced systems.

Table 1. The MTTF values for both the primary and enhanced systems vary across different sets.

System	Sets	θ				
		3	4	5	6	7
Original	—	2.2463	2.9951	3.7438	4.4926	5.2414
Hot	G_1	2.3118	3.0824	3.8530	4.6236	5.3941
	G_2	2.4740	3.2987	4.1233	4.9480	5.7727
	G_3	2.5427	3.3903	4.2378	5.0854	5.9330
	G_4	2.6352	3.5136	4.3920	5.2704	6.1487
	G_5	2.7050	3.6066	4.5082	5.4099	6.3115
	G_6	2.7974	3.7299	4.6623	5.5948	6.5273
Cold (Imperfect S.)	G_1	2.3162	3.0876	3.8832	4.6551	5.4021
	G_2	2.4907	3.3251	4.2607	5.046	5.8466
	G_3	2.5652	3.3977	4.2814	5.1113	5.9466
	G_4	2.6623	3.5396	4.4954	5.2370	6.1730
	G_5	2.7397	3.6087	4.5539	5.4872	6.3153
	G_6	2.8505	3.7933	4.6773	5.753	6.7525
Cold (Perfect S.)	G_1	2.3709	3.1612	3.9516	4.7419	5.5322
	G_2	2.7054	3.6072	4.5090	5.4108	6.3126
	G_3	2.8711	3.8282	4.7852	5.7423	6.6993
	G_4	2.9835	3.9780	4.9724	5.9670	6.9614
	G_5	3.2056	4.2741	5.3427	6.4112	7.4797
	G_6	3.7945	5.0593	6.3241	7.5890	8.8537

Table 2 displays the δ -fractiles for both the primary and upgraded systems when $\theta = 2.5$.

Table 2. The δ -fractiles for the primary and enhanced systems at $\theta = 2.5$.

δ	L	Hot					
		G_1	G_2	G_3	G_4	G_5	G_6
0.1	3.0345	3.0971	3.2668	3.3248	3.4395	3.4597	3.5613
0.2	2.5253	2.5940	2.7469	2.8129	2.9125	2.9408	3.0320
0.3	2.2009	2.2705	2.4136	2.4826	2.5724	2.6078	2.6921
0.4	1.9490	2.0170	2.1532	2.2227	2.3050	2.3474	2.4264
0.5	1.7329	1.7976	1.9284	1.9965	2.0722	2.1223	2.1969
0.6	1.5339	1.5937	1.7201	1.7851	1.8544	1.9134	1.9841
0.7	1.3388	1.3919	1.5139	1.5740	1.6365	1.7066	1.7734
0.8	1.1321	1.1760	1.2930	1.3454	1.3996	1.4849	1.5475
0.9	0.8812	0.9116	1.0203	1.0597	1.1010	1.2109	1.2678

δ	L	Cold (Imperfect)					
		G_1	G_2	G_3	G_4	G_5	G_6
0.1	3.0345	3.1173	3.3403	3.4189	3.5562	3.5890	3.6213
0.2	2.5253	2.6123	2.8069	2.8938	3.0095	3.0534	3.1127
0.3	2.2009	2.2864	2.4646	2.5535	2.6554	2.7087	2.8474
0.4	1.9490	2.0307	2.1972	2.2850	2.3764	2.4388	2.5659
0.5	1.7329	1.8089	1.9662	2.0509	2.1331	2.2051	2.3227
0.6	1.5339	1.6028	1.7521	1.8317	1.9052	1.9881	2.0972
0.7	1.3388	1.3987	1.5402	1.6125	1.6770	1.7731	1.8741
0.8	1.1321	1.1805	1.3134	1.3750	1.4288	1.5425	1.6351
0.9	0.8812	0.9137	1.0337	1.0784	1.1171	1.2574	1.3394

δ	L	Cold (Perfect)					
		G_1	G_2	G_3	G_4	G_5	G_6
0.1	3.0345	3.1965	3.6369	3.8202	4.0029	4.1094	4.8881
0.2	2.5253	2.6776	3.0354	3.2234	3.3648	3.4972	4.1456
0.3	2.2009	2.3402	2.6511	2.8331	2.9490	3.1009	3.6655
0.4	1.9490	2.0744	2.3520	2.5235	2.6202	2.7895	3.2907
0.5	1.7329	1.8436	2.0946	2.2525	2.3332	2.5193	2.9682
0.6	1.5339	1.6292	1.8570	1.9983	2.0645	2.2681	2.6710
0.7	1.3388	1.4174	1.6232	1.7444	1.7966	2.0191	2.3791
0.8	1.1321	1.1921	1.3743	1.4707	1.5084	1.7521	2.0690
0.9	0.8812	0.9188	1.0705	1.1337	1.1551	1.4228	1.6895

Figure 6 depicts the reliability functions for both primary and upgraded systems.

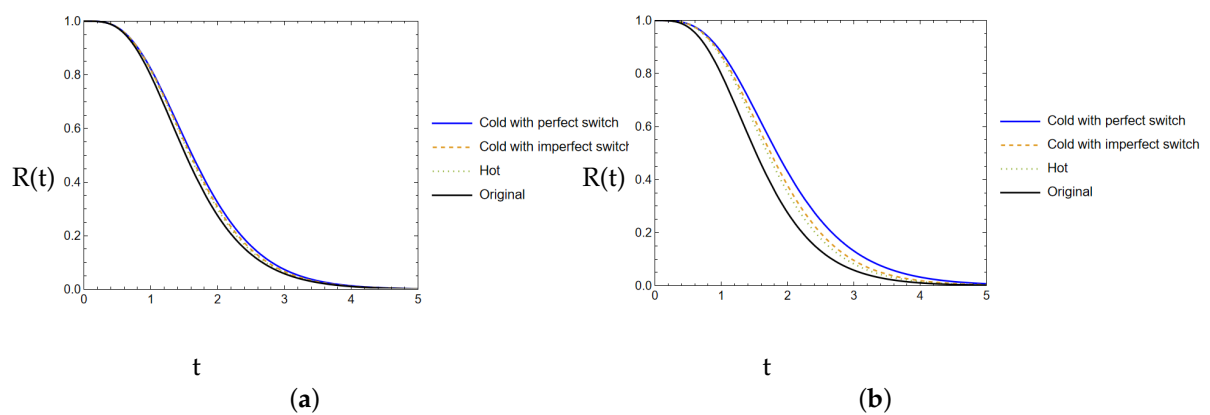


Figure 6. Cont.

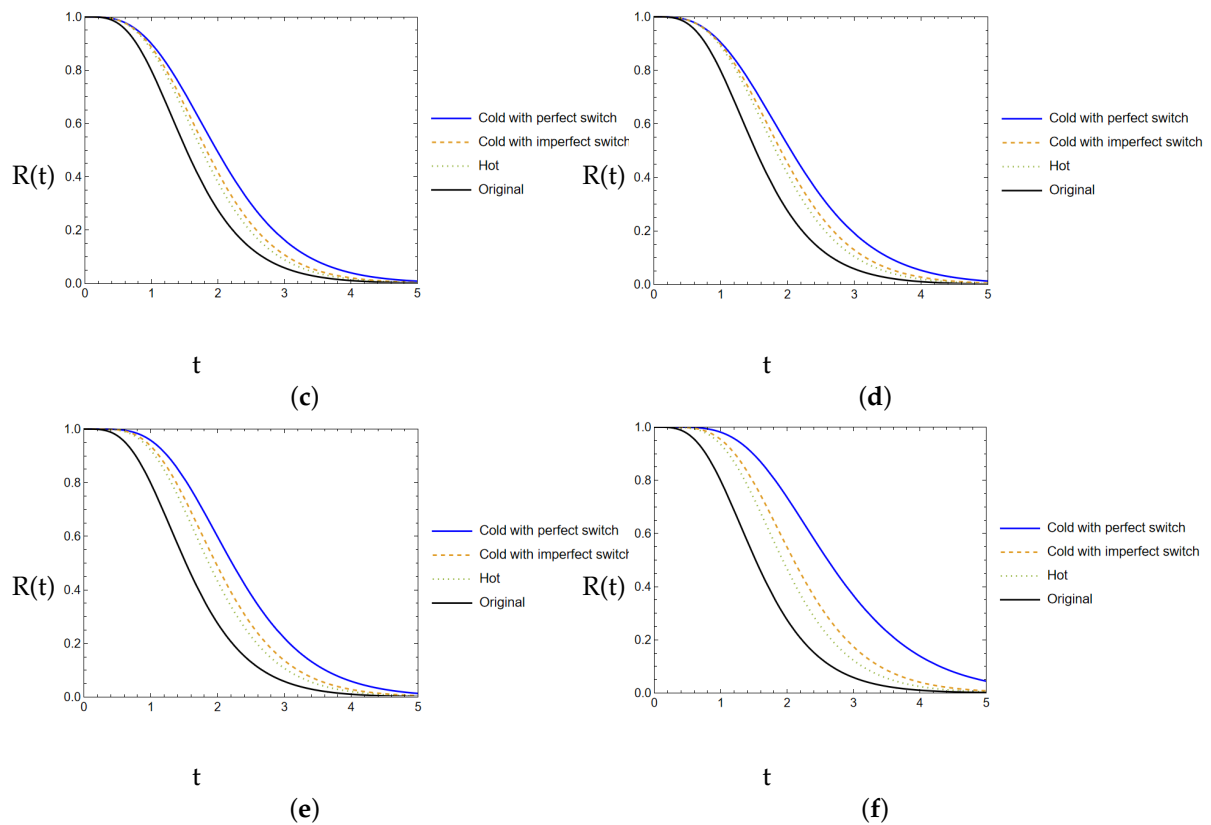


Figure 6. The reliability functions for the primary and upgraded systems when B belongs to the set G_i , $i = 1, \dots, 6$. (a) $B \in G_1 = \{\{3\}\}$; (b) $B \in G_2 = \{\{1\}, \{2\}, \{4\}, \{5\}\}$; (c) $B \in G_3 = \{\{1,3\}, \{2,3\}, \{3,4\}, \{3,5\}\}$; (d) $B \in G_4 = \{\{1,2\}, \{4,5\}\}$; (e) $B \in G_5 = \{\{1,5\}, \{2,4\}\}$; (f) $B \in G_6 = \{\{1,4\}, \{2,5\}\}$.

Additionally, Figure 7 illustrates the reliability functions for primary and enhanced systems based on hot and cold duplication methods.

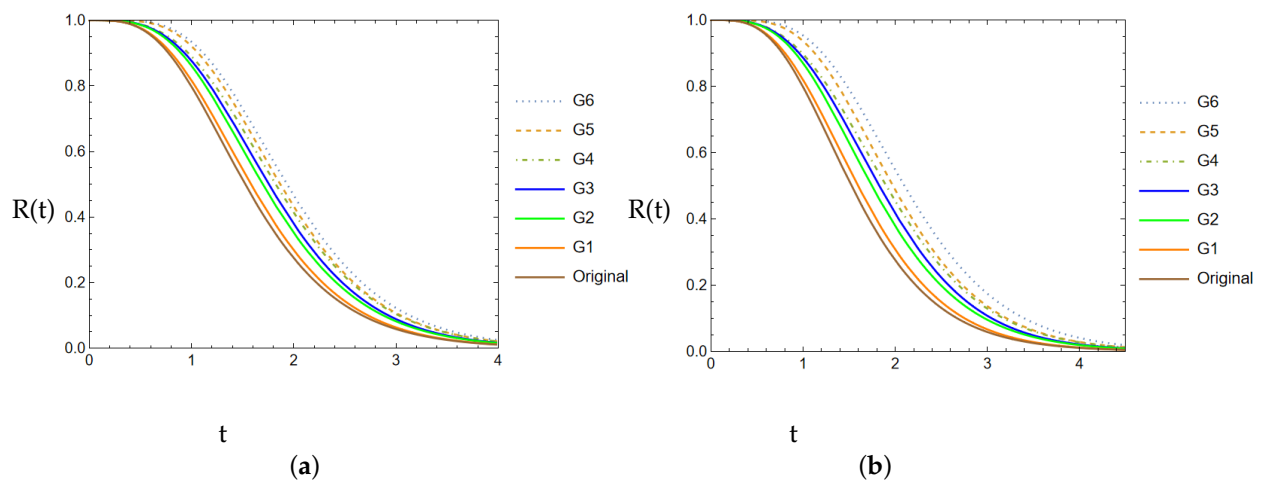


Figure 7. Cont.

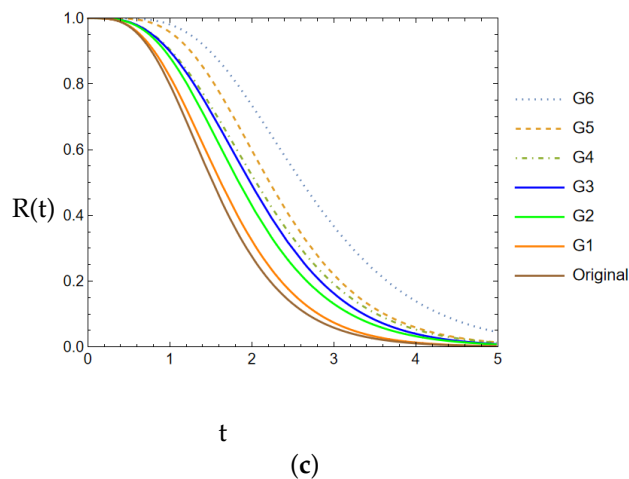


Figure 7. The reliability curves of the primary and upgraded systems. (a) The hot enhancing technique; (b) the imperfect-switch enhancing technique; (c) the cold enhancing technique with a perfect switch.

The findings presented in Tables 1 and 2, along with Figures 6 and 7, indicate that

- $MTTF < MTTF_B^h < MTTF_B^{im} < MTTF_B^c$.
- $MTTF < MTTF_{G_1}^h < MTTF_{G_2}^h < MTTF_{G_3}^h < MTTF_{G_4}^h < MTTF_{G_5}^h < MTTF_{G_6}^h$.
- $MTTF < MTTF_{G_1}^{im} < MTTF_{G_2}^{im} < MTTF_{G_3}^{im} < MTTF_{G_4}^{im} < MTTF_{G_5}^{im} < MTTF_{G_6}^{im}$.
- $MTTF < MTTF_{G_1}^c < MTTF_{G_2}^c < MTTF_{G_3}^c < MTTF_{G_4}^c < MTTF_{G_5}^c < MTTF_{G_6}^c$.
- With an increase in θ , the $MTTF$ also increases.
- $R < R_{G_1}^E < R_{G_2}^E < R_{G_3}^E < R_{G_4}^E < R_{G_5}^E < R_{G_6}^E$, $E = h, im, c$.
- $L_B(\delta) < L_B^h(\delta) < L_B^{im}(\delta) < L_B^c(\delta)$.
- The most effective method for enhancing the system's components is the cold duplication technique with a perfect switch, followed by cold duplication with an imperfect switch, which outperforms hot duplication; all of these methods yield better results compared to components without improvement.

The following tables present the values of the REFs required to determine the equivalence between the primary system and the improved systems using the methods of hot duplication (Table 3), cold duplication with an imperfect switch (Table 4), and cold duplication with a perfect switch (Table 5).

Table 3. The REFs for both the primary and the hot-improved systems at $\theta = 2.5$.

δ	A	Hot, B					
		G ₁	G ₂	G ₃	G ₄	G ₅	G ₆
0.1	G ₁	0.6323	0.1102	0.3781	0.6612	NE	NE
	G ₂	0.8883	0.6506	0.5851	0.4724	0.4544	0.3716
	G ₃	0.9097	0.7120	0.6566	0.5606	0.5452	0.4745
	G ₄	0.9414	0.7967	0.7515	NE	0.6539	0.5866
	G ₅	0.9419	0.8025	0.7601	0.6827	0.6698	0.6087
	G ₆	0.9438	0.8225	0.7889	0.7309	0.7217	0.6793
0.3	G ₁	0.5182	0.2908	0.3116	NE	NE	NE
	G ₂	0.8212	0.5413	0.4359	0.3190	0.2782	0.1908
	G ₃	0.8622	0.6383	0.5519	0.4555	0.4217	0.3493
	G ₄	0.9051	0.7309	NE	0.5652	0.8654	0.4564
	G ₅	NE	NE	0.6773	0.5978	0.5703	0.5072
	G ₆	0.9097	0.7652	0.7102	0.6491	0.6277	0.5820

Table 3. Cont.

δ	A	Hot, B					
		G_1	G_2	G_3	G_4	G_5	G_6
0.5	G_1	0.4289	NE	NE	0.2240	0.1302	0.1519
	G_2	0.7822	0.4557	0.3273	0.2080	0.1394	0.0504
	G_3	0.8466	0.5739	0.4690	0.3705	0.3139	0.2403
	G_4	0.8838	0.6764	0.5810	0.4833	NE	0.3388
	G_5	0.9068	0.7443	0.6186	0.5395	0.4924	0.4287
	G_6	0.8898	0.7203	0.6525	0.5892	0.5529	0.5056
0.6	G_1	0.3837	0.1189	0.2709	NE	NE	0.4537
	G_2	0.7689	0.4122	0.2760	0.1553	0.0682	NE
	G_3	0.8336	0.5382	0.4263	0.3263	0.2538	0.1796
	G_4	0.8765	0.6473	0.5431	0.4408	0.3597	NE
	G_5	0.8868	0.6992	0.5916	0.5125	0.4527	0.3890
	G_6	0.8830	0.6975	0.6253	0.5609	0.5143	0.4666
0.7	G_1	0.3347	0.7110	0.9001	0.2087	0.1673	0.1422
	G_2	0.7605	0.3648	0.2234	0.1019	NE	NE
	G_3	0.8152	0.4965	0.3791	0.2774	0.1829	0.1084
	G_4	0.8720	0.6141	0.5019	0.3936	0.2803	0.1788
	G_5	0.8802	0.6766	0.5648	0.4859	0.4093	0.3458
	G_6	0.8788	0.6728	0.5975	0.5324	0.4716	0.4238
0.8	G_1	0.2776	0.3390	0.2109	0.1007	NE	NE
	G_2	0.7601	0.3091	0.1666	0.0454	NE	NE
	G_3	0.8120	0.4431	0.3228	0.2195	0.0904	0.0159
	G_4	0.8721	NE	0.4532	NE	0.1613	0.0324
	G_5	0.8768	0.6243	0.5373	0.4598	0.3578	0.2952
	G_6	0.8788	0.6439	0.5679	0.5026	0.4206	0.3729
0.9	G_1	0.2022	0.8761	0.3215	0.5095	NE	NE
	G_2	0.7788	0.2338	0.1001	0.0912	NE	NE
	G_3	0.8210	0.3615	0.2450	0.1426	NE	NE
	G_4	0.8828	0.5101	0.3862	NE	0.1142	NE
	G_5	0.8874	0.5881	0.5085	0.4363	0.2871	0.2275
	G_6	0.8885	0.6057	0.5344	0.4716	0.3485	0.3024

Table 4. The REFs for both the primary systems and the systems improved with the cold method and an imperfect switch at $\theta = 2.5$.

δ	A	Cold with an Imperfect Switch, B					
		G_1	G_2	G_3	G_4	G_5	G_6
0.1	G_1	0.5449	0.3279	0.1892	0.5173	0.1160	0.1393
	G_2	0.8554	0.5687	0.4912	0.3755	0.3510	0.2335
	G_3	0.8828	0.6426	0.5766	0.4778	0.4569	NE
	G_4	0.9230	0.7398	0.6824	0.5899	0.5692	0.4645
	G_5	0.9239	0.7492	0.6960	0.6116	0.5929	0.4995
	G_6	0.9271	0.7804	0.7406	0.6813	0.6687	0.6089
0.3	G_1	0.4350	0.1438	0.7329	0.6623	NE	NE
	G_2	0.7848	0.4618	0.3420	0.2272	0.2514	0.0581
	G_3	0.8336	0.5732	0.4745	0.3795	0.3362	0.2398
	G_4	0.8843	0.6747	0.5836	0.4884	0.4422	0.3319
	G_5	0.8869	0.6941	0.6145	0.5334	0.4955	0.4060
	G_6	0.8911	0.7237	0.6611	0.5455	0.5738	0.5132

Table 4. Cont.

δ	A	Cold with an Imperfect Switch, B					
		G_1	G_2	G_3	G_4	G_5	G_6
0.5	G_1	0.3528	0.1742	NE	NE	NE	NE
	G_2	0.7486	0.3817	0.2393	0.1257	0.0414	NE
	G_3	0.8075	0.5135	0.3965	0.3025	0.2329	0.1392
	G_4	0.8644	0.6225	0.5099	0.4103	NE	0.2107
	G_5	0.8686	0.6533	0.5608	0.4827	0.4221	0.3360
	G_6	0.8726	0.6812	0.6059	0.5455	0.5009	0.4406
0.6	G_1	0.3123	NE	NE	0.2291	0.4262	0.8821
	G_2	0.7381	0.3420	0.1925	0.0797	NE	NE
	G_3	0.7990	0.4807	0.3571	0.2633	0.1758	0.0179
	G_4	0.8588	0.5950	0.4734	0.3707	NE	0.1360
	G_5	0.8636	0.6334	0.5373	0.4607	0.3855	0.3012
	G_6	0.8672	0.6603	0.5808	0.5205	0.4641	0.4043
0.7	G_1	0.2692	0.2455	NE	NE	0.9060	0.1752
	G_2	0.7335	0.2994	0.1459	0.0344	NE	NE
	G_3	0.7940	0.4424	0.3143	0.1562	NE	NE
	G_4	0.8565	0.5639	0.4343	NE	0.1792	0.0332
	G_5	0.8619	0.6124	0.5150	0.4404	0.3460	0.2634
	G_6	0.8650	0.6380	0.5560	0.4960	0.4240	0.3650
0.8	G_1	0.2200	0.1186	0.2214	0.7322	NE	NE
	G_2	0.7384	0.2503	0.0976	NE	NE	NE
	G_3	0.7948	0.3937	0.2640	0.1711	0.0213	NE
	G_4	0.8596	NE	0.3888	NE	0.0430	0.1651
	G_5	0.8653	0.5889	0.4936	0.4224	0.2999	0.2209
	G_6	0.8677	0.6127	0.5308	0.4720	0.3763	0.3189
0.9	G_1	0.1570	0.1177	0.0921	NE	NE	NE
	G_2	0.7648	0.1856	0.4457	NE	NE	NE
	G_3	0.8096	0.3197	0.1962	0.1074	NE	NE
	G_4	0.8479	0.4678	0.1370	0.2755	NE	NE
	G_5	0.8801	0.5598	0.4744	0.4109	0.2377	0.1646
	G_6	0.8814	0.5801	0.5045	0.4500	0.3102	0.2562

Table 5. The REFs for both the primary systems and the systems improved with the cold method and an imperfect switch at $\theta = 2.5$.

δ	A	Cold with a Perfect Switch, B					
		G_1	G_2	G_3	G_4	G_5	G_6
0.1	G_1	0.2809	0.8221	0.4718	0.6185	NE	NE
	G_2	0.7396	0.3168	0.2028	0.1095	0.0622	NE
	G_3	0.7868	0.4277	0.3308	0.2522	0.2129	0.01404
	G_4	0.8542	0.5397	NE	NE	0.2957	0.0213
	G_5	0.8573	0.5665	0.4741	0.3939	0.3518	0.1186
	G_6	0.0861	0.6513	0.5934	0.5465	0.5229	0.4023
0.3	G_1	0.2013	0.4992	0.5102	0.4481	NE	NE
	G_2	0.6729	0.2317	0.2025	NE	NE	NE
	G_3	0.7444	0.3832	0.2488	0.1813	0.1084	NE
	G_4	0.8169	0.4923	0.3426	0.2593	0.1619	NE
	G_5	0.8233	0.5371	0.4146	0.3484	0.2727	0.0668
	G_6	0.8333	0.6034	0.5188	0.4764	0.4541	0.4305

Table 5. Cont.

δ	A	Cold with a Perfect Switch, B					
		G_1	G_2	G_3	G_4	G_5	G_6
0.5	G_1	0.1497	NE	NE	0.5316	0.2070	0.9611
	G_2	0.6528	0.1762	NE	NE	NE	0.2641
	G_3	0.7321	0.3444	0.1925	0.1318	0.0227	NE
	G_4	0.8069	0.4558	0.2802	0.2006	0.0379	0.0021
	G_5	0.8153	0.5179	0.3857	0.3290	0.2191	0.0333
	G_6	0.8232	0.5724	0.4749	0.4359	0.3652	0.2610
0.6	G_1	0.1266	0.7761	0.9223	0.3008	0.1635	0.2269
	G_2	0.5534	0.1512	0.7180	0.2855	NE	NE
	G_3	0.7323	0.3228	0.1661	0.1088	NE	NE
	G_4	0.8081	0.4370	0.2515	0.1731	NE	NE
	G_5	0.8170	0.5096	0.3770	0.3250	0.1941	0.0180
	G_6	0.8236	0.5587	0.4579	0.4208	0.3341	0.2336
0.7	G_1	0.1038	NE	NE	0.1168	0.3902	0.1853
	G_2	0.6623	0.1260	NE	0.9127	NE	0.4038
	G_3	0.7376	0.2976	0.1391	0.0854	NE	NE
	G_4	0.8142	0.4161	0.1150	0.27143	NE	NE
	G_5	0.8233	0.5019	0.3723	0.3256	0.1683	0.0026
	G_6	0.8284	0.5453	0.4436	0.4089	0.3014	0.2052
0.8	G_1	0.07985	0.7544	0.7423	0.0991	NE	NE
	G_2	0.6838	0.0992	NE	0.8300	0.7713	0.1108
	G_3	0.7510	0.2654	0.1095	0.0608	NE	NE
	G_4	0.8276	0.3904	NE	0.1136	0.6801	NE
	G_5	0.8362	0.4946	0.3734	0.3333	0.1399	NE
	G_6	0.8397	0.5316	0.4328	0.4017	0.2640	0.1735
0.9	G_1	0.0524	0.1441	0.8331	0.2322	0.4016	NE
	G_2	0.7318	0.0673	NE	NE	0.2266	0.5580
	G_3	0.7824	0.2162	0.0738	0.0334	NE	NE
	G_4	0.8560	0.3522	NE	0.0760	NE	0.1792
	G_5	0.8628	0.4884	0.3863	0.3562	0.1044	NE
	G_6	0.8646	0.5168	0.4293	0.4043	0.2145	0.1332

Based on the information provided in Tables 2–5, the following can be observed:

- Referring to the data presented in Table 2, it is evident that duplicating the B set within G_6 results in an increase in the value of $L(0.5)$ from 1.7329 to 2.1969. A similar impact on $L(0.5)$ is observed when the failure rates of the components in set A are reduced. Specifically, this effect is observed when the failure rates of the components in the following groups are modified: (1) G_1 with a reduction factor of $\eta^h = 0.8898$, (2) G_2 with a reduction factor of $\eta^h = 0.7203$, (3) G_3 with a reduction factor of $\eta^h = 0.6525$, (4) G_4 with a reduction factor of $\eta^h = 0.5892$, (5) G_5 with a reduction factor of $\eta^h = 0.5529$, and (6) G_6 with a reduction factor of $\eta^h = 0.5056$. These changes are detailed in Table 3.
- In addition to the information provided in Table 2, it is worth noting that the cold duplication of the B set within G_5 leads to an increase in the value of $L(0.7)$, which changes from 0.13388 to 1.7731. This same effect is observed when the failure rates of the components in set A are reduced. Specifically, this effect is evident when the failure rates of the components within the following groups are adjusted: (1) G_1 with a reduction factor of $\eta^{im} = 0.8619$, (2) G_2 with a reduction factor of $\eta^{im} = 0.6124$, (3) G_3 with a reduction factor of $\eta^{im} = 0.5150$, (4) G_4 with a reduction factor of $\eta^{im} = 0.4404$, (5) G_5 with a reduction factor of $\eta^{im} = 0.3460$, and (6) G_6 with a reduction factor of $\eta^{im} = 0.2634$. These modifications are detailed in Table 4.
- Furthermore, as indicated in Table 2, the cold duplication of the B set within G_4 results in an increase in the value of $L(0.5)$, which moves from 1.7329 to 2.3332. This same

effect is observed when the failure rates of the components in set A are reduced. Specifically, this effect is evident when the failure rates of the components within the following groups are adjusted: (1) G_1 with a reduction factor of $\eta^c = 0.8069$, (2) G_2 with a reduction factor of $\eta^c = 0.4558$, (3) G_3 with a reduction factor of $\eta^c = 0.2802$, (4) G_4 with a reduction factor of $\eta^c = 0.2006$, (5) G_5 with a reduction factor of $\eta^c = 0.0379$, and (6) G_6 with a reduction factor of $\eta^c = 0.0021$. These changes are detailed in Table 5.

- The notation “NE” signifies that there is no equivalence found between the system enhanced through the reduction method and those enhanced through duplication methods.

8. Conclusions

In this research paper, the concept of reliability is employed to assess a bridge system with components that adhere to a Bilal distribution. The study investigates four improvement methods: reduction, hot duplication, cold duplication with a perfect switch, and cold duplication with an imperfect switch. The findings reveal that the enhanced systems outperform the original system, with cold duplication using a perfect switch demonstrating the highest effectiveness. To illustrate these theoretical findings, this paper includes a numerical example.

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