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Functional Forms for Lorentz Invariant Velocities

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Abstract: Lorentz invariance lies at the very heart of Einstein's special relativity, and both the energy formula and the relative velocity formula are well-known to be invariant under a Lorentz transformation. Here, we investigate the spatial and temporal dependence of the velocity field itself $\mathbf{u}(\mathbf{x}, t)$ and we pose the problem of the determination of the functional form of those velocity fields $\mathbf{u}(\mathbf{x}, t)$ which are automatically invariant under a Lorentz transformation. For a single spatial dimension, we determine a first-order partial differential equation for the velocity $u(x, t)$, which appears to be unknown in the literature, and we investigate its main consequences, including demonstrating that it is entirely consistent with many of the familiar outcomes of special relativity and deriving two new partial differential relations connecting energy and momentum that are fully compatible with the Lorentz invariant energy–momentum relations.

Keywords: special relativity; Lorentz invariance; functional forms; partial differential equation identities

MSC: 35q75

1. Introduction

In special relativity, the word "special" alludes to invariance under transformations relating constant relative velocity frames of reference, which are known as Lorentz transformations. A Lorentz invariant quantity is one which assumes an identical form under a Lorentz transformation. While Lorentz invariance and its consequences are well established in special relativity, it seems to have gone unnoticed that the imposition of a Lorentz invariant velocity field $\mathbf{u}(\mathbf{x}, t)$ restricts the functional form of the velocity $\mathbf{u}(\mathbf{x}, t)$ to the solution of a certain partial differential equation. For example, for a single Cartesian spatial dimension, the requirement that the equation $dx/dt = u(x, t)$ remains invariant under a Lorentz transformation implies that the velocity $u(x, t)$ satisfies the partial differential Equations (11) or (16).

For those problems involving partial differential equations and boundary or initial conditions, invariance of both the equation and the associated conditions under a one-parameter Lie group of transformations generally implies a major simplification of the problem (see, for example, [1]). In the present context, solutions of either (11) or (16) will generate solutions of those one-dimensional special relativistic problems provided that any boundary or initial conditions also remain invariant under Lorentz transformation. For a single space dimension x, this means that any associated boundary or initial condition must be assumed to be expressible in terms of the characteristic coordinates $\alpha = ct + x$ and $\beta = ct - x$ and have the general form $f(\alpha\beta) = constant$ for some function f.

Special relativity has become a standard subject such that almost every text on physics or mechanics has a dedicated chapter. The older texts are closer to the original motivating issues and the developments that gave birth to the subject. Dingle [2] and McCrea [3] are student texts, while more comprehensive accounts can be found in Bohm [4], French [5] Resnick [6]. Both Moller [7] and Tolman [8] are standard works of reference, and the reader may wish to consult [9], which contains the original papers of Einstein, Lorentz, Minkowski and Weyl, with additional notes by Arnold Sommerfeld. In the following

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section we summarise the essential results of special relativity theory that are needed in order to deduce the partial differential Equation (11) for the velocity field $u(x,t)$, which is derived in the section thereafter for a single spatial dimension x.

2. Essential Results of Special Relativity

The notion of invariance with respect to frames moving with constant relative velocity underlies special relativistic mechanics, and in particular those transformations of space and time leaving the wave equation unchanged, and are referred to as Lorentz transformations. We consider a rectangular Cartesian frame (X,Y,Z) and another frame (x,y,z) moving with constant velocity v relative to the first frame, and the motion is assumed to be in the aligned X and x directions, as indicated in Figure 1. We view the relative velocity v as a parameter measuring the departure of the current frame (x,y,z) from the rest frame (X,Y,Z), and for this purpose, we adopt a notation employing the lower case for variables associated with the moving (x,y,z) frame and the upper case or capitals for those variables associated with the rest (X,Y,Z) frame. Accordingly, time is measured from the (X,Y,Z) frame with the variable T and from the (x,y,z) frame with the variable t. Following normal practice, we assume that $y=Y$ and $z=Z$, so that (X,T) and (x,t) are the variables of principal interest.

For $0 \leqslant v < c$, the standard Lorentz transformations are

$$x = \frac{X - vT}{[1-(v/c)^2]^{1/2}}, \quad t = \frac{T - vX/c^2}{[1-(v/c)^2]^{1/2}}, \tag{1}$$

with the inverse and identity transformations characterised by $-v$ and $v=0$, respectively. Derivations of these equations can be found in many standard textbooks such as Feynmann et al. [10] and Landau and Lifshitz [11], and other novel derivations are given by Lee and Kalotas [12] and Levy-Leblond [13]. We note that on using the chain rule for partial derivatives, from Equation (1) we may obtain the differential relations

$$\frac{\partial}{\partial x} = \frac{1}{[1-(v/c)^2]^{1/2}}\left\{\frac{\partial}{\partial X} + \frac{v}{c^2}\frac{\partial}{\partial T}\right\}, \quad \frac{\partial}{\partial t} = \frac{1}{[1-(v/c)^2]^{1/2}}\left\{\frac{\partial}{\partial T} + v\frac{\partial}{\partial X}\right\}. \tag{2}$$

We adopt the notation $\alpha = ct + x$ and $\beta = ct - x$ for the characteristic variables, and from the above equations, by direct substitution, we may readily deduce the relations

$$ct + x = \left(\frac{1 - v/c}{1 + v/c}\right)^{1/2}(cT + X), \quad ct - x = \left(\frac{1 + v/c}{1 - v/c}\right)^{1/2}(cT - X). \tag{3}$$

With the notation $\mathscr{A} = cT + X$, $\mathscr{B} = cT - X$ and $\lambda = [(1 - v/c)/(1 + v/c)]^{1/2}$, (3) becomes simply $\alpha = \lambda\mathscr{A}$ and $\beta = \mathscr{B}/\lambda$, so that in particular we may confirm the simple Lorentz invariance $(ct)^2 - x^2 = (cT)^2 - X^2$.

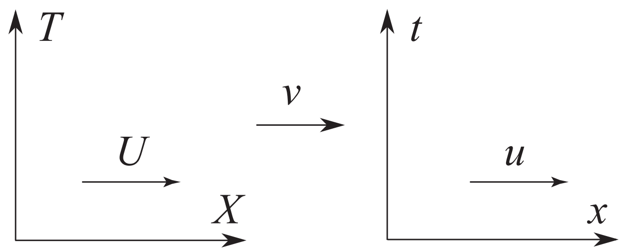

Figure 1. Two inertial frames moving along x axis with relative velocity v.

The Einstein addition of velocities law is an immediate consequence of the notion of invariance under Lorentz transformations. Since the relative frame velocity v is assumed to be constant, on taking the differentials of both equations in (1) with velocities $U = dX/dT$ and $u = dx/dt$, we obtain

$$dx = \frac{dX - vdT}{[1-(v/c)^2]^{1/2}}, \quad dt = \frac{dT - vdX/c^2}{[1-(v/c)^2]^{1/2}}, \tag{4}$$

and dividing the first differential by the second yields the Einstein addition-of-velocity law

$$u = \frac{U - v}{1 - Uv/c^2}. \tag{5}$$

Immediate consequences of (5) are the two equivalent alternative identities

$$[1-(u/c)^2]^{1/2}(1 - Uv/c^2) = [1-(v/c)^2]^{1/2}[1-(U/c)^2]^{1/2}, \tag{6}$$

and

$$\left(\frac{1+U/c}{1-U/c}\right) = \left(\frac{1+u/c}{1-u/c}\right)\left(\frac{1+v/c}{1-v/c}\right), \tag{7}$$

so that with velocity variables $(\Theta, \theta, \epsilon)$ defined by

$$\Theta = \tanh^{-1}(U/c), \quad \theta = \tanh^{-1}(u/c), \quad \epsilon = \tanh^{-1}(v/c), \tag{8}$$

Equation (7) becomes simply the translation $\Theta = \theta + \epsilon$, noting that within the context of special relativity, v and therefore ϵ are both constants. The angle θ is sometimes referred to as the rapidity, and it is the angle in which the Lorentz invariance appears through a translational invariance. We have the elementary relations

$$\theta = \frac{1}{2}\log\left(\frac{1+u/c}{1-u/c}\right) = \tanh^{-1}(u/c), \quad \left(\frac{1+u/c}{1-u/c}\right)^{1/2} = e^{\theta}, \tag{9}$$

and e^{θ} is the Doppler shift.

The particular form of the identity (6) may be established using (5) in the left-hand side of (6), and this version is fundamental to the development of special relativity in establishing the Lorentz invariance of certain quantities such as the Einstein energy expression $e = e_0/[1-(u/c)^2]^{1/2}$ discussed below. The importance of the particular form (7) is that there exist special relativity theories which apply for relative velocities greater than the speed of light, such as that proposed in [14,15]. This theory is complementary to the Einstein special theory of relativity that applies to relative velocities less than the speed of light. The authors in [14,15] derive Lorentz transformations corresponding to (1) for superluminal relative velocities and show that Einstein's addition-of-velocities law still applies. The two Formulae (5) and (6), when expressed in the form (7), reveal that at least one of the velocities u, v or U must not exceed the speed of light, and in terms of taking square roots or logarithms, all need appropriate rearrangement depending upon the particular values of the three velocities u, v and U.

Here, we suppose that energy and momentum are given, respectively, by the usual expressions $e = mc^2$ and $p = mu$, where m denotes the mass given by $m = m_0/[1-(u/c)^2]^{1/2}$, u is the velocity in the (x,t) frame and m_0 is the rest mass, and we use $e_0 = m_0c^2$ to designate the rest energy. The Lorentz invariant energy–momentum relations are given by

$$p = \frac{P - Ev/c^2}{[1-(v/c)^2]^{1/2}}, \quad e = \frac{E - Pv}{[1-(v/c)^2]^{1/2}}. \tag{10}$$

where $E = Mc^2$ and $P = MU$ and where $M = m_0/[1-(U/c)^2]^{1/2}$. It is clear from the Lorentz invariant energy–momentum relations (10) themselves that the quantity $e^2 -$

$(pc)^2 = E^2 - (Pc)^2$ is invariant, while explicit use of the Einstein energy–mass expression $e = e_0/[1-(u/c)^2]^{1/2}$ further prescribes this quantity to be the constant e_0^2.

3. Lorentz Invariant Velocity Fields $u(x,t)$

We now determine the most general one-dimensional velocity field $u(x,t)$ that remains invariant under the Lorentz transformation (1). Equivalently, under the Lorentz transformation (1), we determine the velocity field $u(x,t)$, which is such that the differential problem $dx/dt = u(x,t)$ transforms into $dX/dT = u(X,T)$ for the same function $u(x,t)$. Since the Lorentz transformation forms a one-parameter group of transformations in the frame velocity v, we need only expand (1) and equate the corresponding infinitesimals for $dx/dt = u(x,t)$ to obtain the following first-order partial differential equation for $u(x,t)$, thus

$$t\frac{\partial u}{\partial x} + \frac{x}{c^2}\frac{\partial u}{\partial t} = 1 - \left(\frac{u}{c}\right)^2. \tag{11}$$

This partial differential arises as follows. On taking the differentials of (1), we obtain Equation (4), and by division there follows Einstein's expression for velocities in Equation (5). However, if the velocity equation $dx/dt = u(x,t)$ itself remains invariant under the Lorentz transformation, the implication is that the velocity equation for $U(X,T)$ enjoys the same dependence on the upper-case variables as that of $u(x,t)$ on the lower-case variables, namely, $dX/dT = u(X,T)$, where the lower-case function u is the same in both instances. From $dx/dt = u(x,t)$ and by expanding both sides for infinitesimal values of the velocity v, we thus have

$$\frac{dx}{dt} = \frac{U - v}{1 - Uv/c^2} \approx U - v\left(1 - \left(\frac{U}{c}\right)^2\right), \tag{12}$$

$$u(x,t) \approx u\left(X - vT, T - \frac{vX}{c^2}\right) \approx u(X,T) - v\left(T\frac{\partial u}{\partial X} + \frac{X}{c^2}\frac{\partial u}{\partial T}\right), \tag{13}$$

so that if $U(X,T) = u(X,T)$, then the partial differential Equation (11) follows on reverting to the lower-case variables (x,t).

From (1) and the formulae in (2) for the partial derivatives, the operator L defined below is a Lorentz invariant operator, namely

$$L = ct\frac{\partial}{\partial x} + \frac{x}{c}\frac{\partial}{\partial t} = cT\frac{\partial}{\partial X} + \frac{X}{c}\frac{\partial}{\partial T}. \tag{14}$$

Further, the partial differential Equation (11) remains unchanged by the particle–wave space–time transformation $x^* = ct$, $t^* = x/c$ and $u^* = c^2/u$.

We comment that in a single space dimension x with (x,t) referring to the particle and (x^*, t^*) referring to the wave, the de Broglie particle–wave duality revolves around the space–time transformation $x^* = ct$ and $t^* = x/c$ so that the wave velocity $w = dx^*/dt^*$ is connected to the particle velocity $u = dx/dt$ through the relation $wu = c^2$. In the *Nature* article [16] and elsewhere [17,18], de Broglie was first to supplement Planck's particle energy expression $e = h\nu$ with the equation for particle momentum $p = h/\lambda$ to establish the dual particle–wave nature of matter for all particles and extend the principle of duality to the laws of nature, where h is the Planck constant and ν and λ are the frequency and wavelength, respectively, of the associated wave, having a wave speed $w = \nu\lambda$. By making use of Einstein's energy expression, the group velocity of the wave w_g may be shown to coincide with the particle velocity u, giving rise to de Broglie's idea of the wave guiding the motion of the particle [19,20].

The invariance of the operator L is also apparent in terms of the characteristic coordinates $\alpha = ct + x$ and $\beta = ct - x$, since from $x = (\alpha - \beta)/2$, $t = (\alpha + \beta)/2c$ and the definition of L, we have

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \alpha} - \frac{\partial}{\partial \beta}, \qquad \frac{1}{c}\frac{\partial}{\partial t} = \frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta}, \qquad L = \alpha\frac{\partial}{\partial \alpha} - \beta\frac{\partial}{\partial \beta}, \tag{15}$$

and the first-order partial differential Equation (11) becomes

$$\alpha\frac{\partial u}{\partial \alpha} - \beta\frac{\partial u}{\partial \beta} = c\left(1 - \left(\frac{u}{c}\right)^2\right), \tag{16}$$

and from which it is clear that the above transformations $\alpha = \lambda\mathscr{A}$ and $\beta = \mathscr{B}/\lambda$ ensure invariance under the Lorentz group (1).

In terms of θ defined by (9), the partial differential Equation (16) becomes simply

$$\alpha\frac{\partial \theta}{\partial \alpha} - \beta\frac{\partial \theta}{\partial \beta} = 1. \tag{17}$$

This equation may be solved using Lagrange's characteristic method, which introduces a characteristic parameter s through the three equations

$$\frac{d\alpha}{ds} = \alpha, \qquad \frac{d\beta}{ds} = -\beta, \qquad \frac{d\theta}{ds} = 1, \tag{18}$$

and then the general solution is obtained by taking one integral of these equations to be an arbitrary function of a second independent integral. Thus, for example, by division of the first equation, we have

$$\frac{d\beta}{d\alpha} = -\frac{\beta}{\alpha}, \qquad \frac{d\theta}{d\alpha} = \frac{1}{\alpha}, \tag{19}$$

which can both be integrated to yield $\alpha\beta = C_1$ and $\theta = \log\alpha + C_2$, where C_1 and C_2 denote arbitrary constants. From (19), we might deduce that the general solution of (17) may be determined from $C_2 = \Phi(C_1)$, where Φ denotes an arbitrary function, thus $\theta(\alpha, \beta) = \log\alpha + \Phi(\alpha\beta)$.

We observe that the expression for $\theta(\alpha, \beta)$ is entirely consistent with the Einstein addition-of-velocity law in its various forms (5), (6) or (7), since under the transformations $\alpha = \lambda\mathscr{A}$ and $\beta = \mathscr{B}/\lambda$, we have as required $\Theta = \theta + \epsilon$ arising from

$$\Theta(\mathscr{A}, \mathscr{B}) = \log\mathscr{A} + \Phi(\mathscr{A}\mathscr{B}) = \log\alpha + \Phi(\alpha\beta) - \log\lambda = \theta + \epsilon, \tag{20}$$

and $\epsilon = -\log\lambda$.

From $\theta(\alpha, \beta) = \log\alpha + \Phi(\alpha\beta)$ and the expressions $e = mc^2$, $p = mu$, $m = m_0/[1 - (u/c)^2]^{1/2}$ and (9), the velocity u, energy e and momentum p can be shown to be given, respectively, by

$$\frac{u}{c} = \frac{(\alpha\phi)^2 - 1}{(\alpha\phi)^2 + 1}, \qquad e = \frac{e_0}{2}\left(\alpha\phi + \frac{1}{\alpha\phi}\right), \qquad pc = \frac{e_0}{2}\left(\alpha\phi - \frac{1}{\alpha\phi}\right), \tag{21}$$

where $\phi(\alpha\beta) = \exp\Phi(\alpha\beta)$. We observe that the particular structure of e and p in (21) ensures $e^2 - (pc)^2 = e_0^2$, and that $u(x,t)$ is given explicitly by

$$u(x,t) = c\tanh[\log(ct + x) + \Phi((ct)^2 - x^2)], \tag{22}$$

where $\Phi((ct)^2 - x^2)$ denotes an arbitrary function. Alternatively, the velocity $u(x,t)$ can be expressed as

$$\frac{u}{c} = \frac{[(ct + x)\phi((ct)^2 - x^2)]^2 - 1}{[(ct + x)\phi((ct)^2 - x^2)]^2 + 1}, \tag{23}$$

where the two arbitrary functions ϕ and Φ are connected by the relation $\phi((ct)^2 - x^2) = \exp \Phi((ct)^2 - x^2)$.

As a specific simple illustration, the extension of special relativity formulated by the author in [21] assumes that the momentum $p(x,t)$ and the wave energy $\mathscr{E}(x,t) = -e(x,t) - V(x,t)$ both satisfy the classical wave equation

$$\frac{\partial^2 p}{\partial t^2} = c^2 \frac{\partial^2 p}{\partial x^2}, \qquad \frac{\partial^2 \mathscr{E}}{\partial t^2} = c^2 \frac{\partial^2 \mathscr{E}}{\partial x^2}, \tag{24}$$

where $V(x,t)$ denotes the applied external potential. This potential is assumed to generate a conventional spatial force $f(x,t)$ and a nonconventional force in the direction of time $g(x,t)$ such that

$$f = -\frac{\partial V}{\partial x}, \qquad gc^2 = -\frac{\partial V}{\partial t}. \tag{25}$$

In nonlinear continuum mechanics, $g(x,t)$ is more commonly known as the mass or energy production term. We refer the reader to [21] for further details of this extended version of special relativity.

From the functional form for $p(x,t)$ given by the third equation in (21), we obtain

$$c\frac{\partial p}{\partial \alpha} = \frac{e_0}{2}\left(1 + \frac{1}{(\alpha\phi)^2}\right)\left(\phi + \xi\frac{d\phi}{d\xi}\right) = \frac{e_0}{2}\left(1 + \frac{1}{(\alpha\phi)^2}\right)\frac{d(\xi\phi)}{d\xi}, \tag{26}$$

$$c\frac{\partial p}{\partial \beta} = \frac{e_0}{2}\left(\alpha^2\frac{d\phi}{d\xi} + \frac{1}{\phi^2}\frac{d\phi}{d\xi}\right) = \frac{e_0}{2}\left\{\alpha^2\frac{d\phi}{d\xi} - \frac{d}{d\xi}\left(\frac{1}{\phi}\right)\right\}, \tag{27}$$

where $\xi = \alpha\beta$ and $\alpha = ct + x$ and $\beta = ct - x$ are as previously defined. From the first wave equation in (24), we find that

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial x^2} = 4\frac{\partial^2 p}{\partial\alpha\partial\beta} = \frac{2e_0\alpha}{c}\left\{\left(1 + \frac{1}{(\alpha\phi)^2}\right)\frac{d^2(\xi\phi)}{d\xi^2} - \frac{2\alpha}{(\alpha\phi)^3}\frac{d\phi}{d\xi}\frac{d(\xi\phi)}{d\xi}\right\} \tag{28}$$

$$= \frac{2e_0}{c}\left\{\alpha\frac{d^2(\xi\phi)}{d\xi^2} - \beta\frac{d^2}{d\xi^2}\left(\frac{1}{\phi}\right)\right\} = 0. \tag{29}$$

Evidently then, the first term involving α requires that $\xi\phi(\xi) = C_1\xi + C_2$ where C_1 and C_2 denote two arbitrary constants, while the second term involving β requires a similar expression for $1/\phi$. The only way this can happen is that either $C_2 = 0$ and $\phi(\xi) = C_1$ or $C_1 = 0$ and $\phi(\xi) = C_2/\xi$, in which case we obtain from (21) the following two distinct expressions for particle velocity, energy and momentum,

$$\frac{u}{c} = \frac{(C_1\alpha)^2 - 1}{(C_1\alpha)^2 + 1}, \quad e = \frac{e_0}{2}\left(C_1\alpha + \frac{1}{C_1\alpha}\right), \quad pc = \frac{e_0}{2}\left(C_1\alpha - \frac{1}{C_1\alpha}\right), \tag{30}$$

$$\frac{u}{c} = \frac{(C_2/\beta)^2 - 1}{(C_2/\beta)^2 + 1}, \quad e = \frac{e_0}{2}\left(\frac{C_2}{\beta} + \frac{\beta}{C_2}\right), \quad pc = \frac{e_0}{2}\left(\frac{C_2}{\beta} - \frac{\beta}{C_2}\right), \tag{31}$$

where $\alpha = ct + x$ and $\beta = ct - x$.

4. Partial Differential Relations for Energy and Momentum

From the relations for energy and momentum

$$e = \frac{e_0}{[1 - (u/c)^2]^{1/2}}, \qquad p = \frac{m_0 u}{[1 - (u/c)^2]^{1/2}}, \tag{32}$$

we may deduce the following expressions for the partial derivatives,

$$\frac{\partial e}{\partial x} = \frac{m_0 u}{[1 - (u/c)^2]^{3/2}}\frac{\partial u}{\partial x}, \qquad \frac{\partial p}{\partial x} = \frac{m_0}{[1 - (u/c)^2]^{3/2}}\frac{\partial u}{\partial x}, \tag{33}$$

with similar expressions for the partial derivatives with respect to time. On division of (11) by $[1-(u/c)^2]^{3/2}$ and with some rearrangement, we may deduce the two partial differential relations connecting the partial derivatives of the energy and momentum e and p, thus

$$ct\frac{\partial e}{\partial x}+\frac{x}{c}\frac{\partial e}{\partial t}=cp, \qquad ct\frac{\partial p}{\partial x}+\frac{x}{c}\frac{\partial p}{\partial t}=\frac{e}{c}. \tag{34}$$

In terms of the invariant operator L, these relations become simply $L(e)=cp$ and $L(cp)=e$, giving rise to the apparent relations

$$L(e+cp)=e+cp, \qquad L(e-cp)=-(e-cp), \qquad L^2(e)=e, \qquad L^2(p)=p. \tag{35}$$

For example, formally, we may apply the operator L to the product $e^2-(pc)^2=(e+cp)(e-cp)$ to obtain

$$L(e^2-(pc)^2)=(e-cp)L(e+cp)+(e+cp)L(e-cp) \tag{36}$$
$$=(e^2-(pc)^2)-(e^2-(pc)^2)=0, \tag{37}$$

as might be anticipated. The two partial differential relations (34) are fully compatible with the two Lorentz invariant energy–momentum relations (10), so that combined with (10), one of the partial differential equations in (34) is a consequence of the other.

5. Conclusions

For a single spatial dimension x, we have showed that the requirement for the velocity equation $dx/dt=u(x,t)$ to remain invariant under the Lorentz transformation (1) implies that the velocity $u(x,t)$ satisfies the partial differential Equations (11) or (16) and in consequence inherits the particular functional form $u(x,t)=c\tanh[\log(ct+x)+\Phi((ct)^2-x^2)]$, where $\Phi((ct)^2-x^2)$ denotes an arbitrary function of the indicated argument. As far as the author is aware, neither the partial differential Equation (11) nor the fact that the energy and momentum satisfy the partial differential relations (34) have been mentioned previously in the literature. For the particular special relativistic model proposed in [21], explicit formulae are given for the one-dimensional particle velocity, energy and momentum, which are the only expressions for the proposed model which give rise to Lorentz invariant velocities.

For particles moving in both two and three spatial dimensions, there are corresponding implications that are currently under investigation by the author, and planar Lorentz invariant velocities are examined in [22]. For the higher spatial dimensions, the constraints appear to be less restrictive so that the outcomes are more varied and more physically interesting. For example, for planar motion, the two velocity components satisfy corresponding partial differential equations to (11) with general solutions for the velocity components in terms of two arbitrary functions $\mathscr{F}$ and $\mathscr{G}$. These partial differential equations admit the singular case $\mathscr{G}=(1-\mathscr{F}^2)^{1/2}$, such that for all arbitrary functions $\mathscr{F}$, the magnitude of the particle velocity is the speed of light. This means that there are infinitely many families of paths with particles moving at the speed of light. In addition, there are associated partial differential relations connecting energy and momentum corresponding to (34), and these partial differential relations are fully compatible with the planar Lorentz invariant energy–momentum relations and appear not to have been given previously in the literature.

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