mathematics

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Existence and Nonexistence of Positive Solutions for Semilinear Elliptic Equations Involving Hardy–Sobolev Critical Exponents

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Abstract: The following semi-linear elliptic equations involving Hardy–Sobolev critical exponents $-\Delta u - \mu \frac{u}{|x|^2} = \frac{|u|^{2^*(s)-2}}{|x|^s} u + g(x,u), x \in \Omega \setminus \{0\}, u = 0, x \in \partial\Omega$ have been investigated, where Ω is an open-bounded domain in $\mathbb{R}^N (N \geq 3)$, with a smooth boundary $\partial\Omega$, $0 \in \Omega$, $0 \leq \mu < \bar{\mu} := \left(\frac{N-2}{2}\right)^2$, $0 \leq s < 2$, and $2^*(s) = 2(N-s)/(N-2)$ is the Hardy–Sobolev critical exponent. This problem comes from the study of standing waves in the anisotropic Schrödinger equation; it is very important in the fields of hydrodynamics, glaciology, quantum field theory, and statistical mechanics. Under some deterministic conditions on g, by a detailed estimation of the extremum function and using mountain pass lemma with $(PS)_c$ conditions, we obtained that: (a) If $\mu \leq \overline{\mu} - 1$, and $\lambda < \lambda_1(\mu)$, then the above problem has at least a positive solution in $H_0^1(\Omega)$; (b) If $\overline{\mu} - 1 < \mu < \overline{\mu}$, then when $\lambda_*(\mu) < \lambda < \lambda_1(\mu)$, the above problem has at least a positive solution in $H_0^1(\Omega)$; (c) if $\overline{\mu} - 1 < \mu < \overline{\mu}$ and $\Omega = B(0,R)$, then the above problem has no positive solution for $\lambda \leq \lambda_*(\mu)$. These results are extensions of E. Jannelli's research $(g(x,u) = \lambda u)$.

Keywords: semilinear elliptic equation; Hardy–Sobolev critical exponents; mountain pass lemma; $(PS)_c$ condition

MSC: 35B09; 35J20; 35J60; 35J75

Citation: Wang, L.-L.; Fan, Y.-H. Existence and Nonexistence of Positive Solutions for Semilinear Elliptic Equations Involving Hardy–Sobolev Critical Exponents. *Mathematics* **2024**, *12*, 1616. https://doi.org/10.3390/math12111616

Academic Editor: Alberto Ferrero

Received: 23 March 2024
Revised: 13 May 2024
Accepted: 16 May 2024
Published: 21 May 2024

1. Introduction

First, we present some symbols and notations. As usual, suppose $p \geq 1, k, m$ are two nonnegative integers, N, L are two positive integers, Ω is a subset of $\mathbb{R}^N$, and $C^m(\Omega, \Omega_1)$ denotes the function space, such that U and $D^\alpha U(|\alpha| \leq m)$ are all continuous, and $C^m(\Omega)$ for short. $C^\infty(\Omega) = \cap_{m=0}^{\infty} C^m(\Omega)$, $C_0^\infty(\Omega)$ denotes all spaces of $C^\infty(\Omega)$ that have compact support sets in Ω, and $H_0^1(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $W^{1,2}(\Omega)$. $C^{0,\mu}(\overline{\Omega})$ denotes the space of all μ-Holder continuous functions in $\overline{\Omega}$. $W_0^{1,p}(\Omega)$ denotes the normal Sobolev space, and $W_0^{1,2}(\Omega) = H_0^1(\Omega)$. The norm in $L_p(\Omega)$ is defined as follows:

$$\|u\|_p = \left(\int_\Omega |u|^p\right)^{1/p}.$$

For any $u \in H_0^1(\Omega)$, the norm in $H_0^1(\Omega)$ is simply written as $\|u\|$. And we denote $\mathbb{R}^N$ as the Euclidean space of dimension N and set $\mathbb{R} = \mathbb{R}^1$ simply. For $x_0 \in \mathbb{R}^N$ and $r > 0$, let $B(x_0, r) = \{x||x - x_0| < r\}$ denote the open ball in $\mathbb{R}^N$, which is centered at x_0 with radius r. Let E_1 and E_2 denote two Banach spaces. An operator A: $E_1 \rightarrow E_2$ is "bounded" if it maps bounded subsets of E_1 onto bounded sets in E_2. It is "compact" if it maps bounded subsets of E_1 onto relatively compact sets in E_2. In what follows, "$\rightarrow$" ("$\rightharpoonup$") denotes strong (weak) convergence; $O(x)$ is used to denote an infinitely small or large quantity that has the same order as x under some limiting process; o(x) refers to an infinitely small or large quantity that is of higher order than x under some limiting process; and o(1) represents an infinitely small quantity under some limiting process.

Recently, discussions on nonlinear elliptic equations have attracted more attention from mathematicians, such as [1–4] , etc.

In [5], the authors investigated the following semilinear elliptic problem:

$$\begin{cases} -\Delta u - \mu \dfrac{u}{|x|^2} = \dfrac{|u|^{2^*(s)-2}}{|x|^s} u + g(x,u), & x \in \Omega \setminus \{0\}, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{1}$$

where Ω is an open-bounded domain in $\mathbb{R}^N (N \geq 3)$, with a smooth boundary $\partial\Omega$ and $0 \in \Omega$, $0 \leq \mu < \bar{\mu} := \left(\frac{N-2}{2}\right)^2$, $0 \leq s < 2$, $2^*(s) = 2(N-s)/(N-2)$ is the Hardy–Sobolev critical exponent. This problem comes from the consideration of standing waves in the anisotropic Schrödinger equation; also, it is very important in the fields of hydrodynamics, glaciology, quantum field theory, and statistical mechanics (see [6–9]).

Assume that $g \in C(\Omega \times \mathbb{R}, \mathbb{R})$, $G(x,t) = \int_0^t g(x,s)ds$, such that

(g_1) There exist constants $c_1, c_2 > 0$, and $p \in (2, 2^*)$ (here, $2^* = 2^*(0) = 2N/(N-2)$ is the Sobolev critical exponent), such that

$$|g(x,t)| \leq c_1|t| + c_2|t|^{p-1}, \text{ for any } (x,t) \in \Omega \times \mathbb{R}.$$

(g_2) There exists a constant $K > 0$, large enough such that we have the following:

$$G(x,t) \geq Kt^2, \text{ for any } (x,t) \in \Omega \times \mathbb{R}.$$

(g_3) There are constants $\rho > 2$ and $\nu > 0$, such that

$$\rho G(x,t) \leq g(x,t)t + \nu t^2, \text{ for any } (x,t) \in \Omega \times \mathbb{R}.$$

Theorem 1 ([5], Theorem 1.1)**.** *Suppose that* $N \geq 3$, $0 \leq \mu < \overline{\mu} - 1$, $0 \leq s < 2$, $g(x,t)$ *satisfies* (g_1)*,* (g_2)*,* (g_3)*, then* (1) *has at least one nonnegative solution.*

From the proof of Theorem 1, one can see that it also needs c_1 to be sufficiently small (see page 222, line-8 in [5]). So we may wonder how small c_1 should be. This problem will be solved in this paper.

Many authors investigated problem (1) when $g(x,u)$ was a special function, such as [1,10–18]. Some authors have concentrated on when $g(x,u)$ was a general function that satisfied some definite conditions, such as [5,19,20], and so on. For the study on problem (1), the classical method is a variational method (see [21–25]); we should point out that the classical mountain pass lemma cannot be applied directly to (1), because (1) contains the Hardy–Sobolev critical exponent $\frac{|u|^{2^*(s)-2}}{|x|^s}u$. As we all know, the essential reason is that the embedding from $H_0^1(\Omega)$ into $L^{2^*}(\Omega)$ is continuous but not compact. In order to overcome the difficulty caused by this non-compact embedding, one can use the principle of concentrated compactness proposed by P.L. Lions [26–29]; also, we could use the mountain pass lemma with $(PS)_c$ conditions proposed by H. Brezis and L. Nirenberg (one can refer to [10,23]), and so on. This principle has also been successfully applied to the study of nonlinear elliptic systems [30]. The discovery of these theoretical methods has greatly promoted the development of nonlinear functional analysis and many excellent results have been obtained; for convenience, we will list some that are useful for our study.

The following two theorems describe the existence of positive solutions for (1) when the function $g(x,t)$ grows linearly at $t = 0$.

Theorem 2 ([1], Theorem 1.A-1.C)**.** *In* (1)*, let* $g(x,u) = \lambda u$ *and* $s = 0$.

(B1) If $\mu \le \overline{\mu} - 1$, then when $\lambda < \lambda_1(\mu)$, problem (1) has at least a positive solution in $H_0^1(\Omega)$, where

$$\lambda_1(\mu) = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega \left(|\nabla u|^2 - \mu \frac{u^2}{|x|^2} \right) dx}{\int_\Omega |u|^2 dx}. \tag{2}$$

(B2) If $\overline{\mu} - 1 < \mu < \overline{\mu}$, then when $\lambda_*(\mu) < \lambda < \lambda_1(\mu)$, problem (1) has at least a positive solution in $H_0^1(\Omega)$, where

$$\beta = \sqrt{\overline{\mu}} + \sqrt{\overline{\mu} - \mu} \text{ and } \lambda_*(\mu) = \min_{\varphi \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla \varphi|^2 / |x|^{2\beta} dx}{\int_\Omega \varphi^2 / |x|^{2\beta} dx}. \tag{3}$$

(B3) If $\overline{\mu} - 1 < \mu < \overline{\mu}$ and $\Omega = B(0, R)$, then (1) has no positive solution for $\lambda \le \lambda_*(\mu)$.

One can easily see that when $g(x,u) = \lambda u$, $g(x,u)$ satisfies (g_1), but when $\lambda < 2K$, (g_2) is invalid.

In [19], D.S. Kang and Y.B. Deng obtained the existence of the solution to the following problem:

$$-\Delta u - \mu \frac{u}{|x|^2} = |u|^{2^*-2}u + a(x)|u|^{r-2}u, \; x \in \mathbb{R}^N, \tag{4}$$

where $a(x)$ is a nonnegative function and locally bounded in $\mathbb{R}^N \setminus \{0\}$, $a(x) = O\left(|x|^{-s}\right)$ in the bounded neighborhood of the origin, $a(x) = O\left(|x|^{-t}\right)$ as $|x| \to \infty$, $0 \le s < t < 2$, $2^*(t) < r < 2^*(s)$. They obtained the following: If $r > \max\left\{ \frac{N-s}{\sqrt{\overline{\mu}} + \sqrt{\overline{\mu} - \mu}}, \frac{N-s-2\sqrt{\overline{\mu}-\mu}}{\sqrt{\overline{\mu}}} \right\}$, then (4) has at least one solution.

On the other hand, when $g(x,u) = \lambda u$ and $s = 0$, as E. Jannelli [1] said, "The space dimension N plays a fundamental role when one seeks the positive solutions of (1)". In [10], the authors studied the following:

$$\begin{aligned} -\Delta u &= \lambda u + u^{\frac{N+2}{N-2}} \text{ in } \Omega, \\ u &> 0 \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{5}$$

and obtained.

Theorem 3. *(a) When $N \ge 4$, problem (5) has a solution for every $\lambda \in (0, \lambda_1)$, where λ_1 denotes the first eigenvalue of $-\Delta$ with the zero Dirichlet boundary condition; moreover, it has no solution if $\lambda \notin (0, \lambda_1)$ and Ω is star-shaped; (b) when $N = 3$ and Ω is a ball, problem (5) has a solution if and only if $\lambda \in (\frac{1}{4}\lambda_1, \lambda_1)$.*

Motivated by Theorems 1–3, the following problems also need to be solved:

(P1) In Theorem 1, what would happen if $\overline{\mu} - 1 \le \mu < \overline{\mu}$?

(P2) Could we further weaken the conditions (g_1), (g_2), and (g_3)?

(P3) For problem (1), if g is a general function, does the space dimension N still play an important role?

Theorem 4. *Suppose that $N \ge 3$, $0 \le \mu < \bar{\mu}$, $0 \le s < 2$, $a(x)$ is nonnegative and continuous in $\overline{\Omega} \setminus \{0\}$ (for short, Ω^0), there exists a neighborhood of the origin $U(0) \subset \Omega$ and $0 \le q < 2$ such that $a(x) = O(|x|^{-q})$ for $x \in U(0) \setminus \{0\}$, and $g(x,t)$ satisfy*

(g_1') $g \in C(\Omega^0 \times \mathbb{R}^+, \mathbb{R}^+)$ and there exist constants $\lambda > 0$ and $2 < p < 2^*(1 - \frac{q}{N})$ such that

$$\lambda t \le g(x,t) \le \lambda t + a(x)t^{p-1}, \text{ for any } (x,t) \in \Omega^0 \times \mathbb{R}^+.$$

(g_2') *Assume that there exist two nonnegative constants, as follows:*

$$\rho > 2 \text{ and } 0 \leq \nu \leq \frac{(\theta-2)\rho\lambda}{2\theta}(\theta = \min\{\rho, 2^*(s)\})$$

such that

$$\rho G(x,t) \leq g(x,t)t + \nu t^2, \text{ for any } (x,t) \in \Omega^0 \times \mathbb{R}^+.$$

We can obtain the following:

(i) If $0 \leq \mu \leq \overline{\mu} - 1$, *then when* $\lambda < \lambda_1(\mu)$, *problem (1) has at least one positive solution in* $H_0^1(\Omega)$.

(ii) If $\overline{\mu} - 1 < \mu < \overline{\mu}$, *then when* $\lambda_*(\mu) < \lambda < \lambda_1(\mu)$, *problem (1) has at least one positive solution in* $H_0^1(\Omega)$.

Theorem 5. *Suppose that* $N \geq 3$, $\overline{\mu} - 1 < \mu < \bar{\mu}$, $0 \leq s < 2$, $\Omega = B(0,R)$, $g(x,t)$ *satisfy*

(g_3') $g \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ *and* $g(x,t) = g(|x|,t)$ *for* $(x,t) \in \overline{\Omega} \times R$ *and* $g(r,t)$ *is decreasing in* r, *where* $r = |x|$;

(g_4') *There exist two positive constants*

$$c_1' \leq \frac{N-2}{2N},\ c_2' \leq \frac{1}{N}$$

such that

$$G(x,t) \leq c_1' g(x,t)t + c_2'\lambda t^2, \text{ for any } x \in \overline{\Omega} \times \mathbb{R}^+,$$

then when $\lambda \leq \lambda_*(\mu)$, *problem (1) has no positive solution in* $H_0^1(\Omega)$.

Remark 1. *Comparing the above two theorems with Theorems 1 and 2, one can easily see that* (g_1) *and* (g_1') *are exactly the same. Here, we do not need the condition* (g_2). *And compared with* (g_3), (g_2') *only constricts the range of parameter* ν. *According to Theorem 1, conclusion (ii) in Theorem 4 is new. We can also see that all the conclusions in Theorem 2 are included in Theorems 4 and 5.*

As applications of Theorems 4 and 5, we provide an example.

Example 1. *Consider the following elliptic problem:*

$$\begin{cases} -\Delta u - \mu \dfrac{u}{|x|^2} = \dfrac{|u|^{2^*(s)-2}}{|x|^s}u + \lambda u + a(|x|)u^\alpha, & x \in \Omega \setminus \{0\}, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{6}$$

where $a(r) \in C^1([0,+\infty),[0,+\infty))$ *and* $a'(r) \leq 0$, $N \geq 3$, $0 \leq s < 2$, $\Omega = B(0,R)$, *by Theorems 4 and 5, we have the following:*

(i) If $0 \leq \mu \leq \overline{\mu} - 1$, $1 < \alpha < \frac{N+2}{N-2}$, *then when* $\lambda < \lambda_1(\mu)$, *problem (6) has at least one positive solution in* $H_0^1(\Omega)$.

(ii) If $\overline{\mu} - 1 < \mu < \overline{\mu}$, $1 < \alpha < \frac{N+2}{N-2}$, *then when* $\lambda_*(\mu) < \lambda < \lambda_1(\mu)$, *problem (6) has at least one positive solution in* $H_0^1(\Omega)$.

(iii) If $\overline{\mu} - 1 < \mu < \overline{\mu}$, $\alpha \geq \frac{N+2}{N-2}$ *and* $\lambda \leq \lambda_*(\mu)$, *then (6) has no positive solution in* $H_0^1(\Omega)$.

Corollary 1. *For the following elliptic problem:*

$$\begin{cases} -\Delta u - \mu \dfrac{u}{|x|^2} = \dfrac{|u|^{2^*(s)-2}}{|x|^s}u + \lambda u + a(|x|)u^{\frac{N+2}{N-2}}, & x \in \Omega \setminus \{0\}, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{7}$$

where $a(r) \in C^1([0,+\infty),[0,+\infty))$ *and* $a'(r) \le 0$, $N \ge 3$, $0 \le s < 2$, $\Omega = B(0,R)$. *If* $\overline{\mu}-1 < \mu < \overline{\mu}$ *and* $\lambda \le \lambda_*(\mu)$, *then* (7) *has no positive solution in* $H_0^1(\Omega)$. *Especially, for any* $a \ge 1$, $\overline{\mu}-1 < \mu < \overline{\mu}$ *and* $\lambda \le \lambda_*(\mu)$, *the equations*

$$\begin{cases} -\Delta u - \mu\dfrac{u}{|x|^2} = \lambda u + au^{\frac{N+2}{N-2}}, & x \in \Omega \setminus \{0\}, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{8}$$

has no positive solution in $H_0^1(\Omega)$.

Remark 2. *To the best of our knowledge, the conclusions in Corollary 1 are new. Conclusion (b) in Theorem 3 is included in Corollary 1 and conclusion (a) in Theorem 3 is included in Example 1.*

Corollary 2. *For the following elliptic problem, we have the following:*

$$\begin{cases} -\Delta u = \dfrac{|u|^{2^*(s)-2}}{|x|^s}u + \lambda u + a(|x|)u^{\alpha}, & x \in \Omega \setminus \{0\}, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{9}$$

where $a(r) \in C^1([0,+\infty),[0,+\infty))$ *and* $a'(r) \le 0$, $N \ge 4$, $0 \le s < 2$, *if* $1 < \alpha < \frac{N+2}{N-2}$, *then when* $\lambda < \lambda_1(0)$, *problem* (9) *has at least one positive solution in* $H_0^1(\Omega)$. *In particular, for any* $a \ge 0$, $N \ge 4$, $1 < \alpha < \frac{N+2}{N-2}$ *and* $\lambda < \lambda_1(0)$, *the equations*

$$\begin{cases} -\Delta u = \lambda u + u^{\frac{N+2}{N-2}} + au^{\alpha}, & x \in \Omega \setminus \{0\}, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{10}$$

have at least one positive solution in $H_0^1(\Omega)$.

Remark 3. *From Corollary 2, we can easily see that the existence of a positive solution for Equation* (10) *is independent of the subcritical terms* au^{α} $(a > 0, 1 < \alpha < \frac{N+2}{N-2})$.

We organize the rest of the paper as follows. In Section 2, we provide some preliminaries about the Hardy inequality, the properties of variational functionals corresponding to Equation (1), and the properties of extremal functions. In Section 3, by using the mountain pass lemma with $(PS)_c$ conditions, we provide a detailed proof of Theorem 4. In Section 4, by establishing the Pohozaev-type identity and using the properties of the Bessel function, we provide a detailed proof of Theorem 5.

2. Preliminaries

In this section, we provide some lemmas that will be useful for our study; for more details, one can refer to the references cited therein.

Lemma 1 ([31,32])**.** *Assume that* $1 < p < N$ *and* $u \in W_0^{1,p}(\Omega)$. *Then*

$$\int_\Omega \frac{|u|^p}{|x|^p}dx \le \left(\frac{p}{N-p}\right)^p \int_\Omega |\nabla u|^p dx.$$

By Lemma 1, we can define the equivalent norm and inner product in $H_0^1(\Omega)$ as follows, for $0 \le \mu < \bar{\mu}$:

$$\|u\| := \left[\int_\Omega \left(|\nabla u|^2 - \mu\frac{u^2}{|x|^2}\right)dx\right]^{\frac{1}{2}},\ (u,v) := \int_\Omega \left(\nabla u \nabla v - \mu\frac{uv}{|x|^2}\right)dx,\ \forall u,v \in H_0^1(\Omega).$$

Notice that the values of $g(x,t)$ are irrelevant for $t<0$ in Theorem 4, so we define the following:

$$g(x,t)=0,\ \text{for } (x,t)\in\Omega^0\times(-\infty,0).$$

To study the existence of a positive solution for (1), we first consider the existence of nontrivial solutions to the following problem:

$$\begin{cases} -\Delta u-\mu\dfrac{u}{|x|^2}=\dfrac{(u^+)^{2^*(s)-1}}{|x|^s}+g(x,u), & x\in\Omega^0,\\ u=0, & x\in\partial\Omega, \end{cases} \tag{11}$$

where

$$u^+=\max\{u,0\}.$$

Obviously, the existence of a positive solution for (1) is equivalent to the existence of a positive solution for (11).

The energy functional $J:H_0^1(\Omega)\to\mathbb{R}$ corresponding to (11) is given by the following:

$$J(u)=\frac{1}{2}\|u\|^2-\frac{1}{2^*(s)}\int_\Omega\frac{(u^+)^{2^*(s)}}{|x|^s}dx-\int_\Omega G(x,u)dx,\ u\in H_0^1(\Omega), \tag{12}$$

$J(u)$ is well-defined with $J\in C^1\left(H_0^1(\Omega),\mathbb{R}\right)$, and for any $v\in H_0^1(\Omega)$,

$$\left\langle J'(u),v\right\rangle=(u,v)-\int_\Omega\frac{(u^+)^{2^*(s)-1}}{|x|^s}vdx-\int_\Omega g(x,u)vdx.$$

For $0\le\mu<\bar{\mu}$, we define the best constant (see [16,32])

$$A_{\mu,s}(\Omega)\triangleq\inf_{u\in H_0^1(\Omega)\setminus\{0\}}\frac{\|u\|^2}{\left(\int_\Omega\frac{|u|^{2^*(s)}}{|x|^s}dx\right)^{\frac{2}{2^*(s)}}}. \tag{13}$$

The following two lemmas could be found in [16].

Lemma 2 ([16])**.** *Suppose $0\le s<2$ and $0\le\mu<\bar{\mu}$. Then we have the following:*
(i) $A_{\mu,s}(\Omega)$ is independent of Ω.
(ii) $A_{\mu,s}(\Omega)$ is attained when $\Omega=\mathbb{R}^N$ by the functions:

$$y_\varepsilon(x)=\frac{(2\varepsilon(\bar{\mu}-\mu)(N-s)/\sqrt{\bar{\mu}})^{\sqrt{\bar{\mu}}/(2-s)}}{|x|^{\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu}}\left(\varepsilon+|x|^{(2-s)\sqrt{\bar{\mu}-\mu}/\sqrt{\bar{\mu}}}\right)^{(N-2)/(2-s)}},$$

for all $\varepsilon>0$. Moreover, the extremal functions $y_\varepsilon(x)$ solve the following equation:

$$-\Delta u-\mu\frac{u}{|x|^2}=\frac{|u|^{2^*(s)-2}}{|x|^s}u,\ x\in\mathbb{R}^N\setminus\{0\}$$

and satisfy

$$\int_{\mathbb{R}^N}\left(|\nabla y_\varepsilon|^2-\mu\frac{|y_\varepsilon|^2}{|x|^2}\right)dx=\int_{\mathbb{R}^N}\frac{|y_\varepsilon|^{2^*(s)}}{|x|^s}dx=A_{\mu,s}^{\frac{N-s}{2-s}}. \tag{14}$$

Let

$$C_\varepsilon=\left(\frac{2\varepsilon(\bar{\mu}-\mu)(N-s)}{\sqrt{\bar{\mu}}}\right)^{\sqrt{\bar{\mu}}/(2-s)},\ U_\varepsilon(x)=\frac{y_\varepsilon(x)}{C_\varepsilon}, \tag{15}$$

and define a cut-off function $\varphi(x) \in C_0^\infty(\Omega)$, such that we have the following:

$$\varphi(x) = \begin{cases} 1, & |x| \leq R, \\ 0, & |x| \geq 2R, \end{cases}$$

where $B(0, 2R) \subset \Omega, 0 \leq \varphi(x) \leq 1$, for $R < |x| < 2R$ ($R \leq R_0$, R_0 will be defined later), we set the following:

$$u_\varepsilon(x) = \varphi(x) U_\varepsilon(x), \;\; v_\varepsilon(x) = u_\varepsilon(x) / \left(\int_\Omega |u_\varepsilon(x)|^{2^*(s)} |x|^{-s} dx \right)^{\frac{1}{2^*(s)}}, \tag{16}$$

then $\int_\Omega \frac{|v_\varepsilon|^{2^*(s)}}{|x|^s} dx = 1$.

Lemma 3 ([16])**.** *Let $v_\varepsilon(x)$ be defined as in (16), then $v_\varepsilon(x)$ satisfies the following estimates:*

$$\|v_\varepsilon\|^2 = A_{\mu,s} + O\left(\varepsilon^{\frac{N-2}{2-s}}\right), \tag{17}$$

$$\int_\Omega |v_\varepsilon|^q dx = \begin{cases} O\left(\varepsilon^{\frac{\sqrt{\bar{\mu}} q}{2-s}}\right), & 1 \leq q < \frac{N}{\sqrt{\bar{\mu}} + \sqrt{\bar{\mu} - \mu}}, \\ O\left(\varepsilon^{\frac{\sqrt{\bar{\mu}} q}{2-s}} |ln\varepsilon|\right), & q = \frac{N}{\sqrt{\bar{\mu}} + \sqrt{\bar{\mu} - \mu}}, \\ O\left(\varepsilon^{\frac{\sqrt{\bar{\mu}}(N - q\sqrt{\bar{\mu}})}{(2-s)\sqrt{\bar{\mu} - \mu}}}\right), & \frac{N}{\sqrt{\bar{\mu}} + \sqrt{\bar{\mu} - \mu}} < q < 2^*. \end{cases} \tag{18}$$

Lemma 4 ([1])**.** *Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain, $N \geq 3$, $0 \in \Omega$. Then, $\lambda_*(\mu)$ is attained for a positive $\overline{\varphi} \in H_0^1(\Omega)$, and $0 < \lambda_*(\mu) < \lambda_1(\mu)$.*

Remark 4. *Lemma 4 shows that the interval $(\lambda_*(\mu), \lambda_1(\mu))$ is not empty.*

Lemma 5. *Let $\|u_\varepsilon\|$, $A_{\mu,s}$, C_ε be defined as above, then we have the following:*

$$\|u_\varepsilon\|^2 = C_\varepsilon^{-2} A_{\mu,s}^{\frac{N-s}{2-s}} + D, \;\; \int_\Omega \frac{|u_\varepsilon|^{2^*(s)}}{|x|^s} dx = C_\varepsilon^{-2^*(s)} A_{\mu,s}^{\frac{N-s}{2-s}} + E,$$

where

$$D = \int_{R \leq |x| \leq 2R} \left(|\nabla u_\varepsilon|^2 - \mu \frac{u_\varepsilon^2}{|x|^2} \right) dx - \int_{|x| \geq R} \left(|\nabla U_\varepsilon|^2 - \mu \frac{U_\varepsilon^2}{|x|^2} \right) dx,$$

$$E = -\int_{|x| \geq R} \frac{|U_\varepsilon|^{2^*(s)}}{|x|^2} dx + \int_{R \leq |x| \leq 2R} \frac{|u_\varepsilon|^{2^*(s)}}{|x|^s} dx.$$

Moreover, let $\xi = \varepsilon^{\frac{\sqrt{\bar{\mu}}}{2-s}}$, then

$$\lim_{\xi \to 0^+} \xi \frac{\partial D}{\partial \xi} = 0, \;\; \lim_{\xi \to 0^+} \xi^{\frac{N-s}{N-2}} \frac{\partial E}{\partial \xi} = 0. \tag{19}$$

And there exists $R_0 > 0$, when $R \leq R_0$,

$$\overline{\lim_{\varepsilon \to 0^+}} D < \int_\Omega \frac{|\nabla \varphi(x)|^2}{|x|^{2\beta}} dx. \tag{20}$$

Proof. By (14) and (15), we know the following:

$$\begin{aligned}
\|u_\varepsilon\|^2 &= \int_\Omega \left(|\nabla u_\varepsilon|^2 - \mu \frac{u_\varepsilon^2}{|x|^2} \right) dx \\
&= \int_{|x|\le R} \left(|\nabla U_\varepsilon|^2 - \mu \frac{U_\varepsilon^2}{|x|^2} \right) dx + \int_{R\le|x|\le 2R} \left(|\nabla u_\varepsilon|^2 - \mu \frac{u_\varepsilon^2}{|x|^2} \right) dx \\
&= \int_{\mathbb{R}^N} \left(|\nabla U_\varepsilon|^2 - \mu \frac{U_\varepsilon^2}{|x|^2} \right) dx + D \\
&= C_\varepsilon^{-2} A_{\mu,s}^{\frac{N-s}{2-s}} + D.
\end{aligned}$$

It follows from $\varphi \in C_0^\infty(\Omega)$ and $0 \le \varphi(x) \le 1$ that there exists $M > 0$, for any $x \in \Omega$,

$$|\varphi(x) \cdot \nabla\varphi(x)| \le M. \tag{21}$$

By $\lim\limits_{\varepsilon\to 0^+} U_\varepsilon = \frac{1}{|x|^\beta}$, $\int_{|x|\ge 2R} \frac{|\nabla\varphi(x)|^2}{|x|^{2\beta}} dx \ge 0$, $\beta^2 - \mu > 0$ and (21), we have the following:

$$\begin{aligned}
\lim_{\varepsilon\to 0^+} D &= \int_{R\le|x|\le 2R} \left(\left| \nabla\left(\frac{\varphi(x)}{|x|^\beta} \right) \right|^2 - \mu \frac{|\varphi(x)|^2}{|x|^{2\beta+2}} \right) dx \\
&\quad - \int_{|x|\ge R} \left(\left| \nabla\left(\frac{1}{|x|^\beta} \right) \right|^2 - \mu \frac{1}{|x|^{2\beta+2}} \right) dx \\
&= \int_{R\le|x|\le 2R} \left| \frac{\nabla\varphi(x)}{|x|^\beta} + \varphi(x)\nabla\left(\frac{1}{|x|^\beta} \right) \right|^2 dx \\
&\quad - \mu \int_{R\le|x|\le 2R} \frac{|\varphi(x)|^2}{|x|^{2\beta+2}} dx - \left(\beta^2 - \mu\right) \int_{|x|\ge R} \frac{1}{|x|^{2\beta+2}} dx \\
&\le \int_\Omega \frac{|\nabla\varphi(x)|^2}{|x|^{2\beta}} dx - \left(\beta^2 - \mu\right) \int_{|x|\ge 2R} \frac{1}{|x|^{2\beta+2}} dx + 2\beta M \int_{R\le|x|\le 2R} \frac{1}{|x|^{2\beta+1}} dx.
\end{aligned}$$

Let $r = |x|$ and make an N-dimensional spherical coordinate transformation, then

$$\begin{aligned}
&-\left(\beta^2 - \mu\right) \int_{|x|\ge 2R} \frac{1}{|x|^{2\beta+2}} dx + 2\beta M \int_{R\le|x|\le 2R} \frac{1}{|x|^{2\beta+1}} dx \\
= \quad &-\left(\beta^2 - \mu\right) S_{N-1} \int_{2R}^{+\infty} \frac{1}{r^{2\beta+3-N}} dr + 2\beta M S_{N-1} \int_R^{2R} \frac{1}{r^{2\beta+2-N}} dr,
\end{aligned}$$

where S_{N-1} denotes the N-dimensional unit spherical surface area. Since $2\beta + 3 - N = 1 + 2\sqrt{\bar{\mu} - \mu} > 1$, we have

$$\int_{2R}^{+\infty} \frac{1}{r^{2\beta+3-N}} dr = \frac{(2R)^{N-2-2\beta}}{2\beta + 2 - N}.$$

If $2\beta + 2 - N = 1$, then $\int_R^{2R} \frac{1}{r^{2\beta+2-N}} dr = \ln 2$ and $\lim\limits_{R\to 0^+} \frac{(2R)^{N-2-2\beta}}{2\beta+2-N} = +\infty$. Thus, there exists $R_1 > 0$, such that $-\left(\beta^2 - \mu\right) \int_{|x|\ge 2R} \frac{1}{|x|^{2\beta+2}} dx + 2\beta M \int_{R\le|x|\le 2R} \frac{1}{|x|^{2\beta+1}} dx < 0$, for $R \le R_1$.

If $2\beta + 2 - N < 1$, then $\lim\limits_{R\to 0^+} \int_R^{2R} \frac{1}{r^{2\beta+2-N}} dr = \lim\limits_{R\to 0^+} \frac{R^{N-1-2\beta} - (2R)^{N-1-2\beta}}{2\beta+1-N} = 0$, thus there exists a positive constant $R_2 > 0$, such that $-\left(\beta^2 - \mu\right) \int_{|x|\ge 2R} \frac{1}{|x|^{2\beta+2}} dx + 2\beta M \int_{R\le|x|\le 2R} \frac{1}{|x|^{2\beta+1}} dx < 0$, for $R \le R_2$.

If $2\beta+2-N>1$, then $\lim\limits_{R\to 0^+}\int_R^{2R}\frac{1}{r^{2\beta+2-N}}dr=\lim\limits_{R\to 0^+}\frac{R^{N-1-2\beta}-(2R)^{N-1-2\beta}}{2\beta+1-N}=+\infty$, and $N-2-2\beta<N-1-2\beta<0$ implies that there exists some positive constant $R_3>0$, such that $-(\beta^2-\mu)\int_{2R}^{+\infty}\frac{1}{r^{2\beta+3-N}}dr+2\beta M\int_R^{2R}\frac{1}{r^{2\beta+2-N}}dr<0$, for $R\le R_3$.

As mentioned above, when $R\le R_0=\min\{R_1,R_2,R_3\}$,

$$\lim_{\varepsilon\to 0+} D<\int_\Omega \frac{|\nabla\varphi(x)|^2}{|x|^{2\beta}}dx.$$

Furthermore, by (14) and (15),

$$\begin{aligned}\int_\Omega \frac{|u_\varepsilon|^{2^*(s)}}{|x|^s}dx &= \int_{|x|\le R}\frac{|U_\varepsilon|^{2^*(s)}}{|x|^s}dx+\int_{R\le|x|\le 2R}\frac{|u_\varepsilon|^{2^*(s)}}{|x|^s}dx\\ &=\int_{\mathbb{R}^N}\frac{|U_\varepsilon|^{2^*(s)}}{|x|^s}dx+E\\ &=C_\varepsilon^{-2^*(s)}\int_{\mathbb{R}^N}\frac{|y_\varepsilon|^{2^*(s)}}{|x|^s}dx+E\\ &=C_\varepsilon^{-2^*(s)}A_{\mu,s}^{\frac{N-s}{2-s}}+E.\end{aligned}$$

The proof of (19) is similar to the above, and we omit it. □

Definition 1 ([33])**.** *Let E be a Banach space. Given $c\in\mathbb{R}$, we will say that $I\in C^1(E,\mathbb{R})$ satisfies the $(PS)_c$ condition if any sequence $\{u_n\}\subset E$, such that $I(u_n)\to c$ and $I'(u_n)\to 0$ as $n\to\infty$ possesses a convergent subsequence.*

Lemma 6. *Assume that $a(x)$ is nonnegative and continuous in Ω^0, there exists a neighborhood of the origin $U(0)\subset\Omega$ and $0\le q<2$, such that $a(x)=O(|x|^{-q})$ for $x\in U(0)\setminus\{0\}$, then for any $2<p<2^*(1-\frac{q}{N})$ and $\frac{pN}{N-q}<\gamma<2^*$, $a(x)\in L_{\frac{\gamma}{\gamma-p}}(\Omega)$.*

Proof. Without loss of generality, we only need to prove $a(x)\in L_{\frac{\gamma}{\gamma-p}}(U(0))$, notice that $a(x)=O(|x|^{-q})$ for $x\in U(0)\setminus\{0\}$; thus, there exists a positive constant C_1 such that

$$a(x)\le C_1|x|^{-q} \text{ for } x\in U(0)\setminus\{0\},$$

therefore

$$\int_{U(0)} a(x)^{\frac{\gamma}{\gamma-p}}dx\le C_2\int_0^\delta \frac{dr}{r^{\frac{\gamma q}{\gamma-p}-N+1}},$$

where,

$$C_2=C_1^{\frac{\gamma}{\gamma-p}}\omega_N,$$

notice that $\gamma>\frac{pN}{N-q}$ implies that

$$\frac{\gamma q}{\gamma-p}<N,$$

thus,

$$\int_{U(0)} a(x)^{\frac{\gamma}{\gamma-p}}dx<+\infty.$$

This completes the proof. □

3. The Existence of Positive Solutions for (1)

In this section, we give the proof of Theorem 4. First, we have the following:

Lemma 7. *Suppose* (g_1'), (g_2') *and* $\lambda < \lambda_1(\mu)$ *hold, then for any*

$$c \in \left(0, \frac{2-s}{2(N-s)} A_{\mu,s}^{\frac{N-s}{2-s}}\right),$$

J defined as in (12) satisfies the $(PS)_c$ *condition.*

Proof. Let $\{u_n\} \subset H_0^1(\Omega)$ be any sequence, such that $J(u_n) \to c$ and $J'(u_n) \to 0$ as $n \to \infty$, first we prove that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Arguing by contradiction, without loss of generality, suppose that $\|u_n\| \to \infty$. From $J(u_n) \to c$, $J'(u_n) \to 0$ and (g_2'), also notice that for any $(x,t) \in \Omega^0 \times \mathbb{R}$, $\rho G(x,t) \leq g(x,t)t + \nu t^2$, we have

$$\begin{aligned} c + 1 + o(1)\|u_n\| &\geq J(u_n) - \frac{1}{\theta}\langle J'(u_n), u_n\rangle \\ &\geq \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|^2 + \int_\Omega \left(\frac{1}{\theta} g(x,u_n)u_n - G(x,u_n)\right)dx \\ &\quad + \left(\frac{1}{\theta} - \frac{1}{2^*(s)}\right)\int_\Omega \frac{(u_n^+)^{2^*(s)}}{|x|^s}dx \\ &\geq \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|^2 - \frac{\nu}{\rho}\int_\Omega u_n^2 dx \\ &\geq \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|^2 - \left(\frac{1}{2} - \frac{1}{\theta}\right)\frac{\lambda}{\lambda_1(\mu)}\|u_n\|^2 \\ &= \left(\frac{1}{2} - \frac{1}{\theta}\right)\left(1 - \frac{\lambda}{\lambda_1(\mu)}\right)\|u_n\|^2. \end{aligned}$$

Which is a contradiction. Hence, $\{u_n\}$ is a bounded sequence in $H_0^1(\Omega)$ and there exists a u, such that $u_n \rightharpoonup u (n \to \infty)$, up to a subsequence. Hereafter, without loss of generality, we say $u_n \rightharpoonup u (n \to \infty)$ or $u_n \to u (n \to \infty)$, it means maybe one of the subsequences $\{u_{n_k}\}$ of $\{u_n\}$ satisfies $u_{n_k} \rightharpoonup u (k \to \infty)$ or $u_{n_k} \to u (k \to \infty)$. Furthermore, by the weak continuity of J', $J'(u) = 0$. From $u_n \in H_0^1(\Omega)$, $u_n \rightharpoonup u (n \to \infty)$, by the embedding theorem, we have $u_n \to u (n \to \infty)$ in $L^\gamma(\Omega)$, for any $1 < \gamma < 2^*$. Let $g_1(x,t) = g(x,t)t$, from (g_1'), we have $|g_1(x,t)| \leq \lambda t^2 + a(x)|t|^p$, by Hǒlder inequality, when $\gamma > p$, we have the following:

$$\int_\Omega a(x)|\varphi(x)|^p dx \leq \left(\int_\Omega a^{\frac{\gamma}{\gamma-p}}(x)dx\right)^{(\gamma-p)/\gamma}\left(\int_\Omega |\varphi(x)|^\gamma dx\right)^{p/\gamma}, \tag{22}$$

if we choose $\frac{pN}{N-q} < \gamma < 2^*$, then by Lemma 6, we have $a(x) \in L_{\frac{\gamma}{\gamma-p}}(\Omega)$, which implies that $g_1 : L^\gamma(\Omega) \to L^1(\Omega)$ is a continuous and bounded operator. Therefore, we have

$$\lim_{n\to\infty}\int_\Omega (g_1(x,u_n) - g_1(x,u))dx = 0,$$

that is,

$$\lim_{n\to\infty}\int_\Omega g(x,u_n)u_n dx = \int_\Omega g(x,u)u dx. \tag{23}$$

Similarly, we have

$$\lim_{n\to\infty}\int_\Omega G(x,u_n)dx = \int_\Omega G(x,u)dx. \tag{24}$$

Thus,

$$\frac{1}{2}\|u_n\|^2 - \frac{1}{2^*(s)}\int_\Omega \frac{(u_n^+)^{2^*(s)}}{|x|^s}dx - \int_\Omega G(x,u)dx = c + o(1) \tag{25}$$

and

$$\|u_n\|^2 - \int_\Omega \frac{(u_n^+)^{2^*(s)}}{|x|^s}dx - \int_\Omega g(x,u)u dx = o(1), \tag{26}$$

which imply that

$$\int_\Omega \frac{(u_n^+)^{2^*(s)}}{|x|^s}dx = \frac{2(N-s)}{2-s}\left[\int_\Omega G(x,u)dx - \frac{1}{2}\int_\Omega g(x,u)udx + c\right] + o(1), \tag{27}$$

therefore,

$$\|u_n\|^2 = \frac{2(N-s)}{2-s}\left[\int_\Omega G(x,u)dx - \frac{1}{2}\int_\Omega g(x,u)udx + c\right] + \int_\Omega g(x,u)udx + o(1). \tag{28}$$

We claim that

$$\|u\|^2 = \frac{2(N-s)}{2-s}\left[\int_\Omega G(x,u)dx - \frac{1}{2}\int_\Omega g(x,u)udx + c\right] + \int_\Omega g(x,u)udx, \tag{29}$$

in fact, notice that $J(u) = c$ and $J'(u) = 0$, repeat the above derivation, (29) obviously holds true. By (28),

$$\|u_n\| \to \|u\|,\ n \to \infty. \tag{30}$$

By (30) and $u_n \rightharpoonup u(n \to \infty)$, we have

$$u_n \to u \text{ in } H_0^1(\Omega),\ n \to \infty.$$

□

Lemma 8. *Suppose* (g_1'), (g_2'), $\lambda < \lambda_1(\mu)$, *and*

$$c \in \left(0, \frac{2-s}{2(N-s)}A_{\mu,s}^{\frac{N-s}{2-s}}\right),$$

then u defined in the proof of Lemma 7 is a positive solution of (11).

Proof. From the definition of u, also notice the relationship between the functional J and problem (11), we can easily know that u is a solution of (11).

If $u \equiv 0$ in Ω, from $\langle J'(u_n), u_n\rangle = o(1)$ and (23), we have the following:

$$\|u_n\|^2 - \int_\Omega \frac{(u_n^+)^{2^*(s)}}{|x|^s}dx = o(1). \tag{31}$$

By the definition of $A_{\mu,s}$, we have the following:

$$\|u_n\|^2 \geq A_{\mu,s}\left(\int_\Omega \frac{(u_n^+)^{2^*(s)}}{|x|^s}dx\right)^{\frac{2}{2^*(s)}}. \tag{32}$$

From (31) and (32), we have the following:

$$o(1) \geq \|u_n\|^2\left(1 - A_{\mu,s}^{-\frac{2^*(s)}{2}}\|u_n\|^{2^*(s)-2}\right).$$

If $\|u_n\| \to 0$, then $J(u_n) \to 0$, which contradicts $c > 0$. Therefore, we have the following:

$$\|u_n\|^2 \geq A_{\mu,s}^{\frac{N-s}{2-s}} + o(1). \tag{33}$$

By (g_2'), (31), and (33), we have the following:

$$\begin{aligned}J(u_n) &= \frac{1}{2}\|u_n\|^2 - \frac{1}{2^*(s)}\int_\Omega \frac{(u_n^+)^{2^*(s)}}{|x|^s}dx + o(1)\\ &= \frac{2-s}{2(N-s)}\|u_n\|^2 + o(1)\\ &\geq \frac{2-s}{2(N-s)}A_{\mu,s}^{\frac{N-s}{2-s}} + o(1),\end{aligned}$$

which contradicts $c < \frac{2-s}{2(N-s)}A_{\mu,s}^{\frac{N-s}{2-s}}$. Thus, $u \not\equiv 0$.

Notice that for any $v \in H_0^1(\Omega)$, we have the following:

$$\langle J'(u), v\rangle = (u,v) - \int_\Omega \frac{(u^+)^{2^*(s)-1}}{|x|^s}vdx - \int_\Omega g(x,u)vdx = 0.$$

Therefore,

$$(u,u^-) - \int_\Omega \frac{(u^+)^{2^*(s)-1}}{|x|^s}u^-dx - \int_\Omega g(x,u)u^-dx = 0$$

and

$$(u,u^+) - \int_\Omega \frac{(u^+)^{2^*(s)}}{|x|^s}dx - \int_\Omega g(x,u)u^+dx = 0,$$

where $u^- = \max\{-u, 0\}$. According to the definitions of g, u^+, and u^-, we have the following:

$$\int_\Omega g(x,u)u^+dx = \int_\Omega g(x,u)udx, \quad \int_\Omega \frac{(u^+)^{2^*(s)-1}}{|x|^s}u^-dx = 0, \quad \int_\Omega g(x,u)u^-dx = 0,$$

so

$$||u^-||^2 = 0,$$

thus $u \geq 0$. Moreover, by the strong maximum principle, $u > 0$, which completes the proof. □

Lemma 9. *Assume that (g_1'), (g_2'), and $\lambda < \lambda_1(\mu)$ hold, then the functional J admits a $(PS)_c$ sequence at level*

$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t)),$$

where

$$\Gamma = \Big\{\gamma \in C\Big([0,1],\ H_0^1(\Omega)\Big);\ \gamma(0) = 0,\ J(\gamma(1)) < 0\Big\}.$$

Proof. By (g_1') and (22), for $\frac{pN}{N-q} < \gamma < 2^*$, we have

$$\begin{aligned}J(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{2^*(s)}\int_\Omega \frac{(u^+)^{2^*(s)}}{|x|^s}dx - \int_\Omega G(x,u)dx\\ &\geq \frac{1}{2}\|u\|^2 - \frac{A_{u,s}^{-2^*(s)/2}}{2^*(s)}\|u\|^{2^*(s)} - \frac{1}{p}||a(x)||_{\frac{\gamma}{\gamma-p}}\|u\|_\gamma^p - \frac{\lambda}{2}\|u\|_2^2\\ &\geq \frac{1-\lambda/\lambda_1(\mu)}{2}\|u\|^2 - \frac{A_{u,s}^{-2^*(s)/2}}{2^*(s)}\|u\|^{2^*(s)} - \frac{1}{p}||a(x)||_{\frac{\gamma}{\gamma-p}}\|u\|_\gamma^p,\end{aligned}$$

here, $||\cdot||_r$ represents the usual norm of space $L^r(\Omega)$. Notice that $\gamma < 2^*$, then by the Sobolev Embedding Theorem, there exists a positive constant C_3, such that

$$\|u\|_\gamma^p \leq C_3\|u\|^p,$$

thus,

$$J(u) \geq \frac{1-\lambda/\lambda_1(\mu)}{2}\|u\|^2 - \frac{A_{u,s}^{-2^*(s)/2}}{2^*(s)}\|u\|^{2^*(s)} - C_4\|u\|^p, \tag{34}$$

where

$$C_4 = \frac{C_3}{p}||a(x)||_{\frac{\gamma}{\gamma-p}}.$$

Notice that $\lambda < \lambda_1(\mu)$ implies that $\frac{1-\lambda/\lambda_1(\mu)}{2} > 0$, and $p > 2$, thus, there exist some positive constants α, r, such that

$$J(u) \geq \alpha > 0,\ \forall u \in \Big\{u \in H_0^1(\Omega) | \|u\| = r\Big\}.$$

Furthermore, from the nonnegativity of $G(x,u)$, we have the following:

$$\begin{aligned} J(tv_\varepsilon) &= \frac{t^2}{2}\|v_\varepsilon\|^2 - \frac{t^{2^*(s)}}{2^*(s)} - \int_\Omega G(x, tv_\varepsilon)dx \\ &\leq \frac{t^2}{2}\|v_\varepsilon\|^2 - \frac{t^{2^*(s)}}{2^*(s)}, \end{aligned}$$

as $t \to +\infty$, $\lim\limits_{t\to+\infty} J(tv_\varepsilon) = -\infty$, thus there exists $t_0 > 0$, such that $\|t_0 v_\varepsilon\| > r$ and $J(t_0 v_\varepsilon) < 0$. By the mountain pass lemma with $(PS)_c$ conditions, we infer that J admits a $(PS)_c$ sequence at the level c, that is, there exists a sequence $\{u_n\} \subset H_0^1(\Omega)$ such that

$$J(u_n) \to c \geq \alpha \text{ and } J'(u_n) \to 0.$$

This completes the proof. □

Lemma 10. *Suppose* $0 \leq s < 2$, (g_1') *and* (g_2') *hold. If one of the following conditions*

(i) $0 \leq \mu \leq \overline{\mu} - 1, 0 < \lambda < \lambda_1(\mu)$.

(ii) $\overline{\mu} - 1 < \mu < \overline{\mu},\ \lambda_*(\mu) < \lambda < \lambda_1(\mu)$.

is true, then

$$0 < c < \frac{2-s}{2(N-s)}A_{\mu,s}^{\frac{N-s}{2-s}}.$$

Proof. Consider the functions

$$h(t) := J(tv_\varepsilon) = \frac{t^2}{2}\|v_\varepsilon\|^2 - \frac{t^{2^*(s)}}{2^*(s)} - \int_\Omega G(x, tv_\varepsilon)dx,\ t \geq 0$$

and

$$\bar{h}(t) := \frac{t^2}{2}\|v_\varepsilon\|^2 - \frac{t^{2^*(s)}}{2^*(s)},\ t \geq 0,$$

where v_ε is defined as in (16). We have $h(0) = 0$, $\lim\limits_{t\to+\infty} h(t) = -\infty$.

Notice that (g_1') and (g_2') imply (34), therefore

$$J(tv_\varepsilon) \geq \frac{1-\lambda/\lambda_1(\mu)}{2}\|v_\varepsilon\|^2 t^2 - \frac{A_{u,s}^{-2^*(s)/2}}{2^*(s)}\|v_\varepsilon\|^{2^*(s)}t^{2^*(s)} - C_4\|v_\varepsilon\|^p t^p, \tag{35}$$

$\lambda < \lambda_1(\mu)$, $2^*(s) > 2$, $p > 2$ show that $h(t) > 0$ when t is small enough, thus there exists some $t_\varepsilon > 0$, such that $h(t_\varepsilon) = \sup_{t\geq 0} h(t) > 0$, thus $c > 0$ and $h'(t_\varepsilon) = 0$; that is

$$h'(t_\varepsilon) = t_\varepsilon\|v_\varepsilon\|^2 - t_\varepsilon^{2^*(s)-1} - \int_\Omega g(x, t_\varepsilon v_\varepsilon)v_\varepsilon dx = 0, \tag{36}$$

which implies that

$$\|v_\varepsilon\|^2 = t_\varepsilon^{2^*(s)-2} + \frac{1}{t_\varepsilon}\int_\Omega g(x, t_\varepsilon v_\varepsilon)v_\varepsilon dx \geq t_\varepsilon^{2^*(s)-2}. \tag{37}$$

Therefore, if we set $\bar{t}_\varepsilon := \|v_\varepsilon\|^{\frac{2}{2^*(s)-2}}$, then

$$t_\varepsilon \leq \bar{t}_\varepsilon. \tag{38}$$

Obviously, the function $\bar{h}(t)$ reaches its maximum at $\bar{t}_\varepsilon = \|v_\varepsilon\|^{\frac{2}{2^*(s)-2}}$ and is increasing in the interval $\left[0, \bar{t}_\varepsilon\right]$, then from (17), (18), (38) and (g_1'), we have the following:

$$\begin{aligned} h(t_\varepsilon) = \bar{h}(t_\varepsilon) - \int_\Omega G(x, t_\varepsilon v_\varepsilon)dx &\leq \bar{h}(\bar{t}_\varepsilon) - \int_\Omega G(x, t_\varepsilon v_\varepsilon)dx \\ &\leq \frac{2-s}{2(N-s)}\|v_\varepsilon\|^{\frac{2(N-s)}{2-s}} - \int_\Omega G(x, t_\varepsilon v_\varepsilon)dx \\ &\leq \frac{2-s}{2(N-s)}A_{\mu,s}^{\frac{N-s}{2-s}} + O(\varepsilon^{\frac{N-2}{2-s}}) - \frac{\lambda}{2}t_\varepsilon^2||v_\varepsilon||_2^2. \end{aligned}$$

Here, we use the following facts:

$$\lim_{x\to 0}\frac{(1+x)^\vartheta - 1}{\vartheta x} = 1.$$

Under case (i), by (g_1'), we have the following:

$$g(x, t_\varepsilon v_\varepsilon)v_\varepsilon \leq \lambda v_\varepsilon^2 t_\varepsilon + a(x)v_\varepsilon^p t_\varepsilon^{p-1}.$$

Thus, for $\frac{pN}{N-q} < \gamma < 2^*$, by (37), we have the following:

$$\left|\|v_\varepsilon\|^2 - t_\varepsilon^{2^*(s)-2}\right| \leq \lambda\|v_\varepsilon\|_2^2 + \bar{t}_\varepsilon^{p-2}||a||_{\frac{\gamma}{\gamma-p}}\|v_\varepsilon\|_\gamma^p. \tag{39}$$

By Lemma 3, (39) implies the following:

$$\lim_{\varepsilon\to 0^+} t_\varepsilon = A_{\mu,s}^{\frac{1}{2^*(s)-2}}. \tag{40}$$

From Lemma 3, when $\mu < \overline{\mu} - 1$,

$$\|v_\varepsilon\|_2^2 = O\left(\varepsilon^{\frac{N-2}{(2-s)\sqrt{\overline{\mu}-\mu}}}\right), \tag{41}$$

when $\mu = \overline{\mu} - 1$,

$$\|v_\varepsilon\|_2^2 = O\left(\varepsilon^{\frac{N-2}{2-s}}|ln\varepsilon|\right). \tag{42}$$

Thus for $\lambda > 0$,

$$h(t_\varepsilon) \leq \begin{cases} \frac{2-s}{2(N-s)}A_{\mu,s}^{\frac{N-s}{2-s}} + O(\varepsilon^{\frac{N-2}{2-s}}) - O(\varepsilon^{\frac{N-2}{(2-s)\sqrt{\overline{\mu}-\mu}}}) \text{ for } \mu < \overline{\mu} - 1, \\ \frac{2-s}{2(N-s)}A_{\mu,s}^{\frac{N-s}{2-s}} + O(\varepsilon^{\frac{N-2}{2-s}}) - O\left(\varepsilon^{\frac{N-2}{2-s}}|ln\varepsilon|\right) \text{ for } \mu = \overline{\mu} - 1. \end{cases}$$

The above inequalities show that if we choose ε small enough, then we have the following:

$$c \leq h(t_\varepsilon) < \frac{2-s}{2(N-s)}A_{\mu,s}^{\frac{N-s}{2-s}}.$$

Under case (ii), from (39) and (g_1'), we can obtain the following:

$$h(t_\varepsilon) \le \frac{2-s}{2(N-s)}\|v_\varepsilon\|^{\frac{2(N-s)}{2-s}} - \frac{\lambda}{2}t_\varepsilon^2||v_\varepsilon||_2^2. \tag{43}$$

By (40) and (43), we have the following:

$$h(t_\varepsilon) \le \frac{2-s}{2(N-s)}\|v_\varepsilon\|^{\frac{2(N-s)}{2-s}} - \frac{\lambda}{2}A_{\mu,s}^{\frac{2}{2^*(s)-2}}||v_\varepsilon||_2^2 + o\left(\varepsilon^{\frac{N-2}{2-s}}\right).$$

Notice that $\lambda > \lambda_*(\mu)$, then by Lemma 3, we can choose ε small enough, such that we have the following:

$$h(t_\varepsilon) < \frac{2-s}{2(N-s)}\|v_\varepsilon\|^{\frac{2(N-s)}{2-s}} - \frac{\lambda_*(\mu)}{2}A_{\mu,s}^{\frac{2}{2^*(s)-2}}||v_\varepsilon||_2^2.$$

We claim that when ε is small enough, then we have the following:

$$\frac{2-s}{2(N-s)}\|v_\varepsilon\|^{\frac{2(N-s)}{2-s}} - \frac{\lambda_*(\mu)}{2}A_{\mu,s}^{\frac{2}{2^*(s)-2}}||v_\varepsilon||_2^2 \le \frac{2-s}{2(N-s)}A_{\mu,s}^{\frac{N-s}{2-s}}.$$

In fact, by Lemma 5, we have the following:

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+}\frac{\|v_\varepsilon\|^{\frac{2(N-s)}{2-s}} - A_{\mu,s}^{\frac{N-s}{2-s}}}{A_{\mu,s}^{\frac{N-2}{2-s}}\int_\Omega|v_\varepsilon|^2dx}\\
=\;&\lim_{\varepsilon\to 0^+}\frac{\left(D + C_\varepsilon^{-2}A_{\mu,s}^{\frac{N-s}{2-s}}\right)^{\frac{N-s}{2-s}} - A_{\mu,s}^{\frac{N-s}{2-s}}\left(E + C_\varepsilon^{-2^*(s)}A_{\mu,s}^{\frac{N-s}{2-s}}\right)^{\frac{N-2}{2-s}}}{A_{\mu,s}^{\frac{N-2}{2-s}}\left(E + C_\varepsilon^{-2^*(s)}A_{\mu,s}^{\frac{N-s}{2-s}}\right)^{\frac{N-2}{2-s}\frac{2}{2^*(s)}}\int_\Omega|u_\varepsilon|^2dx}\\
=\;&\frac{N-s}{2-s}\lim_{\xi\to 0^+}\frac{Q_1\left(D + \xi\frac{\partial D}{\partial\xi}\right) - A_{\mu,s}^{\frac{N-s}{2-s}}Q_2C^{2^*(s)-2}\left(E\xi^{\frac{2-s}{N-2}} + \frac{N-2}{N-s}\frac{\partial E}{\partial\xi}\xi^{\frac{N-s}{N-2}}\right)}{A_{\mu,s}^{\frac{(N-s)(N-2)}{(2-s)^2}}\int_\Omega\frac{|\varphi(x)|^2}{|x|^{2\beta}}dx}\\
=\;&\frac{N-s}{2-s}\frac{\lim\limits_{\xi\to 0^+}D}{\int_\Omega\frac{|\varphi(x)|^2}{|x|^{2\beta}}dx},
\end{aligned}\tag{44}$$

where $C = \left(\frac{2(\overline{\mu}-\mu)(N-s)}{\sqrt{\overline{\mu}}}\right)^{\frac{\sqrt{\overline{\mu}}}{2-s}}$, $Q_1 = \left(C^2D\xi + A_{\mu,s}^{\frac{N-s}{2-s}}\right)^{\frac{N-2}{2-s}}$, $Q_2 = \left(EC^{2^*(s)}\xi^{\frac{N-s}{N-2}} + A_{\mu,s}^{\frac{N-s}{2-s}}\right)^{\frac{N-4+s}{2-s}}$.

By (20) and (44), we have the following:

$$\lim_{\varepsilon\to 0^+}\frac{\|v_\varepsilon\|^{\frac{2(N-s)}{2-s}} - A_{\mu,s}^{\frac{N-s}{2-s}}}{A_{\mu,s}^{\frac{N-2}{2-s}}\int_\Omega|v_\varepsilon|^2dx} < \frac{N-s}{2-s}\frac{\int_\Omega\frac{|\nabla\varphi(x)|^2}{|x|^{2\beta}}dx}{\int_\Omega\frac{|\varphi(x)|^2}{|x|^{2\beta}}dx},$$

so, when we choose ε small enough, $c \le g(t_\varepsilon) < \frac{2-s}{2(N-s)}A_{\mu,s}^{\frac{N-s}{2-s}}$. □

Proof of Theorem 4. By Lemmas 8–10, the conclusion is obvious. □

4. The Nonexistence of a Positive Solution for (1)

In this section, we consider the nonexistence of a solution for (1). To this end, we assume $\overline{\mu} - 1 < \mu < \overline{\mu}$, $g(x,t) = g(|x|,t)$ for $(x,t) \in \overline{\Omega}\times\mathbb{R}$, $g(r,t)$ is decreasing in r, $\Omega = B(0,R)$, $\lambda \le \lambda_*(\mu)$, and (g_4') hold; that is, all the conditions in Theorem 5 hold true.

The following lemma can be found in [1].

Lemma 11. *Let $J_\tau(z)$ be the Bessel function, as follows:*

$$J_\tau(z) = \sum_{i=0}^{\infty} \frac{(-1)^i (z/2)^{\tau+2i}}{i!\Gamma(i+\tau+1)}.$$

Then,

(a) $z^2 J''_\tau(z) + zJ'_\tau(z) + (z^2 - \tau^2)J_\tau(z) = 0$;
(b) $\forall \tau > -1, \exists\, z_\tau > 0$ *such that* $J_\tau(z) > 0$ *for* $z \in (0, z_\tau)$ *and* $J_\tau(z_\tau) = 0$;
(c) if $-1 < \tau' < \tau''$, *then* $0 < z_{\tau'} < z_{\tau''}$;
(d) $J'_\tau(z) = \frac{\tau}{z} J_\tau(z) - J_{\tau+1}(z)$;
(e) $J_{\tau+1}(z) = \frac{2\tau}{z} J_\tau(z) - J_{\tau-1}(z)$.

Proof of Theorem 5. From [34], we can easily see that under the condition (g'_3), any solution of (1) must be spherically symmetric. The radial equation for (1) is as follows:

$$u'' + \frac{N-1}{r}u' + \mu\frac{u}{r^2} + \frac{|u|^{2*(s)-2}}{r^s}u + g(r,u) = 0. \tag{45}$$

If (1) has a positive solution u, then $u(R) = u'(0) = 0$ and we have the following:

$$u'' + \frac{N-1}{r}u' + \mu\frac{u}{r^2} + \frac{u^{2*(s)-1}}{r^s} + g(r,u) = 0. \tag{46}$$

Let $\psi(r)$ and $w(r)$ be two smooth functions, such that $\psi(0) = 0$, $\psi'(0) > 0$, $\psi''(0) = 0$. Multiply both sides of equation (46) by $r^{N-1}u'(r)\psi(r)$ and $-r^{N-1}u(r)w(r)$, respectively, then sum them and integrate them on $[0,1]$ (without loss of generality, assume $R = 1$), we have the following:

$$\begin{aligned}
& \int_0^1 r^{N-1}(u')^2\left[\frac{\psi'}{2} - \frac{(N-1)\psi}{2r} - w\right]dr \\
& +\frac{1}{2}\int_0^1 r^{N-1}u^2\left[w'' - \frac{(N-1)w'}{r} + \frac{\mu}{r^2}\left(2w + \psi' + \frac{(N-3)\psi}{r}\right)\right]dr \\
& +\frac{\lambda}{2}\int_0^1 r^{N-1}u^2\left[2w + \psi' + \frac{(N-1)\psi}{r}\right]dr \\
& +\frac{1}{2^*(s)}\int_0^1 r^{N-s-1}u^{2^*(s)}\left[2^*(s)w + \psi' + \frac{(N-s-1)\psi}{r}\right]dr \\
& -\int_0^1 r^{N-1}[g(r,u) - \lambda u]u'\psi dr + \int_0^1 r^{N-1}[g(r,u) - \lambda u]uw dr \\
= \; & \frac{1}{2}\psi(1)\left[u'(1)\right]^2.
\end{aligned} \tag{47}$$

Choosing

$$w = \frac{\psi'}{2} - \frac{(N-1)\psi}{2r},$$
$$\psi(r) = \varphi(\sqrt{\lambda}r),$$

where the function $\varphi(r)$ is the solution of the following Cauchy problem:

$$\begin{cases} \varphi''' + [(N-1)(3-N) + 4\mu]\left(\dfrac{\varphi'}{r^2} - \dfrac{\varphi}{r^3}\right) + 4\varphi' = 0; \\ \varphi(0) = 0; \varphi'(0) = 1; \varphi''(0) = 0. \end{cases}$$

From the works in [1], we have the following:

$$\varphi(r) = rJ_{-\zeta}(r)J_\zeta(r),\ \zeta = \sqrt{\bar{\mu} - \mu}, \tag{48}$$

and $\psi(r)$ satisfies the following:

$$\psi''' + [(N-1)(3-N)+4\mu]\left(\frac{\psi'}{r^2} - \frac{\psi}{r^3}\right) + 4\lambda\psi' = 0.$$

Which implies that the Pohozaev-type identity (47) can be simplified as follows:

$$\begin{aligned} & \frac{2N-s-2}{2(N-s)}\int_0^1 r^{N-s-1}u^{2^*(s)}\left[\psi' - \frac{\psi}{r}\right]dr \\ & + \int_0^1 r^{N-1}\left[G(r,u) - \frac{\lambda u^2}{2}\right]\left[\psi' + \frac{(N-1)\psi}{r}\right]dr \\ & + \frac{1}{2}\int_0^1 r^{N-1}[g(r,u) - \lambda u]u\left[\psi' - \frac{(N-1)\psi}{r}\right]dr \\ = \; & \frac{1}{2}\psi(1)\left[u'(1)\right]^2. \end{aligned}$$

By Lemma 11, if $\lambda \le \lambda_*(\mu) = z_{-\zeta}^2$, then

$$\psi(r) = \varphi\left(\sqrt{\lambda}r\right) \ge 0, \text{ for } 0 \le r \le 1 \tag{49}$$

and

$$\psi' - \frac{\psi}{r} < 0 \text{ on } [0,1). \tag{50}$$

Notice that

$$\begin{aligned} & \left[G(r,u) - \frac{\lambda u^2}{2}\right]\left[\psi' + \frac{(N-1)\psi}{r}\right] \\ & + \frac{1}{2}[g(r,u) - \lambda u]u\left[\psi' - \frac{(N-1)\psi}{r}\right] \\ = \; & (H_1 + H_2)\psi' + (H_1 - H_2)(N-1)\frac{\psi}{r} \\ = \; & (H_1 + H_2)\sqrt{\lambda}\varphi'(\sqrt{\lambda}r) + (H_1 - H_2)(N-1)\sqrt{\lambda}J_{-\zeta}(\sqrt{\lambda}r)J_{\zeta}(\sqrt{\lambda}r), \end{aligned}$$

where

$$H_1 = \left[G(r,u) - \frac{\lambda u^2}{2}\right] \text{ and } H_2 = \frac{1}{2}[g(r,u) - \lambda u]u.$$

From (48) and Lemma 11, for any $y \in (0, z_{-\zeta})$, we have the following:

$$\begin{aligned} \varphi'(y) &= J_{-\zeta}(y)J_{\zeta}(y) + yJ'_{-\zeta}(y)J_{\zeta}(y) + yJ_{-\zeta}(y)J'_{\zeta}(y) \\ &= J_{-\zeta}(y)J_{\zeta}(y) - yJ_{1-\zeta}(y)J_{\zeta}(y) - yJ_{-\zeta}(y)J_{\zeta+1}(y) \\ &\le J_{-\zeta}(y)J_{\zeta}(y), \end{aligned}$$

thus

$$\begin{aligned} & (H_1 + H_2)\sqrt{\lambda}\varphi'(\sqrt{\lambda}r) + (H_1 - H_2)(N-1)\sqrt{\lambda}J_{-\zeta}(\sqrt{\lambda}r)J_{\zeta}(\sqrt{\lambda}r) \\ \le \; & [H_1 + H_2 + (N-1)(H_1 - H_2)]\sqrt{\lambda}J_{-\zeta}(\sqrt{\lambda}r)J_{\zeta}(\sqrt{\lambda}r). \end{aligned} \tag{51}$$

From (g_4'), (47) and (51), we have the following:

$$\frac{1}{2}\psi(1)\left[u'(1)\right]^2 < 0.$$

Which contradicts (49). This completes the proof. □

5. Conclusions

In this paper, we investigate a class of semi-linear elliptic equations involving Hardy–Sobolev critical exponents. By a detailed estimation of the extremum function and using the mountain pass lemma with $(PS)_c$ conditions, the existence of positive solutions is obtained. On the other hand, by establishing the Pohozaev-type identity [35] (which can be applied to the study of differential inequality theory, one can refer to [36,37]) and using the properties of the Bessel function, the nonexistence of the positive solution is also obtained. We completely generalize E. Jannelli's linearity research [1] to the nonlinearity case. It also improves Wang's conclusion [5] and generalizes the main results in [10].

If $g(x,t)$ grows superlinearly at $t = 0$ for any $x \in \Omega^0$, from the work of Ding [6], we can easily see that for any $0 \leq \mu < \overline{\mu}$, problem (1) has at least one positive solution. However, if the function $g(x,t)$ grows linearly at $t = 0$ for any $x \in \Omega^0$, from Theorems 4 and 5, we can easily see that $\mu = \overline{\mu} - 1$ is a critical value, this shows that the condition (g_1') is an essential condition. Notice that $\overline{\mu}$ only depends on the space dimension N; thus, the branch value $\overline{\mu} - 1$ only depends on the space dimension N, which shows why the existence of the solution for (1) depends on the spatial dimension N.

Recall that the domain $\Omega \subset \mathbb{R}^N$ is star-shaped with respect to point $x_0 \in \Omega$, it means that if every ray starting from x_0 intersects the boundary $\partial\Omega$ only at one point. Obviously, $B(0,R)$ is star-shaped with respect to the point 0. So we may have the following:

Conjecture 1. *If we replace $\Omega = B(0,R)$ in Theorem 5 with an $\Omega \subset \mathbb{R}^N$ which is star-shaped with respect to point 0, and other conditions remain unchanged, then the conclusion in Theorem 5 still holds true.*

If we compare our conclusion in Theorem 5 the conclusion in Theorem 3, we may also have

Conjecture 2. *If $\overline{\mu} - 1 < \mu < \overline{\mu}$ and $\Omega = B(0,R)$, then*

$$\frac{\lambda_*(\mu)}{\lambda_1(\mu)} = \frac{1}{4}.$$

We leave the above two conjectures for further study.

Author Contributions: Writing—original draft, L.-L.W. and Y.-H.F.; Writing—review & editing, L.-L.W. and Y.-H.F. All authors have read and agreed to the published version of the manuscript.

Funding: This work is supported by the National Natural Science Foundation of China (grant nos. 11201213 and 11371183), the National Natural Science Foundation of Shandong Province, China (grant nos. ZR2015AM026 and ZR2013AM004), and the Project of Shandong Provincial Higher Educational Science and Technology (grant nos. J15LI07).

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Acknowledgments: The authors thank the reviewers for their insightful and detailed comments to improve this manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

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