

Article

Well-Posedness of the Fixed Point Problem of Multifunctions of Metric Spaces

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Abstract: We consider a class of metrics which are equivalent to the Hausdorff metric in some sense to establish the well-posedness of fixed point problems associated with multifunctions of metric spaces, satisfying various generalized contraction conditions. Examples are provided to justify the applicability of new results.

Keywords: sequentially equivalent metrics; well-posedness; multifunction; fixed point problem (FPP)

MSC: 47H10; 47H04; 47H07



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1. Introduction and Preliminaries

Because of its various applications in mathematics and many other related fields, fixed-point theory has evolved as an important and active area of research. For instance, fixed-point theory has been used to solve various geometric problems on Riemann surfaces, see [1,2] and references therein. In metric fixed point theory, the Banach contraction principle (known as BCP) [3] is among the earliest tools that, due to its usefulness, has been developed further in many different directions. For the historical background of the topic, we refer to [4–17].

Samet et al. [15] generalized the BCP for self mappings and Nadler [18] extended it for multivalued mappings. Gabeleh and Markin [19] obtained some very useful consequences of Nadler's result in connection with the best proximity point theory of multivalued mappings. Hasanzade et al. [4] generalized Nadler's theorem by extending Samet's result for multivalued mappings. Kirk and Shahzad [12] proved fixed-point results via metrics sequentially equivalent to the Hausdorff metric. Ali et al. [20] obtained multivalued versions of results in [12] via metrics which are (strongly) semi-sequentially equivalent to the Hausdorff metric.

Suzuki [16] generalized the BCP and characterized the completeness of the underlying metric space. Note that the BCP does not have this property; that is, it does not characterize the completeness of underlying metric space ([8], Example 3). Ali and Abbas [21] introduced generalized multivalued Suzuki-type contractions and proved some fixed-point theorems in the settings of b-metric spaces.

The concept of well-posedness holds great significance as it plays a very important role in the theory of nonlinear analysis. Well-posedness of problems is important in establishing the convergence of algorithms in various problems, for instance, variational inequalities, equilibrium, and inverse problems. Due to its usefulness, this concept has been studied in various contexts, and we refer the reader to [22] and the references therein. Petrusel and Rus [23] considered the well-posedness of the FPP. Moreover, Petrusel et al. [24] (compare [25]) defined the well-posedness of the FPP in a generalized sense.

We fix certain notations and present some introductory concepts that will be utilized to prove the main conclusions presented in this article. The symbols $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{R}_+$, $\mathbb{N}$, $\mathbb{N}_1$, U denote the set of reals, nonnegative reals, positive reals, positive integers, nonnegative integers, and a nonempty set, respectively. The collections of subsets of U are defined as follows.

- $P(U) = \{V : V \subset U\}$.
- $P_1(U) = \{V \in P(U) : V \neq \emptyset\}$.
- $P_c(U) = \{V \in P_1(U) : V \text{ is closed}\}$.
- $P_{cb}(U) = \{V \in P_1(U) : V \text{ is bounded and closed}\}$.

Let (U, p) be a metric space. If

$$\delta(V, W) = \sup_{v \in V} \inf_{w \in W} p(v, w)$$

for all $V, W \in P_1(U)$, then the functions $\mathcal{D}, H, H^+ : P(U) \times P(U) \rightarrow [0, \infty]$ are defined as follows:

$$\mathcal{D}(V, W) = \begin{cases} \inf_{v \in V, w \in W} p(v, w), & \text{if } V \neq \emptyset \text{ and } W \neq \emptyset, \\ 0, & \text{if } V = W = \emptyset, \\ \infty, & \text{otherwise,} \end{cases}$$

$$H(V, W) = \begin{cases} \max\{\delta(V, W), \delta(W, V)\}, & \text{if } V \neq \emptyset, W \neq \emptyset, \\ 0, & \text{if } V = W = \emptyset, \\ \infty, & \text{otherwise,} \end{cases}$$

and

$$H^+(V, W) = \begin{cases} \frac{\delta(W, V) + \delta(V, W)}{2}, & \text{if } V \neq \emptyset, W \neq \emptyset, \\ 0, & \text{if } W = V = \emptyset, \\ \infty, & \text{otherwise.} \end{cases}$$

Note that H and H^+ are metrics on $P(U)$, and $\mathcal{D}$ and H are known as the gap function and Pompeiu Hausdorff generalized function, respectively. Suppose $S : U \rightarrow P(U)$ is the mapping, then $F(S)$ (a set of fixed points of the mapping S) and $F^*(S)$ (a strict fixed point set of S) are provided as follows:

$$F(S) = \{z \in U : z \in Sz\}, \text{ and } F^*(S) = \{y \in U : \{y\} = Sy\}.$$

Definition 1 (compare [12]). Let (U, p) be a metric space. A metric H_1 on $P_{cb}(U)$ is sequentially equivalent to the Hausdorff metric H (Shortly SE-H) if, for $V \in P_{cb}(U)$ and a sequence $\{V_q\} \subset P_{cb}(U)$, we have the following:

$$\lim_{q \rightarrow \infty} H_1(V_q, V) = 0 \text{ iff } \lim_{q \rightarrow \infty} H(V_q, V) = 0.$$

Kirk and Shahzad [12] generalized the Nadler result using metrics SE-H.

Theorem 1 ([12]). Let (U, p) be a complete metric space, and H_1 is SE-H on $P_{cb}(Y)$. Suppose $S : U \rightarrow P_{cb}(U)$ satisfies the following:

1. $0 < r < 1$, such that:

$$H_1(Sx, Sy) \leq rp(x, y)$$

for all $x, y \in U$.

2. For all $x \in U$ and $y \in Sx$,

$$\mathcal{D}(y, Sy) \leq H_1(Sy, Sx).$$

Then, $F(S)$ is nonempty.

Consider the slight modification in Definition 1.

Definition 2 ([20]). A metric H_1 on $P_{cb}(U)$ is semi-sequentially equivalent to H (Shortly SSE-H) for $V \in P_{cb}(U)$ and sequence $\{V_q\} \subset P_{cb}(U)$, we obtain the following:

$$\lim_{q \rightarrow \infty} H_1(V_q, V) = 0 \text{ implies } \lim_{q \rightarrow \infty} H(V_q, V) = 0.$$

Definition 3 ([20]). A metric H_1 on $P_{cb}(U)$ is strongly semi-sequentially equivalent to the Hausdorff metric H (known as SSSE-H) for $V \in P_{cb}(U)$ and a sequence of sets $\{V_q\} \subset P_{cb}(U)$,

$$H(V_q, V) \leq H_1(V_q, V).$$

Remark 1. Let H_1 be a metric on $P_{cb}(U)$. Note the following:

$$\begin{aligned} H_1 \text{ is SE-H} &\implies H_1 \text{ is SSE-H and} \\ H_1 \text{ is SSSE-H} &\implies H_1 \text{ is SSE-H.} \end{aligned}$$

Also, note that

$$\frac{1}{2}H(V, W) \leq H^+(V, W) \leq H(V, W)$$

for all V, W in $P(U)$; hence, H^+ is SE-H and SSSE-H

Example 1. Let $U = \mathbb{R}$ and H be a Hausdorff metric on $P_{cb}(U)$ induced by the metric $d(x, y) = |x - y|$. Further, define metric $\mathcal{H}$ on $P_{cb}(U)$ as follows:

$$\mathcal{H}(V, W) = \begin{cases} 1, & \text{if } V \neq W \\ 0, & \text{if } V = W \end{cases}$$

for V, W in $P_{cb}(U)$. Consider that $V_q = [0, 1 + \frac{1}{q}]$ and $V = [0, 1]$ for $q \in \mathbb{N}$. Also, note the following:

$$\lim_{q \rightarrow \infty} H(V_q, V) = 0 \text{ and } \lim_{q \rightarrow \infty} \mathcal{H}(V_q, V) \neq 0.$$

Hence, H is not SSE- $\mathcal{H}$, and $\mathcal{H}$ is SSE- H .

If (U, p) is a metric space, mapping $S : U \rightarrow P_1(U)$ is continuous at point $u \in U$ for any sequence $\{u_n\}$ in U , where $\lim_{n \rightarrow \infty} p(u_n, u) = 0$ implies $\lim_{n \rightarrow \infty} H(Su_n, Su) = 0$. Let $S : U \rightarrow P_1(U)$ and $\alpha : U \times U \rightarrow [0, \infty)$, define the following:

$$\alpha_*(Sx, Sy) = \inf\{\alpha(v, q) : v \in Sx, q \in Sy\}.$$

Mapping $S : U \rightarrow P_1(U)$ is α_* -admissible if

$$\alpha(x, y) \geq 1 \implies \alpha_*(Sx, Sy) \geq 1.$$

Consider the following class of functions:

$$\Phi = \left\{ \begin{array}{l} \phi : \phi : [0, \infty) \rightarrow [0, \infty), \text{ such that} \\ \sum_{q=1}^{\infty} \phi^q(s) < \infty, \text{ for each } s > 0 \text{ and,} \\ \phi \text{ is strictly increasing.} \end{array} \right\}$$

where ϕ^q is the q^{th} iterate of ϕ . Further, if $\phi \in \Phi$, then ϕ is continuous for all $\alpha \in \mathbb{R}^+$, $\lim_{n \rightarrow \infty} \phi^n(\alpha) = 0$, and $\alpha \in \mathbb{R}_+$, $\phi(\alpha) < \alpha$ (see the details [26]).

Samet et al. [15] extended the BCP by defining $(\alpha - \phi)$ contraction condition. Hasanzade et al. [4] generalized the results of Samet for multivalued mappings of metric spaces.

Definition 4. Let (U, p) be a complete metric space, where $\alpha : U \times U \rightarrow \mathbb{R}^+$ is function $\phi \in \Phi$. Mapping $S : U \rightarrow P_c(U)$ is an $(\alpha_* - \phi)$ contractive multifunction if

$$\alpha_*(Sx, Sy)H(Sx, Sy) \leq \phi(p(x, y))$$

for all $x, y \in U$.

Further, they obtained the following theorem.

Theorem 2 ([4]). Let (U, p) be a complete metric space and mapping $S : U \rightarrow P_c(U)$ a $(\alpha_* - \phi)$ contractive multifunction. Moreover, if

- (i) mapping S is α_* -admissible;
- (ii) there is $z_0 \in U$ and $z_1 \in Sz_0$, such that $\alpha(z_0, z_1) \geq 1$;
- (iii) there is a sequence $\{z_q\}$ in U , such that $\alpha(z_q, z_{q+1}) \geq 1$ for $q \in \mathbb{N}_1$, and $z_q \rightarrow z$ as $q \rightarrow \infty$ implies $\alpha(z_q, z) \geq 1$ for all $q \in \mathbb{N}_1$;

then, $F(S)$ is nonempty.

Ali and Abbas [21] introduced generalized contractive multifunctions as follows.

Definition 5 ([21]). Let (U, p) be a metric space. Mapping $S : U \rightarrow P_c(U)$ is generalized Suzuki-type $(\alpha_* - \phi)$ -contractive multifunction if there is a $\phi \in \Phi$, such that

$$\frac{1}{2}\mathcal{D}(x, Sx) \leq p(x, y) \text{ implies } \alpha_*(Sx, Sy)H(Sx, Sy) \leq \phi(\mathcal{M}_1(x, y))$$

for all $x, y \in U$, where

$$\mathcal{M}_1(x, y) = \max \left\{ p(x, y), \mathcal{D}(x, Sx), \mathcal{D}(y, Sy), \frac{\mathcal{D}(x, Sy) + \mathcal{D}(y, Sx)}{2} \right\}.$$

Theorem 3. Let (U, p) be a complete metric space and $S : U \rightarrow P_c(U)$ be a generalized Suzuki-type $(\alpha_* - \phi)$ -contractive multifunction. Moreover,

- (i) S is α_* -admissible and there exist $z_0 \in U$, $z_1 \in Sz_0$, such that $\alpha(z_0, z_1) \geq 1$;
- (ii) if there is a sequence $\{z_q\}$ in U such that $\alpha(z_q, z_{q+1}) \geq 1$ for all $q \in \mathbb{N}_1$ and $z_q \rightarrow z$ as $q \rightarrow \infty$ implies $\alpha(z_q, z) \geq 1$ for all $q \in \mathbb{N}_1$;

then, $F(S)$ is nonempty.

In [27], the following notion was introduced.

Definition 6 ([27]). Mapping $S : U \rightarrow P_{cb}(U)$ is a H^+ contraction if there is an $0 < r < 1$, such that

$$H^+(Sx, Sy) \leq rp(x, y)$$

for all $x, y \in U$, and for every $x \in U$ and $y \in Sx$ and $\varepsilon > 0$, there is $z \in Sy$ and we have

$$p(y, z) \leq H^+(Sx, Sy) + \varepsilon.$$

We present the following notions from [23] with a slight modification for metrics which are SSE-H.

Definition 7 ([23]). If (U, p) is a metric space, H_1 is an SSE-H, and $Z \in P_1(U)$, the FPP of mapping $S : Z \rightarrow P_c(U)$ is well posed w.r.t.

1. $\mathcal{D}$ if $F(S) = \{y^*\}$ and if $\{y_q\} \subset Z$ and $\lim_{q \rightarrow \infty} \mathcal{D}(y_q, Sy_q) = 0$ then $\lim_{q \rightarrow \infty} y_q = y^*$.

2. H_1 if $F^*(S) = \{y^*\}$ and if $\{y_q\} \subset Z$ and $\lim_{q \rightarrow \infty} H_1(y_q, Sy_q) = 0$ then $\lim_{q \rightarrow \infty} y_q = y^*$.

We present the following notions from [24] with a slight modification for metrics which are SSE-H.

Definition 8 ([24]). If (U, p) is a metric space, H_1 is an SSE-H, and $Z \in P_1(U)$, the FPP of mapping $S : Z \rightarrow P_c(U)$ is well-posed in the generalized sense w.r.t.

1. $\mathcal{D}$ if $F(S) \neq \emptyset$ and if $\{y_q\} \subset Z$ and $\lim_{q \rightarrow \infty} \mathcal{D}(y_q, Sy_q) = 0$, then there exists a subsequence $\{y_{q_i}\}$ of $\{y_q\}$, such that $\lim_{i \rightarrow \infty} y_{q_i} = y^* \in F(S)$.
2. H_1 if $F^*(S) \neq \emptyset$ and if $\{y_q\} \subset Z$ and $\lim_{q \rightarrow \infty} H_1(y_q, Sy_q) = 0$, then there exists a subsequence $\{y_{q_i}\}$ of $\{y_q\}$, such that $\lim_{i \rightarrow \infty} y_{q_i} = y^* \in F(S)$.

Dung and Le Hang proved the well-posedness of the FPP of multivalued mappings ([25], Theorem 2.2).

Consider the following notations for multifunction $S : U \rightarrow P_{cb}(U)$:

$$\begin{aligned}\mathcal{M}(x, y) &= \max \left\{ \frac{p(x, y), \mathcal{D}(x, Sx), \mathcal{D}(y, Sy)}{\mathcal{D}(x, Sy) + \mathcal{D}(y, Sx)}, \frac{\mathcal{D}(x, Sx)\mathcal{D}(y, Sy)}{1 + p(x, y)} \right\}, \\ \mathcal{M}^*(x, y) &= \max \left\{ p(x, y), \mathcal{D}(x, Sx), \frac{\mathcal{D}(x, Sx)\mathcal{D}(y, Sy)}{1 + \mathcal{D}(x, y)} \right\}, \text{ and} \\ \mathcal{N}(x, y) &= \max \left\{ \frac{p(x, y), \mathcal{D}(x, Sx), \frac{\mathcal{D}(y, Sy)}{2}}{\mathcal{D}(x, Sy) + \mathcal{D}(y, Sx)}, \frac{\mathcal{D}(x, Sx)\mathcal{D}(y, Sy)}{1 + p(x, y)} \right\}\end{aligned}$$

for all $x, y \in U$.

2. Existence of Fixed Points of Multifunctions

We start with the following result, in which we establish the existence of solutions for the FPP of multifunctions.

Theorem 4. If (U, p) is a complete metric space, H_1 is an SSE-H on $P_c(U)$, and the mapping $S : U \rightarrow P_c(U)$ satisfies the following:

- (i) $\frac{1}{2}\mathcal{D}(x, Sx) \leq p(x, y)$ implies

$$\alpha_*(Sx, Sy)H_1(Sx, Sy) \leq \phi(\mathcal{M}^*(x, y))$$

for all $x, y \in U$ and some $\phi \in \Phi$.

- (ii) For $x \in U$ and $y \in Sx$,

$$\mathcal{D}(y, Sy) \leq H_1(Sy, Sx).$$

- (iii) Mapping S is α_* -admissible.

- (iv) $z_0 \in U$ and $z_1 \in Sz_0$, such that $\alpha(z_0, z_1) \geq 1$.

- (v) Either of the following are applicable: (a) if there is a sequence $\{z_q\}$ in U , such that $\alpha(z_q, z_{q+1}) \geq 1$ for all $q \in \mathbb{N}$ and $z_q \rightarrow z$ when $q \rightarrow \infty$, this implies $\alpha(z_q, z) \geq 1$ for all $q \in \mathbb{N}$, or (b) mapping S forms a continuous hold.

Then, $F(S)$ is nonempty.

Proof. As in (iv), $z_0 \in U$ and $z_1 \in Sz_0$, such that $\alpha(z_0, z_1) \geq 1$, which implies $\alpha_*(Sz_0, Sz_1) \geq 1$. If $z_1 = z_0$, then $z_0 = z_1 \in Sz_0$; that is, $F(S)$ is nonempty and there is nothing further to prove. So, assume $z_0 \neq z_1$. Then, by (i) and (ii), we obtain the following:

$$\frac{1}{2}\mathcal{D}(z_0, Sz_0) \leq p(z_0, z_1);$$

thus, we have

$$\begin{aligned}\mathcal{D}(z_1, Sz_1) &\leq \alpha_*(Sz_0, Sz_1)\mathcal{D}(z_1, Sz_1) \\ &\leq \alpha_*(Sz_0, Sz_1)H_1(Sz_0, Sz_1) \\ &\leq \phi(\mathcal{M}^*(z_0, z_1)).\end{aligned}$$

If $\mathcal{D}(z_1, Sz_1) = 0$, then $z_1 \in Sz_1$ implies that $F(S)$ is nonempty. So, assume $\mathcal{D}(z_1, Sz_1) > 0$. Further, from the given condition, we obtain the following:

$$\begin{aligned}0 &< \mathcal{D}(z_1, Sz_1) \leq \phi(\mathcal{M}^*(z_0, z_1)) \\ &= \phi\left(\max\left\{p(z_0, z_1), \mathcal{D}(z_0, Sz_0), \frac{\mathcal{D}(z_0, Sz_0)p(z_1, Sz_1)}{1 + p(z_0, z_1)}\right\}\right) \\ &\leq \phi(\max\{p(z_0, z_1), \mathcal{D}(z_1, Sz_1)\}).\end{aligned}$$

That is,

$$0 < \mathcal{D}(z_1, Sz_1) \leq \phi(\max\{p(z_0, z_1), \mathcal{D}(z_1, Sz_1)\}).$$

If

$$\max\{p(z_0, z_1), \mathcal{D}(z_1, Sz_1)\} = \mathcal{D}(z_1, Sz_1),$$

then

$$0 < \mathcal{D}(z_1, Sz_1) \leq \phi(\mathcal{D}(z_1, Sz_1)).$$

As $\mathcal{D}(z_1, Sz_1) > 0$ and $\phi \in \Phi$,

$$0 < \mathcal{D}(z_1, Sz_1) < \mathcal{D}(z_1, Sz_1)$$

is a contradiction. Hence,

$$0 < \mathcal{D}(z_1, Sz_1) \leq \phi(p(z_0, z_1)).$$

Choose $z_2 \in Sz_1$ and $v_1 > 1$, such that

$$0 < \mathcal{D}(z_1, Sz_1) \leq p(z_1, z_2) < v_1 \mathcal{D}(z_1, Sz_1).$$

Thus,

$$0 < p(z_1, z_2) < v_1 \mathcal{D}(z_1, Sz_1) \leq v_1 \phi(p(z_0, z_1)) = v_1 \phi(b_0),$$

where $b_0 = p(z_0, z_1)$. Note that, $z_2 \neq z_1$ and

$$\alpha(z_1, z_2) \geq \alpha_*(Sz_0, Sz_1) \geq 1.$$

Thus, $\alpha(z_1, z_2) \geq 1$, and hence, $\alpha_*(Sz_1, Sz_2) \geq 1$. As $\phi \in \Phi$,

$$\phi(p(z_1, z_2)) < \phi(v_1 \phi(b_0)).$$

Set

$$v_2 = \frac{\phi(v_1 \phi(b_0))}{\phi(p(z_1, z_2))} > 1.$$

Now, if $z_2 \in Sz_2$, then the proof is finished. Let $z_2 \notin Sz_2$. Proceeding as above, we obtain the following:

$$0 < \mathcal{D}(z_2, Sz_2) \leq \phi(p(z_1, z_2))$$

and $z_3 \in Sz_2$, such that

$$0 < p(z_2, z_3) < v_2 \mathcal{D}(z_2, Sz_2) \leq v_2 \phi(p(z_1, z_2)) = \phi(v_1 \phi(b_0)). \quad (1)$$

If $z_3 = z_2$, then $z_2 \in F(S)$. So, consider $z_3 \neq z_2$. As S is α_* -admissible and

$$\alpha(z_2, z_3) \geq \alpha_*(Sz_1, Sz_2) \geq 1,$$

$\alpha_*(Sz_2, Sz_3) \geq 1$. Using (1), we obtain the following:

$$\phi(p(z_2, z_3)) < \phi^2(v_1(\phi(b_0))).$$

Set

$$v_3 = \frac{\phi^2(v_1 \phi(b_0))}{\phi(p(z_2, z_3))} > 1.$$

Now, if $z_3 \in Sz_3$, then the proof is finished. Let $z_3 \notin Sz_3$. Continuing, we obtain sequence $\{z_q\}$ in U , which satisfies $z_{q+1} \in Sz_q$, $z_{q+1} \neq z_q$ and $\alpha(z_{q+1}, z_{q+2}) \geq 1$ such that

$$0 < p(z_{q+1}, z_{q+2}) \leq \phi^{q+1}(v_1 \phi(b_0)).$$

As $\phi \in \Phi$, so for every $\varepsilon > 0$, and $k > q$, there is an $n_1 \in \mathbb{N}$ such that

$$\begin{aligned} p(z_q, z_k) &\leq p(z_q, z_{q+1}) + p(z_{q+1}, z_{q+2}) + \cdots + p(z_{k-1}, z_k) \\ &= \sum_{i=q}^{k-1} p(z_i, z_{i+1}) \leq \sum_{i=q}^{k-1} \phi^i(v_1 \phi(b_0)) \leq \sum_{i=q}^{\infty} \phi^i(v_1 \phi(b_0)) < \varepsilon \end{aligned}$$

for all $k, q \geq n_1$. This proves that $\{z_q\}$ is Cauchy, so z_q converges to some z in U . Assume (a) holds true in (v) above. In order to show $z \in F(S)$, assume $w \notin Sw$ (on contrary) for all $w \in U$; that is, $\mathcal{D}(w, Sw) > 0$ for all $w \in U$. We claim that either

$$\frac{1}{2} \mathcal{D}(z_q, Sz_q) \leq p(z_q, z) \quad (2)$$

or

$$\frac{1}{2} \mathcal{D}(z_{q+1}, Sz_{q+1}) \leq p(z_{q+1}, z) \quad (3)$$

holds for all $q \in \mathbb{N}$. If not, there is a $q_0 \in \mathbb{N}_0$ so that

$$\frac{1}{2} \mathcal{D}(z_{q_0}, Sz_{q_0}) > p(z_{q_0}, z) \quad (4)$$

and

$$\frac{1}{2} \mathcal{D}(z_{q_0+1}, Sz_{q_0+1}) > p(z_{q_0+1}, z). \quad (5)$$

Now, (4) and (5) imply

$$\begin{aligned}
 p(z_{q_0}, z_{q_0+1}) &\leq p(z_{q_0}, z) + p(z, z_{q_0+1}) \\
 &< \frac{1}{2} \mathcal{D}(z_{q_0}, Sz_{q_0}) + \frac{1}{2} \mathcal{D}(z_{q_0+1}, Sz_{q_0+1}) \\
 &\leq \frac{1}{2} p(z_{q_0}, z_{q_0+1}) + \frac{1}{2} \phi(p(z_{q_0}, z_{q_0+1})) \\
 &< \frac{1}{2} p(z_{q_0}, z_{q_0+1}) + \frac{1}{2} p(z_{q_0}, z_{q_0+1}) \\
 &= p(z_{q_0}, z_{q_0+1}),
 \end{aligned}$$

which is a contradiction. Hence, either (2) or (3) holds for $\mathbb{N}_1$, where $\mathbb{N}_1$ is an infinite subset of $\mathbb{N}$. Assume (2) holds for all $n \in \mathbb{N}_1$, then there is a subsequence $\{z_{q_k}\}$ of $\{z_q\}$, such that

$$\frac{1}{2} \mathcal{D}(z_{q_k}, Sz_{q_k}) \leq p(z_{q_k}, z).$$

As $\alpha(z_q, z_{q+1}) \geq 1$ and $z_q \rightarrow z$, this provides $\alpha(z_{q_k}, z_{q_{k+1}}) \geq 1$ and $z_{q_k} \rightarrow z$ when $k \rightarrow \infty$; hence, by given assumption, $\alpha(z_{q_k}, z) \geq 1$ for all $k \in \mathbb{N}$. As mapping S is α_* -admissible so $\alpha_*(Sz_{q_k}, Sz) \geq 1$,

$$\frac{1}{2} p(z_{q_k}, Sz_{q_k}) \leq p(z_{q_k}, z)$$

implies

$$\begin{aligned}
 H_1(Sz_{q_k}, Sz) &\leq \alpha_*(Sz_{q_k}, Sz) H_1(Sz_{q_k}, Sz) \\
 &\leq \phi(\mathcal{M}^*(z_{q_k}, z)) \\
 &= \phi\left(\max\left\{p(z_{q_k}, z), \mathcal{D}(z_{q_k}, Sz_{q_k}), \frac{\mathcal{D}(z_{q_k}, Sz_{q_k}) \mathcal{D}(z, Sz)}{1 + p(z_{q_k}, z)}\right\}\right) \\
 &\leq \phi\left(\max\left\{p(z_{q_k}, z), p(z_{q_k}, z_{q_{k+1}}), \frac{\mathcal{D}(z_{q_k}, z_{q_{k+1}}) \mathcal{D}(z, Sz)}{1 + p(z_{q_k}, z)}\right\}\right).
 \end{aligned}$$

As q tends to ∞ , we have

$$\lim_{k \rightarrow \infty} H_1(Sz_{q_k}, Sz) \leq \lim_{k \rightarrow \infty} \phi\left(\max\left\{\frac{p(z_{q_k}, z), p(z_{q_k}, z_{q_{k+1}})}{p(z_{q_k}, z_{q_{k+1}}) \mathcal{D}(z, Sz)}, \frac{\mathcal{D}(z, Sz)}{1 + p(z_{q_k}, z)}\right\}\right).$$

That is, $\lim_{k \rightarrow \infty} H_1(Sz_{q_k}, Sz) = 0$ as ϕ is continuous at 0. As H_1 is SSE- H ,

$$\lim_{k \rightarrow \infty} H(Sz_{q_k}, Sz) = 0.$$

Hence,

$$\mathcal{D}(z, Sz) = \lim_{k \rightarrow \infty} \mathcal{D}(z_{q_{k+1}}, Sz) \leq \lim_{k \rightarrow \infty} H(Sz_{q_k}, Sz) = 0.$$

That is, $z \in Sz$. Assume (b) holds true in (v). That is, S is continuous. Then, $\lim_{q \rightarrow \infty} p(z_q, z) = 0$ implies $\lim_{k \rightarrow \infty} H(Sz_{q_k}, Sz) = 0$. Hence,

$$\mathcal{D}(z, Sz) = \lim_{q \rightarrow \infty} \mathcal{D}(z_{q+1}, Sz) \leq \lim_{k \rightarrow \infty} H(Sz_{q_k}, Sz) = 0.$$

That is, $z \in Sz$. $\square$

From the above theorem, we obtain important corollaries as follows.

Corollary 1. If (U, p) is a complete metric space, H_1 is an SSE-H on $P_c(U)$, and mapping $S : U \rightarrow P_c(U)$ satisfies

$$\frac{1}{2}\mathcal{D}(x, Sx) \leq p(x, y), \text{ , which implies } \alpha_*(Sx, Sy)H_1(Sx, Sy) \leq \phi(p(x, y))$$

for every $x, y \in U$ and for some $\phi \in \Phi$. Further, for $x \in U$ and $y \in Sx$

$$\mathcal{D}(y, Sy) \leq H_1(Sy, Sx),$$

and mapping S is α_* -admissible; therefore, $z_0 \in U$ and $z_1 \in Sz_0$, such that $\alpha(z_0, z_1) \geq 1$. Moreover, (a) if there is a sequence $\{z_q\}$ in U , such that $\alpha(z_q, z_{q+1}) \geq 1$ for all $q \in \mathbb{N}$ and $z_q \rightarrow z$ when $q \rightarrow \infty$, this implies $\alpha(z_q, z) \geq 1$ for all $q \in \mathbb{N}$ or (b) S is continuous. Then, $F(S)$ is nonempty.

Corollary 2. If (U, p) is a complete metric space, H_1 is an SE-H on $P_c(U)$, and mapping $S : U \rightarrow P_c(U)$ satisfies

$$\frac{1}{2}\mathcal{D}(x, Sx) \leq p(x, y), \text{ , which implies } \alpha_*(Sx, Sy)H_1(Sx, Sy) \leq \phi(p(x, y))$$

for every $x, y \in U$ and for some $\phi \in \Phi$. Further, for $x \in U$ and $y \in Sx$,

$$\mathcal{D}(y, Sy) \leq H_1(Sy, Sx),$$

and mapping S is α_* -admissible; therefore, $z_0 \in U$ and $z_1 \in Sz_0$, such that $\alpha(z_0, z_1) \geq 1$. Moreover, (a) if there is a sequence $\{z_q\}$ in U , such that $\alpha(z_q, z_{q+1}) \geq 1$ for all $q \in \mathbb{N}$ and $z_q \rightarrow z$ when $q \rightarrow \infty$, this implies $\alpha(z_q, z) \geq 1$ for all $q \in \mathbb{N}$ or (b) S is continuous. Then, $F(S)$ is nonempty.

Corollary 3. If (U, p) is a complete metric space, H_1 is an SSSE-H on $P_c(U)$, and mapping $S : U \rightarrow P_c(U)$ satisfies

$$\alpha_*(Sx, Sy)H_1(Sx, Sy) \leq \phi(\mathcal{M}^*(x, y))$$

for every $x, y \in U$ and for some $\phi \in \Phi$. Further, for $x \in U$ and $y \in Sx$,

$$\mathcal{D}(y, Sy) \leq H_1(Sy, Sx),$$

and mapping S is α_* -admissible; therefore, $z_0 \in U$ and $z_1 \in Sz_0$, such that $\alpha(z_0, z_1) \geq 1$. Moreover, (a) if there is a sequence $\{z_q\}$ in U , such that $\alpha(z_q, z_{q+1}) \geq 1$ for all $q \in \mathbb{N}$ and $z_q \rightarrow z$ when $q \rightarrow \infty$, this implies $\alpha(z_q, z) \geq 1$ for all $q \in \mathbb{N}$ or (b) S is continuous. Then, $F(S)$ is nonempty.

Corollary 4. If (U, p) is a complete metric space, H_1 is an SE-H on $P_c(U)$, and mapping $S : U \rightarrow P_c(U)$ satisfies

$$\alpha_*(Sx, Sy)H_1(Sx, Sy) \leq \phi(\mathcal{M}^*(x, y))$$

for every $x, y \in U$ and for some $\phi \in \Phi$. Further, for $x \in U$ and $y \in Sx$,

$$\mathcal{D}(y, Sy) \leq H_1(Sy, Sx),$$

and mapping S is α_* -admissible; therefore, $z_0 \in U$ and $z_1 \in Sz_0$, such that $\alpha(z_0, z_1) \geq 1$. Moreover, (a) if there is a sequence $\{z_q\}$ in U , such that $\alpha(z_q, z_{q+1}) \geq 1$ for all $q \in \mathbb{N}$ and $z_q \rightarrow z$ when $q \rightarrow \infty$, this implies $\alpha(z_q, z) \geq 1$ for all $q \in \mathbb{N}$ or (b) S is continuous. Then, $F(S)$ is nonempty.

Theorem 1 for single-valued mappings was originally proposed by Kirk and Shahzad [12]; after slight modification and generalizing, we obtained the following consequence.

Theorem 5. If (U, p) is a complete metric space, ρ is an SSE- p on U , and mapping $S : U \rightarrow U$ satisfies

$$\frac{1}{2}p(x, Sx) \leq p(x, y), \text{ which implies } \alpha_*(Sx, Sy)\rho(Sx, Sy) \leq \phi(\mathcal{M}^*(x, y))$$

for every $x, y \in U$ and for some $\phi \in \Phi$. For $x \in U$,

$$p(Sx, S^2y) \leq \rho(Sx, S^2y).$$

Further, mapping S is α -admissible; therefore, $z_0 \in U$ exist, such that $\alpha(z_0, Sz_0) \geq 1$. Moreover, (a) if there is a sequence $\{z_q\}$ in U , such that $\alpha(z_q, z_{q+1}) \geq 1$ for all $q \in \mathbb{N}$ and $z_q \rightarrow z$ when $q \rightarrow \infty$, this implies $\alpha(z_q, z) \geq 1$ for all $q \in \mathbb{N}$ or (b) mapping S is continuous. Then, $F(S)$ is a singleton.

Proof. The existence of fixed points is guaranteed by Theorem 4. For uniqueness, suppose that there are $u_1, u_2 \in F(S)$, such that $u_1 \neq u_2$. That is, $Su_1 = u_1$ and $Su_2 = u_2$, such that $u_1 \neq u_2$. As

$$\frac{1}{2}p(u_1, Su_1) \leq p(u_1, u_1) = 0 \leq p(u_1, u_2),$$

$$\begin{aligned} p(u_1, u_2) &= p(Su_1, S^2u_2) \leq \alpha(Su_1, Su_2)p(Su_1, S^2u_2) \\ &\leq \alpha(Sz_0, Sz_1)\rho(Su_1, S^2u_2) \\ &= \alpha(Sz_0, Sz_1)\rho(Su_1, Su_2) \\ &\leq \phi(\mathcal{M}^*(u_1, u_2)) \\ &= \phi(p(u_1, u_2)) \\ &< p(u_1, u_2) \end{aligned}$$

is a contradiction. Then, $F(S)$ is a singleton. $\square$

In the next theorem, we prove a more general contraction condition compared to the one we used in Theorem 4 but with metrics which are SSSE- H on $P_c(U)$.

Theorem 6. If (U, p) is a complete metric space, H_1 is an SSSE- H on $P_c(U)$, and mapping $S : U \rightarrow P_c(U)$ satisfies

$$\begin{aligned} \frac{1}{2}\mathcal{D}(x, Sx) &\leq p(x, y) \\ , \text{ which implies} \\ \alpha_*(Sx, Sy)H_1(Sx, Sy) &\leq \phi(\mathcal{M}(x, y)) \end{aligned} \tag{6}$$

for every $x, y \in U$ and for some $\phi \in \Phi$. Therefore, $z_0 \in U$ and $z_1 \in Sz_0$ exist, such that $\alpha(z_0, z_1) \geq 1$. Further, for $x \in U$ and $y \in Sx$,

$$\mathcal{D}(y, Sy) \leq H_1(Sy, Sx),$$

and mapping S is α_* -admissible. Moreover, (a) if there is a sequence $\{z_q\}$ in U , such that $\alpha(z_q, z_{q+1}) \geq 1$ for all $q \in \mathbb{N}$ and $z_q \rightarrow z$ when $q \rightarrow \infty$, this implies $\alpha(z_q, z) \geq 1$ for all $q \in \mathbb{N}$, or (b) mapping S is continuous. Then, $F(S)$ is nonempty.

Proof. Along similar lines, as in the proof of Theorem 4, we obtain a Cauchy sequence $\{z_q\}$ in U , which satisfies $z_{q+1} \in Sz_q$, $z_{q+1} \neq z_q$ such that z_q converges to some $z \in U$. Suppose (a) holds. Further, assume, on the contrary, that there is no fixed point of S . That is,

$\mathcal{D}(z, Sz) > 0$. Again, along similar lines as in the proof of Theorem 4, a subsequence $\{z_{q_k}\}$ of $\{z_q\}$ exists, which satisfies

$$\lim_{k \rightarrow \infty} H_1(Sz_{q_k}, Sz) \leq \phi(\mathcal{D}(z, Sz)).$$

H_1 is a SSSE-H on $P_c(U)$, so

$$\begin{aligned} \mathcal{D}(z, Sz) &= \lim_{k \rightarrow \infty} \mathcal{D}(z_{q_k}, Sz) \\ &\leq \lim_{k \rightarrow \infty} H(Sz_{q_k}, Sz) \\ &\leq \lim_{k \rightarrow \infty} H_1(Sz_{q_k}, Sz) \\ &\leq \phi(\mathcal{D}(z, Sz)) < \mathcal{D}(z, Sz) \end{aligned}$$

is a contradiction. Consequently, $z \in Sz$. As in the proof of Theorem 4, the conclusion holds whenever (b) holds. $\square$

Corollary 5. If (U, p) is a complete metric space, H_1 is an SSSE-H on $P_c(U)$. If mapping $S : U \rightarrow P_c(U)$ satisfies

$$\begin{aligned} \frac{1}{2}\mathcal{D}(x, Sx) &\leq p(x, y), \\ \text{which implies} & \\ \alpha_*(Sx, Sy)H_1(Sx, Sy) &\leq \phi(p(x, y)) \end{aligned} \quad (7)$$

for every $x, y \in U$ and for some $\phi \in \Phi$. Therefore, $z_0 \in U$ and $z_1 \in Sz_0$ exist, such that $\alpha(z_0, z_1) \geq 1$. Further, for $x \in U$ and $y \in Sx$,

$$\mathcal{D}(y, Sy) \leq H_1(Sy, Sx),$$

and mapping S is α_* -admissible. Moreover, (a) if there is a sequence $\{z_q\}$ in U , such that $\alpha(z_q, z_{q+1}) \geq 1$ for all $q \in \mathbb{N}$ and $z_q \rightarrow z$ when $q \rightarrow \infty$, this implies $\alpha(z_q, z) \geq 1$ for all $q \in \mathbb{N}$, or (b) mapping S is continuous. Then, $F(S)$ is nonempty.

Corollary 6. If (U, p) is a complete metric space and mapping $S : U \rightarrow P_c(U)$ satisfies

$$\begin{aligned} \frac{1}{2}\mathcal{D}(x, Sx) &\leq p(x, y), \\ \text{this implies that} & \\ \alpha_*(Sx, Sy)H(Sx, Sy) &\leq \phi(\mathcal{M}^*(x, y)) \end{aligned} \quad (8)$$

for every $x, y \in U$ and for some $\phi \in \Phi$. Therefore, $z_0 \in U$ and $z_1 \in Sz_0$ exist, such that $\alpha(z_0, z_1) \geq 1$. Further, mapping S is α_* -admissible and (a) if there is a sequence $\{z_q\}$ in U , such that $\alpha(z_q, z_{q+1}) \geq 1$ for all $q \in \mathbb{N}$ and $z_q \rightarrow z$ when $q \rightarrow \infty$, this implies $\alpha(z_q, z) \geq 1$ for all $q \in \mathbb{N}$, or (b) mapping S is continuous. Then, $F(S)$ is nonempty.

Corollary 7. If (U, p) is a complete metric space, H_1 is a SSSE-H on $P_c(U)$. If mapping $S : U \rightarrow P_c(U)$ satisfies

$$\alpha_*(Sx, Sy)H_1(Sx, Sy) \leq \phi(\mathcal{M}(x, y)) \quad (9)$$

for every $x, y \in U$ and for some $\phi \in \Phi$. Therefore, $z_0 \in U$ and $z_1 \in Sz_0$ exist, such that $\alpha(z_0, z_1) \geq 1$. Further, for $x \in U$ and $y \in Sx$,

$$\mathcal{D}(y, Sy) \leq H_1(Sy, Sx),$$

and mapping S is α_* -admissible. Moreover, (a) if there is a sequence $\{z_q\}$ in U , such that $\alpha(z_q, z_{q+1}) \geq 1$ for all $q \in \mathbb{N}$ and $z_q \rightarrow z$ when $q \rightarrow \infty$, this implies $\alpha(z_q, z) \geq 1$ for all $q \in \mathbb{N}$, or (b) mapping S is continuous. Then, $F(S)$ is nonempty.

Now we present the following example to explain Theorem 6 and show that some results that already exist are not applicable in this situation.

Example 2. Let $U = \{z_1, z_2, z_3, z_4, z_5\}$. Define $p : U \times U \rightarrow \mathbb{R}^+$ as

$$\begin{aligned} p(z_3, z_5) &= p(z_2, z_5) = p(z_2, z_4) = p(z_3, z_4) = 6, \\ p(z_2, z_3) &= p(z_1, z_4) = p(z_1, z_5) = 8, \\ p(z_1, z_3) &= p(z_1, z_2) = 5, \quad p(z_4, z_5) = 3, \\ p(z, z) &= 0 \text{ and} \\ p(m, n) &= p(n, m) \text{ for all } m, n \in U. \end{aligned}$$

Define $S : U \rightarrow P_c(U)$ by

$$Sz = \begin{cases} \{z_1\} & \text{if } z \neq z_4, z_5, \\ \{z_2\} & \text{if } z = z_4, \\ \{z_3\} & \text{if } z = z_5, \end{cases}$$

and $\alpha : U \times U \rightarrow \mathbb{R}^+$ by $\alpha(z_q, z_k) = 1$ for all $q, k \in \{1, 2, 3, 4, 5\}$. Set $\phi(m) = \frac{8}{9}m$ to $m \in \mathbb{R}^+$, then $\phi \in \Phi$. S is α_* -admissible. Note that H^+ is SSE-H and

$$\frac{1}{2}\mathcal{D}(x, Sx) \leq p(x, z), \text{ which implies } \alpha_*(Sx, Sz)H^+(Sx, Sz) \leq \phi(\mathcal{M}(x, z))$$

holds for all $x, z \in \{z_1, z_2, z_3\}$ as

$$H^+(Sx, Sz) = 0 \leq \phi(\mathcal{M}(x, z)) = \frac{8}{9}p(x, z).$$

If $x = z_1$ and $z = z_4$, then

$$\begin{aligned} H^+(Sz_1, Sz_4) &= 5 \leq \phi(\mathcal{M}(z_1, z_4)) \\ &= \phi \left(\max \left\{ \frac{p(z_1, z_4), \mathcal{D}(z_1, Sz_1), \mathcal{D}(z_4, Sz_4),}{\mathcal{D}(z_1, Sz_4) + \mathcal{D}(z_4, Sz_1)}, \right. \right. \\ &\quad \left. \left. \frac{\mathcal{D}(z_1, Sz_1)\mathcal{D}(z_4, Sz_4)}{1 + p(z_1, z_4)} \right\} \right) \\ &= \phi \left(\max \left\{ 8, 0, 6, \frac{5+8}{2}, 0 \right\} \right) = \phi(8) = 7.11. \end{aligned}$$

If $x = z_1$ and $z = z_5$, then

$$\begin{aligned} H^+(Sz_1, Sz_5) &= 5 \leq \phi(\mathcal{M}(z_1, z_5)) \\ &= \phi \left(\max \left\{ \frac{p(z_1, z_5), \mathcal{D}(z_1, Sz_1), \mathcal{D}(z_5, Sz_5),}{\mathcal{D}(z_1, Sz_5) + \mathcal{D}(z_5, Sz_1)}, \right. \right. \\ &\quad \left. \left. \frac{\mathcal{D}(z_1, Sz_1)\mathcal{D}(z_5, Sz_5)}{1 + p(z_1, z_5)} \right\} \right) \\ &= \phi \left(\max \left\{ 8, 0, 6, \frac{5+8}{2}, 0 \right\} \right) = \phi(8) = 7.11. \end{aligned}$$

If $x = z_2$ and $z = z_4$, then

$$\begin{aligned} H^+(Sz_2, Sz_4) &= p(z_1, z_2) = 5 \leq \phi(\mathcal{M}(z_2, z_4)) \\ &= \phi \left(\max \left\{ \frac{p(z_2, z_4), \mathcal{D}(z_2, Sz_2), \mathcal{D}(z_4, Sz_4),}{\mathcal{D}(z_2, Sz_4) + \mathcal{D}(z_4, Sz_2)}, \right. \right. \\ &\quad \left. \left. \frac{\mathcal{D}(z_2, Sz_2) \mathcal{D}(z_4, Sz_4)}{1 + p(z_2, z_4)} \right\} \right) \\ &= \phi \left(\max \left\{ 6, 5, 6, \frac{0+8}{2}, \frac{5 \times 6}{1+6} \right\} \right) = \phi(6) = 5.333. \end{aligned}$$

If $x = z_3$ and $z = z_5$, then

$$\begin{aligned} H^+(Sz_3, Sz_5) &= p(z_1, z_3) = 5 \leq \phi(\mathcal{M}(z_3, z_5)) \\ &= \phi \left(\max \left\{ \frac{p(z_3, z_5), \mathcal{D}(z_3, Sz_3), \mathcal{D}(z_5, Sz_5),}{\mathcal{D}(z_3, Sz_5) + \mathcal{D}(z_5, Sz_3)}, \right. \right. \\ &\quad \left. \left. \frac{\mathcal{D}(z_3, Sz_3) \mathcal{D}(z_5, Sz_5)}{1 + p(z_3, z_5)} \right\} \right) \\ &= \phi \left(\max \left\{ 6, 5, 6, \frac{0+8}{2}, \frac{5 \times 6}{1+6} \right\} \right) = \phi(6) = \frac{48}{9}. \end{aligned}$$

If $x = z_2$ and $z = z_5$, then

$$\begin{aligned} H^+(Sz_2, Sz_5) &= p(z_1, z_3) = 5 \leq \phi(\mathcal{M}(z_2, z_5)) \\ &= \phi \left(\max \left\{ \frac{p(z_2, z_5), \mathcal{D}(z_2, Sz_2), \mathcal{D}(z_5, Sz_5),}{\mathcal{D}(z_2, Sz_5) + \mathcal{D}(z_5, Sz_2)}, \right. \right. \\ &\quad \left. \left. \frac{\mathcal{D}(z_2, Sz_2) \mathcal{D}(z_5, Sz_5)}{1 + p(z_2, z_5)} \right\} \right) \\ &= \phi \left(\max \left\{ 6, 5, 6, \frac{8+8}{2}, \frac{5 \times 6}{1+6} \right\} \right) = \phi(8) = 7.11. \end{aligned}$$

If $x = z_3$ and $z = z_4$, then

$$\begin{aligned} H^+(Sz_3, Sz_4) &= p(z_1, z_2) = 5 \leq \phi(\mathcal{M}(z_3, z_4)) \\ &= \phi \left(\max \left\{ \frac{p(z_3, z_4), \mathcal{D}(z_3, Sz_3), \mathcal{D}(z_4, Sz_4),}{\mathcal{D}(z_3, Sz_4) + \mathcal{D}(z_4, Sz_3)}, \right. \right. \\ &\quad \left. \left. \frac{\mathcal{D}(z_3, Sz_3) \mathcal{D}(z_4, Sz_4)}{1 + p(z_3, z_4)} \right\} \right) \\ &= \phi \left(\max \left\{ 6, 5, 6, \frac{8+8}{2}, \frac{5 \times 6}{1+6} \right\} \right) = \phi(8) = 7.11. \end{aligned}$$

If $x = z_4$ and $z = z_5$, then

$$\begin{aligned} H^+(Sz_4, Sz_5) &= p(z_2, z_3) = 8 \leq \phi(\mathcal{M}(z_4, z_5)) \\ &= \phi \left(\max \left\{ \frac{p(z_4, z_5), \mathcal{D}(z_4, Sz_4), \mathcal{D}(z_5, Sz_5),}{\mathcal{D}(z_4, Sz_5) + \mathcal{D}(z_5, Sz_4)}, \right. \right. \\ &\quad \left. \left. \frac{\mathcal{D}(z_4, Sz_4) \mathcal{D}(z_5, Sz_5)}{1 + p(z_4, z_5)} \right\} \right) \\ &= \phi \left(\max \left\{ 3, 6, 6, \frac{6+6}{2}, \frac{6 \times 6}{1+3} \right\} \right) = \phi(9) = 8. \end{aligned}$$

Hence, using Theorem 6, S is fixed.

Remark 2. In the above example, please note that

$$\frac{1}{2}\mathcal{D}(z_4, Sz_4) = 3 = p(z_4, z_5),$$

but

$$\begin{aligned} H(Sz_4, Sz_5) &= p(z_2, z_3) = 8 \not\leq \phi(6) \\ &= \phi\left(\max\left\{3, 6, \frac{6+6}{2}\right\}\right) \\ &= \phi\left(\max\left\{\frac{p(z_4, z_5), \mathcal{D}(z_4, Sz_4), \mathcal{D}(z_5, Sz_5),}{\mathcal{D}(z_4, Sz_5) + \mathcal{D}(z_5, Sz_4)}, \frac{2}{2}\right\}\right) \\ &= \phi(\mathcal{M}(z_4, z_5)) \end{aligned}$$

for any $\phi \in \Phi$. Hence, Theorems 2 and 3 are not applicable in this example.

In the following, we present the result for metric H^+ .

Theorem 7. If (U, p) is a complete metric space, $S : U \rightarrow P_c(U)$ is mapping that satisfies

$$\frac{1}{2}\mathcal{D}(x, Sx) \leq p(x, y), \text{ which implies } \alpha_*(Sx, Sy)H^+(Sx, Sy) \leq \phi(\mathcal{N}(x, y))$$

for every $x, y \in U$ and for some $\phi \in \Phi$. Mapping S is α_* -admissible; therefore, $z_0 \in U$ and $z_1 \in Sz_0$ exist, such that $\alpha(z_0, z_1) \geq 1$. Further, for $x \in U$ and $y \in Sx$,

$$\mathcal{D}(y, Sy) \leq H^+(Sy, Sx).$$

Moreover, (a) if there is a sequence $\{z_q\}$ in U , such that $\alpha(z_q, z_{q+1}) \geq 1$ for all $q \in \mathbb{N}$ and $z_q \rightarrow z$ when $q \rightarrow \infty$, this implies $\alpha(z_q, z) \geq 1$ for all q , or (b) S is continuous. Then, $F(S)$ is nonempty.

Proof. As in the proof of Theorem 4, we obtain a Cauchy sequence $\{z_q\}$ in U , which satisfies $z_{q+1} \in Sz_q$, $z_{q+1} \neq z_q$ such that z_q converges to some $z \in U$. Assume there is no fixed point of S . That is, $\mathcal{D}(z, Sz) > 0$. Again, along similar lines to the proof of Theorem 4, a subsequence $\{z_{q_k}\}$ of $\{z_q\}$ exists, such that

$$\frac{1}{2}\mathcal{D}(z_{q_k}, Sz_{q_k}) \leq p(z_{q_k}, z).$$

Assume (a) holds; therefore, $\alpha(z_q, z_{q+1}) \geq 1$ and $z_q \rightarrow z$ as $q \rightarrow \infty$, which implies $\alpha(z_{q_k}, z) \geq 1$ for all $k \in \mathbb{N}$. As mapping S is α_* -admissible, $\alpha_*(Sz_{q_k}, Sz) \geq 1$. Consequently, we obtain

$$\begin{aligned} H(Sz_{q_k}, Sz) &\leq 2H^+(Sz_{q_k}, Sz) \leq 2\alpha_*(Sz_{q_k}, Sz)H^+(Sz_{q_k}, Sz) \\ &\leq 2\phi(\mathcal{N}(z_q, z)) \\ &= 2\phi\left(\max\left\{\frac{p(z_{q_k}, z), \mathcal{D}(z_{q_k}, Sz_{q_k}), \frac{\mathcal{D}(z, Sz)}{2},}{\mathcal{D}(z_{q_k}, Sz) + \mathcal{D}(z, Sz_{q_k})}, \frac{2}{2}\right\}\right) \\ &= 2\phi\left(\max\left\{\frac{\mathcal{D}(z_{q_k}, Sz_{q_k})^2 \mathcal{D}(z, Sz)}{1 + p(z_{q_k}, z)}\right\}\right). \end{aligned}$$

Applying limit $q \rightarrow \infty$, we also obtain the following:

$$\begin{aligned} \mathcal{D}(z, Sz) &\leq 2\phi\left(\frac{\mathcal{D}(z, Sz)}{2}\right) \\ &< \mathcal{D}(z, Sz), \end{aligned}$$

which provides a contradiction, so we obtain $z \in Sz$. As in the proof of Theorem 4, the conclusion holds whenever (b) holds. $\square$

Corollary 8. *If (U, p) is a complete metric space and $S : U \rightarrow P_c(U)$ is mapping that satisfies*

$$\frac{1}{2}\mathcal{D}(x, Sx) \leq p(x, y), \text{ this implies } \alpha_*(Sx, Sy)H(Sx, Sy) \leq \phi(\mathcal{N}(x, y))$$

for every $x, y \in U$ and for some $\phi \in \Phi$. Further, S is α_ -admissible; therefore, $z_0 \in U$ and $z_1 \in Sz_0$ exist, such that $\alpha(z_0, z_1) \geq 1$. Moreover, (a) if there is a sequence $\{z_q\}$ in U , such that $\alpha(z_q, z_{q+1}) \geq 1$ for all $q \in \mathbb{N}$ and $z_q \rightarrow z$ when $q \rightarrow \infty$, this implies $\alpha(z_q, z) \geq 1$ for all q , or (b) S is continuous. Then, $F(S)$ is nonempty.*

Proof. As $H^+(V, W) \leq H(V, W)$, the result follows Theorem 7. $\square$

3. Well-Posedness of the FPP of Multifunctions

Now, we discuss some results about the well-posedness of the FPP.

Theorem 8. *Let (U, p) be a complete metric space, H_1 an SSE-H, and $Y \in P_c(U)$, $S : Y \rightarrow P_c(U)$ satisfies,*

- (i) $F^*(S) \neq \emptyset$;
- (ii) for all $x, y \in Y$,

$$\frac{1}{2}\mathcal{D}(x, Sx) \leq p(x, y) \text{ implies } \alpha_*(Sx, Sy)H_1(Sx, Sy) \leq \phi(\mathcal{M}(x, y));$$

- (iii) for all $x, z \in Y$ and $y \in Sx$,

$$\mathcal{D}(y, Sz) \leq H_1(Sz, Sx);$$

- (iv) $\alpha(x, y) \geq 1$ and $\alpha_*(Sy, Sx_q) \geq 1$ for all $x, y \in F(S)$ and for any sequence $\{x_q\}$ in Y , satisfying $\lim_{q \rightarrow \infty} \mathcal{D}(x_q, Sx_q) = 0$,

then

- (a) $y^* \in U$, such that $F^*(S) = F(S) = \{y^*\}$;
- (b) the given FPP is well posed w.r.t. $\mathcal{D}$ and generalized well posed w.r.t. $\mathcal{D}$ as well;
- (c) the given FPP is well posed w.r.t. H_1 and generalized well posed w.r.t. H_1 as well.

Proof. (a) Let $y^* \in F^*(S)$. We claim that $y^* = y$ for all $y \in F(S)$. On the contrary, if $z^* \in F(S)$, such that $y^* \neq z^*$, then

$$\frac{1}{2}\mathcal{D}(y^*, Sy^*) = \frac{1}{2}p(y^*, y^*) = 0 \leq p(y^*, z^*)$$

and implies

$$\begin{aligned}
 p(y^*, z^*) &= \mathcal{D}(Sy^*, z^*) = \mathcal{D}(z^*, Sy^*) \leq H_1(Sy^*, Sz^*) \\
 &\leq \alpha_*(Sy^*, Sz^*)H_1(Sy^*, Sz^*) \\
 &\leq \phi(\mathcal{M}^*(y^*, z^*)) \\
 &= \phi\left(\max\left\{\frac{p(y^*, z^*), \mathcal{D}(y^*, Sy^*), \mathcal{D}(z^*, Sz^*)}{\mathcal{D}(y^*, Sz^*) + \mathcal{D}(z^*, Sy^*)}, \frac{\mathcal{D}(y^*, Sy^*)\mathcal{D}(z^*, Sz^*)}{1 + \mathcal{D}(y^*, z^*)}\right\}\right) \\
 &= \phi(p(y^*, z^*)) < p(y^*, z^*)
 \end{aligned}$$

is a contradiction. Hence, $y^* = y$ for all $y \in F(S)$. Consequently,

$$F(S) = F^*(S) = \{y^*\}.$$

(b) From (a), we obtain $F(S) = F^*(S) = \{y^*\}$. Suppose $\lim_{q \rightarrow \infty} \mathcal{D}(x_q, Sx_q) = 0$. As

$$\frac{1}{2}\mathcal{D}(y^*, Sy^*) = 0 \leq p(y^*, x_q)$$

for all $x \in Sx_q$, we obtain

$$\begin{aligned}
 p(x_q, y^*) &\leq p(x_q, x) + p(x, y^*) \\
 &= p(x_q, x) + \mathcal{D}(Sy^*, x) \\
 &\leq p(x_q, x) + H_1(Sy^*, Sx_q) \\
 &\leq p(x_q, x) + \alpha_*(Sy^*, Sx_q)H_1(Sy^*, Sx_q),
 \end{aligned}$$

which implies

$$\begin{aligned}
 p(x_q, y^*) &\leq \inf_{x \in Sx_q} p(x_q, x) + \alpha_*(Sy^*, Sx_q)H_1(Sy^*, Sx_q) \\
 &= \mathcal{D}(x_q, Sx_q) + \phi(\mathcal{M}^*(y^*, x_q)) \\
 &\leq \mathcal{D}(x_q, Sx_q) + \phi\left(\max\left\{\frac{p(x_q, y^*), \mathcal{D}(y^*, Sy^*), \mathcal{D}(x_q, Sx_q),}{\mathcal{D}(y^*, Sx_q) + \mathcal{D}(x_q, Sy^*)}, \frac{\mathcal{D}(x_q, Sx_q)\mathcal{D}(y^*, Sy^*)}{1 + \mathcal{D}(x_q, y^*)}\right\}\right) \\
 &= \mathcal{D}(x_q, Sx_q) + \phi\left(\max\left\{\frac{p(x_q, y^*), \mathcal{D}(x_q, Sx_q),}{p(y^*, x_q) + \mathcal{D}(x_q, Sx_q) + p(x_q, y^*)}, \frac{\mathcal{D}(x_q, Sx_q)}{2}\right\}\right) \\
 &= \mathcal{D}(x_q, Sx_q) + \phi\left(\max\left\{\frac{p(x_q, y^*), \mathcal{D}(x_q, Sx_q),}{p(x_q, y^*) + \frac{\mathcal{D}(x_q, Sx_q)}{2}}\right\}\right),
 \end{aligned}$$

which further implies

$$\begin{aligned}
 \limsup_{q \rightarrow \infty} p(x_q, y^*) &\leq \limsup_{q \rightarrow \infty} \phi\left(\max\left\{\frac{p(x_q, y^*), \mathcal{D}(x_q, Sx_q),}{p(x_q, y^*) + \frac{\mathcal{D}(x_q, Sx_q)}{2}}\right\}\right) \\
 &= \phi\left(\limsup_{q \rightarrow \infty} p(x_q, y^*)\right).
 \end{aligned}$$

Since $\phi \in \Phi$; therefore,

$$\limsup_{q \rightarrow \infty} p(x_q, y^*) = 0,$$

hence $\lim_{q \rightarrow \infty} p(x_q, y^*) = 0$. Consequently,

$$\lim_{q \rightarrow \infty} x_q = y^* \in F(S),$$

which implies that the FPP is well-posed w.r.t. $\mathcal{D}$, and particularly, we obtain the generalized well-posedness w.r.t. $\mathcal{D}$ of FPP as well. (c) From (a), we obtain $F(S) = F^*(S) = \{x^*\}$. Assume that $\lim_{q \rightarrow \infty} H_1(\{x_q\}, Sx_q) = 0$. As H_1 is SSE-H, we obtain

$$\lim_{q \rightarrow \infty} H(x_q, Sx_q) = 0.$$

Consequently, we obtain

$$\lim_{q \rightarrow \infty} \mathcal{D}(x_q, Sx_q) = 0.$$

Along similar lines as in part (b), we obtain

$$\lim_{q \rightarrow \infty} x_q = y^* \in F(S).$$

Hence, the FPP is well-posed w.r.t. H_1 , and particularly, we obtain the generalized well-posedness w.r.t. H_1 of the FPP as well. $\square$

Theorem 9. If (U, p) is a complete metric space, H_1 is an SSE-H, and $Y \in P_c(U)$, $S : Y \rightarrow P_c(U)$ satisfies the following:

- (i) $F^*(S) \neq \emptyset$;
- (ii) for all $x, y \in Y$,

$$\frac{1}{2} \mathcal{D}(x, Sx) \leq p(x, y) \text{ implies } \alpha_*(Sx, Sy) H_1(Sx, Sy) \leq \phi(\mathcal{M}^*(x, y));$$

- (iii) for all $x, z \in Y$ and $y \in Sx$,

$$\mathcal{D}(y, Sz) \leq H_1(Sz, Sx);$$

- (iv) $\alpha(x, y) \geq 1$, $\alpha_*(Sy, Sx_q) \geq 1$ for all $x, y \in F(S)$ and for any sequence $\{x_q\}$ in Y satisfying $\lim_{q \rightarrow \infty} \mathcal{D}(x_q, Sx_q) = 0$,

then

- (a) $y^* \in U$ exists, such that $F^*(S) = F(S) = \{y^*\}$;
- (b) the given FPP is well posed w.r.t. $\mathcal{D}$ and generalized well posed w.r.t. $\mathcal{D}$ as well;
- (c) the given FPP is well posed w.r.t. H_1 and generalized well posed w.r.t. H_1 as well.

Proof. As

$$\mathcal{M}^*(x, y) \leq \mathcal{M}(x, y),$$

the result follows from Theorem 8. $\square$

4. Conclusions

In this paper, classes of metrics which are semi-sequentially equivalent to the Hausdorff metric (SSE-H) were considered. An example of a metric was provided for which the Hausdorff metric was not SSE. Further, the problem of the existence of fixed points of multifunctions of metrics, which are SSE-H, was solved. An example has been provided to

show that the obtained results properly generalize some existing results in the literature. Moreover, we established the well-posedness of fixed point problems of multifunctions of SSE- H metrics. In the future, we intend to extend the exploration in the context of quasi-metric spaces and asymmetric normed spaces. As asymmetry has become an essential part of many natural phenomena, asymmetric distance structures can play important roles in solving problems in the context of asymmetry.

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References

1. Antón-Sancho, A. Fixed points of automorphisms of the vector bundle moduli space over a compact Riemann surface. *Mediterr. J. Math.* **2024**, *21*, 20.
2. Antón-Sancho, A. F_4 and $P_{SP}(8, \mathbb{C})$ -Higgs pairs understood as fixed points of the moduli space of E_6 -Higgs bundles over a compact Riemann surface. *Open Math.* **2022**, *20*, 1723–1733.
3. Banach, S. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fund. Math.* **1922**, *3*, 133–181.
4. Asl, J.H.; Rezapour, S.; Shahzad, N. On fixed points of α - ψ -contractive multifunctions. *Fixed Point Theory Appl.* **2012**, *2012*, 212.
5. Beg, I.; Butt, A.R. Fixed point for set-valued mappings satisfying an implicit relation in partially ordered metric spaces. *Nonlinear Anal. Theory Methods Appl.* **2009**, *71*, 3699–3704.
6. Choudhury, B.S.; Kundu, A. A coupled coincidence point result in partially ordered metric spaces for compatible mappings. *Nonlinear Anal. Theory Methods Appl.* **2010**, *73*, 2524–2531.
7. Ćirić, L. Non-self mappings satisfying non-linear contractive condition with applications. *Nonlinear Anal. Theory Methods Appl.* **2009**, *71*, 2927–2935.
8. Connell, E.H. Properties of fixed point spaces. *Proc. Am. Math. Soc.* **1959**, *10*, 974–979.
9. Edelstein, M. An extension of Banach’s contraction principle. *Proc. Am. Math. Soc.* **1961**, *12*, 7–10.
10. Hu, T.; Kirk, W.A. Local contractions in metric spaces. *Proc. Am. Math. Soc.* **1978**, *68*, 121–124.
11. Jachymski, J. Equivalent conditions for generalized contractions on (ordered) metric spaces. *Nonlinear Anal. Theory Methods Appl.* **2011**, *74*, 768–774.
12. Kirk, W.A.; Shahzad, N. Remarks on metric transforms and fixed-point theorems. *Fixed Point Theory Appl.* **2013**, *2013*, 106.
13. Lakshmikantham, V.; Ćirić, L. Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces. *Nonlinear Anal. Theory Methods Appl.* **2009**, *70*, 4341–4349.
14. Petrusel, A.; Rus, I.A.; Serban, M.A. The role of equivalent metrics in fixed point theory. *Topol. Methods Nonlinear Anal.* **2013**, *41*, 85–112.
15. Samet, B.; Vetro, C.; Vetro, P. Fixed point theorems for $\alpha - \psi$ -contractive type mappings. *Nonlinear Anal. Theory Methods Appl.* **2012**, *75*, 2154–2165.
16. Suzuki, T. A generalized Banach contraction principle that characterizes metric completeness. *Proc. Am. Math. Soc.* **2008**, *136*, 1861–1869.
17. Tan, K.K. Fixed point theorems for nonexpansive mappings. *Pac. J. Math.* **1972**, *41*, 829–842.
18. Nadler, S.B. Multi-valued contraction mappings. *Pac. J. Math.* **1969**, *30*, 475–488.
19. Gabeleha, M.; Markin, J. On some consequences of Nadler’s fixed point problem. *Filomat* **2024**, *38*, 129–133.
20. Ali, B.; Nazir, T.; Sundus, N. Fixed points of multivalued $(\alpha_* - \phi)$ -contractions and metric transforms. In *Methods of Mathematical Modelling and Computation for Complex Systems*; Springer: Cham, Switzerland, 2022; pp. 375–411.
21. Ali, B.; Abbas, M. Existence and Ulam-Hyers stability of fixed point problem of generalized Suzuki type (α_*, ψ_ϕ) -contractive multivalued operators. *Rev. Real Academia Cienc. Exactas Físicas y Nat. Ser. A. Math.* **2017**, *111*, 1129–1146.

22. Zhang, W.Y.; Li, S.J.; Teo, K.L. Well-posedness for set optimization problems. *Nonlinear Anal. Theory, Methods Appl.* **2009**, *71*, 3769–3778.
23. Petrusel, A.; Rus, I.A. Well-posedness of the fixed point problem for multivalued operators. In *Applied Analysis and Differential Equations*; World Scientific Publishing Company: Singapore, 2007; pp. 295–306.
24. Petrusel, A.; Rus, I.A.; Yao, J.-C. Well-posedness in the generalized sense of the fixed point problems for multivalued operators. *Taiwan. J. Math.* **2007**, *11*, 903–914.
25. Dung, N.V.; Hang, V.T.L. Well-posedness in the generalized sense of the multivalued fixed point problem. *Quaest. Math.* **2018**, *41*, 799–810.
26. Berinde, V. *Iterative Approximation of Fixed Points*; Springer: Berlin/Heidelberg, Germany, 2007; Volume 1912.
27. Pathak, H.K.; Shahzad, N. A generalization of Nadler’s fixed point theorem and its application to nonconvex integral inclusions. *Topol. Methods Nonlinear Anal.* **2013**, *41*, 207–227.

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