A Study of Some Generalized Results of Neutral Stochastic Differential Equations in the Framework of Caputo–Katugampola Fractional Derivatives

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Abstract: Inequalities serve as fundamental tools for analyzing various important concepts in stochastic differential problems. In this study, we present results on the existence, uniqueness, and averaging principle for fractional neutral stochastic differential equations. We utilize Jensen, Burkholder–Davis–Gundy, Grönwall–Bellman, Hölder, and Chebyshev–Markov inequalities. We generalize results in two ways: first, by extending the existing result for $p = 2$ to results in the $L^p$ space; second, by incorporating the Caputo–Katugampola fractional derivatives, we extend the results established with Caputo fractional derivatives. Additionally, we provide examples to enhance the understanding of the theoretical results we establish.

Keywords: fractional calculus; inequalities; neutral stochastic differential equations; averaging principle; Caputo–Katugampola derivatives

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1. Introduction

The concept of fractional calculus (FC) revolves around extending traditional calculus operations such as differentiation and integration. Rather than confining differentiation and integration to integer orders, FC permits operations with fractional orders, such as $\frac{1}{2}$, $\frac{2}{3}$, or $\frac{3}{2}$.

The origins of FC date back to the early 17th century. Here is a brief overview [1,2]:

- During the 17th and 18th centuries, mathematicians like Leibniz, L’Hôpital, and Euler laid the groundwork for FC by investigating non-integer-order derivatives and integrals, although the concept remained incomplete at that time.
- During the 19th century, FC underwent notable advancements. The mathematician Liouville played a pivotal role by introducing what are now acknowledged as the Riemann–Liouville fractional derivatives (RLFDs) and integrals, thereby laying the groundwork for the theoretical underpinnings of FC.
- Further Progress (20th Century): The significance of FC grew in the 20th century, driven by the input of mathematicians such as Riemann, Grünwald, Letnikov, and others. Their endeavors expanded the scope of both the theory and practical use of FC.
- Contemporary Uses (Late 20th Century to Present): FC has expanded beyond its initial role as a purely mathematical concept, venturing into various practical fields.
decades, FC has emerged as a powerful tool with diverse applications across various fields. FC offers a robust framework for representing intricate systems characterized by memory, long-range dependence, and nonlocal behaviors.

Here are some notable applications that demonstrate the versatility and importance of FC in understanding and modeling complex systems across various scientific, engineering, and socio-economic domains. These applications provide valuable insights and tools for analysis, prediction, and optimization [3–6]:

1. Viscoelastic Materials: Many materials exhibit viscoelastic behavior, meaning they exhibit both viscous (flow-like) and elastic (spring-like) properties. FC models are used to accurately describe the behavior of viscoelastic materials, which is crucial in industries like polymer science, aerospace engineering (for designing composite materials), and civil engineering (for designing structures subjected to dynamic loads).

2. Signal Processing: FC is used in signal processing to model and analyze signals with fractal properties. Applications include noise reduction, signal denoising, time-series analysis (for financial data, environmental data, etc.), and image processing (for edge detection, image enhancement, etc.).

3. Electrochemical Systems: FC is applied in modeling electrochemical systems, such as batteries, fuel cells, and corrosion processes. It helps in understanding processes involving diffusion, charge transfer, and nonlinear dynamics, leading to advancements in energy storage technology and corrosion prevention.

4. Biological Systems: FC is used to model biological systems with memory effects, such as neuronal systems, biological tissues, and drug release from delivery systems. It provides more accurate models for understanding physiological processes, drug kinetics, and disease dynamics.

5. Control Systems: FC plays a role in designing control systems with non-integer order dynamics. Applications include robotics, automotive control (for anti-lock braking systems, traction control), aerospace (for aircraft stabilization), and process control (for chemical processes, industrial automation).

6. Finance: FC is used in modeling financial time-series data to analyze long-range dependence, volatility clustering, and risk management. Applications include pricing financial derivatives, portfolio optimization, risk assessment, and algorithmic trading.

7. Heat Conduction: FC models are used to describe anomalous heat conduction in materials with fractal or disordered structures. This has applications in the thermal management of electronic devices, designing materials for thermal insulation, and optimizing heat exchangers.

8. Fluid Dynamics: FC is applied to modeling fluid flow in porous media, non-Newtonian fluids, and turbulent flows. Applications include petroleum reservoir engineering (for enhanced oil recovery), groundwater hydrology (for modeling flow in aquifers), and environmental fluid dynamics (for pollutant dispersion).

The primary tool of FC is indeed fractional derivatives (FDs). FDs introduce memory effects into mathematical models by extending the concept of differentiation beyond integer orders. Unlike classical derivatives, which only consider the current state of a system, FDs take into account the entire history of the system up to the present time. This memory property allows FDs to capture long-term dependencies and persistence in the system’s behavior. In these systems, past states influence current behavior, and FD provides a mathematical tool to account for this memory-dependent behavior.

There are several types of FD, each with its own characteristics and applications [7–10]. RLFDs are one of the earliest definitions of FDs. RLFDs involve integration from zero to \( \mu \), making them nonlocal. They capture memory effects in systems due to its integral form. For a function \( \mathcal{Y}(\mu) \), the RLFD of order \( \eta \) is given by [11]:

\[
D_0^\eta \mathcal{Y}(\mu) = \frac{1}{\Gamma(\rho - \eta)} \frac{d^\rho}{d\mu^\rho} \int_0^\mu (\mu - \xi)^{\rho - \eta - 1} \mathcal{Y}(\xi)d\xi
\]
Caputo fractional derivatives (CFDs) are a modification of RLFDs that deals with initial conditions more conveniently for some applications. CFDs are defined as follows [12]:

\[
D_\eta^\mu Y(\mu) = \begin{cases} 
\frac{1}{\Gamma(\rho - \eta)} \int_0^\mu (\mu - \xi)^{\rho - \eta - 1} Y(\xi) d\xi, & \text{if } \rho - 1 \leq \eta < \rho, \\
\frac{d}{d\mu} Y(\mu), & \text{if } \eta = \rho,
\end{cases}
\]

where \( \rho - 1 \leq \eta < \rho \in \mathbb{Z}^+ \).

Grünwald–Letnikov fractional derivatives (GLFD) are one of the most commonly used definitions of FDs, especially in numerical computations and practical applications. They provide a straightforward and computationally feasible way to calculate the FD of functions. Compared to other definitions, such as RLFDs or CFDs, the Grünwald–Letnikov approach has the advantage of being easier to implement numerically due to its direct summation form. This makes it particularly suitable for numerical simulations and solving DEs of fractional order in engineering, physics, and other fields. It is defined as follows [13]:

\[
D_\eta^\mu Z(\mu) = \lim_{h \to 0} \frac{1}{h^\eta} \sum_{j=0}^{\infty} (-1)^j \binom{\eta}{j} Z(\mu - jh),
\]

where \( j \in \mathbb{N} \), and the gamma function are used to determine the binomial coefficient,

\[
\binom{\eta}{j} = \frac{\eta(\eta - 1)(\eta - 2)(\eta - 3)\ldots(\eta - j + 1)}{j!} = \frac{\Gamma(\eta + 1)}{j!\Gamma(\eta - j + 1)}.
\]

On the other hand, the Katugampola fractional operators provide a unified representation of both the Hadamard and Riemann–Liouville operators. By incorporating aspects of both the Hadamard and Riemann–Liouville integrals, the Katugampola integral offers a consolidated approach. Additionally, it shares close connections with the Erdelyi–Kober operator, which extends the Riemann–Liouville fractional integral.

Caputo–Katugampola fractional derivatives (CKFDs) are an advanced theory introduced by Katugampola [14,15]. They address the shortcomings of both Hadamard fractional derivatives (HFDs) and RLFDs. In other words, they are a modification of the Katugampola derivative that bridges the gap between CFDs and Caputo–Hadamard fractional derivatives (CHFDs). This new derivative concept is defined by two parameters: \( 0 < \eta < 1 \) and \( \ell > 0 \). Notably, when \( \ell = 1 \), the CKFD simplifies to the conventional CFD [16]. Another notable scenario occurs when \( \ell \) approaches 0, yielding the CHFD [17]. Researchers showed great interest in studying this FD due to its unique properties. For instance, the authors of [18–22] investigated CKFDs in relation to existence, uniqueness, stability, numerical approximation solutions, and continuous dependence.

The Caputo–Katugampola fractional integral is characterized by [23]

\[
\mathcal{I}_b^\eta Z(\mu) = \frac{\ell^{1-\mu}}{\Gamma(\eta)} \int_b^\mu \frac{Z^{1-\eta}(\xi)}{(\mu^{1-\eta} - \xi^{1-\eta})^\eta} d\xi.
\]

(1)

The CKFD of order \( \eta \) is defined by [23]:

\[
\mathcal{D}_b^{\eta, \ell} Z(\mu) = \frac{\ell^{\eta}}{\Gamma(1-\eta)} \int_b^\mu \frac{Z^{\eta}(\xi)}{(\mu^{1-\eta} - \xi^{1-\eta})^\eta} d\xi.
\]

(2)

The selection of FD depends on various factors, including [24–27]:

1. Mathematical Model: The specific mathematical model being studied may dictate the choice of FD. Different models may require different types of FD to accurately represent their behavior.
2. Physical Phenomena: The underlying physical phenomena being modeled can influence the choice of FD. For example, certain processes may exhibit memory effects that are better captured by one type of FD than another.
3. Initial Conditions: The nature of the initial conditions can play a role. Some FDs, such as the Caputo derivative, allow for a more convenient imposition of initial conditions compared to others.

4. Computational Efficiency: In numerical simulations or computational studies, the computational efficiency of different types of FD may be a consideration. Some derivatives may be easier or faster to compute than others.

5. Analytical Properties: The analytical properties of the FD, such as its continuity, differentiability, and behavior under certain operations, may be important depending on the analysis being conducted.

6. Historical Context: The historical context and previous research in the field may influence the choice of FD. Researchers may select derivatives that have been commonly used in similar studies or that have well-established properties.

7. Application-Specific Requirements: The specific requirements of the application or problem being studied can also guide the selection of FD. Different derivatives may be more suitable for certain applications or may better capture the desired characteristics of the system being modeled.

FC is indeed utilized to model various phenomena through fractional differential equations (FDEs). On the other hand, fractional stochastic differential equations (FSDEs) combine three fundamental mathematical concepts: deterministic processes, stochastic processes, and FC. By integrating these elements, FSDEs offer a comprehensive mathematical framework for modeling complex systems with both deterministic and stochastic dynamics, as well as memory effects.

Compared to traditional deterministic DEs, FSDEs offer several advantages [28–30]:
1. FSDEs incorporate FDs, allowing them to capture memory effects and long-range dependencies in system dynamics. This is especially useful in modeling phenomena where past states have a non-negligible impact on the current state, such as in financial markets or biological systems.
2. FSDEs include stochastic terms driven by random processes such as Wiener processes or general Gaussian processes. These stochastic components capture uncertainties and random fluctuations present in real-world systems, making FSDEs suitable for modeling noisy or unpredictable phenomena.
3. FSDEs offer a versatile structure for representing intricate systems comprising deterministic and stochastic elements. They can represent a wide range of dynamical behaviors, from purely deterministic systems to systems dominated by random fluctuations, allowing for more accurate and realistic modeling of real-world phenomena.
4. By incorporating both deterministic and stochastic components, FSDEs can provide more accurate predictions and insights into system behavior compared to purely deterministic models. They can capture the combined effects of system dynamics and random fluctuations, leading to improved understanding and prediction of system behavior.
5. FSDEs find applications in various fields, including finance, biology, physics, engineering, and more. Their ability to capture both deterministic and stochastic aspects of system dynamics makes them versatile tools for modeling and analyzing complex systems across different domains.

After extensive examination, scholars determined that many dynamic systems exhibit intricate connections, not only with their current and past states but also with the delay function itself. Consequently, their focus shifts towards fractional neutral stochastic differential equations (FNSEDEs), which are a specific subset of FSDEs. FNSEDEs incorporate four fundamental mathematical concepts: deterministic processes, stochastic processes, FC, and neutral terms or delay terms. The presence of neutral or delay terms in FNSEDEs introduces memory effects into the system, where the dynamics depend not only on the current state but also on past states with a time delay. This can capture phenomena such as feedback mechanisms, distributed delays, or memory-dependent processes.
Here are several reasons why FNSDEs are essential: [31–33]:

1. Memory Effects and Delayed Responses: Many real-world systems exhibit memory effects, where the current state depends not only on past states but also on delayed states. FNSDEs allow for the modeling of such systems more accurately compared to traditional stochastic differential equations, as they incorporate neutral terms representing delayed responses. This is particularly important in fields such as biology, communication networks, and control systems.

2. Stochastic Fluctuations: Stochastic fluctuations are inherent in many natural and engineered systems due to noise, random perturbations, and environmental variability. FNSDEs with stochastic terms capture these fluctuations, enabling the modeling of systems under uncertainty and providing insights into their probabilistic behavior. This is crucial in fields such as finance, environmental science, and signal processing.

3. Long-Range Dependencies: FC introduces long-range dependencies and nonlocal interactions into mathematical models. FNSDEs with fractional operators capture these dependencies, allowing for the modeling of systems where distant events influence current behavior. This is important in fields such as finance, where market dynamics can be influenced by past events over extended time periods.

4. Flexibility and Versatility: FNSDEs provide a flexible and versatile framework for modeling a wide range of complex systems. By combining neutral terms with FD and stochastic terms, FNSDEs can accurately represent diverse phenomena across different disciplines, including biology, finance, communication networks, environmental science, and control systems.

5. Predictive Power and Analysis: FNSDEs enable researchers to gain deeper insights into the behavior of complex systems and make predictions about their future evolution. By accurately capturing memory effects, delayed responses, and stochastic fluctuations, FNSDEs can provide valuable information for decision-making, risk assessment, and system optimization in various applications.

6. Advancements in Mathematical Theory: The study of FNSDEs has led to advancements in mathematical theory, including the development of new analytical techniques, numerical methods, and stability criteria. This ongoing research contributes to a deeper understanding of FC and its applications in diverse disciplines.

The topic of existence and uniqueness (EU) of solutions for FSDEs is a fascinating subject in mathematics. This property ensures that solutions to FSDEs can be found for given initial conditions. In other words, it guarantees that there is at least one solution that satisfies the given equation and initial values. Uniqueness guarantees that the solution remains well-defined and unaffected by multiple potential outcomes. In the realm of FSDEs, it indicates that there exists a unique solution corresponding to a specific set of initial conditions.

Proving the EU of FSDEs is crucial for several reasons [34,35]:

1. Theoretical Foundations: Establishing the EU of theorems provides the theoretical foundation for studying FSDEs. It ensures that solutions to these equations exist and are unique under certain conditions, which is fundamental for further analysis and applications.

2. Reliability of Solutions: Knowing that solutions to FSDEs exist and are unique helps ensure the reliability of numerical methods used to approximate these solutions. Without such guarantees, numerical simulations might produce unreliable or erroneous results.

3. Physical Interpretation: FSDEs are often used to model complex systems in various fields, including physics, biology, finance, and engineering. Having EU results allows researchers to interpret the solutions obtained from these equations with confidence, enhancing the understanding of the underlying phenomena.

4. Predictive Power: For applications such as stochastic control, option pricing, and risk management, it is essential to have confidence in the solutions provided by FSDEs. EU
theorems provide assurance that these models can accurately predict future behavior under uncertain conditions.

5. Mathematical Rigor: Establishing EU results for FSDEs contributes to the mathematical rigor of the field. It ensures that FSDEs are well defined and behave consistently within the framework of stochastic analysis, enhancing the credibility of research in this area.

6. Applications in Control and Optimization: In control theory and optimization problems involving FSDEs, EU results are fundamental for ensuring the stability and optimality of the solutions obtained.

7. Applications in Engineering and Science: FSDEs are used to model a wide range of phenomena in engineering, physics, biology, finance, and other sciences. Establishing EU ensures the accuracy and applicability of these models in diverse fields.

Some authors have been actively researching FSDEs in various directions; for instance, the authors of [36] constructed the existence, uniqueness, and stability of the solution to the stochastic backward equation. To develop these findings, they developed new mapping and applied the norm approach. For a class of nonlinear FSDEs where the coefficients satisfy Lipschitz continuity and a linear growth condition, the authors of [37] utilized the Euler–Maruyama (EM) approach for the solutions of these equations. Using the Banach contraction principle, the authors of [38] proved the EU of solutions for FNSDEs in the Riemann–Liouville sense. They also examined these equations for Hyers–Ulam stability. The authors of [39] demonstrated that a certain category of Caputo FSDEs is well-posed under specific assumptions. The researchers in [40] examined a group of FSDEs with impulses governed by the Rosenblatt process. They explored the controllability results for this system by employing the FC and Krasnoselskii’s fixed-point theorem (FPT). Additionally, the authors of [41] investigated a category of conformable FSDEs driven by the Rosenblatt process. They discussed the controllability outcomes for the situations under examination using FC, FPT, and stochastic analysis. The authors of [42] discussed the controllability of FSDEs inclusions with an order of $1 \leq \eta < 2$. Results regarding multivalued maps, the FPT, and principal arguments are also provided through the use of FC. The authors of [43] initially focused on the system’s controllability. The authors established stability results in the Ulam–Hyers sense for a system of nonlinear FNSDEs. The authors of [44] studied differential equations (DEs) with stochastic processes involving conformable FD. To provide an explicit description of solutions for linear DEs, they first constructed the Itô formula. Secondly, they utilized the Picard iteration approach to demonstrate the EU of solutions for nonlinear conformable stochastic DEs. Additionally, they provided exponential estimates of solutions and illustrated how solutions depend continuously on initial values via the Gronwall inequality. The writers of [45] addressed the mean square and asymptotic stability of stochastic DEs of fractional order $1 < \eta \leq 2$. They examined a class of stochastic DEs with varying delays in the state. To illustrate the main findings, they applied the Banach FPT.

The averaging approach is widely applied in stochastic fractional dynamical systems and is very useful for analyzing the applications of FSDEs in many different domains. A strong technique for achieving a balance between more realistic, complicated models and more manageable, simpler models that are easier to analyze and simulate is the averaging method. The fundamental concept of the averaging method is to use a simpler system to approximate the original one. Put differently, the averaging principle offers a practical and straightforward method for understanding the properties of complex equations by enabling analysis using related averaging equations. Recently, some scholars have shown interest in the concept of averaging in FSDEs. For example, the mean square-based averaging principle for FSDEs was studied by the authors [46] using CFD. Pei et al. [47] also discussed the averaging principle for FSDEs with delays. Likewise, Shen et al. [48] used CFD to present a similar theory. Moreover, the authors [49] examined the averaging approach in the sense of mean square with CFD for FSDEs that included a Poisson random measure. Xu et al. [50] discussed the average principle of FSDEs with Lévy noise. Xu et al. [51] also
established the result of the average principle using fractional Brownian motion, specifically for the case when $p = 2$. Furthermore, Wang et al. [52] presented the result in the $L^2$ space. Additionally, Luo et al. [53] studied the average principle of delay FSDEs, focusing on the $L^2$ space. Moreover, Wang et al. [54] established the result of the average principle for FNSDEs in the $p$th moment. For further details regarding the averaging approach, see [55–63].

Motivated by the preceding findings, we present results regarding the EU of solutions for a specific set of FNSDEs using the Banach FPT. We also establish the averaging principle by employing various mathematical inequalities, including the Grönwall–Bellman inequality (GB-In), Chebyshev–Markov inequality (CM-In), Jensen inequality (Je-In), Burkholder–Davis–Gundy inequality (BDG-In), and Hölder inequality (Hö-In), alongside the interval translation technique. We generalize the results for $p = 2$ and CFD by establishing results in $L^p$ space and the framework of CKFD. To understand the theoretical results, we also provided numerical examples.

The key contributions of this paper encompass at least the following three aspects:

1. The system under investigation is broader in scope, as this paper presents findings regarding FNSDEs, which are more encompassing than FSDEs.
2. We effectively demonstrate the outcomes within the $L^p$ space. Previously, numerous articles have focused primarily on the case where $p = 2$.
3. We generalized the result of the averaging principle established in the context of CFD [54] by employing CKFD. We attained the result [54] by setting $\ell = 1$ in our established result.
4. Before this research, FNSDEs involving CKFD had not been thoroughly examined. This study represents the first exploration into the existence and uniqueness (EU) and averaging principles concerning these FNSDEs in the existing literature.

We examined the following FNSDEs:

$$
\begin{align*}
\mathcal{D}_\mu^{\eta,\ell} [\mathcal{H} (\mu) - \mathcal{D} \{\mu, \mathcal{H} (\mu)\}] &= \mathcal{L}_1 (\mu, \mathcal{H} (\mu)) + \mathcal{L}_2 (\mu, \mathcal{H} (\mu)) \frac{dW_\mu}{d\mu}, \quad \mu \in [0, \Theta], \\
\mathcal{H} (0) &= \mathcal{H}_0,
\end{align*}
$$

(3)

where $\mathcal{D}_\mu^{\eta,\ell}$ is the CKFD of orders $\eta \in (\frac{1}{2}, 1)$, $\ell > 0$ and $\mathcal{L}_1 : [0, \Theta] \times \mathbb{R}^q \rightarrow \mathbb{R}^r$, $\mathcal{L}_2 : [0, \Theta] \times \mathbb{R}^q \times \mathbb{R}^q \rightarrow \mathbb{R}^{q \times r}$ are continuous functions of measurable. A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ specifies the $r$-dimensional Brownian motion.

The structure of this research work is outlined as follows: In the upcoming section, we present a definition, a lemma, and some useful assumptions that form the foundation of the results we establish concerning FNSDEs. In the first subsection of Section 3, we establish the results of the EU by utilizing the Banach FPT for FNSDEs. In the second subsection, we delve into the most important concept, the averaging principle. Furthermore, examples are provided to enhance the understanding of our theoretical findings in Section 4. Finally, the conclusion is presented in Section 5.

2. Preliminaries

In this section, we present the definition, lemma, and assumptions that serve as pillars for the results of the EU and the averaging principle for a class of FNSDEs established in this paper.
Definition 1. If $\mathcal{H}(\mu)$ is $\mathcal{F}(\mu)$-adapted and $\mathbb{E}\left[\int_0^\Theta \|\mathcal{H}(\mu)\|d\mu\right] < \infty$, $\mathcal{H}(0) = \mathcal{H}_0$, and satisfy the following conditions, then an $\mathbb{R}$-value stochastic process $\{\mathcal{H}(\mu)\}_{0 \leq \mu \leq \Theta}$ is a unique solution to Equation (3).

\[
\mathcal{H}(\mu) = \mathcal{H}_0 - \mathcal{D}(0, \mathcal{H}_0) + \mathcal{D}(\mu, \mathcal{H}(\mu)) + \int_0^{\mu - \eta} \xi_t^{\eta - 1}(\mu_t - \xi_t)^{1-\eta} \mathcal{Z}_2(\xi, \mathcal{H}(\xi)) d\xi + \int_0^{\mu - \eta} \xi_t^{\eta - 1}(\mu_t - \xi_t)^{1-\eta} \mathcal{Z}_2(\xi, \mathcal{H}(\xi)) d\mathcal{W}_\mu, \mu \in [0, \Theta],
\]

\[
\mathcal{H}(0) = \mathcal{H}_0.
\]

We now assume that the coefficient $\mathcal{D}$ with $\|\mathcal{D}(0, \mathcal{H}_0)\| < \infty$, and the uniformly continuous functions $\mathcal{Z}_1$ and $\mathcal{Z}_2$ in Equation (3) when $\forall \varphi, \sigma \in \mathbb{R}$, $\mu \in [0, \Theta]$ there are $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3 > 0$ that meet the following requirements:

(}\mathcal{J}_1\): \[
\|\mathcal{D}(\mu, \varphi) - \mathcal{D}(\mu, \sigma)\| \leq \mathcal{Q}_1 \|\varphi - \sigma\|.
\]

(}\mathcal{J}_2\): \[
\|\mathcal{Z}_1(\mu, \varphi) - \mathcal{Z}_1(\mu, \sigma)\| \vee \|\mathcal{Z}_2(\mu, \varphi) - \mathcal{Z}_2(\mu, \sigma)\| \leq \mathcal{Q}_2 \|\varphi - \sigma\|.
\]

where $\mathcal{Z}_1 \vee \mathcal{Z}_2 = \max(\mathcal{Z}_1, \mathcal{Z}_2)$.

(}\mathcal{J}_3\): \[
\|\mathcal{Z}_1(\mu, \varphi)\| \vee \|\mathcal{Z}_2(\mu, \varphi)\| \leq \mathcal{Q}_3 (1 + \|\varphi\|).
\]

(}\mathcal{J}_4\): Functions $\mathcal{Z}_1$ and $\mathcal{Z}_2$ exist and for $\Theta_1 \in [0, \Theta]$, $\mu \in [0, \Theta]$, and $p \geq 2$, we can identify non-negative bounded functions $\mathcal{K}_1(\Theta_1)$ that satisfy

\[
\frac{1}{\Theta_1} \int_0^{\Theta_1} \|\mathcal{Z}_1(\mu, \varphi) - \mathcal{Z}_1(\varphi)\|^p d\mu \vee \frac{1}{\Theta_1} \int_0^{\Theta_1} \|\mathcal{Z}_2(\mu, \varphi) - \mathcal{Z}_2(\varphi)\|^p d\mu \leq \mathcal{K}_1(\Theta_1) (1 + \|\varphi\|^p),
\]

with $\lim_{\Theta_1 \to \infty} \mathcal{K}_1(\Theta_1) = 0$.

Lemma 1 ([64]). Let real numbers $\lambda_1, \lambda_2, \ldots, \lambda_\gamma$, where $\gamma \in \mathbb{N}$, and satisfy $\lambda_v \geq 0$ for $v = 1, 2, \ldots, \gamma$. Then,

$$\left(\sum_{v=1}^\gamma \lambda_v\right)^p \leq \gamma^{p-1} \sum_{v=1}^\gamma \lambda_v^p, \forall p > 1.$$  

3. Existence and Uniqueness

This section delves into the EU of solutions for a class of FNSDEs presented in Equation (3), utilizing the Banach FPT. The Banach FPT serves as a cornerstone in the theoretical framework for proving the EU of solutions for FNSDEs. If the conditions of the Banach FPT are satisfied in the context of FNSDEs, it implies that the mapping defined by the FNSDE is a contraction mapping, which, in turn, ensures the EU of solutions. Therefore, verifying the conditions of the Banach FPT confirms that the mapping defined by the FNSDE is a contraction mapping.

Theorem 1. If conditions (}\mathcal{J}_1\) and (}\mathcal{J}_2\) hold true, then Equation (3) possesses a sole solution through satisfying the subsequent condition:

$$\Lambda = \left(3^{p-1} \mathcal{O}_1^p + 3^{p-1} \mathcal{O}_2^p \left(\frac{\ell^{1-\eta}}{\Gamma(\eta)}\right)^p \left(\frac{\mu^{(p-1)}\gamma}{\ell}\right)^{p-1} \left(\frac{p-1}{2p-1}\right)^{p-1}\right)^{p-1}.$$
\[ + \left( \frac{\ell_{1-\eta}}{\Gamma(\eta)} \right)^p \frac{3^{p-1} \theta_p}{\nu} \left( \frac{\mu_{p-1}}{2(p-1)^{p-1}} \right)^p, \]  

The value of \( \Lambda \) is non-negative and satisfies the condition \( \Lambda < 1 \).

**Proof.** We establish an operator \( \mathcal{D} : \mathbb{U} \to \mathbb{U} \) by \( \mathcal{H}(0) = \mathcal{H}_0 \) and the subsequent equality is valid.

\[
\mathcal{D}(\mathcal{H}(\mu)) = \mathcal{H}_0 - \mathcal{D}(0, \mathcal{H}_0) + \mathcal{D}(\mu, \mathcal{H}(\mu)) \\
+ \frac{\ell_{1-\eta}}{\Gamma(\eta)} \int_0^\mu \varsigma_{\ell_1} - (\ell_{1-\eta})^\eta - 2 \mathcal{L}_1(\xi, \mathcal{H}(\xi))d\xi \\
+ \frac{\ell_{1-\eta}}{\Gamma(\eta)} \int_0^\mu \varsigma_{\ell_1} - (\ell_{1-\eta})^\eta - 2 \mathcal{L}_2(\xi, \mathcal{H}(\xi))dW_\xi. \tag{6}
\]

**Step 1:** Initially, we will show that \( \mathcal{D} \) transforms \( \mathbb{U} \) into itself. Let \( \mathcal{H}(\mu) \) be an arbitrary element of \( \mathbb{U} \), where \( \mu \) belongs to the interval \( [0, \Theta] \). We establish this for all \( \mu \) using the definition of \( \mathcal{D}(\mathcal{H}(\mu)) \) as outlined in Equation (6) and Je-In. So, we obtain the following result:

\[
\Xi \left[ \| \mathcal{D}(\mathcal{H}(\mu)) \|_p \right] \leq 4^{p-1} \Xi \left[ \| \mathcal{H}_0 \|_p \right] \\
+ 4^{p-1} \Xi \left[ \| \mathcal{D}(\mu, \mathcal{H}(\mu)) - \mathcal{D}(0, \mathcal{H}_0) \|_p \right] \\
+ \left( \frac{\ell_{1-\eta}}{\Gamma(\eta)} \right)^p \Xi \left[ \int_0^\mu \varsigma_{\ell_1} - (\ell_{1-\eta})^\eta - 2 \mathcal{L}_1(\xi, \mathcal{H}(\xi))d\xi \right]_p^p \\
+ 4^{p-1} \left( \frac{\ell_{1-\eta}}{\Gamma(\eta)} \right)^p \Xi \left[ \int_0^\mu \varsigma_{\ell_1} - (\ell_{1-\eta})^\eta - 2 \mathcal{L}_2(\xi, \mathcal{H}(\xi))dW_\xi \right]_p^p \tag{7}
\]

By employing (3.1), we achieve the following:

\[
\mathcal{B}_2 = 4^{p-1} \Xi \left[ \| \mathcal{D}(\mu, \mathcal{H}(\mu)) - \mathcal{D}(0, \mathcal{H}_0) \|_p \right] \\
\leq 2 \cdot 4^{p-1} \Xi \left[ \| \mathcal{H}_0 \|_p \right]. \tag{8}
\]

By utilizing Hö-In, Je-In, and condition (3.3), we derive the following results:

\[
\mathcal{B}_3 = 4^{p-1} \left( \frac{\ell_{1-\eta}}{\Gamma(\eta)} \right)^p \Xi \left[ \int_0^\mu \varsigma_{\ell_1} - (\ell_{1-\eta})^\eta - 2 \mathcal{L}_1(\xi, \mathcal{H}(\xi))d\xi \right]_p^p \\
\leq 4^{p-1} \left( \frac{\ell_{1-\eta}}{\Gamma(\eta)} \right)^p \left( \int_0^\mu \varsigma_{\ell_1} - (\ell_{1-\eta})^\eta - 2 \mathcal{L}_1(\xi, \mathcal{H}(\xi))d\xi \right) \Xi \left[ \int_0^\mu \| \mathcal{L}_1(\xi, \mathcal{H}(\xi)) \|_p d\xi \right] \\
\leq 4^{p-1} \left( \frac{\ell_{1-\eta}}{\Gamma(\eta)} \right)^p \mathcal{B}_3 \left( \frac{\mu_{p-1}}{\mu_{p-1}} \right) \Xi \left[ \mathcal{B}_3 \left( \frac{\mu_{p-1}}{\mu_{p-1}} \right) \mu \Xi \left[ \int_0^\mu \| \mathcal{L}_1(\xi, \mathcal{H}(\xi)) \|_p d\xi \right] \right] \\
\leq 8^{p-1} \mu \left( \frac{\ell_{1-\eta}}{\Gamma(\eta)} \right)^p \mathcal{B}_3 \left( \frac{\mu_{p-1}}{\mu_{p-1}} \right) \Xi \left[ \mathcal{B}_3 \left( \frac{\mu_{p-1}}{\mu_{p-1}} \right) \mu \Xi \left[ \int_0^\mu \| \mathcal{L}_1(\xi, \mathcal{H}(\xi)) \|_p d\xi \right] \right]. \tag{9}
\]

Utilizing BDG-In, Je-In, and condition (3.3), we obtain

\[
\mathcal{B}_4 \leq 4^{p-1} \left( \frac{\ell_{1-\eta}}{\Gamma(\eta)} \right)^p \Xi \left[ \sup_{\mu \in [0, \Theta]} \int_0^\mu \varsigma_{\ell_1} - (\ell_{1-\eta})^\eta - 2 \mathcal{L}_2(\xi, \mathcal{H}(\xi))dW_\xi \right]_p^p \\
\leq \left( \frac{\mu_{p-1}}{2(p-1)^{p-1}} \right)^p \left( \frac{\ell_{1-\eta}}{\Gamma(\eta)} \right)^p \Xi \left[ \int_0^\mu \varsigma_{\ell_1} - (\ell_{1-\eta})^\eta - 2 \mathcal{L}_2(\xi, \mathcal{H}(\xi))dW_\xi \right]_p^p \right]^{\frac{1}{p}} 4^{p-1} \tag{10}
\]
\[
\leq 4^{p-1} \left( \frac{\ell - 1}{\Gamma(\eta)} \right)^p \left( \frac{\mu (2\eta - 1)}{\ell} \right)^p \Gamma_3 \left( \frac{(2p - 1)^1 - p \eta}{\eta - 1} \right)^p \Xi \left( 1 + \Xi \| \mathcal{H} \| \right)^p
\]

\[
\leq 8^{p-1} \Gamma_3 \left( \frac{\ell - 1}{\Gamma(\eta)} \right)^p \left( 2(2p - 1)^1 - p \eta \right)^p \\left( \frac{\mu (2\eta - 1)}{\ell} \right)^p \Xi \left( 1 + \Xi \| \mathcal{H} \| \right)^p.
\]

By utilizing Equations (8)–(10) in (7), we achieve the following outcomes:

\[
\Xi \| \mathcal{D}(\mathcal{H}(\mu)) \| ^p \leq 4^{p-1} \Xi \| \mathcal{H} \| ^p + 2 \cdot 4^{p-1} \Gamma_3 \Xi \| \mathcal{H} \| ^p +
\]

\[
8^{p-1} \mu \left( \frac{\ell - 1}{\Gamma(\eta)} \right)^p \left( \frac{\mu (2\eta - 1)}{\ell} \right)^p \left( \frac{p - 1}{\eta - 1} \right)^p \Xi \left( 1 + \Xi \| \mathcal{H} \| \right)^p +
\]

\[
8^{p-1} \Gamma_3 \left( \frac{\ell - 1}{\Gamma(\eta)} \right)^p \left( 2(2p - 1)^1 - p \eta \right)^p \\left( \frac{\mu (2\eta - 1)}{\ell} \right)^p \Xi \left( 1 + \Xi \| \mathcal{H} \| \right)^p.
\]

Combining the insights from the preceding discussion, it becomes evident that the following condition is fulfilled by a constant \( \mathcal{V} \).

\[
\Xi \| \mathcal{D}(\mathcal{H}(\mu)) \| ^p \leq \mathcal{V} (1 + \Xi \| \mathcal{H} \| ^p).
\]

In other words, \( \mathcal{D} \) maps \( \mathcal{U} \) into \( \mathcal{U} \).

**Step 2:** Next, we aim to demonstrate that the mapping \( \mathcal{D} \) is contractive. To do so, consider arbitrary functions \( \mathcal{H}(\mu) \) and \( \mathcal{E}(\mu) \). From Equation (6) and Je-In, we derive the following for all \( \mu \) within the interval \([0, \Theta]\):

\[
\Xi \| \mathcal{D}(\mathcal{H}(\mu)) - \mathcal{D}(\mathcal{E}(\mu)) \| ^p \leq \Xi \| \mathcal{H}(\mu) - \mathcal{E}(\mu) \| ^p +
\]

\[
3^{p-1} \Xi \| \mathcal{D}(\mathcal{H}(\mu)) - \mathcal{H}(\mu) \| ^p + \Xi \| \mathcal{H}(\mu) - \mathcal{E}(\mu) \| ^p +
\]

\[
3^{p-1} \sup_{\mu \in [0, \Theta]} \Xi \| \mathcal{H}(\mu) - \mathcal{E}(\mu) \| ^p.
\]

By utilizing Hö-In and condition \((\mathcal{Z}_2)\), we acquire

\[
\Xi \| \mathcal{D}(\mathcal{H}(\mu)) - \mathcal{H}(\mu) \| ^p \leq \Xi \| \mathcal{D}(\mathcal{H}(\mu)) - \mathcal{D}(\mathcal{E}(\mu)) \| ^p +
\]

\[
\Xi \| \mathcal{H}(\mu) - \mathcal{E}(\mu) \| ^p + \Xi \| \mathcal{H}(\mu) - \mathcal{E}(\mu) \| ^p +
\]

\[
\Xi \| \mathcal{H}(\mu) - \mathcal{E}(\mu) \| ^p.
\]

However, using \((\mathcal{Z}_2)\) and BDG-In, we have

\[
\Xi \| \mathcal{D}(\mathcal{H}(\mu)) - \mathcal{H}(\mu) \| ^p \leq \Xi \| \mathcal{D}(\mathcal{H}(\mu)) - \mathcal{D}(\mathcal{E}(\mu)) \| ^p +
\]

\[
\Xi \| \mathcal{H}(\mu) - \mathcal{E}(\mu) \| ^p + \Xi \| \mathcal{H}(\mu) - \mathcal{E}(\mu) \| ^p +
\]

\[
\Xi \| \mathcal{H}(\mu) - \mathcal{E}(\mu) \| ^p.
\]
\[
\leq 3^{p-1} \left( \frac{\ell^{1-\eta}}{\Gamma(\eta)} \right)^p \left( \frac{(p)_{p+1}}{2(p-1)^{p-1}} \right) \sup_{\mu \in [0,1]} \left[ \| \mathcal{H}(\mu) - \mathcal{E}(\mu) \|_P \right] \]

By utilizing Equations (13)–(15) in (12), we extract the following outcomes:

\[
\left[ \| \mathcal{H}(\mu) - \mathcal{E}(\mu) \|_P \right] \leq 3^{p-1} \left( \frac{\ell^{1-\eta}}{\Gamma(\eta)} \right)^p \left( \frac{(p)_{p+1}}{2(p-1)^{p-1}} \right) \sup_{\mu \in [0,1]} \left[ \| \mathcal{H}(\mu) - \mathcal{E}(\mu) \|_P \right]
\]

So, from above, we have the required result as follows:

\[
\left[ \| \mathcal{H}(\mu) - \mathcal{E}(\mu) \|_P \right] \leq \Lambda \| \mathcal{H}(\mu) - \mathcal{E}(\mu) \|_P.
\]

We obtain \( \Lambda < 1 \) from Equation (5), indicating that the operator \( \mathcal{D} \) acts as a contractive mapping. Consequently, there exists a unique fixed point \( \mathcal{H}(\mu) \in U \) for this map. \( \square \)

In the context of FNSDEs, proving that a certain constant is less than one is crucial for applying the Banach FPT to establish the EU of solutions. The importance of the constant being less than one lies in its role in guaranteeing the contraction property necessary for the theorem’s application. When this condition holds, the Banach FPT guarantees the EU of a fixed point for the mapping. Specifically, it ensures that the mapping defined by the system is a contraction mapping, wherein the function contracts distances between points in the underlying space. Mathematically, this implies that for any two points in the function’s domain, the distance between their images under the function is always less than the distance between the original points. In summary, ensuring that the constant in Equation (5) is less than one is essential because it establishes the contraction property necessary for applying the Banach FPT. This, in turn, guarantees the EU of solutions to the considered system, providing a powerful mathematical tool for analysis and computation.

### Averaging Principle Result

In this section, we focus on establishing the result of the averaging principle for FNSDEs within the framework of the \( p \)th space in the sense of CKFD. By using CKFD, we extended the outcome of the averaging principle that was developed in the context of CFD \([54]\). By adjusting \( \ell = 1 \) in our established result, we are able to obtain the desired outcome. The standard form of the FNSDEs is established as follows:

\[
\mathcal{H}_c(\mu) = \mathcal{H}_0 - \mathcal{D}(0, \mathcal{H}_0) + \mathcal{D}(\mu, \mathcal{H}_c(\mu)) \\
+ \varepsilon \frac{\ell^{1-\eta}}{\Gamma(\eta)} \int_0^\mu \xi^{\ell-1} (\mu' - \xi^{\ell-1-\eta} \mathcal{F}_2(\xi, \mathcal{H}_c(\xi)) \, d\xi)
\]

\[
+ \sqrt{\varepsilon} \frac{\ell^{1-\eta}}{\Gamma(\eta)} \int_0^\mu \xi^{\ell-1} (\mu' - \xi^{\ell-1-\eta} \mathcal{F}_2(\xi, \mathcal{H}_c(\xi)) \, d\mathcal{H}_c(\xi),
\]

\(^{(17)}\)
where $\epsilon \in (0, \epsilon_0]$ denotes a small positive parameter, with $\epsilon_0$ being a fixed point. Moreover, $\mathcal{D}$, $\mathcal{Z}_1$, and $\mathcal{Z}_2$ meet the conditions specified by $(\mathcal{J}_1)$, $(\mathcal{J}_2)$, and $(\mathcal{J}_3)$. As a result, the averaged representation of Equation (17) is provided below.

$$
\mathcal{H}_\epsilon^a(\mu) = \mathcal{H}_0 - \mathcal{D}(0, \mathcal{H}_0) + \mathcal{D}(\mu, \mathcal{H}_\epsilon^a(\mu))
+ \epsilon \frac{\ell^{1-\eta}}{\Gamma(\eta)} \int_0^\mu \xi^{\ell-1}(\mu^{\ell} - \xi^{\ell})^{1-\eta} \mathcal{Z}_1'(\xi, \mathcal{H}_\epsilon^c(\xi))d\xi
+ \sqrt{\epsilon} \frac{\ell^{1-\eta}}{\Gamma(\eta)} \int_0^\mu \xi^{\ell-1}(\mu^{\ell} - \xi^{\ell})^{1-\eta} \mathcal{Z}_2'(\xi, \mathcal{H}_\epsilon^c(\xi))d\xi, \quad (18)
$$

where $\mathcal{Z}_1 : \mathbb{R}^q \to \mathbb{R}^q$, $\mathcal{Z}_2 : \mathbb{R}^q \to \mathbb{R}^q \times \mathbb{R}$.

**Lemma 2.** Given any $\Theta_1$ belonging to the interval $[0, \Theta]$, we can establish the subsequent growth conditions for $\mathcal{Z}_2$ by employing assumptions $(\mathcal{J}_3)$, $(\mathcal{J}_4)$:

$$
\|\mathcal{Z}_2(\varphi)\|^p \leq Q_4(1 + \|\varphi\|^p),
$$

where $Q_4 = (2^{p-1} \mathcal{H}_1(\Theta_1 + Q_3^p))$.

**Proof.** Taking into account Je-In and assumptions $(\mathcal{J}_3)$, $(\mathcal{J}_4)$, we obtain the following outcome:

$$
\|\mathcal{Z}_2(\varphi, \sigma)\|^p \leq 2^{p-1} \frac{1}{\Theta_1} \int_0^{\Theta_1} \|\mathcal{Z}_2(\mu, \varphi) - \mathcal{Z}_2(\varphi)\|^p d\mu + 2^{p-1} \frac{1}{\Theta_1} \int_0^{\Theta_1} \|\mathcal{Z}_2(\mu, \varphi)\|^p d\mu
\leq 2^{p-1} \mathcal{H}_1(\Theta_1)(1 + \|\varphi\|^p) + 2^{p-1} Q_3^p (1 + \|\varphi\|^p)
\leq 2^{p-1} \left(\mathcal{H}_1(\Theta_1) + Q_3^p\right)(1 + \|\varphi\|^p).
$$

**Theorem 2.** Assume that conditions $(\mathcal{J}_1)$ to $(\mathcal{J}_4)$ are satisfied. For any arbitrarily small value $\Theta > 0$ and $\varphi \in [2, (1 - \eta)^{-1}]$, there exist $F > 0$, $\epsilon_1 \in (0, \epsilon_0)$, $\omega \in (0, 1)$ such that

$$
\mathbb{D}\left[\sup_{\mu \in [0, F]} \|\mathcal{H}_\epsilon(\mu) - \mathcal{H}_\epsilon^c(\mu)\|^p\right] \leq \mathbb{D}, \quad \epsilon \in (0, \epsilon_1]. \quad (19)
$$

**Proof.** For any $\mu \in [0, \Theta]$, we obtain the following result using Equations (17) and (18).

$$
\mathcal{H}_\epsilon(\mu) - \mathcal{H}_\epsilon^c(\mu) = \mathcal{D}(\mu, \mathcal{H}_\epsilon(\mu)) - \mathcal{D}(\mu, \mathcal{H}_\epsilon^c(\mu))
+ \epsilon \frac{\ell^{1-\eta}}{\Gamma(\eta)} \int_0^\mu \xi^{\ell-1}(\mu^{\ell} - \xi^{\ell})^{1-\eta} \mathcal{Z}_1'(\xi, \mathcal{H}_\epsilon(\xi), \mathcal{H}_\epsilon^c(\xi))d\xi
+ \sqrt{\epsilon} \frac{\ell^{1-\eta}}{\Gamma(\eta)} \int_0^\mu \xi^{\ell-1}(\mu^{\ell} - \xi^{\ell})^{1-\eta} \mathcal{Z}_2'(\xi, \mathcal{H}_\epsilon(\xi), \mathcal{H}_\epsilon^c(\xi))d\xi. \quad (20)
$$

When $N \in (0, 1)$, $\beta_1, \beta_2 \in \mathbb{R}^q, \varphi \geq 2$, we have

$$
\|\beta_1 + \beta_2\|^p \leq \frac{\|\beta_1\|^p}{N^{p-1}} + \frac{\|\beta_2\|^p}{(1 - N)^{p-1}}. \quad (21)
$$

Let $N = \Theta_1$. Employing Equation (20) in Equation (21), and subsequently utilizing $(\mathcal{J}_1)$ and Je-In, yields the following outcome:

$$
\|\mathcal{H}_\epsilon(\mu) - \mathcal{H}_\epsilon^c(\mu)\|^p \leq \Theta_1^{1-p} \|\mathcal{D}(\mu, \mathcal{H}_\epsilon(\mu)) - \mathcal{D}(\mu, \mathcal{H}_\epsilon^c(\mu))\|^p
+ \frac{2^{p-1}}{(1 - \Theta_1)^{p-1}} \left[\frac{\ell^{1-\eta}}{\Gamma(\eta)} \int_0^\mu \xi^{\ell-1}(\mu^{\ell} - \xi^{\ell})^{1-\eta} \left(\mathcal{Z}_1'(\xi, \mathcal{H}_\epsilon(\xi), \mathcal{H}_\epsilon^c(\xi)) - \mathcal{Z}_1'(\mathcal{H}_\epsilon^c(\xi))\right)d\xi\right]^p.
$$
\[
+ \frac{2^{p-1}}{(1 - \Theta_1)^{p-1}} \left( \frac{\ell^1 - \eta}{\Gamma(\eta)} \right)^p \left( \int \xi^2 \left( \frac{u^p(\ell - 1) - (u^p - 1)(\ell - 1)^2}{\ell^2} \right)^{p-1} \right)
\]

\[
\leq \Theta_1 \left[ \mathcal{H}(\xi) - \mathcal{H}^*_{\xi}(\xi) \right] \right]^{\mathbb{P}}
\]

Utilizing Equation (23) in Equation (19).

Utilizing Equation (23) in Equation (19).

\[
\mathcal{V}_1 \leq \frac{2^{p-2}}{(1 - \Theta_1)^{p-1}} \left( \frac{\ell^1 - \eta}{\Gamma(\eta)} \right)^p \left[ \sup_{0 < u \leq \xi} \left[ \mathcal{H}(\xi) - \mathcal{H}^*_{\xi}(\xi) \right]^{\mathbb{P}} \right]
\]

Applying Hö-In and (32) to (23) yields the following result:

\[
\mathcal{V}_1 \leq (2^{p-2} - 2p) \left[ \frac{\ell^1 - \eta}{\Gamma(\eta)} \right] \left[ \sup_{0 < u \leq \xi} \left[ \mathcal{H}(\xi) - \mathcal{H}^*_{\xi}(\xi) \right]^{\mathbb{P}} \right]
\]

where

\[
\mathcal{V}_{21} = \frac{2^{3p-3} \Theta_2}{(1 - \Theta_1)^{p-1} (1 - 2^{p-1} \Theta_1)} \left( \frac{p - 1}{\ell \eta - 1} \right)^{p-1} \left( \frac{\ell^1 - \eta}{\Gamma(\eta)} \right)^{p-1}
\]

Applying Hö-In and (34) to (22) yields the following result:
\[\mathcal{V}_2 \leq \frac{2^{p-2}}{(1 - \Theta_1)^p} \left( \int_0^u \frac{(\ell - \xi^{p-1})(u - \xi^p)}{\ell^{p-1}} \, d\xi \right)^p \left( \ell^\frac{1-\eta}{\Gamma(\eta)} \right)^p \]

\[\mathcal{V}_2 \leq \frac{2^{p-2} u^{p-1}}{(1 - \Theta_1)^p} \left( \int_0^{\mu} \frac{\xi^{2p-2}(u - \xi^{p-1})^2}{\ell^{p-1}} \, d\xi \right)^p \left( \ell^\frac{1-\eta}{\Gamma(\eta)} \right)^p \]

By employing Je-In, \(\mathcal{V}_2\) yields the following:

\[\mathcal{V}_2 \leq \frac{2^{p-2} \mathcal{X}_2(u) (1 + \Xi \|H_{e}^{\ast}(\xi)\|^p)}{(1 - \Theta_1)^p} \left( -\frac{p - 1}{\ell (\eta - 1)^p} \right) \left( \ell \frac{1-\eta}{\Gamma(\eta)} \right)^p \]

By employing (22), Hö-In, and BDG-in on \(\mathcal{V}_{21}\), we arrive at the following conclusions:

\[\mathcal{V}_{21} \leq \frac{2^{p-2}}{(1 - \Theta_1)^p} \left( \int_0^u \xi^{(\ell - 1)}(u - \xi^p)^{\eta - 1} \, d\xi \right)^p \left( \ell^\frac{1-\eta}{\Gamma(\eta)} \right)^p \]

\[\mathcal{V}_{22} \leq \frac{2^{p-3}}{(1 - \Theta_1)^p} \left( \int_0^u \xi^{(\ell - 1)}(u - \xi^p)^{(\eta - 1)p} \, d\xi \right)^p \left( \ell^\frac{1-\eta}{\Gamma(\eta)} \right)^p \]

Again using (22), Hö-In, and BDG-In on \(\mathcal{V}_{22}\), we arrive at the following conclusions:
\[\left(\left\| \mathcal{L}_2(\xi, \mathcal{H}_e^*(\xi), \right\|_p + \left\| \mathcal{L}_2(\mathcal{H}_e^*(\xi)) \right\|_p^p \right)^{1/\ell} \leq \left( \frac{\ell_1 - \eta}{\Gamma(\eta)} \right)^p \frac{2^p - 4 - p}{(1 - \mathcal{G}_3 + \mathcal{G}_4)^p} \left( \frac{1}{(\ell - 1)p + 1} \right) (2^{-1}(p - 1)^{1-p}p^{p-1})^p (1 + \mathbb{E}\left[\left\| \mathcal{H}_e^*(\xi) \right\|_p^p \right]) \right) \]
\[= \mathcal{X}_{22}^\mathcal{G} u^p \eta^p \mathcal{Z}, \quad \text{where} \]
\[\mathcal{X}_{22} = \left( \frac{\ell_1 - \eta}{\Gamma(\eta)} \right)^p \frac{1}{1 - \mathcal{G}_3} \mathcal{X}_{22}^\mathcal{G} u^p \eta^p \mathcal{Z} \quad \text{and} \quad \mathcal{Z} = \left( \frac{1}{1 - \mathcal{G}_4} \right)^p \frac{1}{(\ell - 1)p + 1} (2^{-1}(p - 1)^{1-p}p^{p-1})^p (1 + \mathbb{E}\left[\left\| \mathcal{H}_e^*(\xi) \right\|_p^p \right]). \quad (30)\]

By employing Equations (25) through (30) in Equation (24), consequently, we obtain the following results:

\[\mathbb{E}\left[\sup_{\xi \in \mathbb{L}} \left\| \mathcal{H}_e(\xi) - \mathcal{H}_e^*(\xi) \right\|^p \right] \leq \mathcal{X}_{22}^\mathcal{G} u^p \eta^p + \mathcal{X}_{22}^\mathcal{G} u^p \eta^p \mathcal{Z} \quad \text{and} \quad \sup_{\xi \in \mathbb{L}} \left\| \mathcal{H}_e(\xi) - \mathcal{H}_e^*(\xi) \right\|^p \mathcal{D} = \left( \mathcal{X}_{22}^\mathcal{G} u^p \eta^p + \mathcal{X}_{22}^\mathcal{G} u^p \eta^p \mathcal{Z} \right) \exp \left( \frac{2^p - 4 - p}{(1 - \mathcal{G}_3 + \mathcal{G}_4)^p} (1 + \mathbb{E}\left[\left\| \mathcal{H}_e^*(\xi) \right\|_p^p \right]) \right). \quad (31)\]

Based on the GB-In condition, we obtain the following:

\[\mathbb{E}\left[\sup_{\xi \in \mathbb{L}} \left\| \mathcal{H}_e(\xi) - \mathcal{H}_e^*(\xi) \right\|^p \right] \leq \mathcal{X}_{22}^\mathcal{G} u^p \eta^p + \mathcal{X}_{22}^\mathcal{G} u^p \eta^p \mathcal{Z} \quad \text{and} \quad \sup_{\xi \in \mathbb{L}} \left\| \mathcal{H}_e(\xi) - \mathcal{H}_e^*(\xi) \right\|^p \mathcal{D} \leq \mathcal{X}_{22}^\mathcal{G} u^p \eta^p \mathcal{Z} \exp \left( \frac{2^p - 4 - p}{(1 - \mathcal{G}_3 + \mathcal{G}_4)^p} (1 + \mathbb{E}\left[\left\| \mathcal{H}_e^*(\xi) \right\|_p^p \right]) \right). \quad (32)\]

So, \(\forall \eta > 0\), when \(\varepsilon_1 \in (0, \varepsilon_0)\) that \(\forall \varepsilon \in (0, \varepsilon_1)\) and \(\mu \in [0, F - \varepsilon^\omega]\) give the following result:

\[\mathbb{E}\left[\sup_{\xi \in \mathbb{L}} \left\| \mathcal{H}_e(\xi) - \mathcal{H}_e^*(\xi) \right\|^p \right] \leq \mathcal{X}_{22}^\mathcal{G} u^p \eta^p \mathcal{Z} \exp \left( \frac{2^p - 4 - p}{(1 - \mathcal{G}_3 + \mathcal{G}_4)^p} (1 + \mathbb{E}\left[\left\| \mathcal{H}_e^*(\xi) \right\|_p^p \right]) \right). \]

\[\Box\]

Our result matches the result established in [54] when we substituted \(\ell = 1\) in Equation (32).

**Corollary 1.** Suppose that the conditions stated in (31) and (34) hold true. Considering \(\mathcal{X}_1 > 0\), when \(\omega \in (0, 1), F > 0\) and \(\varepsilon_1 \in (0, \varepsilon_0)\) for all \(\varepsilon \in (0, \varepsilon_1)\), we have

\[\lim_{\varepsilon \to 0} \mathcal{P}\left( \sup_{\mu \in [0, F - \varepsilon^\omega]} \left\| \mathcal{H}_e(\xi) - \mathcal{H}_e^*(\xi) \right\| > \mathcal{X}_1 \right) = 0. \quad (33)\]

**Proof.** Combining Theorem 2 with CM-In, the following can be established for \(\mathcal{X}_1\).

\[\mathcal{P}\left( \sup_{\mu \in [0, F - \varepsilon^\omega]} \left\| \mathcal{H}_e(\xi) - \mathcal{H}_e^*(\xi) \right\| > \mathcal{X}_1 \right) \leq \frac{1}{\mathcal{X}_1} \mathbb{E}\left[\sup_{\mu \in [0, F - \varepsilon^\omega]} \left\| \mathcal{H}_e(\xi) - \mathcal{H}_e^*(\xi) \right\|^2 \right] \leq \frac{\varepsilon_1 - \varepsilon}{\mathcal{X}_1} \leq 0 \text{ as } \varepsilon \to 0,\]
To derive the average formulation related to Equation (34), the subsequent expressions denote the averages of $\varepsilon$. Hence, all conditions outlined in Theorem 2 are satisfied. Consequently, in the limit as $\varepsilon \to 0$, the original solution $\mathcal{H}_e(\mu)$ and the averaged solution $\mathcal{H}_e^*(\mu)$ are equivalent in the $\mathbb{L}^p$ sense.

4. Examples

The average behavior of a complex system can be derived using the principles of averaging, as demonstrated in the following two numerical examples:

Example 1. Consider the following FrNSDE:

$$
\begin{align*}
\mathcal{D}_\mu^{0,8,1.5} & \left( \mathcal{H}_e(\mu) - \frac{1}{2} \mu^\frac{1}{3} - \sin^4(\mu) \mathcal{H}_e(\mu) \right) = \varepsilon \left( 4 \sin^2(\mu) \cos(\mathcal{H}_e(\mu)) \right) \\
+ & \sqrt{\varepsilon} \mathcal{H}_e(\mu) \frac{dW_\mu}{d\tau}, \mu \in [0, \Theta], \\
\mathcal{H}(0) & = 0.2.
\end{align*}
$$

(34)

Clearly, the conditions of Theorem 1 are satisfied by Equation (34). Therefore, the solution to Equation (34) exists and is unique. Based on the aforementioned system, we obtain the following:

$$
\begin{align*}
\mathcal{D}(\mu, \mathcal{H}(\mu)) & = \mathcal{H}_e(\mu) - \frac{1}{2} \mu^\frac{1}{3} - \sin^4(\mu) \mathcal{H}_e(\mu), \\
\mathcal{D}_1(\mu, \mathcal{H}(\mu)) & = 4 \sin^2(\mu) \cos(\mathcal{H}_e(\mu)), \\
\mathcal{D}_2(\mu, \mathcal{H}(\mu)) & = \mathcal{H}_e(\mu).
\end{align*}
$$

The subsequent expressions denote the averages of $\mathcal{D}_1$ and $\mathcal{D}_2$:

$$
\begin{align*}
\mathcal{F}_1(\mathcal{H}_e(\mu)) & = \frac{1}{\pi} \int_0^\pi \left( 4 \sin^2(\mu) \cos(\mathcal{H}_e(\mu)) \right) d\mu \\
& = 2 \cos(\mathcal{H}_e^*(\mu)), \\
\mathcal{F}_2(\mathcal{H}_e(\mu)) & = \frac{1}{\pi} \int_0^\pi \mathcal{H}_e(\mu) d\mu \\
& = \mathcal{H}_e^*(\mu).
\end{align*}
$$

To derive the average formulation related to Equation (34), substitute the simplified solution $\mathcal{H}_e^*(\mu)$ for the original solution $\mathcal{H}_e(\mu)$. Consequently, the simplified averaged equation is expressed as follows:

$$
\begin{align*}
\mathcal{D}_\mu^{0,8,1.5} & \left( \mathcal{H}_e^*(\mu) - \frac{1}{2} \mu^\frac{1}{3} - \sin^4(\mu) \mathcal{H}_e^*(\mu) \right) = \varepsilon \left( 2 \cos(\mathcal{H}_e^*(\mu)) \right) \\
+ & \sqrt{\varepsilon} \mathcal{H}_e^*(\mu) \frac{dW_\mu}{d\tau}, \mu \in [0, \Theta], \\
\mathcal{H}(0) & = 0.2.
\end{align*}
$$

(35)

Hence, all conditions outlined in Theorem 2 are satisfied. Consequently, in the limit as $\varepsilon \to 0$, the original solution $\mathcal{H}_e(\mu)$ and the averaged solution $\mathcal{H}_e^*(\mu)$ are equivalent in the $\mathbb{L}^p$ sense.

Example 2. Consider the subsequent FNSDE:

$$
\begin{align*}
\mathcal{D}_\mu^{0.9,1.3} & \left( \mathcal{H}_e(\mu) - \frac{1}{2} \mu^\frac{1}{3} - \cos^3(\mu) \mathcal{H}_e(\mu) \right) = \varepsilon \left( \frac{1}{4} \sin(\mathcal{H}_e(\mu)) \right) \\
+ & \sqrt{\varepsilon} \cos^2(\mu) \mathcal{H}_e(\mu) \frac{dW_\mu}{d\tau}, \mu \in [0, \Theta], \\
\mathcal{H}(0) & = 0.4.
\end{align*}
$$

(35)
It is clear that Equation (35) satisfies the requirements of Theorem 1. Consequently, the solution to Equation (35) exists and is unique. Based on the aforementioned system, we obtain the following:

\[ \mathcal{D}(\mu, \mathcal{H}(\mu)) = \mathcal{H}_e(\mu) - \frac{1}{3} \mu^2 - \cos^3(\mu) \mathcal{H}_e(\mu), \]

\[ \mathcal{L}_1(\mu, \mathcal{H}(\mu)) = \frac{1}{4} \sin(\mathcal{H}(\mu)), \]

\[ \mathcal{L}_2(\mu, \mathcal{H}(\mu)) = \cos^2(\mu) \mathcal{H}_e(\mu). \]

The subsequent expressions denote the averages of \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \):

\[ \mathcal{F}_1(\mathcal{H}(\mu)) = \frac{1}{\pi} \int_0^\pi \left( \frac{1}{4} \sin(\mathcal{H}(\mu)) \right) d\mu = \frac{1}{4} \sin(\mathcal{H}_e^*(\mu)), \]

\[ \mathcal{F}_2(\mathcal{H}(\mu)) = \frac{1}{\pi} \int_0^\pi \cos^2(\mu) \mathcal{H}_e(\mu) d\mu = \frac{1}{2} \mathcal{H}_e^*(\mu). \]

To derive the average formulation related to Eq. (35), substitute the simplified solution \( \mathcal{H}_e^*(\mu) \) for the original solution \( \mathcal{H}_e(\mu) \). Consequently, the simplified averaged equation is expressed as follows:

\[
\begin{cases}
\mathcal{P}^{0.9,1.3}_\mu \left( \mathcal{H}_e^*(\mu) - \frac{1}{12} \mu^2 - \cos^3(\mu) \mathcal{H}_e^*(\mu) \right) = \epsilon \left( \frac{1}{4} \sin(\mathcal{H}_e^*(\mu)) \right) \\
+ \frac{1}{4} \sqrt{\mathcal{H}_e^*(\mu)} \frac{d\mathcal{H}_e}{d\mu}, \quad \mu \in [0, \Theta], \\
\mathcal{H}(0) = 0.4.
\end{cases}
\]

Hence, all conditions outlined in Theorem 2 are satisfied. Consequently, in the limit as \( \epsilon \to 0 \), the original solution \( \mathcal{H}_e(\mu) \) and the averaged solution \( \mathcal{H}_e^*(\mu) \) are equivalent in the \( \mathbb{L}^p \) sense.

5. Conclusions

In this research, we have established the results of the EU and the averaging principle for FNSDEs. We have extended these results for \( p = 2 \) and CFD by establishing them within the \( \mathbb{L}^p \) space and within the framework of CKFD. The contraction mapping concept has been applied to explore the EU of the problem under discussion. Moreover, we have illustrated the averaging principle for FNSDEs using GB-In, CM-In, Je-In, BDG-In, Hö-In, and an interval translation approach. Ultimately, two examples have been provided to aid in understanding the established results and to demonstrate the effectiveness of our findings.

In our upcoming research, we aim to utilize numerical methods to address various real-world challenges modeled by FNSDEs.


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