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# On Summations of Generalized Hypergeometric Functions with Integral Parameter Differences

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**Abstract:** In this paper, we present an extension of the Karlsson–Minton summation formula for a generalized hypergeometric function with integral parameter differences. Namely, we extend one single negative difference in Karlsson–Minton formula to a finite number of integral negative differences, some of which will be repeated. Next, we continue our study of the generalized hypergeometric function evaluated at unity and with integral positive differences (IPD hypergeometric function at the unit argument). We obtain a recurrence relation that reduces the IPD hypergeometric function at the unit argument to ${}_4F_3$. Finally, we note that Euler–Pfaff-type transformations are always based on summation formulas for finite hypergeometric functions, and we give a number of examples.

**Keywords:** generalized hypergeometric function; summation formulas; hypergeometric identity; Miller–Paris transformations; Euler–Pfaff type transformations

**MSC:** 33C20

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## 1. Introduction

Let us fix some notation and terminology first. The standard symbols $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ will be used to denote the natural, real and complex numbers, respectively; $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $\mathbf{a} = (a_1, \dots, a_p) \in \mathbb{C}^p$, $\mathbf{n} = (n_1, \dots, n_p) \in \mathbb{N}^p$, we define

$$\Gamma(\mathbf{a}) = \Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_p), \quad (\mathbf{a})_{\mathbf{n}} = (a_1)_{n_1}(a_2)_{n_2}\cdots(a_p)_{n_p},$$
$$\mathbf{a} + \mu = (a_1 + \mu, a_2 + \mu, \dots, a_p + \mu), \quad \mathbf{a} > 0 \Leftrightarrow a_k > 0 \text{ for } k = 1, \dots, p,$$
$$\mathbf{a}_{[k]} = (a_1, \dots, a_{k-1}, a_{k+1}, ..., a_p), \Delta(a, l) = (a/l, (a+1)/l, \dots, (a+l-1)/l).$$

Here, $(a)_n = a(a+1)...(a+n-1)$ is the Pochhammer symbol. Furthermore, we will follow the standard definition of the generalized hypergeometric function ${}_pF_q$ ([1] Section 2.1), ([2,3] Chapter 12), as the sum of the series

$${}_pF_q\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\middle| z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n\cdots(a_p)_n}{n!(b_1)_n(b_2)_n\cdots(b_q)_n} z^n,$$

provided that this series converges. If $p = q + 1$, the above series has unit radius of convergence and ${}_pF_q(z)$ is defined as an analytic continuation of its sum to $\mathbb{C} \setminus [1, \infty)$. We will omit the argument 1 from the notation of the hypergeometric series and this convention will be adopted throughout the paper. Generalized hypergeometric functions occur in a wide variety of problems in theoretical physics, applied mathematics, statistics and engineering sciences, let alone pure mathematics itself. In particular, the functions ${}_3F_2$ and ${}_4F_3$ evaluated at the unit argument are related to the Clebsch–Gordan and Racah coefficients, respectively, see ([4] Sections 8.2.5 and 9.2.3) and [5,6]. In recent years, special functions also play an important role in the theory of approximation. Their growing importance is attributed to their versatility [7,8]. In a series of joint works by the second author of this paper and Dmitrii Karp [9–12], transformation and summation formulas for the generalized hypergeometric functions with integral parameter differences were studied. Recall that the generalized hypergeometric functions with integral parameter differences are functions containing the parameter pairs $\left[\begin{array}{c} f+m \\ f \end{array}\right]$ (known as the positive integral parameter difference) and/or $\left[\begin{array}{c} b \\ b+n \end{array}\right]$ (known as the negative integral parameter difference) for arbitrary positive integers $m$, $n$.

In this paper, we continue our research in this direction and discuss three comments on the articles [9–12].

Our first remark ( 3) refers to the paper [9] and concerns the generalization of Minton's and Karlsson's summation formulas. In 1970, Minton [13] proved the summation formula

$$ {}_pF_{p-1}\left(\begin{array}{c} -k, b, \mathbf{f}+\mathbf{m} \\ b+1, \mathbf{f} \end{array}\right) = \frac{k!}{(b+1)_k}\frac{(\mathbf{f}-b)_{\mathbf{m}}}{(\mathbf{f})_{\mathbf{m}}}, $$

valid for $k \geq m, k \in \mathbb{N}$, where $\mathbf{f} \in \mathbb{C}^{p-2}$ and $\mathbf{m} \in \mathbb{N}^{p-2}$. Soon thereafter, his result was generalized by Karlsson [14], who replaced $-k$ with an arbitrary complex number $a$ satisfying $Re(1-a-m) > 0$ to obtain

$$ {}_pF_{p-1}\left(\begin{array}{c} a, b, \mathbf{f}+\mathbf{m} \\ b+1, \mathbf{f} \end{array}\right) = \frac{\Gamma(b+1)\Gamma(1-a)}{\Gamma(b+1-a)}\frac{(\mathbf{f}-b)_{\mathbf{m}}}{(\mathbf{f})_{\mathbf{m}}}. \tag{1} $$

These celebrated formulas for the generalized hypergeometric series with integral parameter differences motivated a stream of works dedicated to this type of hypergeometric series. Extensions in many directions were found. Gasper [15] deduced a $q$-analogue and a generalization of Minton's and Karlsson's formulas; Chu [16,17] found extensions to bilateral hypergeometric and $q$-hypergeometric series; their results were re-derived by simpler means and further generalized by Schlosser [18], who also found multidimensional extensions to hypergeometric functions associated with root systems [19]. In [9], we obtained a generalization of (1), replacing the parameter $b$ with the finite sequence $\mathbf{b} = (b_1, b_2, \ldots, b_l)$ of parameters. Namely, the formula

$$ \frac{1}{\Gamma(1-a)}\,{}_{p+l-1}F_{p+l-2}\left(\begin{array}{c} a, \mathbf{b}, \mathbf{f}+\mathbf{m} \\ \mathbf{b}+\mathbf{p}, \mathbf{f} \end{array}\right) = \frac{(\mathbf{b})_{\mathbf{p}}}{(\mathbf{f})_{\mathbf{m}}}\sum_{q=1}^{n}\frac{\Gamma(\beta_q)(\mathbf{f}-\beta_q)_{\mathbf{m}}}{B_q\Gamma(1+\beta_q-a)}, \tag{2} $$

was proven ([9] (2.11)). Here, $\mathbf{b} = (b_1, b_2, \ldots, b_l) \in \mathbb{C}^l$, $\mathbf{p} = (p_1, p_2, \ldots p_l)$ is a sequence of positive integers, $n = p_1 + p_2 + \cdots + p_l$, and all elements of $\beta := (b_1, b_1+1, \ldots, b_1+p_1-1, \ldots, b_l, b_l+1, \ldots, b_l+p_l-1) = (\beta_1, \beta_2, \ldots, \beta_n)$ are distinct, $B_q = \prod_{v=1, v\neq q}^{n}(\beta_v - \beta_q)$. Adding parameters according to the principle

$$ {}_pF_{p-1}\left(\begin{array}{c} a, b, \mathbf{f}+\mathbf{m} \\ b+3, \mathbf{f} \end{array}\right) = {}_{p+2}F_{p+1}\left(\begin{array}{c} a, b, b+1, b+2, \mathbf{f}+\mathbf{m} \\ b+1, b+2, b+3, \mathbf{f} \end{array}\right), $$

it is enough to consider the case $\mathbf{p} = \mathbf{1} = (1, 1, \ldots, 1)$. Then, (2) takes the form

$$ {}_{p+l-1}F_{p+l-2}\left(\begin{array}{c} a, \mathbf{b}, \mathbf{f}+\mathbf{m} \\ \mathbf{b}+\mathbf{1}, \mathbf{f} \end{array}\right) = \frac{\Gamma(1-a)(\mathbf{b})_1}{(\mathbf{f})_{\mathbf{m}}}\sum_{q=1}^{l}\frac{\Gamma(b_q)(\mathbf{f}-b_q)_{\mathbf{m}}}{\alpha_q\Gamma(1+b_q-a)}, \tag{3} $$

where $\mathbf{b} = (b_1, b_2, \ldots, b_l)$ and $\alpha_q = \prod_{v=1, v\neq q}^{l}(b_v - b_q)$. Formula (2) has found application in the study of multiple orthogonal polynomials of the hypergeometric type with respect to two measures supported on the positive real line [20,21]. Examples of using Formula (3) can be found in the paper [22]. We would like to emphasize that the validity of Formula (3) indeed requires that the parameters of the sequence **b** are all *distinct*. We explain what happens instead if some of those parameters repeat. The formula in that case becomes more complicated and involves derivatives (see Theorem 1).

Our second remark (4) refers to the article [10]. In [10], we gave a complete description of the group of two-term relations for the function ${}_4F_3\left(\begin{matrix} a,b,c,f+1 \\ d,e,f \end{matrix}\right)$. Namely, we studied a group of transformations of the form

$$ {}_4F_3(\mathbf{r},f) = M(\mathbf{r})\frac{\varepsilon f+\lambda(\mathbf{r})}{f}\,{}_4F_3(D\mathbf{r},\eta), \quad \eta = \frac{\varepsilon f+\lambda(\mathbf{r})}{\alpha(\mathbf{r})f+\beta(\mathbf{r})}, \tag{4} $$

where $a,b,c,d,e,f \in \mathbb{C}$, $\mathbf{r} = (a,b,c,d,e,1)^T$ are the column vectors; $M(\mathbf{r})$ is a function of the $\Gamma$ type; $\varepsilon \in \{0,1\}$, $\lambda(\mathbf{r})$, and $\alpha(\mathbf{r})$, $\beta(\mathbf{r})$ are rational functions of the arguments $a,b,c,d,e$ (possibly vanishing but with $\lambda = 1$ if $\varepsilon = 0$); $D$ is a unit determinant $6\times 6$ matrix with integer entries and the bottom row $(0,0,0,0,0,1)$; and we define

$$ {}_4F_3(\mathbf{r},f) = {}_4F_3\left(\begin{matrix} a,b,c,f+1 \\ d,e,f \end{matrix}\right). \tag{5} $$

Now, we replace the parameter $f$ on the finite sequence $\mathbf{f}_k = (f_1,\ldots,f_k) \in \mathbb{C}^k$ and denote

$$ {}_{k+3}F_{k+2}(\mathbf{r},\mathbf{f}_k) = {}_{k+3}F_{k+2}\left(\begin{matrix} a,b,c,\mathbf{f}_k+1 \\ d,e,\mathbf{f}_k \end{matrix}\right). \tag{6} $$

Let us raise the question on the existence of transformations similar to (4) for functions (6). In 4, we show that each function of the form (6) can be reduced to some ${}_4F_3$ of the form (5). Thus, each transformation (4) generates some transformation for functions of the type (6). In [10,12], we obtained formulas for summing functions of the type ${}_4F_3(a,b,c,f+m,d,e,f)$ with nonlinear constraints on the parameters. Reduction (6) to the function (5) now allows us to obtain some summation formulas for functions ${}_{k+3}F_{k+2}\left(\begin{matrix} a,b,c,\mathbf{f}_k+1 \\ d,e,\mathbf{f}_k \end{matrix}\right)$. As an example, we have given the formula for ${}_5F_4$.

Finally, our third remark ( 5) is caused by the desire to understand which summation formulas are hidden in transformations of hypergeometric functions of the form

$$ F\left(\begin{matrix} \mathbf{a} \\ \mathbf{b} \end{matrix}\middle| Mx^w\right) = V(1-x)^{\lambda} F\left(\begin{matrix} \mathbf{c} \\ \mathbf{d} \end{matrix}\middle| \frac{Dx^u}{(1-x)^v}\right), \ \ x \in G, \tag{7} $$

where $\mathbf{c},\mathbf{d}$, $V$, and $\lambda$ are functions of $\mathbf{a},\mathbf{b}$, $w,u \in \mathbb{N}$, and $v \in \mathbb{Z}$; $M,D$ are constants; and $G$ is some domain of the complex plane $\mathbb{C}_x$, (here we have omitted the indices of the hypergeometric function $F$). Transformations of this type go back to the well-known Euler–Pfaff transformation. Multiplying transformation (7) by Meijer's G function and performing subsequent integration lead (in conjunction with a suitable summation formula) to a number of transformations of hypergeometric functions evaluated at unity [11,23]. In the articles [11,23], one can also find a list of some of the known transformations of the type (7). Obviously, in any transformation $f(x) = g(x)h(x)$, $x \in G$, some summation formula is hidden in the case of the existence of a decomposition $f(x)$, $g(x)$, $h(x)$ into the power ranks. In this paper, we write it out explicitly and make sure that it is a summation formula for finite hypergeometric functions.

## 2. The Generalization of Minton's and Karlsson's Summation Formulas

In this section, we consider the extension of (3) to the case where the parameters in the sequence **b** are not necessarily distinct. Our result below involves higher-order derivatives, depending on the multiplicities of the repeated entries in the sequence **b**.

**Theorem 1.** *Suppose that all elements of* $\mathbf{b} = (b_1, b_2, \ldots, b_l)$ *are distinct,* $n = n_1 + n_2 + \ldots + n_l$, $\boldsymbol{f} \in \mathbb{C}^{p-2}$, $\boldsymbol{m} \in \mathbb{N}^{p-2}$, $m = m_1 + m_2 + \ldots + m_{p-2}$, $Re(n - a - m) > 0$, $-1 + a \notin \mathbb{N}_0$, $-b_i \notin \mathbb{N}_0$, $-b_i + a - 1 \notin \mathbb{N}_0$, $-f_r \notin \mathbb{N}_0$ *for any* $i = 1, \ldots, l$, $r = 1, \ldots, p-2$. *Then,*

$$ {}_{p+n-1}F_{p+n-2}\left( \begin{matrix} a, \; b_1, \; \ldots \; , b_1, \quad b_2, \; \ldots \; , b_2, \; \ldots \quad , b_l, \; \ldots, \; b_l, \; \mathbf{f}+\mathbf{m} \\ \underbrace{b_1+1, \ldots, b_1+1}_{n_1-times}, \underbrace{b_2+1, \ldots, b_2+1}_{n_2-times}, \ldots, \underbrace{b_l+1, \ldots, b_l+1}_{n_l-times}, \mathbf{f} \end{matrix} \right) = $$

$$ \frac{\Gamma(1-a)}{(\mathbf{f})_{\mathbf{m}}} \sum_{i=1}^{l} \sum_{k_i=1}^{n_i} \frac{(-1)^{k_i+1}\alpha^i_{k_i}}{(k_i-1)!} \left( \frac{\Gamma(x)(\mathbf{f}-x)_{\mathbf{m}}}{\Gamma(x-a+1)} \right)^{(k_i-1)}_{x=b_i}, \quad (8) $$

*where* $\alpha^i_{k_i}$ *are the coefficients of decomposition into simple fractions for* $\prod_{i=1}^{l} \frac{b_i^{n_i}}{(b_i+x)^{n_i}}$,

$$ \prod_{i=1}^{l} \frac{b_i^{n_i}}{(b_i+x)^{n_i}} = \sum_{i=1}^{l} \sum_{k_i=1}^{n_i} \frac{\alpha^i_{k_i}}{(b_i+x)^{k_i}}. \quad (9) $$

**Proof.** Assume that $\mathbf{a} = (a_1, a_2, \ldots, a_{p-1})$, $\mathbf{d} = (d_1, d_2, \ldots, d_{p-2})$, $b \in \mathbb{C}$, $n > 1$. Following the definition of the hypergeometric function, we obtain the equality $(b)_l/(b+1)_l = b/(b+l)$ and the following simple calculations:

$$ \left( {}_pF_{p-1}\left( \begin{matrix} \mathbf{a}-1, x-1 \\ \mathbf{d}-1, x \end{matrix} \middle| z \right) \right)^{(n-1)}_{x=b} = \sum_{l=0}^{\infty} \frac{(\mathbf{a}-1)_l}{l!(\mathbf{d}-1)_l} \left( \left( \frac{x-1}{x-1+l} \right) \right)^{(n-1)}_{x=b} z^l = $$

$$ \sum_{l=1}^{\infty} \frac{(-1)^n(n-1)!(\mathbf{a}-1)_l}{(l-1)!(\mathbf{d}-1)_l} \left( \frac{1}{b-1+l} \right)^n z^l = $$

$$ \frac{(-1)^n(n-1)!(\mathbf{a}-1)z}{b^n(\mathbf{d}-1)} \sum_{k=o}^{\infty} \frac{(\mathbf{a})_k}{k!(\mathbf{d})_k} \left( \frac{b}{b+k} \right)^n z^k = $$

$$ \frac{(-1)^n(n-1)!(\mathbf{a}-1)z}{b^n(\mathbf{d}-1)} {}_{p+n-1}F_{p+n-2}\left( \begin{matrix} \mathbf{a}, b, \ldots, b \\ \mathbf{d}, \underbrace{b+1, \ldots, b+1}_{n-times} \end{matrix} \middle| z \right). \quad (10) $$

Applying (10) in the case of $\mathbf{a} = (a, \mathbf{f}+\mathbf{m})$, $\mathbf{d} = \mathbf{f}$, $z = 1$ and the summation formula

$$ {}_pF_{p-1}\left( \begin{matrix} a-1, x-1, \mathbf{f}+\mathbf{m}-1 \\ x, \mathbf{f}-1 \end{matrix} \right) = \frac{\Gamma(2-a)}{(\mathbf{f}-1)_{\mathbf{m}}} \frac{\Gamma(x)}{\Gamma(x-a+1)} \frac{\Gamma(\mathbf{f}-x+\mathbf{m})}{\Gamma(\mathbf{f}-x)}, $$

we have, under the assumption $Re(2-a-m) > 0$,

$$ {}_{p+n-1}F_{p+n-2}\left( \begin{matrix} a, b, \ldots, b, \mathbf{f}+\mathbf{m} \\ \underbrace{b+1, \ldots, b+1}_{n-times}, \mathbf{f} \end{matrix} \right) = \frac{(-1)^{n+1}b^n\Gamma(1-a)}{(n-1)!(\mathbf{f})_{\mathbf{m}}} \left( \frac{\Gamma(x)(\mathbf{f}-x)_{\mathbf{m}}}{\Gamma(x-a+1)} \right)^{(n-1)}_{x=b}. \quad (11) $$

Luckily, Equation (11) still holds for $n = 1$ by (1). In addition, Formula (11) is valid under the condition $Re(n-a-m) > 0$ according to the principle of analytical continuation. Taking into account (9), we write

$$
{}_{p+n-1}F_{p+n-2}\left(\begin{array}{c} a, \quad b_1, \ \ldots \ , b_1, \quad b_2, \ \ldots \ , b_2, \ \ldots \quad b_l, \ \ldots, \quad b_l, \ \mathbf{f}+\mathbf{m} \\ \underbrace{b_1+1,\ldots,b_1+1}_{n_1-times}, \underbrace{b_2+1,\ldots,b_2+1}_{n_2-times}, \ldots, \underbrace{b_l+1,\ldots,b_l+1}_{n_l-times}, \mathbf{f} \end{array}\right) =
\sum_{i=1}^{l}\sum_{k_i=1}^{n_i}\frac{\alpha^{i}_{k_i}}{b_i^{k_i}}\, {}_{p+k_i-1}F_{p+k_i-2}\left(\begin{array}{c} a, \quad b_i, \ \ldots \ , b_i, \mathbf{f}+\mathbf{m} \\ \underbrace{b_i+1,\ldots,b_i+1}_{k_i-times}, \mathbf{f} \end{array}\right).
$$

Now, to complete the proof, we use (11). □

It is easy to see that $\prod_{q=1}^{l}(b_q+x)^{-1} = \sum\limits_{q=1}^{l}(\alpha_q(b_q+x))^{-1}$, where $\alpha_q$ is defined by (3). Thus, the Formulas (3) and (8) coincide for $n_1 = \ldots = n_l = 1$.

In Examples 1–4 below, we assume that the conditions of Theorem 1 are fulfilled.

**Example 1.** *For $b - f_i \notin \mathbb{N}_0$, $i = 1, \ldots p-2$, the following relation is true:*

$$
{}_{p+1}F_p\left(\begin{array}{c} a, b, b, \mathbf{f}+\mathbf{m} \\ b+1, b+1, \mathbf{f} \end{array}\right) =
\frac{b\Gamma(1-a)\Gamma(b+1)(\mathbf{f}-b)_{\mathbf{m}}}{(\mathbf{f})_{\mathbf{m}}\Gamma(b+1-a)}\Big(\psi(b+1-a)-\psi(b)+\psi(\mathbf{f}+\mathbf{m}-b)-\psi(\mathbf{f}-b)\Big),
$$

*where $\psi(a) = \Gamma'(a)/\Gamma(a)$ is digamma function, $\psi(\mathbf{a}) = \psi(a_1, \ldots, a_p) = \sum_{i=1}^{p}\psi(a_i)$.*

*If $a = -k$, $k \in \mathbb{N}$, $k \geq m-1$, then*

$$
{}_{p+1}F_p\left(\begin{array}{c} -k, b, b, \mathbf{f}+\mathbf{m} \\ b+1, b+1, \mathbf{f} \end{array}\right) =
\frac{k!b(\mathbf{f}-b)_{\mathbf{m}}}{(b+1)_k(\mathbf{f})_{\mathbf{m}}}\Big(\psi(b+k+1)-\psi(b)-\psi(\mathbf{f}-b)+\psi(\mathbf{f}+\mathbf{m}-b)\Big).
$$

**Example 2.** *Suppose that $b - f_i \notin \mathbb{N}_0$, $i = 1, \ldots p-2$. Then, we have*

$$
{}_{p+2}F_{p+1}\left(\begin{array}{c} a, b, b, b, \mathbf{f}+\mathbf{m} \\ b+1, b+1, b+1, \mathbf{f} \end{array}\right) = \frac{b^2\Gamma(1-a)(\mathbf{f}-b)_{\mathbf{m}}\Gamma(b+1)}{2(\mathbf{f})_{\mathbf{m}}\Gamma(b+1-a)}\times
[(\psi(b+1-a)-\psi(b-1)+\psi(\mathbf{f}+\mathbf{m}-b)-\psi(\mathbf{f}-b))^2 -
-\psi'(b+1-a)+\psi'(b-1)-\psi'(\mathbf{f}-b)+\psi'(\mathbf{f}+\mathbf{m}-b)].
$$

**Example 3.** *For $b - f_i, c - f_i \notin \mathbb{N}_0$, $i = 1, \ldots p-2$, $c \neq b$, the following relation is true:*

$$
{}_{p+2}F_{p+1}\left(\begin{array}{c} a, b, b, c, c, \mathbf{f}+\mathbf{m} \\ b+1, b+1, c+1, c+1, \mathbf{f} \end{array}\right) = \frac{\Gamma(1-a)}{(\mathbf{f})_{\mathbf{m}}}\frac{b^2c^2}{(b-c)^3}\Bigg[\frac{\Gamma(b)(\mathbf{f}-b)_{\mathbf{m}}}{\Gamma(b-a+1)}\times
\Big(2+(c-b)(\psi(b)-\psi(\mathbf{f}+\mathbf{m}-b)+\psi(\mathbf{f}-b)-\psi(b-a+1))\Big)-
\frac{\Gamma(c)(\mathbf{f}-c)_{\mathbf{m}}}{\Gamma(c-a+1)}\Big(2+(b-c)(\psi(c)-\psi(\mathbf{f}+\mathbf{m}-c)+\psi(\mathbf{f}-c)-\psi(c-a+1))\Big)\Bigg]
$$

**Example 4.** *When some i-th component of the vector* $\mathbf{f}$ *satisfies the condition* $f_i = b - p$, $p \in \mathbb{N}_0$, *it is impossible to apply the formula* $((f_i - x)_{m_i})'_{x=b} = (f_i - x)_{m_i}(\psi(f_i - x) - \psi(f_i + m_i - x))$ *for differentiation of the* (11). *But if in addition* $m_i - p - 1 = m > 0$, *then we have the opportunity to write a hypergeometric function in the form*

$$ {}_{p+n-1}F_{p+n-2}\left(\begin{array}{c} a,\ b,\ \ldots, b,\ \mathbf{f}+\mathbf{m} \\ \underbrace{b+1,\ldots,b+1}_{n-times}, \mathbf{f} \end{array}\right) = $$

$$ {}_{p+n-1}F_{p+n-2}\left(\begin{array}{ccc} a,\ b,\ \ldots,\ b, & (b-p)+p, b+1+m, \mathbf{f}_{[i]} + \mathbf{m}_{[i]} \\ \underbrace{b+1,\ldots,b+1}_{(n-1)-times}, & b-p,\quad b+1,\quad \mathbf{f}_{[i]} \end{array}\right). $$

*Then, we apply* (11) *while replacing the vector* $\mathbf{f}$ *with the vector* $(b-p, b+1, \mathbf{f}_{[i]})$ *and the vector* $\mathbf{m}$ *with* $(p, m, \mathbf{m}_{[i]})$. *For example, assuming in* (11) $a = -k$, $\mathbf{f} = (b-1, b+1)$, $\mathbf{m} = (1, m)$, $n = 2$, *we obtain*

$$ {}_5F_4\left(\begin{array}{c} -k, b, b, b, b+m+1 \\ b+1, b+1, b+1, b-1 \end{array}\right) = $$

$$ \frac{k!m!b}{(b+1)_m(b+1)_k(b-1)}\Big(\psi(1) + \psi(b) - \psi(m+1) - \psi(b+k+1) + 1\Big). $$

## 3. On the Reduction of the Hypergeometric Function ${}_{p+1}F_p\left(\begin{array}{c} a, b, c, \mathbf{f}+\mathbf{m} \\ d, e, \mathbf{f} \end{array}\right)$ to the Function ${}_4F_3\left(\begin{array}{c} a, b, c, f+1 \\ d, e, f \end{array}\right)$

Recall that $a, b, c, d, e \in \mathbb{C}$, $\mathbf{r} = (a, b, c, d, e, 1)^T$, $\mathbf{f}_k = (f_1, \ldots, f_k) \in \mathbb{C}^k$, and we denote

$$ {}_{k+3}F_{k+2}(\mathbf{r}, \mathbf{f}_k) = {}_{k+3}F_{k+2}\left(\begin{array}{c} a, b, c, \mathbf{f}_k + 1 \\ d, e, \mathbf{f}_k \end{array}\right). $$

**Theorem 2.** *For* $k \geq 2$, *there exist rational functions* $V_k = V_k(\mathbf{r}, \mathbf{f}_k)$ *and* $\mu_k = \mu_k(\mathbf{r}, \mathbf{f}_k)$ *such that*

$$ {}_{k+3}F_{k+2}(\mathbf{r}, \mathbf{f}_k) = V_k(\mathbf{r}, \mathbf{f}_k)\,{}_4F_3(\mathbf{r}, \mu_k(\mathbf{r}, \mathbf{f}_k)). \tag{12} $$

*The functions* $V_k$ *and* $\mu_k$ *are computed recursively by*

$$ \mu^*_{k-1} = \mu_{k-1}(a+1, b+1, c+1, d+1, e+1, \boldsymbol{f}_{k-1}+1), $$

$$ V^*_{k-1} = V_{k-1}(a+1, b+1, c+1, d+1, e+1, \boldsymbol{f}_{k-1}+1), $$

$$ U_k = \frac{V^*_{k-1}}{V_{k-1}} \times \frac{abc(\boldsymbol{f}_{k-1}+1)}{(\boldsymbol{f}_k)(s-2)\mu^*_{k-1}}, \quad s = d + e - a - b - c, $$

$$ \eta_k = \frac{abc}{ab + ac + bc - de + d + e + (s-2)(\mu^*_{k-1} - 1) - 1}, $$

$$ V_k = V_{k-1}(1 + U_k),\ \mu_k = \frac{\mu_{k-1}\eta_k(1+U_k)}{\eta_k + \mu_{k-1}U_k}, $$

*and the initial values are given by* $V_1 = 1$, $\mu_1 = f$.

**Proof.** To prove the theorem, we apply the method of mathematical induction by $k$. Obviously, the theorem is valid when $k = 1$, $V_1 = 1$, $\mu_1 = f$. By definition of the hypergeometric function, we have

$$ {}_{k+4}F_{k+3}\begin{pmatrix} a,b,c,\mathbf{f}_{k+1}+1 \\ d,e,\mathbf{f}_{k+1} \end{pmatrix} = \sum_{n=0}^{+\infty} \frac{(a)_n(b)_n(c)_n(\mathbf{f}_k+1)_n}{n!(d)_n(e)_n(\mathbf{f}_k)_n}\left(1+\frac{n}{f_{k+1}}\right) = $$
$$ {}_{k+3}F_{k+2}\begin{pmatrix} a,b,c,\mathbf{f}_k+1 \\ d,e,\mathbf{f}_k \end{pmatrix} + \frac{abc(\mathbf{f}_k+1)}{f_{k+1}de(\mathbf{f}_k)}{}_{k+3}F_{k+2}\begin{pmatrix} a+1,b+1,c+1,\mathbf{f}_k+2 \\ d+1,e+1,\mathbf{f}_k+1 \end{pmatrix}. $$

Assume that the Formula (15) is correct for $k$,

$$ {}_{k+3}F_{k+2}\begin{pmatrix} a,b,c,\mathbf{f}_k+1 \\ d,e,\mathbf{f}_k \end{pmatrix} = V_k \times {}_4F_3\begin{pmatrix} a,b,c,\mu_k+1 \\ d,e,\mu_k \end{pmatrix}. $$

Assumption of mathematical induction yields

$$ {}_{k+4}F_{k+3}\begin{pmatrix} a,b,c,\mathbf{f}_{k+1}+1 \\ d,e,\mathbf{f}_{k+1} \end{pmatrix} = V_k \times {}_4F_3\begin{pmatrix} a,b,c,\mu_k+1 \\ d,e,\mu_k \end{pmatrix} + $$
$$ \frac{abc(\mathbf{f}_k+1)}{f_{k+1}de(\mathbf{f}_k)} V_k^* \times {}_4F_3\begin{pmatrix} a+1,b+1,c+1,(\mu_k^*-1)+2 \\ d+1,e+1,(\mu_k^*-1)+1 \end{pmatrix}, \quad (13) $$

where $\mu_k^* = \mu_k(a+1,b+1,c+1,d+1,e+1,\mathbf{f}_k+1)$, $V_k^* = V_k(a+1,b+1,c+1,d+1,e+1,\mathbf{f}_k+1)$. Transformation for ${}_4F_3$ with one single step ([10] formula at the bottom of page 10 and Formula (A2)) implies

$$ {}_4F_3\begin{pmatrix} a+1,b+1,c+1,(\mu_k^*-1)+2 \\ d+1,e+1,(\mu_k^*-1)+1 \end{pmatrix} = \frac{de}{(s-2)\mu_k^*}{}_4F_3\begin{pmatrix} a,b,c,\eta_k+1 \\ d,e,\eta_k \end{pmatrix}, $$

where

$$ \eta_k = \frac{abc}{ab+ac+bc-de+d+e+(s-2)(\mu_k^*-1)-1}, \quad s = d+e-a-b-c. $$

Substituting this expression into the right side of the Formula (13), we have

$$ {}_{k+4}F_{k+3}\begin{pmatrix} a,b,c,\mathbf{f}_{k+1}+1 \\ d,e,\mathbf{f}_{k+1} \end{pmatrix}(V_k)^{-1} = $$
$$ {}_4F_3\begin{pmatrix} a,b,c,\mu_k+1 \\ d,e,\mu_k \end{pmatrix} + \frac{V_k^*}{V_k}\frac{abc(\mathbf{f}_k+1)}{f_{k+1}(\mathbf{f}_k)(s-2)\mu_k^*}{}_4F_3\begin{pmatrix} a,b,c,\eta_k+1 \\ d,e,\eta_k \end{pmatrix}. $$

We will introduce the notation

$$ U_{k+1} = \frac{V_k^*}{V_k}\frac{abc(\mathbf{f}_k+1)}{f_{k+1}(\mathbf{f}_k)(s-2)\mu_k^*} $$

to obtain

$$ {}_{k+4}F_{k+3}\begin{pmatrix} a,b,c,\mathbf{f}_{k+1}+1 \\ d,e,\mathbf{f}_{k+1} \end{pmatrix}(V_k)^{-1} = $$
$$ {}_4F_3\begin{pmatrix} a,b,c,\mu_k+1 \\ d,e,\mu_k \end{pmatrix} + U_{k+1} \times {}_4F_3\begin{pmatrix} a,b,c,\eta_k+1 \\ d,e,\eta_k \end{pmatrix}. $$

From the definition of the hypergeometric function, it then follows

$$ {}_{k+4}F_{k+3}\binom{a,b,c,\mathbf{f}_{k+1}+1}{d,e,\mathbf{f}_{k+1}}(V_k)^{-1} = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n(c)_n}{(d)_n(e)_n n!}(1+U_{k+1})\times \left(1+\frac{\eta_k+\mu_k U_{k+1}}{\mu_k\eta_k(1+U_{k+1})}n\right). \quad (14) $$

We can see that Equation (14) is equivalent to

$$ {}_{k+4}F_{k+3}\binom{a,b,c,\mathbf{f}_{k+1}+1}{d,e,\mathbf{f}_{k+1}} = V_{k+1}\times {}_4F_3\binom{a,b,c,\mu_{k+1}+1}{d,e,\mu_{k+1}}, \quad (15) $$

where

$$ V_{k+1} = V_k(1+U_{k+1}), \quad \mu_{k+1} = \frac{\mu_k\eta_k(1+U_{k+1})}{\eta_k+\mu_k U_{k+1}}. $$

Thus, the theorem is proven by induction. □

**Example 5.** *Assume that $s = d+e-a-b-c$. Then,*

$$ {}_5F_4\left(\begin{array}{c} a,b,c,f+1,g+1 \\ d,e,f,g \end{array}\right) = \left(1+\frac{abc}{(s-2)fg}\right){}_4F_3\left(\begin{array}{c} a,b,c,\nu+1 \\ d,e,\nu \end{array}\right), \quad (16) $$

*where*

$$ \nu = \frac{abc+(s-2)fg}{ab+ac+bc-de+d+e+(s-2)(f+g)-1}. $$

**Example 6.** *We have*

$$ {}_6F_5\left(\begin{array}{c} a,b,c,f_1+1,f_2+1,f_3+1 \\ d,e,f_1,f_2,f_3 \end{array}\right) = \left(\frac{a^2bc(2+b+c)+w(-3+s)(-2+s)}{w(-3+s)(-2+s)}+ \frac{abc(2c+b(2+c)-de-3(1+z)+(2+z)s)}{w(-3+s)(-2+s)}\right){}_4F_3\left(\begin{array}{c} a,b,c,\mu_3+1 \\ d,e,\mu_3 \end{array}\right), $$

*where $w = f_1f_2f_3$, $z = f_1+f_2+f_3$,*

$$ \mu_3 = \frac{\mu_2\eta_3(1+U_3)}{\eta_3+\mu_2U_3}, \quad \mu_2 = \frac{abc+(s-2)f_1f_2}{\psi+(s-2)(f_1+f_2)}, \quad \psi = -1+ab+ac+bc+d+e-de, $$

$$ \eta_3 = \frac{abc}{\psi+(s-2)K}, $$

$$ K = \left(-1+\frac{(1+a)(1+b)(1+c)+(1+f_1)(1+f_2)(-3+s)}{2c+b(2+c)+a(2+b+c)-de-3(1+f_1+f_2)+(2+f_1+f_2)s}\right), $$

$$ U_3 = \frac{abc(2c+b(2+c)+a(2+b+c)-de-3(1+f_1+f_2)+(2+f_1+f_2)s)}{f_3(abc+f_1f_2(-2+s))(-3+s)}. $$

**Example 7.** *Assume that $\phi = d-b-a+1$ and $\mathbf{r}$, $f$, $g$ satisfy the condition*

$$ \frac{abc+(\phi-1)fg}{ab+ac+bc-dc-d+c+(\phi-1)(f+g)+1} = -\frac{c(a+b-d)}{\phi} + \frac{(d-a)(d-b)c}{\phi((d-a)(d-b+c)+(d-b)c+(d-1)(a+b-d-c-1))}. $$

*Then, the following identity holds true:*

$$
{}_5F_4\left(\begin{array}{c} a,b,c,f+1,g+1 \\ d,c+2,f,g \end{array}\right) = \left(1+\frac{abc}{(\phi-1)fg}\right)\left(\frac{(c+1)\Gamma(d)\Gamma(\phi+1)(\nu\phi+c(a+b-d))}{\Gamma(d-a+1)\Gamma(d-b+1)\phi\nu}\right). \quad (17)
$$

*To prove* (17)*, we apply* (16) *with the replacement* $e = c+2$*:*

$$
{}_5F_4\left(\begin{array}{c} a,b,c,f+1,g+1 \\ d,c+2,f,g \end{array}\right) = \left(1+\frac{abc}{(\phi-1)fg}\right){}_4F_3\left(\begin{array}{c} a,b,c,\nu+1 \\ d,c+2,\nu \end{array}\right),
$$

$$
\nu = \frac{abc+(\phi-1)fg}{ab+ac+bc-dc-d+c+(\phi-1)(f+g)+1}.
$$

*Then, we use the summation formula ([10] page 16)*

$$
{}_4F_3\left(\begin{array}{c} a,b,c,\nu+1 \\ d,c+2,\nu \end{array}\right) = \frac{(c+1)\Gamma(d)\Gamma(d-a-b+2)(\nu\phi+c(a+b-d))}{\Gamma(d-a+1)\Gamma(d-b+1)\phi\nu}.
$$

To formulate the following result, we need the transformation

$$
{}_4F_3\left(\begin{array}{c} a,b,c,f+1 \\ d,e,f \end{array}\right) = W^j \times {}_4F_3\left(\begin{array}{c} \alpha+j,\beta,\gamma,\nu^j+1 \\ \delta,\sigma,\nu^j \end{array}\right),\ j\in\mathbb{N}, \quad (18)
$$

with rational functions $W^j$, $\nu^j$, depending on $\mathbf{r}, f$. This transformation can be obtained by applying

$$
{}_4F_3\left(\begin{array}{c} a,b,c,f+1 \\ d,e,f \end{array}\right) = \left(1-\frac{bc}{(d-a-1)(e-a-1)}\right)\left(1+\frac{\lambda}{f}\right){}_4F_3\left(\begin{array}{c} a+1,b,c,\eta+1 \\ d,e,\eta \end{array}\right),
$$

$$
\lambda = \frac{abc}{a(2+a-d-e)-bc+(d-1)(e-1)},\quad \eta = \frac{abc+((a+1)(a+1-d-e)-bc+de)f}{(a-f)(2+a+b+c-d-e)},
$$

([10] Formula (A1)) $j$-times.

**Theorem 3.** *For each transformation* (4)*, there are rational function* $W(\mathbf{r},\mathbf{f}_k)$ *and polynomial* $T_p(x)$ *of degree* $p$ *with rational coefficients* $a_i(\mathbf{r},\mathbf{f}_k)$, $T_p(x) = \sum_{i=0}^{p} a_i(\mathbf{r},\mathbf{f}_k)x^i$*, such that*

$$
{}_{k+3}F_{k+2}(\mathbf{r},\mathbf{f}_k) = M(\mathbf{r})W(\mathbf{r},\mathbf{f}_k)\times{}_{p+3}F_{p+2}(D\mathbf{r},-\mathbf{g}_p(\mathbf{r},\mathbf{f}_k)).
$$

*Here,* $\mathbf{g}_p(\mathbf{r},\mathbf{f}_k) = (g_1(\mathbf{r},\mathbf{f}_k),\ldots,g_p(\mathbf{r},\mathbf{f}_k))$ *are roots of the polynomial* $T_p(x)$*.*

**Proof.** To find $W(\mathbf{r},\mathbf{f}_k)$ and $T_p(x)$, we can use the transformations (12), (4), (18) in the form

$$
{}_{k+3}F_{k+2}(\mathbf{r},\mathbf{f}_k) = V_k \times_4 F_3(\mathbf{r},\mu_k),
$$

$$
{}_4F_3(\mathbf{r},\mu_k) = M(\varepsilon+\lambda/\mu_k){}_4F_3(D\mathbf{r},\eta_k),\quad \eta_k = \frac{\varepsilon\mu_k+\lambda}{\alpha\mu_k+\beta},
$$

$$
{}_4F_3\left(\begin{array}{c} \alpha,\beta,\gamma,\eta_k+1 \\ \delta,\sigma,\eta_k \end{array}\right) = W_k^j \times {}_4F_3\left(\begin{array}{c} \alpha+j,\beta,\gamma,\nu_k^j+1 \\ \delta,\sigma,\nu_k^j \end{array}\right),
$$

where $\alpha,\beta,\gamma,\delta,\sigma$ are the first five coordinates of the vector $D\mathbf{r}$. We have

$$
{}_{k+3}F_{k+2}(\mathbf{r},\mathbf{f}_k) = \sum_{j=0}^{p-1}\frac{V_k}{p}{}_4F_3(\mathbf{r},\mu_k) = M\frac{V_k(\varepsilon+\lambda/\mu_k)}{p}\sum_{j=0}^{p-1}{}_4F_3(D\mathbf{r},\eta_k). \quad (19)
$$

where

$$ {}_4F_3(D\mathbf{r},\eta_k) = {}_4F_3\left(\begin{matrix}\alpha,\beta,\gamma,\eta_k+1\\ \delta,\sigma,\eta_k\end{matrix}\right). $$

From (19) we obtain, in conjunction with (18) and the formulas $(\alpha+j)_n = (\alpha)_n(\alpha+n)_j/(\alpha)_j$, $(\nu_k^j+1)_n/(\nu_k^j)_n = (\nu_k^j+n)/\nu_k^j$,

$$ {}_{k+3}F_{k+2}(\mathbf{r},\mathbf{f}_k) = M\frac{V_k(\varepsilon+\lambda/\mu_k)}{p}\sum_{j=0}^{p-1} W_k^j \times {}_4F_3\left(\begin{matrix}\alpha+j,\beta,\gamma,\nu_k^j+1\\ \delta,\sigma,\nu_k^j\end{matrix}\right) = $$
$$ M\frac{V_k(\varepsilon+\lambda/\mu_k)}{p}\sum_{j=0}^{p-1}\sum_{n=0}^{\infty}\frac{W_k^j}{(\alpha)_j\nu_k^j}\frac{(\alpha)_n(\beta)_n(\gamma)_n}{n!(\delta)_n(\sigma)_n}(n+\alpha)_j(n+\nu_k^j) = $$
$$ M\frac{V_k(\varepsilon+\lambda/\mu_k)}{p}\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n(\gamma)_n}{n!(\delta)_n(\sigma)_n}T_p(n). \quad (20) $$

Here, $T_p(x) = \sum_{j=0}^{p-1}\frac{W_k^j}{(\alpha)_j\nu_k^j}(x+\alpha)_j(x+\nu_k^j)$ is a polynomial of degree $p$. If $\mathbf{g}_p = (g_1,\ldots,g_p)$ are roots $T_p(x)$, then

$$ T_p(x) = \frac{W_j^{p-1}}{(\alpha)_{p-1}\nu_k^{p-1}}\prod_{j=1}^{p}(x-g_j),\ T_p(n) = \frac{W_j^{p-1}(-\mathbf{g}_p)_1}{(\alpha)_{p-1}\nu_k^{p-1}}\frac{(-\mathbf{g}_p+1)_n}{(-\mathbf{g}_p)_n}. $$

It is follows from (20)

$$ {}_{k+3}F_{k+2}(\mathbf{r},\mathbf{f}_k) = M(\mathbf{r})W(\mathbf{r},\mathbf{f}_k)\times {}_{p+3}F_{p+2}(D\mathbf{r},-\mathbf{g}_p(\mathbf{r},\mathbf{f}_k)), $$

where

$$ W(\mathbf{r},\mathbf{f}_k) = \frac{V_k(\varepsilon+\lambda/\mu_k)}{p}\frac{W_j^{p-1}(-\mathbf{g}_p(\mathbf{r},\mathbf{f}_k))_1}{(\alpha)_{p-1}\nu_k^{p-1}}. $$

□

**Example 8.** *Note that our proposed algorithm for finding the polynomial $T_p(n)$ is quite cumbersome. In some cases, other approaches may be applied. For example, the decomposition follows directly from the definition of the hypergeometric function*

$$ {}_5F_4\left(\begin{matrix}a,b,c,g+1,f+1\\ d,e,g,f\end{matrix}\right) = {}_3F_2\left(\begin{matrix}a,b,c\\ d,e\end{matrix}\right) + \frac{abc(g+f+1)}{degf}\times {}_3F_2\left(\begin{matrix}a+1,b+1,c+1\\ d+1,e+1\end{matrix}\right) + $$
$$ \frac{a(a+1)b(b+1)c(c+1)}{d(d+1)e(e+1)gf}\times {}_3F_2\left(\begin{matrix}a+2,b+2,c+2\\ d+2,e+2\end{matrix}\right). $$

*We apply Thomae's transformation ([10] Formula (3)) to each function ${}_3F_2$ to obtain*

$$ {}_5F_4\left(\begin{matrix}a,b,c,g+1,f+1\\ d,e,g,f\end{matrix}\right) = \frac{\Gamma(e)\Gamma(d)\Gamma(s)}{\Gamma(c)\Gamma(s+a)\Gamma(s+b)}\left({}_3F_2\left(\begin{matrix}d-c,e-c,s\\ s+a,s+b\end{matrix}\right) + \right. $$
$$ \frac{ab(g+f+1)}{(s-1)gf}\times {}_3F_2\left(\begin{matrix}d-c,e-c,s-1\\ s+a,s+b\end{matrix}\right) + $$
$$ \left. \frac{a(a+1)b(b+1)}{(s-1)(s-2)gf}\times {}_3F_2\left(\begin{matrix}d-c,e-c,s-2\\ s+a,s+b\end{matrix}\right)\right). \quad (21) $$

$s = d + e - a - b - c$. *Using the obvious equalities*

$$(s)_n = \frac{(s-2)_n(s+n-2)(s+n-1)}{(s-2)(s-1)},\ (s-1)_n = \frac{(s-2)_n(s+n-2)}{(s-2)},$$

*we transform the* ${}_3F_2$ *sum on the right side of* (21) *into the form*

$$\sum_{n=0}^{\infty} \frac{(d-c)_n(e-c)_n(s-2)_n}{n!(s+a)_n(s+b)_n} T_p(n),$$

*where*

$$T_p(n) = \frac{n^2 + Bn + C}{(s-1)(s-2)}, \qquad B = \frac{(2s-3)gf + (g+f+1)ab}{gf},$$
$$C = (s-2)(s-1) + \frac{ab((g+f+1)(s-2) + (a+1)(b+1))}{gf}.$$

*So, we obtain the transformation between two* ${}_5F_4$ *series*

$${}_5F_4\left(\begin{matrix} a, b, c, g+1, f+1 \\ d, e, g, f \end{matrix}\right) = \frac{\Gamma(e)\Gamma(d)\Gamma(s-2)f_1g_1}{\Gamma(c)\Gamma(s+a)\Gamma(s+b)} \times {}_5F_4\left(\begin{matrix} d-c, e-c, s-2, g_1+1, f_1+1 \\ s+a, s+b, g_1, f_1 \end{matrix}\right),$$

*where*

$$g_1 = \frac{B}{2} + \frac{\sqrt{B^2 - 4C}}{2}, \qquad f_1 = \frac{B}{2} - \frac{\sqrt{B^2 - 4C}}{2}.$$

## 4. Summation Formulas That Are Hidden in Transformations of Hypergeometric Functions with an Arbitrary Argument

**Theorem 4.** *Let* $\mathbf{c}, \mathbf{d}$, $V$, *and* $\lambda$ *be functions of* $\mathbf{a}, \mathbf{b}$, $w, u \in \mathbb{N}$, $v \in \mathbb{Z}$; $M, D$ *are constants; and* $G$ *is a domain of* $\mathbb{C}_x$. *The transformation*

$$F\left(\begin{matrix} \mathbf{a} \\ \mathbf{b} \end{matrix} \middle| Mx^w\right) = V(1-x)^{\lambda} F\left(\begin{matrix} \mathbf{c} \\ \mathbf{d} \end{matrix} \middle| \frac{Dx^u}{(1-x)^v}\right),\ \ x \in G, \tag{22}$$

*is valid iff for arbitrary* $l \in \mathbb{N}_0$ *we have the summation formula*

$$V \sum_{k=0}^{[l/u]} \frac{(\mathbf{c})_k D^k (vk-\lambda)_{l-uk}}{k!(\mathbf{d})_k (l-uk)!} = \frac{(\mathbf{a})_{[l/w]} M^{[l/w]}}{[l/w]!(\mathbf{b})_{[l/w]}} (1 - Sg(l/w - [l/w]), \tag{23}$$

*where* $Sg(\cdot)$ *is the signum function and the hypergeometric functions in* (22) *are the convergent power series at the* $G$.

**Proof.** It is follows directly from the (22) and definition of the hypergeometric function that

$$\sum_{k=0}^{\infty} \frac{(\mathbf{a})_k M^k x^{kw}}{k!(\mathbf{b})_k} = V \sum_{k=0}^{\infty} \frac{(\mathbf{c})_k D^k x^{ku}}{k!(\mathbf{d})_k} (1-x)^{\lambda - vk}. \tag{24}$$

Using the binomial expansion formulas, we obtain the following transformation:

$$(1-x)^{\lambda - vk} = \sum_{n=0}^{\infty} \frac{(vk-\lambda)_n x^n}{n!}$$

Substituting the resulting expression into (24), we have

$$\sum_{k=0}^{\infty} \frac{(\mathbf{a})_k M^k x^{kw}}{k!(\mathbf{b})_k} = V \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\mathbf{c})_k D^k}{k!(\mathbf{d})_k} \frac{(vk-\lambda)_n x^{n+ku}}{n!}. \tag{25}$$

Next, we transform the right side of the equality (25) by making the replacements $l = n + uk$ and $n = l - uk$. Then, the right side of the equality (25) will look like

$$V\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\mathbf{c})_k D^k}{k!(\mathbf{d})_k}\frac{(vk-\lambda)_n x^{n+ku}}{n!} = V\sum_{l=0}^{\infty}\sum_{k=0}^{[l/u]}\frac{(\mathbf{c})_k D^k}{k!(\mathbf{d})_k}\frac{(vk-\lambda)_{l-uk}x^l}{(l-uk)!}. \quad (26)$$

On the left side of the equality (25), we make the replacement $l = kw$ and write

$$V\sum_{k=0}^{\infty}\frac{(\mathbf{a})_k M^k x^{kw}}{k!(\mathbf{b})_k} = \sum_{l=0}^{\infty}\frac{(\mathbf{a})_{[l/w]}M^{[l/w]}x^l}{[l/w]!(\mathbf{b})_{[l/w]}}(1 - Sg(l/w - [l/w]). \quad (27)$$

We substitute (26) and (27) into (25) to obtain

$$\sum_{l=0}^{\infty}\sum_{k=0}^{[l/u]}\frac{(\mathbf{c})_k D^k}{k!(\mathbf{d})_k}\frac{(vk-\lambda)_{l-uk}x^l}{(l-uk)!} = \sum_{l=0}^{\infty}\frac{(\mathbf{a})_{[l/w]}M^{[l/w]}x^l}{[l/w]!(\mathbf{b})_{[l/w]}}(1 - Sg(l/w - [l/w]).$$

Equating the coefficients for $x^l$, we obtain the following formula:

$$V\sum_{k=0}^{[l/u]}\frac{(\mathbf{c})_k D^k(vk-\lambda)_{l-uk}}{k!(\mathbf{d})_k(l-uk)!} = \frac{(\mathbf{a})_{[l/w]}M^{[l/w]}}{[l/w]!(\mathbf{b})_{[l/w]}}(1 - Sg(l/w - [l/w]). \quad (28)$$

□

**Remark 1.** *For $v - u > 0$, $v > 0$, we transform the left side of the equality (28) using the formulas*

$$(l-uk)! = \frac{(-1)^{uk}l!}{(-l)_{uk}}, \quad (\mathbf{c})_k = \frac{\Gamma(\mathbf{c}+k)}{\Gamma(\mathbf{c})}, \ \Gamma(a+uk) = u^{uk}\Gamma(a)(\Delta(a,u))_k. \quad (29)$$

*Then, the left side of the equality (28) will take the form*

$$\sum_{k=0}^{[l/u]}\frac{(\mathbf{c})_k D^k(vk-\lambda)_{l-uk}}{k!(\mathbf{d})_k(l-uk)!} = \sum_{k=0}^{[l/u]}\frac{(\mathbf{c})_k\Gamma((v-u)k+l-\lambda)((-1)^u D)^k\Gamma(-l+uk)}{k!(\mathbf{d})_k l!\Gamma(-l)\Gamma(vk-\lambda)} =$$
$$\sum_{k=0}^{[l/u]}\frac{(\mathbf{c})_k\Gamma(l-\lambda)(\Delta(l-\lambda,v-u))_k(\Delta(-l,u))_k}{k!(\mathbf{d})_k l!\Gamma(-\lambda)(\Delta(-\lambda,v))_k}\left(\frac{(-u)^u(v-u)^{v-u}D}{v^v}\right)^k =$$
$$\frac{\Gamma(l-\lambda)}{l!\Gamma(-\lambda)}F\left(\left.\begin{matrix}\mathbf{c},\Delta(l-\lambda,v-u),\Delta(-l,u)\\ \mathbf{d},\Delta(-\lambda,v)\end{matrix}\right|\frac{(-u)^u(v-u)^{v-u}D}{v^v}\right).$$

*Hence,*

$$F\left(\left.\begin{matrix}\mathbf{c},\Delta(l-\lambda,v-u),\Delta(-l,u)\\ \mathbf{d},\Delta(-\lambda,v)\end{matrix}\right|\frac{(-u)^u(v-u)^{v-u}D}{v^v}\right) =$$
$$\frac{l!}{V(-\lambda)_l}\frac{(\mathbf{a})_{[l/w]}M^{[l/w]}}{[l/w]!(\mathbf{b})_{[l/w]}}(1 - Sg(l/w - [l/w]).$$

**Example 9.** *We consider the transformation (40) from [11]*

$${}_2F_1\left(\left.\begin{matrix}a,b\\ 1-b+a\end{matrix}\right| -x\right) = (1-x)^{-a}{}_2F_1\left(\left.\begin{matrix}a/2,a/2+1/2\\ 1-b+a\end{matrix}\right|\frac{-4x}{(1-x)^2}\right),$$

*where* $(u,v) = (1,2)$, $\lambda = -a, M = -1, D = -4, w = 1$, $\mathbf{a} = (a,b), \mathbf{b} = (1-b+a), \mathbf{c} = (a/2, a/2+1/2), \mathbf{d} = (1-b+a)$. *We obtain the summation formula*

$$ {}_4F_3\left(\begin{matrix} -l, a/2, a/2+1/2, l+a \\ 1-b+a, a/2, a/2+1/2 \end{matrix}\middle| 1\right) = {}_2F_1\left(\begin{matrix} -l, l+a \\ 1-b+a \end{matrix}\right) = \frac{(-1)^l (b)_l}{(1-b+a)_l}. $$

**Remark 2.** *For* $v-u<0$, $v>0$, *we transform the left side of the equality* (28) *using the Formula* (29) *and equality*

$$ \Gamma(a-k) = \frac{\Gamma(a)\Gamma(1-a)}{\Gamma(-a+1+k)(-1)^k}. \tag{30} $$

*We produce*

$$ \sum_{k=0}^{[l/u]} \frac{(\mathbf{c})_k D^k (vk-\lambda)_{l-uk}}{k!(\mathbf{d})_k (l-uk)!} = \sum_{k=0}^{[l/u]} \frac{(\mathbf{c})_k \Gamma((v-u)k+l-\lambda)((-1)^u D)^k \Gamma(-l+uk)}{k!(\mathbf{d})_k l! \Gamma(-l)\Gamma(vk-\lambda)} = $$
$$ \sum_{k=0}^{[l/u]} \frac{(\mathbf{c})_k \Gamma(l-\lambda)\Gamma(1-l+\lambda)((-1)^v D)^k \Gamma(-l+uk)}{k!\Gamma(1+\lambda-l+k(u-v))(\mathbf{d})_k l! \Gamma(-l)\Gamma(vk-\lambda)} = $$
$$ \frac{\Gamma(l-\lambda)}{l!\Gamma(-\lambda)} \sum_{k=0}^{[l/u]} \frac{(\mathbf{c})_k (\Delta(-l,u))_k}{k!(\mathbf{d})_k (\Delta(1+\lambda-l,u-v))_k (\Delta(-\lambda,v))_k} \left(\frac{(-1)^v u^u D}{(u-v)^{u-v} v^v}\right)^k = $$
$$ \frac{\Gamma(l-\lambda)}{l!\Gamma(-\lambda)} F\left(\begin{matrix} \mathbf{c}, \Delta(-l,u) \\ \mathbf{d}, \Delta(1+\lambda-l,u-v), \Delta(-\lambda,v) \end{matrix}\middle| \frac{u^u D}{(u-v)^{u-v}(-v)^v}\right). $$

*Thus, the Formula* (28) *takes the form*

$$ F\left(\begin{matrix} \mathbf{c}, \Delta(-l,u) \\ \mathbf{d}, \Delta(1+\lambda-l,u-v), \Delta(-\lambda,v) \end{matrix}\middle| \frac{u^u D}{(u-v)^{u-v}(-v)^v}\right) = $$
$$ \frac{l!}{V(-\lambda)_l} \frac{(\mathbf{a})_{[l/w]} M^{[l/w]}}{[l/w]!(\mathbf{b})_{[l/w]}} (1 - Sg(l/w - [l/w]). $$

**Example 10.** *We consider the transformation (56) from [15]*

$$ {}_2F_1\left(\begin{matrix} a, b \\ 2b \end{matrix}\middle| x\right) = (1-x)^{-a/2} {}_2F_1\left(\begin{matrix} a/2, b-a/2 \\ b+1/2 \end{matrix}\middle| \frac{-x^2}{4(1-x)}\right), $$

*where* $(u,v) = (2,1)$, $\lambda = -a/2, M = 1, D = -1/4, w = 1$, $\mathbf{a} = (a,b), \mathbf{b} = (2b), \mathbf{c} = (a/2, b-a/2), \mathbf{d} = (b+1/2)$. *We obtain the summation formula*

$$ {}_4F_3\left(\begin{matrix} -l/2, (-l+1)/2, a/2, b-a/2 \\ b+1/2, 1-a/2-l, a/2 \end{matrix}\right) = $$
$$ {}_3F_2\left(\begin{matrix} -l/2, (-l+1)/2, b-a/2 \\ b+1/2, 1-a/2-l \end{matrix}\right) = \frac{(a)_l (b)_l}{(a/2)_l (2b)_l}. $$

**Remark 3.** *For $v-u<0$, $v<0$, we transform the left side of the equality (28) using the Formulas (29) and (30). Then, the left side of the equality (28) is*

$$\sum_{k=0}^{[l/u]}\frac{(\mathbf{c})_k D^k(vk-\lambda)_{l-uk}}{k!(\mathbf{d})_k(l-uk)!}=\sum_{k=0}^{[l/u]}\frac{(\mathbf{c})_k\Gamma((v-u)k+l-\lambda)((-1)^u D)^k\Gamma(-l+uk)}{k!(\mathbf{d})_k l!\Gamma(-l)\Gamma(vk-\lambda)}=$$
$$\sum_{k=0}^{[l/u]}\frac{(\mathbf{c})_k\Gamma(l-\lambda)(\Delta(-l,u))_k(\Delta(1+\lambda,-v))_k}{k!(\mathbf{d})_k l!\Gamma(-\lambda)(\Delta(1+\lambda-l,u-v))_k}\left(\frac{u^u D}{(-v)^v(u-v)^{u-v}}\right)^k=$$
$$\frac{\Gamma(l-\lambda)}{l!\Gamma(-\lambda)}F\left(\begin{matrix}\mathbf{c},\Delta(-l,u),(\Delta(1+\lambda,-v))_k\\ \mathbf{d},\Delta(1+\lambda-l,u-v)\end{matrix}\middle|\frac{u^u D}{(-v)^v(u-v)^{u-v}}\right).$$

*Hence, the summation formula looks like*

$$F\left(\begin{matrix}\mathbf{c},\Delta(-l,u),\Delta(1+\lambda,-v)\\ \mathbf{d},\Delta(1+\lambda-l,u-v)\end{matrix}\middle|\frac{u^u D}{(-v)^v(u-v)^{u-v}}\right)=$$
$$\frac{l!}{V(-\lambda)_l}\frac{(\mathbf{a})_{[l/w]}M^{[l/w]}}{[l/w]!(\mathbf{b})_{[l/w]}}(1-Sg(l/w-[l/w])).$$

**Example 11.** *We now consider (39) from [11]*

$${}_2F_1\left(\begin{matrix}a,1-a\\ b\end{matrix}\middle|x\right)=(1-x)^{b-1}{}_2F_1\left(\begin{matrix}(b-a)/2,(a+b-1)/2\\ b\end{matrix}\middle|4x(1-x)\right).$$

*Here, $(u,v)=(1,-1),\lambda=b-1,M=1,D=4,w=1$, $\mathbf{a}=(a,1-a),\mathbf{b}=(b)$, $\mathbf{c}=((b-a)/2,(a+b-1)/2),\mathbf{d}=(b)$. Thus, we produce*

$${}_4F_3\left(\begin{matrix}-l,b,(b-a)/2,(a+b-1)/2\\ b,(b-l)/2,(1+b-l)/2\end{matrix}\middle|1\right)={}_3F_2\left(\begin{matrix}-l,(b-a)/2,(a+b-1)/2\\ (b-l)/2,(1+b-l)/2\end{matrix}\right)=\frac{(a)_l(1-a)_l}{(1-b)_l(b)_l}.$$

*Similar considerations yield further identities, which we collect in the following remarks and examples.*

**Remark 4.** *If $v=0$, then the Formula (23) takes the form*

$$F\left(\begin{matrix}\mathbf{c},\Delta(-l,u)\\ \mathbf{d},\Delta(1+\lambda-l,u)\end{matrix}\middle|D\right)=\frac{l!}{V(-\lambda)_l}\frac{(\mathbf{a})_{[l/w]}M^{[l/w]}}{[l/w]!(\mathbf{b})_{[l/w]}}(1-Sg(l/w-[l/w])).\quad(31)$$

**Example 12.** *We consider the Miller–Paris transformations, given in [24] (Theorems 3 and 4). The second Miller–Paris transformation generalizes the second Euler transformation for the Gauss hypergeometric function as follows:*

$${}_{r+2}F_{r+1}\left(\begin{matrix}a,b,\mathbf{f}+\mathbf{m}\\ c,\mathbf{f}\end{matrix}\middle|x\right)=(1-x)^{c-a-b-m}{}_{m+2}F_{m+1}\left(\begin{matrix}c-a-m,c-b-m,\mathbf{q}+1\\ c,\mathbf{q}\end{matrix}\middle|x\right),\quad(32)$$

*and holds true for $(c-a-m)_m\neq 0$, $(c-b-m)_m\neq 0$, $(1+a+b-c)_m\neq 0$ and $x\in\mathbb{C}\setminus[1,\infty)$. The vector $\mathbf{q}=\mathbf{q}(a,b,c,\mathbf{f})=(\eta_1,\dots,\eta_m)$ is formed by the roots of the second characteristic polynomial $Q_m(t)=Q_m(a,b,c,\mathbf{f}|t)$ given by*

$$Q_m(t)=\sum_{k=0}^{m}\frac{(-1)^k(a)_k(-b-m)_k(t)_k(c-a-m-t)_{m-k}}{(c-a-m)_m(c-b-m)_k k!}{}_{r+2}F_{r+1}\left(\begin{matrix}-k,b,\mathbf{f}+\mathbf{m}\\ b+m-k+1,\mathbf{f}\end{matrix}\right).$$

*We may bring the transformation (32) into the form (22) by putting* $\mathbf{a} = (a, b, \mathbf{f}+\mathbf{m})$, $\mathbf{b} = (c, \mathbf{f})$, $\mathbf{c} = (c-a-m, c-b-m, \mathbf{q}+1)$ , $\mathbf{d} = (c, \mathbf{q})$, $\lambda = c-a-b-m$, $M = 1$, $w = 1$, $D = 1$, $u = 1$, $v = 0$. *It follows from (31) that*

$$ {}_{r+2}F_{r+1}\left(\begin{matrix}-l, c-a-m, c-b-m, \mathbf{q}+1\\ 1-l+c-a-b-m, c, \mathbf{q}\end{matrix}\right) = \frac{(a)_l(b)_l(\mathbf{f}+\mathbf{m})_l}{(c)_l(\mathbf{f})_l(a+b+m-c)_l}. $$

**Remark 5.** *If* $v > 0$, $v-u = 0$, *then the equality (23) can be rewritten as follows:*

$$ F\left(\begin{matrix}\mathbf{c}, \Delta(-l, u)\\ \mathbf{d}, \Delta(-\lambda, u)\end{matrix}\middle|\frac{(-u)^u D}{v^v}\right) = \frac{l!}{V(-\lambda)_l}\frac{(\mathbf{a})_{[l/w]}M^{[l/w]}}{[l/w]!(\mathbf{b})_{[l/w]}}(1-Sg(l/w-[l/w]). \quad (33) $$

**Example 13.** *The first Miller–Paris transformation is*

$$ {}_{m+2}F_{m+1}\left(\begin{matrix}a, b, \mathbf{f}+\mathbf{1}\\ c, \mathbf{f}\end{matrix}\middle|x\right) = (1-x)^{-a}{}_{m+2}F_{m+1}\left(\begin{matrix}a, c-b-m, \mathbf{h}+1\\ c, \mathbf{h}\end{matrix}\middle|\frac{x}{x-1}\right). \quad (34) $$

*It is true for* $b \neq f_j, j = 1, 2, \ldots, r, (c-b-m)_m \neq 0$ *and* $x \in \mathbb{C} \setminus [1, \infty)$. *The vector* $\mathbf{h} = \mathbf{h}(b, c, \mathbf{f}) = (h_1, \ldots, h_m)$ *is formed by the roots of the first characteristic polynomial* $K_m(t) = K_m(b, c, \mathbf{f}|t)$, *given by*

$$ K_m(t) = \frac{1}{(c-b-m)_m}\sum_{k=0}^{m}\frac{(-1)^k}{k!}{}_{r+1}F_r\left(\begin{matrix}-k, \mathbf{f}+\mathbf{1}\\ \mathbf{f}\end{matrix}\right)(b)_k(t)_k(c-b-m-t)_{m-k}. $$

*We may bring the transformation (34) into the form (22) by putting* $\mathbf{a} = (a, b, \mathbf{f}+\mathbf{1})$, $\mathbf{b} = (c, \mathbf{f})$, $\mathbf{c} = (a, c-b-m, \mathbf{h}+1)$, $\mathbf{d} = (c, \mathbf{h})$, $\lambda = -a$, $M = 1$, $w = 1$, $D = -1$, $u = 1$, $v = 1$. *It follows that*

$$ {}_{m+2}F_{m+1}\left(\begin{matrix}-l, c-b-m, \mathbf{h}+1\\ c, \mathbf{h}\end{matrix}\right) = \frac{(b)_l(\mathbf{f}+\mathbf{1})_l}{(c)_l(\mathbf{f})_l}. \quad (35) $$

*Making the substitution* $t = x/(x-1)$, $\beta = c-b-m$ *in (34), we obtain*

$$ {}_{m+2}F_{m+1}\left(\begin{matrix}a, \beta, \mathbf{h}+1\\ c, \mathbf{h}\end{matrix}\middle|t\right) = (1-t)^{-a}{}_{r+2}F_{r+1}\left(\begin{matrix}a, c-\beta-m, \mathbf{f}+\mathbf{1}\\ c, \mathbf{f}\end{matrix}\middle|\frac{t}{t-1}\right). $$

*Thus , (35) takes the form*

$$ {}_{m+2}F_{m+1}\left(\begin{matrix}-l, \beta, \mathbf{h}+1\\ c, \mathbf{h}\end{matrix}\right) = \frac{(c-\beta-m)_l(\mathbf{f}+1)_l}{(c)_l(\mathbf{f})_l}, $$

*where* $\mathbf{f}$ *is formed by the roots of the polynomial* $K_m(t) = K_m(\beta, c, \mathbf{h}|t)$ *given by*

$$ K_m(t) = \frac{1}{(c-\beta-m)_m}\sum_{k=0}^{m}\frac{(-1)^k}{k!}{}_{m+1}F_m\left(\begin{matrix}-k, \mathbf{h}+1\\ \mathbf{h}\end{matrix}\right)(\beta)_k(t)_k(c-\beta-m-t)_{m-k}. $$

**Example 14.** *We consider (6.1) from [24]*

$$ {}_{r+2}F_{r+1}\left(\begin{matrix}a, a+1/2, \mathbf{f}+\mathbf{m}\\ c, \mathbf{f}\end{matrix}\middle|\frac{x^2}{(1-x)^2}\right) = (1-x)^{2a}{}_{2m+2}F_{2m+1}\left(\begin{matrix}2a, c-m-1/2, \mathbf{z_{2m}}+1\\ 2c-1, \mathbf{z_{2m}}\end{matrix}\middle|2x\right), \quad (36) $$

*where* $\mathbf{z_{2m}}$ *are the nonvanishing zeros of the associated parametric polynomial* $Q_{2m}(t)$ *of degree* $2m$ *given by*

$$ Q_{2m}(t) = \sum_{k=0}^{m}\frac{A_k}{2_k}(t)_{2k}(c-m-1/2-t)_{m-k}, $$

*and the coefficients $A_k$ are defined by (2.9) in [24].*

*By transferring $(1-x)^{2a}$ from (36) in the left part, we obtain the following coefficients and vectors:*

$$\mathbf{a} = (2a, c-m-1/2, \mathbf{z_{2m}}+1), \quad \mathbf{b} = (2c-1, \mathbf{z_{2m}}), \quad \mathbf{c} = (a, a+1/2, \mathbf{f}+\mathbf{m}), \quad \mathbf{d} = (c, \mathbf{f})$$

$$\omega = 1, \quad M = 2, \quad D = 1, \quad u = 2, \quad v = 2, \quad \lambda = -2a.$$

*Using the Formula (33) and by reducing the same multipliers, we obtain the following summation formula:*

$$ {}_{r+2}F_{r+1}\left(\begin{matrix} -l/2, -l/2+1/2, \mathbf{f}+\mathbf{m} \\ c, \mathbf{f} \end{matrix}\right) = \frac{(c-m-1/2)_l(\mathbf{z_{2m}}+1)_l}{(2c-1)_l(\mathbf{z_{2m}})_l}.$$

## 5. Discussion

The first 2 of this paper complements the results that were obtained in [9]. Theorem 1 generalizes the relation (2.11) from [9] to the case of repeated parameters. At the same time, the positivity of the parametric balance is required. With a negative balance, the hypergeometric series evaluated at unit generally diverges. The exception is the case when one upper parameter is a negative integer. As we have noted in the introduction, the Minton summation in Formula (2) is valid when $k \geq m$ (positive parametric balance). We generalized Minton's result (2) to the case of $0 \leq k \leq m-1$ ([9] Theorem 2.1) (negative balance). The formulas obtained are quite complex and use the Norlund coefficients. In this paper, the question remains open about the analogue of Theorem 1 for $a = -k$ and negative parametric balance. The proof we have proposed stops working in this case.

The next section of our paper ( 3) is also devoted to generalized hypergeometric functions with integral parameter differences evaluated at unit. In the articles [10,12], groups of transformations ${}_4F_3(1)$ of the IPD type with one positive integer difference were studied in detail. Does it make sense to consider such groups for hypergeometric functions of higher dimensions? In 3, we showed that all hypergeometric functions of the type we are considering transform into each other using some transformation. We have proposed algorithms for calculating these transformations, but these algorithms are difficult to programatically execute. It would be nice to simplify them further. We also note that Theorem 1 of [12] is a special case of Theorem 2 presented by us now. This fact is easy to prove using the method of additional parameters.

In 4, we started with the general type of transformations (22) and performed simple manipulations with power series. As a result, we conclude that (22) is always based on the summation formula for finite hypergeometric functions. It would be interesting to find an answer to the following question: How, starting with summation formulas for a finite hypergeometric function, can we obtain a transformation formula of the type (22)? Note that a large number of summation formulas are given in [11]. Is it possible to use these formulas or generalizations of the Karlsson–Minton summation formula in order to obtain transformations of the type (22)?

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