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Rota–Baxter Operators on Skew Braces

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Abstract: In this paper, we introduce the concept of Rota–Baxter skew braces, and provide classifications of Rota–Baxter operators on various skew braces, such as \((\mathbb{Z}, +, \circ)\) and \((\mathbb{Z}/(4), +, \circ)\). We also present a necessary and sufficient condition for a skew brace to be a co-inverse skew brace. Additionally, we describe some constructions of Rota–Baxter quasiskew braces, and demonstrate that every Rota–Baxter skew brace can induce a quasigroup and a Rota–Baxter quasiskew brace.

Keywords: Rota–Baxter operator; skew brace; co-inverse skew brace; Rota–Baxter quasiskew brace

MSC: 14L35; 14L99

1. Introduction

In 2007, braces were introduced by Rump in [1] to study the non-degenerate involutive set-theoretic solutions of the Yang–Baxter equation. Braces generalize Jacobson radical rings and can effectively imitate ring theory to discuss braided groups and sets. Moreover, braces possess an inherent left quasigroup structure. The development of brace theory is currently progressing rapidly and with great momentum, as evidenced by [2–7].

In 2017, Guarnieri and Vendramin introduced the concept of skew left braces [8] and proved that every skew left brace provides a non-involutive solution to the Yang–Baxter equation. Skew left braces have been widely applied in various branches of mathematics, including connections to regular subgroups [9], Hopf–Galois extensions [10], triply factorized groups [11], Garside theory [12,13], ring theory [14,15], flat manifolds [16], and pre-Lie algebras [17].

In 2021, Guo, Lang, and Sheng introduced Rota–Baxter groups in [18]. Smooth Rota–Baxter operators on Lie groups were proved to be differentiable, which, in turn, led to the derivation of the factorization theorem of Semenov–Tian–Shansky for Lie groups via the factorization theorem for Rota–Baxter Lie groups.

A Rota–Baxter group is a group \(G\) equipped with a map \(R : G \to G\) satisfying the identity

\[ R(g)R(h) = R(gR(g)hR(g)^{-1}), \]

where \(g, h \in G\). This notion emerged as a group version of Rota–Baxter operators that was defined on an algebra. In the 1960s, Rota–Baxter algebras first appeared in the work of Baxter [19] in the realm of probability theory. In the context of Lie algebras, physicists rediscovered the Rota–Baxter operator of weight zero in the 1980s as the operator form of the classical Yang–Baxter equation [20]. In 2000, Aguiar demonstrated that a Rota–Baxter algebra of weight zero possesses the structure of a dendriform algebra, thereby establishing a connection between Rota–Baxter algebras and dendriform algebras [21]. In 2019, Li Guo and Zongzhu Lin studied the representation and module theory of Rota–Baxter algebras [22]. Subsequently, Zheng, Guo, and Zhang presented a natural generalization of Rota–Baxter modules, namely the concept of Rota–Baxter paired modules [23]. Later research on Rota–Baxter algebra has connections with mathematical physics (classical and quantum Yang–Baxter equations) [24], mixable shuffle product constructions [25], Hopf
algebras [26–28], and renormalization of perturbative quantum field theory [29]. For further details, see [30].

After the seminal work [18], investigations into Rota–Baxter groups progressed through subsequent research [31–34]. In 2022, Bardakov and Gubarev studied Rota–Baxter operators on abstract groups, and established a connection between Rota–Baxter groups and skew left braces [35]. They proved that every Rota–Baxter group can obtain a skew left brace \((G, \circ, \cdot)\) by defining a new binary operation \(a \circ b = aR(a)bR(a)^{-1}\), where \(a, b \in G\). Additionally, they showed that each skew left brace could be embedded into a Rota–Baxter group.

In light of the aforementioned facts, it is natural to ask: can Rota–Baxter operators be directly defined on skew braces, and how does one provide such a definition?

The purpose of this paper is to introduce the concept of Rota–Baxter operators on skew braces and study the structures of Rota–Baxter skew braces.

The research background and content of this paper are summarized in the diagram—the black arrows represent the background of the research, while the red arrows denote the main research content.

2. Skew Brace and Co-Inverse Skew Braces

In this section, we introduce the notion of co-inverse skew braces for the first time and provide the necessary and sufficient condition for every skew brace to be a co-inverse skew brace. We also study the relationships between co-inverse skew braces and two-sided skew braces.

**Definition 1 ([8,36]).** (1) A skew left brace is a set \((A, \circ, \cdot)\) equipped with two group operations “\(\circ\)” and “\(\cdot\)” such that the following equation:

\[
a \circ (b \cdot c) = (a \circ b) \cdot a^{-1} \cdot (a \circ c)
\]

holds for all \(a, b, c \in A\), where \(a^{-1}\) is the inverse of an element \(a\) in \((A, \cdot)\).

(2) A skew left brace \((A, \circ, \cdot)\) is called trivial if \(a \circ b = a \cdot b\) for all \(a, b \in A\).

(3) A skew left brace \((A, \circ, \cdot)\) is called a two-sided skew brace if

\[
(a \cdot b) \circ c = (a \circ c) \cdot c^{-1} \cdot (b \circ c)
\]

for all \(a, b, c \in A\).

For a skew left brace \((A, \circ, \cdot)\), we easily see that the two groups \((A, \circ)\) and \((A, \cdot)\) have the same identity element denoted by \(e\).
Without further comment, we call a skew left brace simply a skew brace, and denote $a \cdot b$ by $ab$ for $a, b \in A$, and denote the inverse of an element $a$ in $(A, \circ)$ by $a^{-1}$.

**Example 1** ([35,37]). (1) Let $(\mathbb{Z}, +)$ be an integer additive group. Then, $(\mathbb{Z}, \circ, +)$ is a skew brace, where

$$a \circ b = a + (-1)^b b,$$

for all $a, b \in \mathbb{Z}$.

(2) The additive group $\mathbb{Z}/(4)$ is a skew brace with circ operation “$\circ$”:

$$x \circ y = x + y + 2xy,$$

for any $x, y \in \mathbb{Z}/(4)$.

**Example 2.** Let $(A, \circ, \cdot)$ be a skew brace with an Abelian group $(A, \circ)$. Then, it is also a two-sided skew brace.

**Proof.** It suffices to show that $(ab) \circ c = (a \circ c) c^{-1} (b \circ c)$, for any $a, b, c \in A$. Since $(A, \circ, \cdot)$ is a skew brace and $(A, \circ)$ is an Abelian group, we can obtain that

$$(ab) \circ c = c \circ (ab) = (c \circ a) c^{-1} (c \circ b) = (a \circ c) c^{-1} (b \circ c).$$

\[ \square \]

**Definition 2.** A skew brace $(A, \circ, \cdot)$ is called a co-inverse skew brace if the inverses $a^{-1}$ and $a^{o^{-1}}$ of every element $a$ in two groups $(A, \cdot)$ and $(A, \circ)$ are identical.

**Remark 1.** (1) It is obvious that every trivial skew brace $(A, \circ, \cdot)$ is a co-inverse skew brace.

(2) Let $(A, \circ, \cdot)$ be an almost trivial skew brace, that is, $(A, \circ, \cdot)$ is a skew brace satisfying the condition “$a \circ b = b \cdot a$” for any $a, b \in A$. Then, it is easy to see that $(A, \circ, \cdot)$ is a co-inverse skew brace.

(3) It is easy to see that the skew brace $(\mathbb{Z}, \circ, +)$ in Example 1 is not a two-sided skew brace. This is since

$$2 \circ 3 = 5 \neq -7 = (1 \circ 3) - 3 + (1 \circ 3).$$

Moreover, the skew brace $(\mathbb{Z}, \circ, \cdot)$ is not a co-inverse skew brace since

$$1^{o^{-1}} = 1 \neq -1 = 1^{-1}.$$

**Proposition 1.** Let $(A, \circ, \cdot)$ be a co-inverse skew brace. Then, it is a two-sided skew brace.

**Proof.** Since $(A, \circ, \cdot)$ is a skew brace, we have

$$a^{-1} \circ (b^{-1} c^{-1}) = (a^{-1} \circ b^{-1}) a (a^{-1} \circ c^{-1})$$

for all $a, b, c \in A$. Hence,

$$(a^{-1} \circ (b^{-1} c^{-1}))^{-1} = (a^{-1} \circ c^{-1})^{-1} a^{-1} (a^{-1} \circ b^{-1})^{-1},$$

that is, we have

$$(cb) \circ a = (c \circ a) a^{-1} (b \circ a).$$

Thus, $(A, \circ, \cdot)$ is a two-sided skew brace. \[ \square \]

**Remark 2.** It is easy to check that the skew brace $(\mathbb{Z}/(4), \circ, +)$ in Example 1 is a two-sided brace by Example 2. But it is not a co-inverse skew brace since

$$I^{-1} = 3 \neq 1 = I^{o^{-1}}.$$
Lemma 1. Let \((A, \circ, \cdot)\) be a two-sided skew brace. Then, for any \(g, h \in A\),

1. \(g^{-1} \circ g^{-1} = g^{-1} \circ g^{-1}\);
2. \((g \circ h)g^{-1}(g \circ h^{-1}) = (h \circ g)g^{-1}(h^{-1} \circ g)\).

Proof. Firstly, since

\[ g^{-1} \circ e = (g^{-1} \circ g)(g^{-1} \circ g^{-1}) = (g^{-1} \circ g^{-1})(g^{-1} \circ g), \]

we obtain that

\[ g^{-1} \circ g^{-1} = g^{-1} \circ g^{-1}. \]

Secondly, since

\[ g = g \circ e = g \circ (h \circ h^{-1}) = (g \circ h)g^{-1}(g \circ h^{-1}), \]

\[ g = e \circ g = (h \circ h^{-1}) \circ g = (h \circ g)g^{-1}(h^{-1} \circ g), \]

we can obtain that

\[ (g \circ h)g^{-1}(g \circ h^{-1}) = (h \circ g)g^{-1}(h^{-1} \circ g). \]

\( \square \)

Lemma 2 ([8]). Let \((A, \circ, \cdot)\) be a skew brace and

\[ \lambda : (A, \circ) \rightarrow \text{Aut}(A, \cdot), a \mapsto \lambda_a(b) = a^{-1}(a \circ b). \]

Then, \(\lambda\) is a group homomorphism, and it follows that

\[ a \circ b = a_{\lambda_a(b)}, ab = a \circ \lambda_{a^{-1}}(b). \]

Proposition 2. Let \((A, \circ, \cdot)\) be a skew brace. Then, \((A, \circ, \cdot)\) is a co-inverse skew brace if, and only if, \(a \circ a = a \cdot a\) for all \(a \in A\).

Proof. Assume that \((A, \circ, \cdot)\) is a co-inverse skew brace. Then, by Lemma 2, we have

\[ ab = a \circ \lambda_{a^{-1}}(b) = a \circ (a(a^{-1} \circ b)) = (a \circ a)a^{-1}(a \circ a^{-1} \circ b) = (a \circ a)a^{-1}b, \]

for any \(a, b \in A\), so \(a \cdot a = a \circ a\).

Conversely, if \(a \cdot a = a \circ a\) for all \(a \in A\), we have

\[ a = a \circ (aa^{-1}) = (a \circ a)a^{-1}(a \circ a^{-1}) = (a \cdot a)a^{-1}(a \circ a^{-1}) = a(a \circ a^{-1}). \]

So, we have \(a \circ a^{-1} = e\), that is, \(a^{-1} = a^c. \)

\( \square \)

Remark 3. The skew brace \((\mathbb{Z}, \circ, +)\) in Example 1 is not a non-trivial co-inverse subskew brace.

In fact, it is easy to see that every skew brace of \((\mathbb{Z}, \circ, +)\) is the form \((n\mathbb{Z}, \circ, \cdot), n \in \mathbb{Z}\).

If \((n\mathbb{Z}, \circ, \cdot)\) is a co-inverse skew brace, then, by Proposition 2, we have

\[ a \circ a = a + a, \]

for all \(a \in n\mathbb{Z}\), that is, \(a + (−1)^{a}a = a + a. \)

However, it is easy to see that \((2\mathbb{Z}, \circ, \cdot)\) is a trivial skew brace.
3. Rota–Baxter Skew Braces

In this section, we first introduce the notion of Rota–Baxter skew braces and give the classification of Rota–Baxter operators on some skew braces \((\mathbb{Z}, +, \circ)\) and \((\mathbb{Z}/(4), +, \circ)\), respectively. Furthermore, we study the structures of Rota–Baxter skew braces.

**Definition 3.** Let \((A, \circ, \cdot)\) be a skew brace.

1. A map \(R : A \rightarrow A\) is called a Rota–Baxter operator of weight 1 if for any \(x, y \in A\),
   \[ R(x) \circ R(y) = R(x \circ (R(x)yR(x)^{-1})). \]  

2. A map \(R : A \rightarrow A\) is called a Rota–Baxter operator of weight \(-1\) if
   \[ R(x) \circ R(y) = R((R(x)yR(x)^{-1}) \circ x). \]

In what follows, we call the quadruple \((A, \circ, \cdot, R)\) a Rota–Baxter skew brace if \(R\) is a Rota–Baxter operator of weight 1 on a skew brace \((A, \circ, \cdot)\).

**Proposition 3.** Let \((A, \circ, \cdot, R)\) be a co-inverse Rota–Baxter skew brace of weight 1 (resp. \(-1\)). Define \(R'(x) = R(x - 1)\), for all \(x \in A\). Then, \((A, \circ, \cdot, R')\) is a co-inverse Rota–Baxter skew brace of weight \(-1\) (resp. 1).

**Proof.** In fact, for any \(x, y \in A\), we have
\[
R'(x) \circ R'(y) = R(x^{-1}) \circ R(y^{-1})
= R(x^{-1} \circ (R(x^{-1})y^{-1}R(x^{-1})^{-1}))
= R'((R(x^{-1})yR(x^{-1})^{-1}) \circ x)
= R'((R'(x)yR'(x)^{-1}) \circ x).
\]
So, \((A, \circ, \cdot, R')\) is a co-inverse Rota–Baxter skew brace of weight \(-1\).

In a similar way, we can prove the other situation. \(\Box\)

**Example 3.** All Rota–Baxter operators on the skew brace \((\mathbb{Z}, \circ, +)\) in Example 1 are given as follows:

1. Let \(R(1) = 0\).
   - (i) if \(R(-1) = 0\), we know that \(R(x) = 0\) for any \(x \in \mathbb{Z}\); so, in this case, \(R = 0\);
   - (ii) if \(R(-1) = a\), where \(a\) is an odd number, we have
     \[
     R(x) = \begin{cases} 
     0, & x \equiv 0, 1 \mod 4, \\
     a, & x \equiv 2, 3 \mod 4.
     \end{cases}
     \]

2. Let \(R(1) = b\), where \(b\) is an odd number.
   - (iii) if \(R(-1) = 0\), we have
     \[
     R(x) = \begin{cases} 
     0, & x \equiv 0, 3 \mod 4, \\
     b, & x \equiv 1, 2 \mod 4.
     \end{cases}
     \]
   - (iv) if \(R(-1) = c\), where \(c\) is an odd number, we have
     \[
     R(x) = \begin{cases} 
     k(b - c), & x = 2k, \\
     b + k(b - c), & x = 2k + 1.
     \end{cases}
     \]
Proof. Assume that $R$ is a Rota–Baxter operator on the skew brace $(\mathbb{Z}, \circ, +)$. Then, it must satisfy the following equation:

$$R(x) + (-1)^{R(x)} R(y) = R(x + (-1)^x y),$$

for any $x, y \in \mathbb{Z}$.

Firstly, we show that $R(0) = 0$. If taking $x = y = 0$, we have

$$R(0) + (-1)^{R(0)} R(0) = R(0),$$

so $R(0) = 0$.

Secondly, we find that $R(a) = 0$ or some odd number for an odd number $a$. Taking $x = y = 2k + 1$ ($k \in \mathbb{Z}$) in the Equation (3), we have

$$R(x) + (-1)^{R(x)} R(x) = R(0).$$

So, by $R(0) = 0$, we know that $R(x) = 0$ or $R(x)$ is an odd number.

Next, we discuss $R(1)$, which can be divided into two cases.

1. If $R(1) = 0$. Taking $x = 1$ in the Equation (3), then, we have

$$R(y) = R(1 - y),$$

for all $y \in \mathbb{Z}$.

   (i) If $R(-1) = 0$. Let $x = -1$ in the Equation (3). Then, we have

   $$R(y) = R(-1 - y),$$

   for all $y \in \mathbb{Z}$, so $R$ is a map with period 2 by Equations (4) and (5).

   Since $R(1) = R(0) = 0$, we know that $R(x) = 0$ for all $x \in \mathbb{Z}$.

   (ii) If $R(-1) = a$, which is an odd number. Let $x = -1$ in the Equation (3). Then, we have

   $$a - R(y) = R(-1 - y),$$

   for all $y \in \mathbb{Z}$; so, $R$ is a map with period 4 by Equations (4) and (6).

   Combining $R(0) = R(1) = 0$ with $R(-1) = a$, we easily obtain that

   $$R(x) = \begin{cases} 0, & x \equiv 0, 1 \text{ mod } 4, \\ a, & x \equiv 2, 3 \text{ mod } 4. \end{cases}$$

2. If $R(1) = b$, where $b$ is an odd number. Let $x = 1$ in the Equation (3). Then, we have

$$b - R(y) = R(1 - y),$$

for all $y \in \mathbb{Z}$.

   (iii) If $R(-1) = 0$. Let $x = -1$ in the Equation (3). Then, we have

   $$R(y) = R(-1 - y),$$

   for all $y \in \mathbb{Z}$; so, $R$ is a map with period 4 by Equations (7) and (8).

   Combining $R(0) = 0$, $R(1) = b$ with $R(-1) = 0$, we easily obtain that

   $$R(x) = \begin{cases} 0, & x \equiv 0, 3 \text{ mod } 4, \\ b, & x \equiv 1, 2 \text{ mod } 4. \end{cases}$$
(iv) If $R(-1) = c$, which is an odd number. Let $x = -1$ in the Equation (3). Then, we have
\[ c - R(y) = R(-1 - y), \tag{9} \]
for all $y \in \mathbb{Z}$, so $R(y) = R(y - 2) + b - c$ by Equations (7) and (9).
Combining $R(0) = 0$ and $R(1) = b$ with $R(-1) = c$, by the induction method, we easily obtain that
\[ R(x) = \begin{cases} 
  k(b - c), & x = 2k, \\
  b + k(b - c), & x = 2k + 1.
\end{cases} \]
\[ \square \]

**Example 4.** All Rota–Baxter operators on the skew brace $\mathbb{Z} / (4)$ in Example 1 are given as follows:
\[ R(x) = \begin{cases} 
  0, & x = \bar{0}, \\
  a, & x = \bar{1}, \\
  b, & x = \bar{2}, \\
  a + b + 2ab, & x = \bar{3},
\end{cases} \]
where $R(\bar{1}) = a$ and $R(\bar{2}) = b$ in $\mathbb{Z} / (4)$.

**Proof.** Assume that $R$ is a Rota–Baxter operator on the skew brace $\mathbb{Z} / (4)$. Then, it must satisfy the following equation:
\[ R(x) + R(y) + 2R(x)R(y) = R(x + y + 2xy), \tag{10} \]
for any $x, y \in \mathbb{Z} / (4)$.

Firstly, we show that $R(\bar{0}) = 0$. If taking $x = y = \bar{0}$, then we have $R(\bar{0}) + 2R(\bar{0})R(\bar{0}) = 0$. So, $R(\bar{0}) = 0$.

Secondly, taking $x = \bar{2}$ in the Equation (10), we have
\[ R(\bar{2}) + R(y) + 2R(\bar{2})R(y) = R(\bar{2} + y) \]
for all $y \in \mathbb{Z} / (4)$.

Next, we discuss $R(\bar{2})$, which can be divided into four cases.

1. If $R(\bar{2}) = \bar{0}$, then we have $R(y) = R(\bar{2} + y)$. So, if $R(\bar{1}) = a$, then $R(\bar{3}) = \bar{a}$. Thus,
\[ R(x) = \begin{cases} 
  0, & x = \bar{0}, \\
  a, & x = \bar{1}, \\
  b, & x = \bar{2}, \\
  0, & x = \bar{3}.
\end{cases} \]

2. If $R(\bar{2}) = \bar{1}$, then we have $\bar{1} + 3R(y) = R(\bar{2} + y)$. So, if $R(\bar{1}) = a$, then $R(\bar{3}) = 3a + \bar{1}$. Thus,
\[ R(x) = \begin{cases} 
  0, & x = \bar{0}, \\
  a, & x = \bar{1}, \\
  1, & x = \bar{2}, \\
  3a + 1, & x = \bar{3}.
\end{cases} \]

3. If $R(\bar{2}) = \bar{2}$, then we have $\bar{2} + R(y) = R(\bar{2} + y)$. So, if $R(\bar{1}) = a$, then $R(\bar{3}) = \bar{a} + \bar{2}$. Thus,
\[ R(x) = \begin{cases} 
  0, & x = \bar{0}, \\
  a, & x = \bar{1}, \\
  \bar{2}, & x = \bar{2}, \\
  a + \bar{2}, & x = \bar{3}.
\end{cases} \]
(4) If $R(1) = 3$, then we have $3 + 3R(y) = R(1 + y)$. So, if $R(1) = 3$, then $R(3) = 3\pi + 3$.

Thus,

$$R(x) = \begin{cases} 
0, & x = 0, \\
\pi, & x = 1, \\
3, & x = 2, \\
3\pi + 3, & x = 3.
\end{cases}$$

\[\square\]

**Lemma 3.** Let $R$ be a Rota–Baxter operator on the skew brace $(A, \circ, \cdot)$. Then, for any $g, h \in A$,

1. $R(e) = e$;
2. $R(g) \circ R(R(g)^{-1} h R(g)) = R(g \circ h)$;
3. $R(g)^{-1} = R(R(g)^{-1} g^{-1} R(g))$;
4. If a group $(A, \cdot)$ is an Abelian group, then $R$ is a homomorphism of a group $(A, \circ)$, and $R(g)^{-1} = R(g^{-1})$.

**Proof.** (1) It follows from the equality (1) considered with $x = y = e$.

(2) It follows from the equality (1) considered with $x = g, y = R(g)^{-1} h R(g)$.

(3) It follows from the equality

$$R(g) \circ R(R(g)^{-1} g^{-1} R(g)) \overset{(1)}{=} R(g \circ (R(g)^{-1} g^{-1} R(g) R(g)^{-1})) = R(g \circ g^{-1}) = R(e) = e.$$ 

(4) The conclusion is obvious when a group $(A, \cdot)$ is Abelian. \[\square\]

**Proposition 4.** Let $(A, \circ, \cdot, R)$ be a Rota–Baxter skew brace. Then, a group $(A, \cdot)$ is Abelian if $R$ is an automorphism of a group $(A, \circ)$.

In particular, the converse holds if $R$ is a bijection.

**Proof.** Since

$$R(x) \circ R(y) = R(x \circ (R(x) y R(x)^{-1})) = R(x) \circ R(R(x) y R(x)^{-1}),$$

for any $x, y \in A$, $R(y) = R(R(x) y R(x)^{-1})$. That is, $y = R(x) y R(x)^{-1}$; so, a group $(A, \cdot)$ is Abelian.

The converse is satisfied by the Equation (1). \[\square\]

**Corollary 1.** Let $(A, \circ, \cdot)$ be a skew brace. Define a map as follows:

$$R : A \to A, g \mapsto a \circ g \circ b,$$

for some $a, b \in A$. Then, $R$ is a Rota–Baxter operator and $b = a^{-1}$, if, and only if, a group $(A, \cdot)$ is Abelian, and $R$ is a homomorphism of a group $(A, \circ)$.

**Proof.** It is easy to prove that $R$ is a bijection. Suppose that a group $(A, \cdot)$ is Abelian, and $R$ is a homomorphism of a group $(A, \circ)$, then it is easy to see that the Equation (1) holds. So, $R$ is a Rota–Baxter operator on the skew brace $A$. Since $R(e) = e$, then $b = a^{-1}$.

Conversely, if $R$ is a Rota–Baxter operator on the skew brace $A$ and $b = a^{-1}$, then for any $x, y \in A$, we have

$$R(x) \circ R(y) = a \circ x \circ b \circ a \circ y \circ b = a \circ x \circ y \circ b = R(x \circ y),$$

That is, $R$ is an automorphism of a group $(A, \circ)$. Again, by Proposition 4, a group $(A, \cdot)$ is Abelian. \[\square\]
Lemma 4. Let $R$ be a Rota–Baxter operator on the skew brace $(A, \circ, \cdot)$. Then, $\text{Ker}(R)$ and $\text{Im}(R)$ are subgroups of a group $(A, \circ)$. Moreover, if $R$ is a homomorphism of a group $(A, \cdot)$, then $\text{Ker}(R)$ and $\text{Im}(R)$ are subskew braces.

Proof. Firstly, we show that $\text{Ker}(R)$ is a subgroup of a group $(A, \circ)$.
As a matter of fact, for all $g \in \text{Ker}(R)$, since
\[
R(g^{o-1}) = R(g) \circ R(g^{-1}) = R(g \circ (R(g)g^{o-1}R(g)^{-1})) = R(g \circ g^{-1}) = R(e) = e,
\]
that is, $g^{o-1} \in \text{Ker}(R)$. Moreover, for all $x, y \in \text{Ker}(R)$, we have
\[
e = R(x) \circ R(y) = R(x \circ (R(x)yR(x)^{-1})) = R(x \circ y),
\]
that is, $x \circ y \in \text{Ker}(R)$; so, $\text{Ker}(R)$ is a subgroup of a group $(A, \circ)$.

Secondly, we show that $\text{Im}(R)$ is a subgroup of a group $(A, \circ)$.
Indeed, if $R(g) \in \text{Im}(R)$, we have $R(g)^{o-1} \in \text{Im}(R)$ by Lemma 3. And $R(g) \circ R(h) \in \text{Im}(R)$ by the Equation (1), for any $R(g), R(h) \in \text{Im}(R)$; thus, $\text{Im}(R)$ is a subgroup of a group $(A, \circ)$.

Moreover, if $R$ is a homomorphism of a group $(A, \cdot)$, it is easy to see that $\text{Ker}(R)$ and $\text{Im}(R)$ are subgroups of a group $(A, \cdot)$; thus, $\text{Ker}(R)$ and $\text{Im}(R)$ are subskew braces of $(A, \circ, \cdot)$.

Lemma 5 ([8]). Let $(A, \circ, \cdot)$ be a skew brace. A subset $I$ of the skew brace $(A, \circ, \cdot)$ is called an ideal if it is both a normal subgroup of a group $(A, \circ)$ and a normal subgroup of a group $(A, \cdot)$ and $\lambda_a(I) \subseteq I$ for all $a \in A$.

Let $I$ be an ideal of the skew brace $(A, \circ, \cdot)$. Then, the following conclusions hold:
1) $a \circ I = aI$ for all $a \in A$;
2) $I$ and $A/I$ are skew braces.

In what follows, we give a differentiated condition for a given subgroup $(\text{Ker}(R), \circ)$ in Lemma 4 to be a Rota–Baxter skew brace.

Lemma 6. Let $R$ be a Rota–Baxter operator on the skew brace $(A, \circ, \cdot)$. Then, we have the following conclusions:

(1) Assume that $R$ is a Rota–Baxter operator of weight 1. Then, $R(g \circ x) = R(x)$ if $g \in \text{Ker}(R)$, for all $x \in A$.
Furthermore, if $A = \bigsqcup_{i \in A} \text{Ker}(R) \circ g_i$ is the decomposition of a group $(A, \circ)$ in the disjoint union of right cosets, then $R(x) = R(y)$ if $x$ and $y$ lie in the same right coset.

(2) Assume that $R$ is a Rota–Baxter operator of weight $-1$. Then, $R(x \circ g) = R(x)$ if $g \in \text{Ker}(R)$, for all $x \in A$.
Furthermore, if $A = \bigsqcup_{i \in A} g_i \circ \text{Ker}(R)$ is the decomposition of a group $(A, \circ)$ in the disjoint union of left cosets, then $R(x) = R(y)$ if $x$ and $y$ lie in the same left coset.

Proof. (1) Since $g \in \text{Ker}(R)$, we obtain that
\[
R(x) = R(g) \circ R(x) = R(g \circ (R(g)xR^{-1}(g))) = R(g \circ x),
\]
for all $x \in A$.

Furthermore, if $x, y \in \text{Ker}(R) \circ g_i$, we know that $x = b \circ g_i, y = a \circ g_i$ for some $a, b \in \text{Ker}(R)$, so
\[
R(x) = R(g_i) = R(y).
\]
The converse is obvious. 
(2) It can be similarly proved. □

By the above lemmas, we can obtain the following:

**Proposition 5.** Let \( (A, o, R) \) be a Rota–Baxter skew brace of weight \(-1\) (resp.1). If a group \((A, o)\) is an Abelian group and \(R\) is a group homomorphism of a group \((A, \cdot)\), then \((A/Ker(R), o, \cdot, R)\) is also a Rota–Baxter skew brace of weight \(-1\) (resp.1)

**Proof.** It is easy to see that \(Ker(R)\) is both a normal subgroup of a group \((A, o)\) and a normal subgroup of a group \((A, \cdot)\).

Firstly, we prove that \(Ker(R)\) is an ideal of \(A\). As a matter of fact, we have

\[
R(a^{-1}(a \circ g)) = R(a^{-1})R(a \circ g) = R(a^{-1})R(a) = e,
\]

for any \(a \in A, g \in Ker(R)\). According to Lemma 2, \(\lambda_a(Ker(R)) \subseteq Ker(R)\); so, \(Ker(R)\) is an ideal of \(A\).

Secondly, we check that \(R\) is also a Rota–Baxter operator on the skew brace \((A/Ker(R), o, \cdot)\).

In fact, for any \(x, y \in (A/Ker(R), o, \cdot)\), we have

\[
R\left((R(x)(R(y)(x)^{-1}) \circ x)\right) = R((R(x)\circ R(x)^{-1}) \circ x) \tag{3}
\]

\[
= R((R(x)\circ R(x)^{-1}) \circ x) \tag{4}
\]

\[
= R(x) \circ R(y) \tag{5}
\]

\[
= R(x) \circ R(y).
\]

So, \(R\) is a Rota–Baxter operator of weight \(-1\) on the skew brace \((A/Ker(R), o, \cdot)\). □

**4. Constructions of Rota–Baxter Quasiskew Braces**

In this section, we discuss the conditions under which a Rota–Baxter operator on a group is also a Rota–Baxter operator on a skew brace induced by a Rota–Baxter group. We also introduce the concept of quasigroups and Rota–Baxter quasiskew braces, and prove that every Rota–Baxter skew brace can be associated with a Rota–Baxter quasiskew brace. Moreover, we provide some constructions of Rota–Baxter quasiskew braces.

**Lemma 7** ([18]). Let \((G, R)\) be a Rota–Baxter group. Then, we have the following conclusions.

1. The pair \((G, o)\), with the multiplication:

\[
g \circ h = gR(g)hR(g)^{-1}
\]

is also a group, called the descendent group of a Rota–Baxter group \((G, \cdot)\).

2. \(R\) is also a Rota–Baxter operator on a group \((G, o)\), and \((G, o, \cdot)\) is a skew brace.

3. The map \(R : (G, o) \rightarrow (G, \cdot)\) is a homomorphism of Rota–Baxter groups.

In the following, we discuss the conditions under which a Rota–Baxter operator on a group is also a Rota–Baxter operator on a skew brace induced by a Rota–Baxter group.

**Proposition 6.** Let \((G, R)\) be a Rota–Baxter group. Define

\[
a \circ b = aR(a)bR(a)^{-1},
\]

for all \(a, b \in G\). Then, \((G, o, \cdot, R)\) is a Rota–Baxter skew brace if \(R\) is a group homomorphism for a group \((G, \cdot)\).

**Proof.** By Lemma 7, we know that \((G, o, \cdot)\) is a skew brace. In what follows, it suffices to show that \(R\) is a Rota–Baxter operator on the skew brace \((G, o, \cdot)\).
In fact, for any \( x, y \in G \), we have

\[
R(x) \circ R(y) = R(x)R(R(x))R(y)R(R(x))^{-1} = R(x)R(x)\frac{yR(x)^{-1}}{1} = R(xR(x)R(x)^{-1})R(x)^{-1} = R(x)\circ (R(x)yR(x)^{-1})).
\]

So, \((G, \circ, \cdot, R)\) is a Rota–Baxter skew brace.  

In [31], it has been established that \( R(g) = g^{-1} \) for any \( g \in G \) is always a Rota–Baxter operator on a group \((G, \cdot)\). However, we know that this result does not hold true for a Rota–Baxter skew brace.

In the following example, we can see that neither \( R(g) = g^{-1} \) nor \( R(g) = g^{o-1} \) is a Rota–Baxter operator on a skew brace \((G, \circ, \cdot)\).

**Example 5.** Let \((Z, +, o)\) be a skew brace introduced in Example 1. Then, neither \( R(g) = g^{-1} \) nor \( R(g) = g^{o-1} \) is a Rota–Baxter operator on the skew brace \((Z, +, o)\).

**Proof.** (1) \( R(g) = g^{-1} \) is not a Rota–Baxter operator.

It is easy to see that \( 1^{-1} = 1, 1^{-1} = -1, 0^{-1} = 0^{-1} = 0 \). If taking \( g = 0 \) and \( h = 1 \), we have

\[
R(g) \circ R(h) = 0 \circ (-1) = 0 + (-1)^01 = 1,
\]

so, \( R(g) \circ R(h) \neq R(g \circ (R(g)hR(g)^{-1})). \)

(2) \( R(x) = x^{o-1} \) is not a Rota–Baxter operator. Since \( 2^{o-1} = -2 = 2^{-1} \), if taking \( g = 1, h = 2 \), we have

\[
R(g) \circ R(h) = R(1) \circ R(2) = 1 \circ -2 = 1 + (-1)^1(-2) = 3,
\]

so, \( R(g) \circ R(h) \neq R(g \circ (R(g)hR(g)^{-1})). \)

Let \((A, \circ, \cdot)\) be a skew brace. Define the two adjoint actions: for any \( g \in A \),

\[
Ad_g^A : A \rightarrow A, Ad_g^A(h) = ghg^{-1},
\]

\[
Ad_g^o : A \rightarrow A, Ad_g^o(h) = g \circ h \circ g^{-1}.
\]

**Proposition 7.** Let \((A, \circ, \cdot)\) be a skew brace, and \( Ad_g^o = Ad_g^A \) for any \( g \in A \). Then, \( R \) is a Rota–Baxter operator on the skew brace \((A, \circ, \cdot)\) if, and only if, \( R \) is a Rota–Baxter operator on a group \((A, \circ)\).

In particular, if \((A, \circ, \cdot)\) is a co-inverse skew brace and \( Ad_g^o = Ad_g^A \) for any \( g \in A \), then \( R'(g) = g^{-1} \) is a Rota–Baxter operator on the skew brace \((A, \circ, \cdot)\).

**Proof.** Let \( R \) be a Rota–Baxter operator on the skew brace \((A, \circ, \cdot)\). Then, for any \( x, y \in A \),

\[
R(x) \circ R(y) = R(x \circ (R(x)yR(x)^{-1})) = R(x \circ (R(x) \circ y \circ R(x)^{o-1})),
\]

that is, \( R \) is a Rota–Baxter operator on a group \((A, \circ)\).

We can similarly prove the converse.
It is obvious that $R'(g) = g^{o^{-1}}$ is a Rota–Baxter operator on a group $(A, o)$. Thus, by the above result, we know that $R'(g) = g^{-1}$ is a Rota–Baxter operator on the co-inverse skew brace $(A, o, \cdot)$.

**Definition 4** ([38]). A left quasigroup is a set $(A, *)$ equipped with a binary operation “*” such that for all $a$ and $b$ in $A$, there is a unique element $c$ such that

$$a * c = b.$$ 

In what follows, a left quasigroup is simply called a quasigroup.

**Definition 5.** (1) A left quasiskew brace is a set $(A, o, \cdot)$ equipped with two binary operations “$o$” and “$\cdot$” such that $(A, o)$ is a quasigroup, $(A, \cdot)$ is a group and satisfies the following equation:

$$a \circ (b \cdot c) = (a \circ b) \cdot a^{-1} \circ (a \circ c)$$

for all $a, b, c \in A$.

(2) A map $R : A \to A$ is called a Rota–Baxter operator on the left quasiskew brace $(A, o, \cdot)$ if for any $x, y \in A$,

$$R(x) \circ R(y) = R(x \circ (R(x)yR(x)^{-1})).$$  \hspace{1cm} (11)

(3) Let $(A, o_{1}, 1, R_{1})$ and $(D, o_{2}, 2, R_{2})$ be two quasiskew braces. A map $f : A \to D$ is called a Rota–Baxter quasiskew brace homomorphism if $f$ is both a quasigroup homomorphism from the $(A, o_{1})$ to $(D, o_{2})$, and a group homomorphism from the $(A, 2)$ to $(D, 2)$ such that $f \circ R_{1} = R_{2} \circ f$.

Without further comment, we call a left quasiskew brace simply a quasiskew brace.

**Example 6.** (1) It is obvious that every Rota–Baxter skew brace is a quasiskew brace.

(2) Let $\mathbb{Z}$ be a set of integers. Then, $(\mathbb{Z}, -, +)$ is a quasiskew brace with two operations subtraction “$-$” and addition “$+$”.

Moreover, all Rota–Baxter operators on the quasiskew brace $(\mathbb{Z}, - , +)$ are all group homomorphisms of the additive group $(\mathbb{Z}, +)$.

**Proof.** (1) It is straightforward. (2) It is easy to see that $(\mathbb{Z}, -)$ is a quasigroup and $(\mathbb{Z}, +)$ is a group.

Moreover, for all $a, b, c \in \mathbb{Z}$, we have

$$a - (b + c) = a - b - c = (a - b) - a + (a - c).$$

So, $(\mathbb{Z}, -, +)$ is a quasiskew brace.

Assume that $R$ is a Rota–Baxter operator on a quasiskew brace $(\mathbb{Z}, -, +)$. Then, by the equality (11), $R$ satisfies the following equation:

$$R(x) - R(y) = R(x - y),$$

for any $x, y \in \mathbb{Z}$. This means that all group homomorphisms of the additive group $(\mathbb{Z}, +)$ are Rota–Baxter operators on the quasiskew brace $(\mathbb{Z}, - , +)$.

In the following, we give a construction of Rota–Baxter quasiskew braces via Rota–Baxter skew braces.

**Lemma 8.** Let $(A, o, \cdot, R)$ be a Rota–Baxter skew brace. Define

$$a * b = a \circ (R(a)bR(a)^{-1}),$$

for any $a, b \in A$. Then, $(A, *)$ is a quasigroup.
Proof. It is easy to see that
\[ a * x = b \iff R(a)xR(a)^{-1} = a^o^{-1} \circ b, \]
for any \( a, b \in A \). So, there is a unique element \( x = R(a)^{-1}(a^o^{-1} \circ b)R(a) \) such that \( a * x = b \). \( \square \)

**Proposition 8.** Let \((A, \circ, \cdot, R)\) be a Rota–Baxter skew brace. Then, we have the following conclusions:

The triple \((A, *, \cdot)\) with the multiplication
\[ a * b = a \circ (R(a)bR(a)^{-1}) \]
is a quasiskew brace, called the descendent quasiskew brace of the Rota–Baxter skew brace \((A, \circ, \cdot)\).

In particular, if \( R \) is a homomorphism on a group \((A, \cdot)\), then \((A, *, \cdot, R)\) is a Rota–Baxter quasiskew brace, and \( R : (A, *, \cdot) \to (A, \circ, \cdot) \) is a homomorphism of Rota–Baxter quasiskew braces.

**Proof.** (1) It follows that \((A, *)\) is a quasigroup by Lemma 8. Moreover, for any \( a, b, c \in A \), we have
\[ a * (bc) = a \circ (R(a)bcR(a)^{-1}) = a \circ ((R(a)bR(a)^{-1})(R(a)cR(a)^{-1})) = (a \circ (R(a)bR(a)^{-1}))a^{-1}(a \circ (R(a)cR(a)^{-1})) = (a \circ b)a^{-1}(a \circ c). \]
So, \((A, *, \cdot)\) is a quasiskew brace.

(2) Suppose that \( R \) is an endomorphism of a group \((A, \cdot)\). Then, for all \( x, y \in A \), we have
\[ R(x) \circ R(y) = R(x) \circ (R(R(x))R(y)R(R(x)^{-1})) = R(x) \circ R(R(x)yR(x)^{-1}) = R(x) \circ (R(x)yR(x)^{-1}R(x)^{-1}) = R(x) \circ (R(x)yR(x)^{-1}). \]
So, \( R \) is a Rota–Baxter operator on the quasiskew brace \((A, *, \cdot)\).

It is easy to see that \((A, \circ, *, \cdot, R)\) is a Rota–Baxter quasiskew brace. In what follows, we need to show that \( R \) is a quasigroup homomorphism from \((A, *)\) to \((A, \circ)\).

In fact, for any \( x, y \in A \), we have
\[ R(x * y) = R(x \circ (R(x)yR(x)^{-1}(x))) = R(x) \circ R(y). \]

\( \square \)

Concluding remarks: The concept of Rota–Baxter skew braces is first introduced, which is a generalization of Rota–Baxter groups. The structures of Rota–Baxter skew braces are studied and investigated. However, we demonstrate that every Rota–Baxter skew brace can only induce a quasiskew brace instead of a skew brace.

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