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Double-Observer-Based Bumpless Transfer Control of Switched Positive Systems

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Abstract: This paper investigates the bumpless transfer control of linear switched positive systems based on state and disturbance observers. First, state and disturbance observers are designed for linear switched positive systems to estimate the state and the disturbance. By combining the designed state observer, the disturbance observer, and the output, a new controller is constructed for the systems. All gain matrices are described in the form of linear programming. By using co-positive Lyapunov functions, the positivity and stability of the closed-loop system can be ensured. In order to achieve the bumpless transfer property, some additional sufficient conditions are imposed on the control conditions. The novelties of this paper lie in that (i) a novel framework is presented for positive disturbance observer, (ii) double observers are constructed for linear switched positive systems, and (iii) a bumpless transfer controller is proposed in terms of linear programming. Finally, two examples are given to illustrate the effectiveness of the proposed results.

Keywords: linear switched positive systems; disturbance observe; bumpless transfer control; linear programming

MSC: 93C28



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1. Introduction

Positive systems are an important class of systems whose states and outputs are always non-negative for any non-negative initial states and inputs. They have been covered in [1–3] and are widely used in applications such as water systems [4], ecology [5], industrial engineering [6], and so on. Linear switched positive systems (SPSs) are a class of hybrid systems consisting of a finite number of positive subsystems and a switching rule. The stability and stabilization of SPSs have attracted great attention [7–10]. For general (nonpositive) switched systems, quadratic Lyapunov functions are commonly used [11–14]. Owing to the positivity requirement, quadratic Lyapunov functions are not the best choice for SPSs. It has also been verified that the co-positive Lyapunov function (CLF) is more suitable for SPSs [7–10]. In [9], a common CLF was constructed for the exponential stability of SPSs. For discrete-time SPSs, Liu constructed a switched CLF to reduce the conservativeness of common CLF [15]. In [16], multiple CLFs and average dwell time (ADT) switching were first introduced for SPSs. The work in [17] discusses the stability in terms of CLF and further designs a state-feedback controller. Existing results verify that CLF is powerful for handling the issues of SPSs.

The observer technique is widely used in switched systems [18], multiagent systems [19], nonlinear systems [20], etc. For positive systems, there are a number of features that are different from general systems. Thus, the utilization of a state observer in positive systems is also different from general systems. In [21], a Luenberger-type observer was designed for positive systems using linear programming. In [22], a novel design approach to the observer and a static state-feedback controller were presented for positive systems characterized by interval uncertainties and time delay. It should be noted that

the observer of positive systems must also exhibit positivity. This point arises from the fact that a negative attribute within the output of observer is inadequate for accurately estimating the non-negative state. Different from positive systems, the observer of SPSs has to consider not only the positivity but also the effects of the switching signal on the observer. In [23], two types of interval switched positive observers were introduced for discrete-time SPSs. In [24], asynchronous observers were presented for SPSs. In the above literature, disturbance factor was assumed to be unmeasurable and set as a disturbance signal. For example, the work in [24] discussed the L_∞ performance of the systems with regard to the disturbance. The disturbance attenuation performance can achieve some prescribed attenuation effects for the disturbance. However, it cannot accurately estimate the disturbance. The lack of disturbance information may deteriorate the system performance. The estimation of disturbance was one of effective strategies to acquire disturbance information in [25]. In [26], the problem of disturbance attenuation in stochastic systems was solved by introducing a disturbance observer. In positive systems, the work in [27] provides descriptions of the utilization of disturbance observers in positive disturbances. For positive Markovian jump systems, a linear-programming-based disturbance observer was designed in [28]. The work in [29] proposed an event-triggered disturbance-observer-based controller to ensure the positivity and stability of SPSs. A situation where both states and disturbances are unmeasurable is not considered in the aforementioned literature. Therefore, it needs to design both state and disturbance observers simultaneously. Some issues arise, such as (i) how to jointly design the state and disturbance observers, (ii) how to construct a double-observer-based controller, and (iii) how to describe the corresponding conditions in a tractable form.

For switched systems, there exist transient bumps when the switching occurs. The signal bump is a specific behavior of switched systems. With increasing demands on the accuracy of switched systems, the effects caused by transient bumps need to be eliminated. As a result, bumpless transfer control was proposed to ensure the smooth switching of the systems by adopting appropriate control strategies [30–32]. This can avoid negative effects caused by the signal bump, such as system performance damage, equipment failure, etc. For switched linear parameter-varying systems, an H_∞ bumpless transfer control method was presented in [33,34]. The work in [35] proposed a control strategy based on mode-dependent ADT and bumpless transfer to achieve stability in switched positive delayed nonlinear systems. By virtue of event-triggered strategies, the work in [36] introduced an event-triggered bumpless transfer controller. The work in [37] applied the bumpless transfer control of switched systems for aerospace engines. In [38], a bumpless transfer feedback controller was proposed for SPSs, which can ensure a lower control signal bump caused by the switching and guarantee the L_1 -gain performance of the systems. Note that few results exist on state and disturbance observer-based bumpless transfer control of SPSs. How to design a double-observer-based bumpless transfer controller of SPSs motivates us to carry out this work.

This paper aims to deal with the problem of the bumpless transfer control of SPSs by building a state observer and a disturbance observer. First, a state observer and a disturbance observer are established based on the matrix decomposition technique, respectively. Then, a controller is designed by means of the estimated states and disturbances. Meanwhile, the bumpless transfer property of the controller is considered by imposing some conditions on the controller. The rest of this paper is presented as follows. Section 2 gives some preliminaries. In Section 3, the main results are presented. Section 4 provides two examples to verify the effectiveness of the presented results. Section 5 concludes this paper.

2. Preliminaries

Consider the following linear switched system:

$$\begin{aligned}\dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}u_{\sigma(t)}(t) + E_{\sigma(t)}w(t), \\ y(t) &= C_{\sigma(t)}x(t) + D_{\sigma(t)}w(t),\end{aligned}\tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $w(t) \in \mathbb{R}_+^r$, and $y(t) \in \mathbb{R}^s$ represent the system state, the system input, the disturbance, and the output, respectively, $\sigma(t)$ is the switching law and takes values from a finite set $\mathcal{S} = \{1, 2, \dots, J\}$, $J \in \mathbb{N}^+$, and it is continuous from the right everywhere for a switching sequence $0 \leq t_0 \leq t_1 \leq \dots$. In this paper, it is assumed that control signals $u(t)$ and disturbances $w(t)$ in SPSs are continuous and the disturbance in the system (1) is non-negative. Assume that $A_{\sigma(t)}$ is a Metzler matrix and $B_{\sigma(t)} \succeq 0$, $C_{\sigma(t)} \succeq 0$, $D_{\sigma(t)} \succeq 0$, $E_{\sigma(t)} \succeq 0$.

Some definitions and lemmas are introduced for the facilitation of later developments.

Definition 1 ([1]). A system is said to be positive if all its states and outputs are non-negative for any nonnegative initial conditions.

Lemma 1 ([1,2]). A matrix A is a Metzler matrix if and only if there exists a scalar δ such that $A + \delta I \succeq 0$.

Lemma 2 ([1,3]). The system (1) is positive if and only if $A_{\sigma(t)}$ is a Metzler matrix, and $B_{\sigma(t)} \succeq 0$, $C_{\sigma(t)} \succeq 0$, $D_{\sigma(t)} \succeq 0$, $E_{\sigma(t)} \succeq 0$.

Lemma 3 ([1,3]). For a positive system $\dot{x}(t) = Ax(t)$, the following statements are equivalent:

- (i) The matrix A is a Hurwitz matrix;
- (ii) There exists a vector $v \succ 0$ such that $A^\top v \prec 0$;
- (iii) The system is stable.

Definition 2 ([1]). For a switching signal $\sigma(t)$ and $0 \leq t_1 \leq t_2$, denote the switching number of $\sigma(t)$ by $N_{\sigma(t)}(t_2, t_1)$. If

$$N_{\sigma(t)}(t_2, t_1) \leq N_0 + (t_2 - t_1) / \tau^*$$

holds for $N_0 \geq 0$ and $\tau^* > 0$, then τ^* is an ADT of the switching signal $\sigma(t)$ and N_0 is the chatter bound.

Definition 3 ([30]). Given a reference controller: $u^*(t) = K_{1\sigma(t)}^* \hat{x}(t) + K_{2\sigma(t)}^* \hat{w}(t) - K_{3\sigma(t)}^* y(t)$, the system (1) is said to have bumpless transfer performance if the condition:

$$|u_j(t) - u_j^*(t)| \leq \alpha \mathbf{1}_o^\top \hat{x}(t) + \beta \mathbf{1}_r^\top \hat{w}(t) + \gamma \mathbf{1}_s^\top y(t)$$

holds, where $\hat{x}(t)$ and $\hat{w}(t)$ are the state and disturbance observer states to be designed later, $\alpha \geq 0$, $\beta \geq 0$, and $\gamma \geq 0$ are given scalars called the bumpless transfer performance level, $K_{1\sigma(t)}^*$, $K_{2\sigma(t)}^*$, and $K_{3\sigma(t)}^*$ are given matrices with compatible dimension, and $u_j(t)$ and $u_j^*(t)$ are the j th element of $u(t)$ and $u^*(t)$.

Notation 1. $\mathbb{R}^n$, $\mathbb{R}_+^n$ and $\mathbb{R}^{n \times m}$ represent the sets of n -dimensional vectors, non-negative vectors, and $n \times m$ real matrices, respectively. $\mathbb{N}$ and $\mathbb{N}_+$ represent the sets of non-negative and positive integers, respectively. A matrix I_n symbolizes the n -dimensions identity matrix. The symbols $\succ$ and $\succeq$ hold for components. Define $\mathbf{1}_n = (\underbrace{1, \dots, 1}_n)^\top$ and $\mathbf{1}_n^{(i)} = (\underbrace{0, \dots, 0}_{i-1}, 1, \underbrace{0, \dots, 0}_{n-i})^\top$. p stands for the switching signal $\sigma(t)$. Symbols p and q denote different subsystems.

Remark 1. The term $u^*(t)$ represents an objective control law and it is dependent on a given gain matrix K^* . One can choose the gain matrix based on additional factors, such as practical desire, limitation of element, etc. The main objectives of the introduction of $u^*(t)$ are to satisfy practical requirements and reduce the fluctuations of the designed controller.

3. Main Results

State and disturbance observers of SPSs are first designed in this section. Then, the bumpless transfer is achieved in the second subsection.

3.1. State and Disturbance Observers

The disturbance is generated by the exogenous system:

$$\dot{\xi}(t) = Y_{\sigma(t)}\xi(t), w(t) = \Gamma_{\sigma(t)}\xi(t),$$

where $\xi(t) \in R_+^r$ is the state of the exogenous system, $\Gamma_{\sigma(t)} \in R^{r \times r}$, $\Gamma_{\sigma(t)} \succeq 0$, and $Y_{\sigma(t)} \in R^{r \times r}$ is a Metzler matrix.

The disturbance observer is constructed as

$$\begin{aligned}\hat{\xi}(t) &= H_{\sigma(t)}\hat{\xi}(t) + F_{\sigma(t)}\hat{x}(t) + L_{d\sigma(t)}y(t), \\ \hat{w}(t) &= \Gamma_{\sigma(t)}\hat{\xi}(t),\end{aligned}\quad (2)$$

where $\hat{\xi}(t) \in R^r$ is the state of the disturbance observer and $H_{\sigma(t)} \in R^{r \times r}$, $F_{\sigma(t)} \in R^{r \times o}$, and $L_{d\sigma(t)} \in R^{r \times s}$ are the observer gains to be designed. By Lemma 2, the exogenous system is positive. The disturbance estimation will be obtained by estimating the state $\xi(t)$.

The state observer of system (1) is designed as

$$\dot{\hat{x}}(t) = G_{\sigma(t)}\hat{x}(t) + B_{\sigma(t)}u(t) + M_{\sigma(t)}\hat{w}(t) + L_{c\sigma(t)}y(t), \quad (3)$$

where $\hat{x}(t) \in R^n$ is the observer state, $\hat{w}(t) \in R^r$ is the estimate of disturbance, and $G_{\sigma(t)} \in R^{n \times n}$, $M_{\sigma(t)} \in R^{n \times r}$, and $L_{c\sigma(t)} \in R^{n \times s}$ are the observer gains to be designed.

Define the errors: $e_x(t) = \hat{x}(t) - x(t)$ and $e_{\xi}(t) = \hat{\xi}(t) - \xi(t)$. Then, the error system can be described as

$$\begin{aligned}\dot{e}_x(t) &= G_{\sigma(t)}\hat{x}(t) - (A_{\sigma(t)} - L_{c\sigma(t)}C_{\sigma(t)})x(t) + M_{\sigma(t)}\hat{w}(t) \\ &\quad - (E_{\sigma(t)} - L_{c\sigma(t)}D_{\sigma(t)})w(t), \\ \dot{e}_{\xi}(t) &= H_{\sigma(t)}\hat{\xi}(t) + (L_{d\sigma(t)}D_{\sigma(t)}\Gamma_{\sigma(t)} - Y_{\sigma(t)})\xi(t) \\ &\quad + F_{\sigma(t)}\hat{x}(t) + L_{d\sigma(t)}C_{\sigma(t)}x(t).\end{aligned}\quad (4)$$

Theorem 1. If there exist constants $\mu_1 > 0$, $\lambda_1 > 1$, $\alpha_1 > 0$, $\delta_1 > 0$, $\delta_2 > 0$, $\delta_3 > 0$, R^n vectors $v_1^{(p)} \succ 0$, $v_1^{(q)} \succ 0$, $q_p^{(i)} \succ 0$, $q_p \succ 0$, $\zeta_p \succ 0$, $\zeta_p^{(i)} \succ 0$, and R^r vectors $v_2^{(p)} \succ 0$, $v_2^{(q)} \succ 0$, $f_p \succ 0$, $f_p^{(i)} \succ 0$, $\eta_p \prec 0$, $\eta_p^{(i)} \prec 0$ such that

$$A_p \mathbf{1}_n^\top v_1^{(p)} - \sum_{i=1}^n \mathbf{1}_n^{(i)} \zeta_p^{(i)\top} C_p - \sum_{i=1}^n \mathbf{1}_n^{(i)} q_p^{(i)\top} + \delta_1 I_n = 0, \quad (5a)$$

$$\sum_{i=1}^r \mathbf{1}_r^{(i)} f_p^{(i)\top} + \sum_{i=1}^r \mathbf{1}_r^{(i)} \eta_p^{(i)\top} C_p = 0, \quad (5b)$$

$$E_p \mathbf{1}_n^\top v_1^{(p)} - \sum_{i=1}^n \mathbf{1}_n^{(i)} \zeta_p^{(i)\top} D_p \succeq 0, \quad (5c)$$

$$\sum_{i=1}^n \mathbf{1}_n^{(i)} q_p^{(i)\top} - \delta_1 I_n + \delta_2 I_n \succeq 0, \quad (5d)$$

$$Y_p \mathbf{1}_r^\top v_2^{(p)} - \sum_{i=1}^r \mathbf{1}_r^{(i)} \eta_p^{(i)\top} D_p \Gamma_p + \delta_3 I_r \succeq 0, \quad (5e)$$

$$q_p + f_p - \alpha_1 v_1^{(p)} \prec 0, \quad (5f)$$

$$(\alpha_1 + \mu_1) \mathbf{1}_o^\top v_1^{(p)} \leq \delta_1, \quad (5g)$$

$$\Gamma_p^\top E_p^\top v_1^{(p)} - \Gamma_p^\top D_p^\top (\zeta_p + \eta_p) + (Y_p^\top + \mu_1 I_r) v_2^{(p)} < 0, \quad (5h)$$

$$q_p^{(i)} \preceq q_p, \zeta_p^{(i)} \succeq \zeta_p, i = 1, 2, \dots, n, \quad (5i)$$

$$f_p^{(i)} \preceq f_p, \eta_p^{(i)} \succeq \eta_p, i = 1, 2, \dots, r, \quad (5j)$$

$$v_i^{(p)} \preceq \lambda_1 v_i^{(q)}, i = 1, 2 \quad (5k)$$

hold $\forall (p, q) \in S, p \neq q$, then the observers (2) and (3) with gain matrices:

$$\begin{aligned} G_p &= \frac{\sum_{i=1}^n \mathbf{1}_n^{(i)} q_p^{(i)\top} - \delta_1 I_n}{\mathbf{1}_n^\top v_1^{(p)}}, F_p = \frac{\sum_{i=1}^r \mathbf{1}_r^{(i)} f_p^{(i)\top}}{\mathbf{1}_r^\top v_2^{(p)}}, M_p = E_p - \frac{\sum_{i=1}^n \mathbf{1}_n^{(i)} \zeta_p^{(i)\top} D_p}{\mathbf{1}_n^\top v_1^{(p)}}, \\ L_{cp} &= \frac{\sum_{i=1}^n \mathbf{1}_n^{(i)} \zeta_p^{(i)\top}}{\mathbf{1}_n^\top v_1^{(p)}}, L_{dp} = \frac{\sum_{i=1}^r \mathbf{1}_r^{(i)} \eta_p^{(i)\top}}{\mathbf{1}_r^\top v_2^{(p)}}, H_p = Y_p - \frac{\sum_{i=1}^r \mathbf{1}_r^{(i)} \eta_p^{(i)\top} D_p \Gamma_p}{\mathbf{1}_r^\top v_2^{(p)}}, \end{aligned} \quad (6)$$

are positive, and the error system (4) is positive and asymptotically stable under the ADT switching satisfying

$$\tau^* \leq \frac{\ln \lambda}{\mu}. \quad (7)$$

Proof. First, the positivity of the observers (2) and (3) is addressed. From (5a,b) and (6), we have $A_p - L_{cp}C_p - G_p = 0, F_p + L_{dp}C_p = 0, E_p - L_{cp}D_p - M_p = 0$, and $Y_p - L_{dp}D_p\Gamma_p - H_p = 0$. Then, the system (4) can be transformed into

$$\begin{pmatrix} \dot{e}_x(t) \\ \dot{e}_\zeta(t) \end{pmatrix} = \begin{pmatrix} G_p & M_p \Gamma_p \\ F_p & H_p \end{pmatrix} \begin{pmatrix} e_x(t) \\ e_\zeta(t) \end{pmatrix}. \quad (8)$$

From (5d,e), it follows that

$$\frac{\sum_{i=1}^n \mathbf{1}_n^{(i)} q_p^{(i)\top} - \delta_1 I_n}{\mathbf{1}_n^\top v_1^{(p)}} + \frac{\delta_2}{\mathbf{1}_n^\top v_1^{(p)}} I_n \succeq 0, Y_p - \frac{\sum_{i=1}^r \mathbf{1}_r^{(i)} \eta_p^{(i)\top} D_p \Gamma_p}{\mathbf{1}_r^\top v_2^{(p)}} + \frac{\delta_3}{\mathbf{1}_r^\top v_2^{(p)}} I_r \succeq 0. \quad (9)$$

Together with (6) gives $G_p + \frac{\delta_2}{\mathbf{1}_n^\top v_1^{(p)}} I_n \succeq 0$ and $H_p + \frac{\delta_3}{\mathbf{1}_r^\top v_2^{(p)}} I_r \succeq 0$. By Lemma 1, G_p and H_p are Metzler matrices. By (5c), we have $M_p \succeq 0$. By $\eta_p^{(i)} < 0, f_p^{(i)} > 0$, and (5b), it is easy to be obtained that $F_p \succeq 0$. By Lemma 2, the error system (4) is positive. Since $e_x(0) \succeq 0$ and $e_\zeta(0) \succeq 0$, then $e_x(t) \succeq 0$ and $e_\zeta(t) \succeq 0$. Thus, the observers (2) and (3) are positive.

Choose a multiple CLF: $V_{\sigma(t)}(X(t)) = X^\top(t) \vartheta^{(\sigma(t))}$, where $X(t) = (e_x^\top(t), e_\zeta^\top(t))^\top$ and $\vartheta^{(\sigma(t))} = (v_1^{(\sigma(t))\top}, v_2^{(\sigma(t))\top})^\top$. Give a switching sequence $0 \leq t_0 \leq t_1 \leq \dots \leq t_m \leq t$, where $t \in [t_m, t_{m+1})$ and $m \in \mathbb{N}$. By (8), it yields that

$$\dot{V}_p(X(t)) = e_x^\top(t) (G_p^\top v_1^{(p)} + F_p^\top v_2^{(p)}) + e_\zeta^\top(t) (\Gamma_p^\top M_p^\top v_1^{(p)} + H_p^\top v_2^{(p)}). \quad (10)$$

Using (5i,j) and (6) gives

$$G_p \preceq \frac{\sum_{i=1}^n \mathbf{1}_n^{(i)} q_p^{(i)\top} - \delta_1 I_n}{\mathbf{1}_n^\top v_1^{(p)}} = \frac{\mathbf{1}_n q_p^\top - \delta_1 I_n}{\mathbf{1}_n^\top v_1^{(p)}}, F_p \preceq \frac{\sum_{i=1}^r \mathbf{1}_r^{(i)} f_p^{(i)\top}}{\mathbf{1}_r^\top v_2^{(p)}} = \frac{\mathbf{1}_r f_p^\top}{\mathbf{1}_r^\top v_2^{(p)}}. \quad (11)$$

Together with (5f,g) and (11), it deduces that

$$\begin{aligned} G_p^\top v_1^{(p)} + F_p^\top v_2^{(p)} + \mu_1 v_1^{(p)} &\preceq q_p - \frac{\delta_1}{\mathbf{1}_n^\top v_1^{(p)}} v_1^{(p)} + f_p + \mu_1 v_1^{(p)} \\ &\preceq q_p + f_p - \alpha_1 v_1^{(p)} < 0. \end{aligned}$$

Then,

$$G_p^\top v_1^{(p)} + F_p^\top v_2^{(p)} \prec -\mu_1 v_1^{(p)}. \quad (12)$$

Using (5i,j) and (6) gives

$$M_p \preceq E_p - \frac{\sum_{i=1}^n \mathbf{1}_n^{(i)} \zeta_p^\top D_p}{\mathbf{1}_n^\top v_1^{(p)}} = E_p - \frac{\mathbf{1}_n \zeta_p^\top D_p}{\mathbf{1}_n^\top v_1^{(p)}}, H_p \preceq Y_p - \frac{\sum_{i=1}^r \mathbf{1}_r^{(i)} \eta_p^\top D_p \Gamma_p}{\mathbf{1}_r^\top v_2^{(p)}} = Y_p - \frac{\mathbf{1}_r \eta_p^\top D_p \Gamma_p}{\mathbf{1}_r^\top v_2^{(p)}}. \quad (13)$$

By (5h), (6) and (13), the following inequality holds:

$$\Gamma_p^\top M_p^\top v_1^{(p)} + H_p^\top v_2^{(p)} + \mu_1 v_2^{(p)} \preceq \Gamma_p^\top E_p^\top v_1^{(p)} - \Gamma_p^\top D_p^\top (\zeta_p + \eta_p) + (Y_p^\top + \mu_1 I_r) v_2^{(p)} \prec 0.$$

Then,

$$\Gamma_p^\top M_p^\top v_1^{(p)} + H_p^\top v_2^{(p)} \prec -\mu_1 v_2^{(p)}. \quad (14)$$

By (10), (12) and (14), we have

$$\dot{V}_{\sigma(t)}(X(t)) \leq -\mu_1 X^\top(t) \vartheta^{(\sigma(t))} \leq 0. \quad (15)$$

Taking integration both sides of (15) yields that

$$V_{\sigma(t)}(X(t)) \leq e^{-\mu_1(t-t_m)} V_{\sigma(t_m)}(X(t_m)) \quad (16)$$

for $t \in [t_m, t_{m+1}]$. Combining (5k) and $X(t) \succeq 0$ gives

$$X^\top(t_m) \vartheta^{(\sigma(t_m))} \leq \lambda_1 X^\top(t_m) \vartheta^{(\sigma(t_{m-1}))}, \quad (17)$$

that is,

$$V_{\sigma(t_m)}(X(t_m)) \leq \lambda_1 e^{-\mu_1(t-t_m)} V_{\sigma(t_{m-1})}(X(t_{m-1})). \quad (18)$$

Repeating (15)–(18), the following relation holds:

$$\begin{aligned} V_{\sigma(t)}(X(t)) &\leq e^{-\mu_1(t-t_m)} V_{\sigma(t_m)}(X(t_m)) \\ &\leq \lambda_1 e^{-\mu_1(t-t_{m-1})} V_{\sigma(t_{m-1})}(X(t_{m-1})) \\ &\leq \dots \leq \lambda_1^m e^{-\mu_1(t-t_0)} V_{\sigma(t_0)}(X(t_0)). \end{aligned}$$

Noting Definition 2 and $\lambda_1 > 1$, the inequality can be transformed into

$$\begin{aligned} V_{\sigma(t)}(X(t)) &\leq e^{N_0 \ln \lambda_1} e^{-\mu_1(t-t_0)} V_{\sigma(t_0)}(X(t_0)) \\ &\leq e^{(N_0 + \frac{t-t_0}{\tau^*}) \ln \lambda_1} e^{-\mu_1(t-t_0)} V_{\sigma(t_0)}(X(t_0)) \\ &\leq \rho e^{\varepsilon(t-t_0)} V_{\sigma(t_0)}(X(t_0)), \end{aligned}$$

where $\rho = e^{N_0 \ln \lambda_1}$ and $\varepsilon = \mu_1 - \frac{\ln \lambda_1}{\tau^*}$. By (7), $\varepsilon < 0$. Then, the error system (4) is asymptotically stable. $\square$

Remark 2. Noting the condition (5b) in Theorem 1, it can be derived that $F_p + L_{dp} C_p = 0$. If the term $F_{\sigma(t)} \hat{x}(t)$ is removed, the condition (5b) becomes the equation $L_{dp} C_p = 0$. The equation is rigorous, and thus the validity of other conditions in (5) may not be guaranteed.

Remark 3. In [21–24], the works focused on state observer of positive systems. The disturbance observer issue of SPSs is open, and few results are devoted to this issue. The obstacles to deal with this issue contain three aspects. First, a framework on double observers needs to be constructed. How to integrate the positivity restriction into the double observers is one of the difficult points. Moreover, the state observer and the disturbance observer interact with each other. Therefore, a Luenberger-type observer may not be directly applied for the double observers. Second, the question of how to design the observer gain matrices arises. For positive systems, it is interesting to present some alternative design approaches to the gain design. Under the double observers framework, the corresponding

gain design refers to the state and disturbance observer gains. They are more complex than the single-state observer design. Finally, how to design the double-observer-based controller is another difficult point. Up to now, there have been no referable approaches to design a double-observer-based controller of SPSs. The key point of the design is to transform all conditions into tractable ones. Thus, some reliable computation methods can be employed to solve these conditions. Therefore, it is not easy to solve the problem of double observers design.

Remark 4. In (2) and (3), the disturbance and state observers are constructed, respectively. It can be found that the two observers are correlated. Such observer forms have differences from the Luenberger-type observer. In (6), all gain matrices are described in the form of vector variables. Thus, the conditions in (5) can be easily formulated in the form of linear programming. Finally, the CLF is chosen to analyze the stability of the error dynamic systems. Theorem 1 develops the CLF integrated with linear programming approach to the double observers design of SPSs. It further verifies that the effectiveness of the linear approach in handling the issues of positive systems.

3.2. Bumpless Transfer Control

In this subsection, a double-observer-based bumpless transfer control design is proposed. For the state-observer-based control of SPSs, it is easy to utilize the traditional control form [23]. However, it fails to develop for the double-observer-based control of SPSs when the disturbance observer is introduced. Therefore, a novel controller form is given as

$$u(t) = K_{1\sigma(t)}\hat{x}(t) + K_{2\sigma(t)}\hat{w}(t) - K_{3\sigma(t)}y(t), \quad (19)$$

where $\hat{x}(t)$ is the estimation of $x(t)$, $\hat{w}(t)$ is the estimation of $w(t)$, and $K_{1\sigma(t)} \in R^{m \times n}$, and $K_{2\sigma(t)} \in R^{m \times r}$ and $K_{3\sigma(t)} \in R^{m \times s}$ are controller gain matrices to be determined later. In general, it is unnecessary to add the output term in the controller (19). Under the double observers framework, the control design will fail without the output term. Therefore, the output is added in (19).

With the controller (19) and the error system (4), the closed-loop system can be described as

$$\begin{aligned} \dot{\hat{x}}(t) &= (A_{\sigma(t)} + B_{\sigma(t)}K_{1\sigma(t)} - B_{\sigma(t)}K_{3\sigma(t)}C_{\sigma(t)})\hat{x}(t) + B_{\sigma(t)}K_{1\sigma(t)}(\hat{x}(t) \\ &\quad - x(t)) + B_{\sigma(t)}K_{2\sigma(t)}\hat{w}(t) - (B_{\sigma(t)}K_{3\sigma(t)}D_{\sigma(t)} - E_{\sigma(t)})w(t), \\ \dot{e}_x(t) &= G_{\sigma(t)}\hat{x}(t) - (A_{\sigma(t)} - L_{c\sigma(t)}C_{\sigma(t)})x(t) + M_{\sigma(t)}\hat{w}(t) \\ &\quad - (E_{\sigma(t)} - L_{c\sigma(t)}D_{\sigma(t)})w(t), \\ \dot{e}_\xi(t) &= H_{\sigma(t)}\hat{\xi}(t) + (L_{d\sigma(t)}D_{\sigma(t)}\Gamma_{\sigma(t)} - Y_{\sigma(t)})\xi(t) + F_{\sigma(t)}\hat{x}(t) + L_{d\sigma(t)}C_{\sigma(t)}x(t). \end{aligned} \quad (20)$$

Theorem 2. If there exist constants $\mu_2 > 0$, $\lambda_2 > 1$, $\alpha_2 > 0$, $\delta_4 > 0$, $\alpha > 0$, $\beta > 0$, $\gamma > 0$, R^s vectors $z_{3p} \succ 0$, $z_{3p}^{(i)} \succ 0$, R^n vectors $\varphi^{(p)} \succ 0$, $\varphi^{(q)} \succ 0$, $v_1^{(p)} \succ 0$, $v_1^{(q)} \succ 0$, $q_p \succ 0$, $q_p^{(i)} \succ 0$, $\zeta_p \succ 0$, $\zeta_p^{(i)} \succ 0$, $z_{1p} \succ 0$, $z_{1p}^{(i)} \succ 0$, and R^r vectors $v_2^{(p)} \succ 0$, $v_2^{(q)} \succ 0$, $f_p \succ 0$, $f_p^{(i)} \succ 0$, $z_{2p} \succ 0$, $z_{2p}^{(i)} \succ 0$, $\eta_p \prec 0$, $\eta_p^{(i)} \prec 0$ such that the conditions (5a–e, i, j), and

$$B_p \sum_{i=1}^m \mathbf{1}_m^{(i)} z_{2p}^{(i)\top} - B_p \sum_{i=1}^m \mathbf{1}_m^{(i)} z_{3p}^{(i)\top} D_p + E_p \mathbf{1}_m^\top B_p^\top \varphi^{(p)} = 0, \quad (21a)$$

$$\mathbf{1}_m^\top B_p^\top \varphi^{(p)} A_p + B_p \sum_{i=1}^m \mathbf{1}_m^{(i)} z_{1p}^{(i)\top} - B_p \sum_{i=1}^m \mathbf{1}_m^{(i)} z_{3p}^{(i)\top} C_p + \delta_4 I_o \succeq 0, \quad (21b)$$

$$B_p \sum_{i=1}^m \mathbf{1}_m^{(i)} z_{3p}^{(i)\top} D_p \Gamma_p - \mathbf{1}_m^\top B_p^\top \varphi^{(p)} E_p \Gamma_p \succeq 0, \quad (21c)$$

$$A_p^\top \varphi^{(p)} + z_{1p} - C_p^\top z_{3p} + \mu_2 \varphi^{(p)} \prec 0, \quad (21d)$$

$$z_{1p} + q_p + f_p - \alpha_2 v_1 \prec 0, \quad (21e)$$

$$(\alpha_2 + \mu_2) \mathbf{1}_n^\top v_1 \leq \delta_1, \quad (21f)$$

$$\Gamma_p^\top z_{2p} + \Gamma_p^\top E_p^\top v_1^{(p)} - \Gamma_p^\top D_p^\top (\zeta_p + \eta_p) + Y_p^\top v_2^{(p)} + \mu_2 v_2^{(p)} \prec 0, \quad (21g)$$

$$z_{1p}^{(i)} \preceq z_{1p}, z_{2p}^{(i)} \preceq z_{2p}, z_{3p}^{(i)} \succeq z_{3p}, i = 1, 2, \dots, m, \quad (21h)$$

$$\varphi^{(p)} \preceq \lambda_2 \varphi^{(q)}, v_i^{(p)} \preceq \lambda_2 v_i^{(q)}, i = 1, 2, \quad (21i)$$

and

$$\mathfrak{R}_{1p}^{j*} \mathbf{1}_m^\top B_p^\top \varphi^{(p)} - \mathbf{1}_n^\top \alpha \mathbf{1}_m^\top B_p^\top \varphi^{(p)} - z_{1p}^{(j)\top} \preceq 0, \quad (22a)$$

$$z_{1p}^{(j)\top} - \mathfrak{R}_{1p}^{j*} \mathbf{1}_m^\top B_p^\top \varphi^{(p)} - \mathbf{1}_n^\top \alpha \mathbf{1}_m^\top B_p^\top \varphi^{(p)} \preceq 0, \quad (22b)$$

$$\mathfrak{R}_{2p}^{j*} \mathbf{1}_m^\top B_p^\top \varphi^{(p)} - \mathbf{1}_r^\top \beta \mathbf{1}_m^\top B_p^\top \varphi^{(p)} - z_{2p}^{(j)\top} \preceq 0, \quad (22c)$$

$$z_{2p}^{(j)\top} - \mathfrak{R}_{2p}^{j*} \mathbf{1}_m^\top B_p^\top \varphi^{(p)} - \mathbf{1}_r^\top \beta \mathbf{1}_m^\top B_p^\top \varphi^{(p)} \preceq 0, \quad (22d)$$

$$z_{3p}^{(j)\top} - \mathfrak{R}_{3p}^{j*} \mathbf{1}_m^\top B_p^\top \varphi^{(p)} - \mathbf{1}_s^\top \gamma \mathbf{1}_m^\top B_p^\top \varphi^{(p)} \preceq 0, \quad (22e)$$

$$\mathfrak{R}_{3p}^{j*} \mathbf{1}_m^\top B_p^\top \varphi^{(p)} - z_{3p}^{(j)\top} - \mathbf{1}_s^\top \gamma \mathbf{1}_m^\top B_p^\top \varphi^{(p)} \preceq 0, \quad (22f)$$

hold for $j = 1, 2, \dots, m$ and $(p, q) \in S, p \neq q$, with p and q denoting the different subsystems in the SPS (1), then the observers (2) and (3) with the gain matrices (6) are positive and the system (1) is positive, asymptotically stable, and reaches the bumpless transfer performance under the controller (19) with the gain matrices:

$$K_{1p} = \frac{\sum_{i=1}^m \mathbf{1}_m^{(i)} z_{1p}^{(i)\top}}{\mathbf{1}_m^\top B_p^\top \varphi^{(p)}}, K_{2p} = \frac{\sum_{i=1}^m \mathbf{1}_m^{(i)} z_{2p}^{(i)\top}}{\mathbf{1}_m^\top B_p^\top \varphi^{(p)}}, K_{3p} = \frac{\sum_{i=1}^m \mathbf{1}_m^{(i)} z_{3p}^{(i)\top}}{\mathbf{1}_m^\top B_p^\top \varphi^{(p)}}, \quad (23)$$

and the ADT switching satisfying (7).

Proof. From (21a), it can be obtained that $B_p K_{2p} = B_p K_{3p} D_p - E_p$. Then, the system (20) can be transformed into

$$\begin{pmatrix} \dot{x}(t) \\ \dot{e}_x(t) \\ \dot{e}_z(t) \end{pmatrix} = \begin{pmatrix} \mathfrak{A}_p & B_p K_{1p} & B_p K_{2p} \Gamma_p \\ 0 & G_p & M_p \Gamma_p \\ 0 & F_p & H_p \end{pmatrix} \begin{pmatrix} x(t) \\ e_x(t) \\ e_z(t) \end{pmatrix}, \quad (24)$$

where $\mathfrak{A}_p = A_p + B_p K_{1p} - B_p K_{3p} C_p$. From (21b), it follows that $\mathfrak{A}_p = A_p + B_p \frac{\sum_{i=1}^m \mathbf{1}_m^{(i)} z_{1p}^{(i)\top}}{\mathbf{1}_m^\top B_p^\top \varphi^{(p)}} - B_p \frac{\sum_{i=1}^m \mathbf{1}_m^{(i)} z_{3p}^{(i)\top}}{\mathbf{1}_m^\top B_p^\top \varphi^{(p)}} C_p + \frac{\delta_4}{\mathbf{1}_m^\top B_p^\top \varphi^{(p)}} I_m \succeq 0$. Together with (23), it gives that $A_p + B_p K_{1p} - B_p K_{3p} C_p + \frac{\delta_4}{\mathbf{1}_m^\top B_p^\top \varphi^{(p)}} I_m \succeq 0$, which implies that $A_p + B_p K_{1p} - B_p K_{3p} C_p$ is Metzler matrix by Lemma 1.

From $B_p \succeq 0$ and $K_{1p} \succeq 0$, it is easy to know that $B_p K_{1p} \succeq 0$. By (21c), $\Gamma_p \succeq 0$, and $z_{2p}^{(i)} \succ 0$, it can be obtained that $B_p K_{2p} \Gamma_p \succeq 0$. By Theorem 1, it is clear that G_p and H_p are Metzler matrices and $F_p \succeq 0, M_p \succeq 0$. Therefore, the error system (20) is positive by Lemma 2.

Choose a multiple CLF $V_{\sigma(t)}(\chi(t)) = \chi(t)^\top \mathcal{V}^{(\sigma(t))}$, where $\chi(t) = (x^\top(t), e_x^\top(t), e_\xi^\top(t))^\top$ and $\mathcal{V}^{(\sigma(t))} = (\varphi^{(\sigma(t))^\top}, v_1^{(\sigma(t))^\top}, v_2^{(\sigma(t))^\top})^\top$. Give a switching sequence $0 \leq t_0 \leq t_1 \leq \dots \leq t_m \leq t$, where $t \in [t_m, t_{m+1})$ and $m \in \mathbb{N}$. Then,

$$\begin{aligned} \dot{V}_{\sigma(t)}(\chi(t)) &= x^\top(t)(A_p^\top + K_{1p}^\top B_p^\top - C_p^\top K_{3p}^\top B_p^\top)\varphi^{(p)} + e_x^\top(t)(K_{1p}^\top B_p^\top \varphi^{(p)} \\ &\quad + G_p^\top v_1^{(p)} + F_p^\top v_2^{(p)})e_\xi^\top(t)(\Gamma_p^\top K_{2p}^\top B_p^\top \varphi^{(p)} + \Gamma_p^\top M_p^\top v_1^{(p)} + H_p^\top v_2^{(p)}). \end{aligned} \quad (25)$$

Using (21h) gives $K_{1p} \preceq \frac{\mathbf{1}_m z_{1p}^\top}{\mathbf{1}_m^\top B_p^\top \varphi^{(p)}}$, $K_{2p} \preceq \frac{\mathbf{1}_m z_{2p}^\top}{\mathbf{1}_m^\top B_p^\top \varphi^{(p)}}$, $K_{3p} \succeq \frac{\mathbf{1}_m z_{3p}^\top}{\mathbf{1}_m^\top B_p^\top \varphi^{(p)}}$. Together with (21d), it yields that

$$A_p^\top \varphi^{(p)} + K_{1p}^\top B_p^\top \varphi^{(p)} - C_p^\top K_{3p}^\top B_p^\top \varphi^{(p)} + \mu_2 \varphi^{(p)} \preceq A_p \varphi^\top + z_{1p} - C_p^\top z_{3p} + \mu_2 \varphi^{(p)} \prec 0.$$

Then, $A_p^\top \varphi^{(p)} + K_{1p}^\top B_p^\top \varphi^{(p)} - C_p^\top K_{3p}^\top B_p^\top \varphi^{(p)} \prec -\mu_2 \varphi^{(p)}$. Together with (11) and (21e,f), it gives

$$\begin{aligned} K_{1p}^\top B_p^\top \varphi^{(p)} + G_p^\top v_1^{(p)} + F_p^\top v_2^{(p)} + \mu_2 v_1^{(p)} &\preceq z_{1p} + \varrho_p - \frac{\delta_1}{\mathbf{1}_n^\top v_1^{(p)}} v_1^{(p)} + f_p + \mu_2 v_1^{(p)} \\ &\preceq z_{1p} + \varrho_p + f_p - \alpha_2 v_1^{(p)} \prec 0. \end{aligned}$$

Then, $K_{1p}^\top B_p^\top \varphi^{(p)} + G_p^\top v_1^{(p)} + F_p^\top v_2^{(p)} \prec -\mu_2 v_1^{(p)}$. By (21g), we can obtain that

$$\begin{aligned} &\Gamma_p^\top K_{2p}^\top B_p^\top \varphi^{(p)} + \Gamma_p^\top M_p^\top v_1^{(p)} + H_p^\top v_2^{(p)} + \mu_2 v_2^{(p)} \\ &\preceq \Gamma_p^\top z_{2p} + \Gamma_p^\top E_p^\top v_1^{(p)} - \Gamma_p^\top D_p^\top \zeta_p + \Upsilon_p^\top v_2^{(p)} - \Gamma_p^\top D_p^\top \eta_p + \mu_2 v_2^{(p)} \\ &\prec 0. \end{aligned}$$

Then, $\Gamma_p^\top K_{2p}^\top B_p^\top \varphi^{(p)} + \Gamma_p^\top M_p^\top v_1^{(p)} + H_p^\top v_2^{(p)} \prec -\mu_2 v_2^{(p)}$. Then, $\dot{V}_p(\chi(t)) \leq -\mu_2 \chi^\top(t) \mathcal{V}^{(p)} < 0$. Taking integration from both sides yields that $V_{\sigma(t)}(\chi(t)) \leq e^{-\mu_2(t-t_m)} V_{\sigma(t_m)}(\chi(t_m))$ for $t \in [t_m, t_{m+1})$. The rest of the proof for the stability can be obtained by using a similar method to Theorem 1 and is omitted.

Finally, consider the bumpless transfer performance. By Definition 3, it is derived that

$$\begin{aligned} &-\mathbf{1}_n^\top \alpha \hat{x}(t) - \mathbf{1}_r^\top \beta \hat{w}(t) - \mathbf{1}_s^\top \gamma y(t) \leq \\ &(\mathfrak{K}_{1\sigma(t)}^{(j)} - \mathfrak{K}_{1\sigma(t)}^{(j)*}) \hat{x}(t) + (\mathfrak{K}_{2\sigma(t)}^{(j)} - \mathfrak{K}_{2\sigma(t)}^{(j)*}) \hat{w}(t) + (\mathfrak{K}_{3\sigma(t)}^{(j)*} - \mathfrak{K}_{3\sigma(t)}^{(j)}) y(t) \\ &\leq \mathbf{1}_n^\top \alpha \hat{x}(t) + \mathbf{1}_r^\top \beta \hat{w}(t) + \mathbf{1}_s^\top \gamma y(t), \end{aligned}$$

where $\mathfrak{K}^{(j)}$ stands for the j th row element of K and $\mathfrak{K}^{(j)*}$ is the j th row element of K^* . By $\hat{x}(t) \succeq x(t)$, $x(t) \succ 0$, (22a), and (22b), we have $-\mathbf{1}_n^\top \alpha - (\mathfrak{K}_{1\sigma(t)}^{(j)} - \mathfrak{K}_{1\sigma(t)}^{(j)*}) \preceq 0$ and $(\mathfrak{K}_{1\sigma(t)}^{(j)} - \mathfrak{K}_{1\sigma(t)}^{(j)*}) - \mathbf{1}_n^\top \alpha \preceq 0$. By $\hat{w}(t) \succeq w(t)$, $w(t) \succ 0$, (22c), and (22d), it is not hard to know that $-\mathbf{1}_r^\top \beta - (\mathfrak{K}_{2\sigma(t)}^{(j)} - \mathfrak{K}_{2\sigma(t)}^{(j)*}) \preceq 0$ and $(\mathfrak{K}_{2\sigma(t)}^{(j)} - \mathfrak{K}_{2\sigma(t)}^{(j)*}) - \mathbf{1}_r^\top \beta \preceq 0$. Based on $x(t) \succeq 0$ and $w(t) \succeq 0$, it can be obtained that $y(t) \succeq 0$. By (22e) and (22f), we obtain that $-\mathbf{1}_s^\top \gamma - (\mathfrak{K}_{3\sigma(t)}^{(j)*} - \mathfrak{K}_{3\sigma(t)}^{(j)}) \preceq 0$ and $(\mathfrak{K}_{3\sigma(t)}^{(j)*} - \mathfrak{K}_{3\sigma(t)}^{(j)}) - \mathbf{1}_s^\top \gamma \preceq 0$. Then, we have $|u_j - u_j^*| \leq \mathbf{1}_n^\top \alpha \hat{x}(t) + \mathbf{1}_r^\top \beta \hat{w}(t) + \mathbf{1}_s^\top \gamma y(t)$. Thus, the considered system (1) has the bumpless transfer performance by Definition 3. $\square$

Remark 5. Bumpless control has been applied for switched systems [30]. In [38], the asynchronous bumpless control was explored for SPSs in terms of linear matrix inequalities. Theorem 2 further utilizes the CLF and linear programming to investigate the bumpless control of SPSs. Different from the L_1 gain performance presented in [39], Theorem 2 introduces the state and disturbance observers. Under the disturbance observer framework, the asymptotic stability and bumpless transfer control are simultaneously reached.

Remark 6. This paper designs a double observer and double-observer-based bumpless transfer controller. There are three main difficulties to handle the considered issues. First, how to construct a novel framework on double observers and the corresponding controller of SPSs? Second, how to design the gain matrices of observer and controller? Third, how to guarantee the positivity and stability of observers and the closed-loop systems? In the existing literature [10,13,14], quadratic the Lyapunov function and linear matrix inequalities are usually used. For positive systems, some novel approaches need to be introduced owing to the positivity requirement. Therefore, it is not easy to solve the considered problems. Moreover, it is also clear that the gain design approach in (6) and the corresponding conditions in (5) are different from those in [26,31,32].

In Theorem 2, the parameters λ_2 , μ_2 , α_2 , and δ_4 are required to be known. How to choose the parameters is important to Theorem 2. To provide a feasible algorithm for choosing parameters $\lambda_2, \mu_2, \alpha_2, \delta_4$ and computing the conditions in Theorem 2, Algorithm 1 is presented. Algorithm 1 aims to search parameters satisfying the conditions (21a–i) in Theorem 2. First, an empty set Ω is defined to store the parameters that satisfy the corresponding conditions and $\lambda_2, \mu_2, \alpha_2$, and δ_4 are initialized to their lower bounds. Then, Algorithm 1 utilizes loops to iterate through all possible values of $\lambda_2, \mu_2, \alpha_2$, and δ_4 . Finally, these parameters will be released when the parameters satisfy conditions in (21).

Algorithm 1 Searching parameters of Theorem 2

Input: $A_{\sigma(t)}, B_{\sigma(t)}, E_{\sigma(t)}, C_{\sigma(t)}, D_{\sigma(t)}; m_i, i = 1, \dots, 4; \underline{\lambda}_2, \bar{\lambda}_2, \underline{\mu}_2, \bar{\mu}_2, \underline{\alpha}_2, \bar{\alpha}_2, \underline{\delta}_4, \bar{\delta}_4;$

Output: $\lambda_2, \mu_2, \alpha_2, \delta_4;$

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1: Define a set  $\Omega = \emptyset$ ; Initialize parameters  $\lambda_2 = \underline{\lambda}_2, \mu_2 = \underline{\mu}_2, \alpha_2 = \underline{\alpha}_2$ , and  $\delta_2 = \underline{\delta}_2$ ;
2: while  $\lambda_2 \leq \bar{\lambda}_2$  do
3:   repeat
4:     repeat
5:       repeat
6:         if (21a)–(21i) are feasible,
7:           then save  $\lambda_2, \mu_2, \alpha_2$ , and  $\delta_4$  to  $\Omega$ 
8:         end
9:          $\delta_4 \leftarrow \delta_4 + m_4$ ;
10:      until  $\delta_4 > \bar{\delta}_4$ ;  $\alpha_2 \leftarrow \alpha_2 + m_3$ ;
11:    until  $\alpha_2 > \bar{\alpha}_2$ ;  $\mu_2 \leftarrow \mu_2 + m_2$ ;
12:  until  $\mu_2 > \bar{\mu}_2$ ;  $\lambda_2 \leftarrow \lambda_2 + m_1$ ;
13: end
```

4. Illustrative Examples

With the increase in urbanization levels, urban populations have been steadily saturating. This leads to a growing demand for water. As a result, urban water supply systems (UWSSs) face significant pressure. In [39], some findings on UWSSs were summarized. Improving the supply ability of UWSSs can reduce resource waste and energy consumption and ensure the stability of water supply pressure [40–42]. Figure 1 shows the relationship between observers, water flow, and residents in a UWSS. Considering the non-negativity of water flow and the switching characteristics exhibited by the changing resident water data, the UWSSs can be abstracted into SPSs. In practice, it is not easy to measure the water consumption of residents. Meanwhile, water supply systems are inevitably affected by disturbances such as damaged pipelines, weather, and so on. Therefore, it is necessary to establish both state and disturbance observers for UWSSs. Finally, it is reasonable to employ a bumpless controller to minimize the effects of water flow bump during controller switching. Considering the points mentioned above, a UWSS model is described as follows:

$$\dot{X}(t) = A_{\sigma(t)}X(t) + B_{\sigma(t)}U(t) + D_{\sigma(t)}W(t), \quad (26)$$

where $X(t) = (X_1^\top(t), X_2^\top(t), X_3^\top(t))^\top$ denotes water consumption data of users, $U(t)$ denotes the double-observer-based bumpless controller, $W(t) = (W_1^\top(t), W_2^\top(t))^\top$ is the external disturbance in UWSS, such as water flow instability caused by insufficient water pressure, and $\sigma(t) \in \{1, 2\}$ represents the $\sigma(t)$ th subsystem. A state observer and a disturbance observer are designed in (2) and (3). For the $\sigma(t)$ th subsystem of the UWSS, the control law is designed as

$$U(t) = K_{2\sigma(t)}\hat{X}(t) + K_{2\sigma(t)}\hat{W}(t) - K_{3\sigma(t)}y(t),$$

where $\hat{X}(t) = (\hat{X}_1^\top(t), \hat{X}_2^\top(t), \hat{X}_3^\top(t))^\top$ denotes the estimate of user data and $\hat{W}(t)$ denotes the estimate of disturbance.

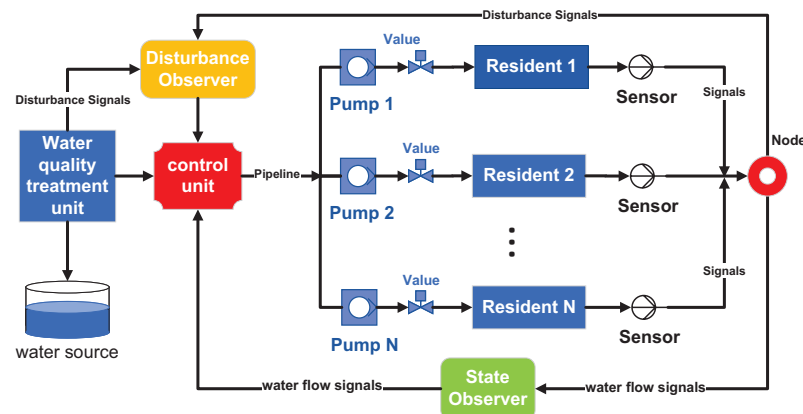


Figure 1. The structure chart of urban water supply systems.

Example 1. The system matrices in system (26) are

$$A_1 = \begin{pmatrix} -2.94 & 0.22 & 0.41 \\ 0.47 & -2.73 & 0.84 \\ 0.60 & 0.64 & -3.64 \end{pmatrix}, B_1 = \begin{pmatrix} 0.88 & 0.56 \\ 0.59 & 0.90 \\ 0.66 & 0.55 \end{pmatrix}, E_1 = \begin{pmatrix} 0.78 & 0.68 \\ 0.69 & 0.70 \\ 0.56 & 0.65 \end{pmatrix},$$

$$C_1 = \begin{pmatrix} 1.23 & 1.25 \\ 1.21 & 1.15 \\ 1.19 & 0.86 \end{pmatrix}^\top, D_1 = \begin{pmatrix} 0.85 & 0.78 \\ 0.78 & 0.88 \end{pmatrix},$$

$$\Upsilon_1 = \begin{pmatrix} -0.51 & 0.41 \\ 0.45 & -0.52 \end{pmatrix}, \Gamma_1 = \begin{pmatrix} 1.10 & 0.10 \\ 0.10 & 0.13 \end{pmatrix},$$

and

$$A_2 = \begin{pmatrix} -2.64 & 0.21 & 0.40 \\ 0.46 & -2.53 & 0.74 \\ 0.57 & 0.58 & -3.62 \end{pmatrix}, B_2 = \begin{pmatrix} 0.68 & 0.43 \\ 0.57 & 0.60 \\ 0.64 & 0.53 \end{pmatrix}, E_2 = \begin{pmatrix} 0.78 & 0.68 \\ 0.69 & 0.69 \\ 0.56 & 0.60 \end{pmatrix},$$

$$C_2 = \begin{pmatrix} 1.22 & 1.24 \\ 1.20 & 1.12 \\ 1.17 & 0.82 \end{pmatrix}^\top, D_2 = \begin{pmatrix} 0.84 & 0.76 \\ 0.72 & 0.86 \end{pmatrix},$$

$$\Upsilon_2 = \begin{pmatrix} -0.54 & 0.39 \\ 0.41 & -0.53 \end{pmatrix}, \Gamma_2 = \begin{pmatrix} 1.08 & 0.09 \\ 0.09 & 0.12 \end{pmatrix}.$$

Choose $\mu_1 = 0.03$, $\lambda_1 = 1.3$, and $\alpha_1 = 1.2$. By Theorem 1, we have $\delta_1 = 172.8$, $\delta_2 = 316.1$, $\delta_3 = 780.8$, and the observer gain matrices in

$$G_1 = \begin{pmatrix} -2.7701 & 0.3366 & 0.5772 \\ 0.5801 & -2.6704 & 0.9508 \\ 0.6956 & 0.6854 & -3.5429 \end{pmatrix}, F_1 = \begin{pmatrix} 0.0152 & 0.0147 & 0.0143 \\ 0.0146 & 0.0140 & 0.0137 \end{pmatrix},$$

$$M_1 = \begin{pmatrix} 0.8915 & 0.7572 & 0.6164 \\ 0.7745 & 0.7499 & 0.6889 \end{pmatrix}^\top, H_1 = \begin{pmatrix} -0.4964 & 0.4126 \\ 0.4630 & -0.5175 \end{pmatrix},$$

$$G_2 = \begin{pmatrix} -2.7440 & 0.4849 & 0.1896 \\ 0.3282 & -2.2745 & 0.5930 \\ 0.3748 & 0.7755 & -3.8266 \end{pmatrix}, F_2 = \begin{pmatrix} 0.0168 & 0.0157 & 0.0159 \\ 0.0159 & 0.0148 & 0.0149 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 0.7300 & 0.5907 & 0.5458 \\ 0.7891 & 0.7810 & 0.6416 \end{pmatrix}^\top, H_2 = \begin{pmatrix} -0.4943 & 0.3942 \\ 0.4248 & -0.5060 \end{pmatrix}.$$

The example provides an SPS with two subsystems. In the figures, a switching signal value of 1 indicates that the first subsystem is working, while a switching signal value of 2 means that the second subsystem is working. Figure 2 shows the simulations of the errors $e_x(t)$ under ADT switching. Figure 3 presents the errors $e_\xi(t)$ under ADT switching. Figures 2 and 3 show that the established state observer and disturbance observer can work. Therefore, it can be concluded that the dual observer framework based on states and disturbance is feasible in SPSs.

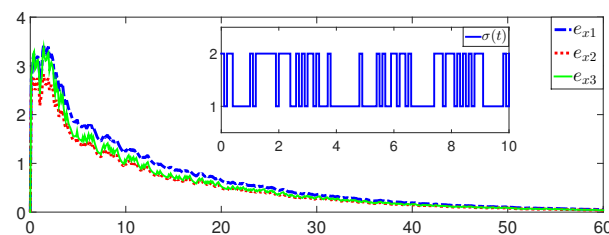


Figure 2. The errors $e_x(t)$ under ADT switching.

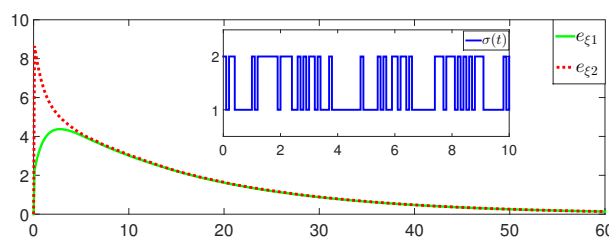


Figure 3. The errors $e_\xi(t)$ under ADT switching.

Example 2. Consider the system (26) with two subsystems:

$$A_1 = \begin{pmatrix} -2.94 & 1.82 & 1.41 \\ 1.87 & -2.73 & 1.84 \\ 1.60 & 1.64 & -3.64 \end{pmatrix}, B_1 = \begin{pmatrix} 1.98 & 1.26 \\ 1.25 & 1.29 \\ 1.26 & 1.65 \end{pmatrix}, E_1 = \begin{pmatrix} 1.88 & 1.68 \\ 1.69 & 1.70 \\ 1.56 & 1.65 \end{pmatrix},$$

$$C_1 = \begin{pmatrix} 1.13 & 1.13 \\ 1.18 & 1.15 \\ 0.65 & 1.76 \end{pmatrix}^\top, D_1 = \begin{pmatrix} 1.18 & 0.87 \\ 0.87 & 1.16 \end{pmatrix},$$

$$Y_1 = \begin{pmatrix} -0.71 & 0.21 \\ 0.25 & -0.72 \end{pmatrix}, \Gamma_1 = \begin{pmatrix} 1.10 & 0.10 \\ 0.10 & 1.12 \end{pmatrix},$$

and

$$A_2 = \begin{pmatrix} -2.91 & 1.80 & 1.38 \\ 1.83 & -2.63 & 1.81 \\ 1.59 & 1.74 & -3.65 \end{pmatrix}, B_2 = \begin{pmatrix} 1.97 & 1.24 \\ 1.28 & 1.87 \\ 1.24 & 1.62 \end{pmatrix}, E_2 = \begin{pmatrix} 1.86 & 1.56 \\ 1.67 & 1.66 \\ 1.55 & 1.63 \end{pmatrix},$$

$$C_2 = \begin{pmatrix} 1.12 & 1.14 \\ 1.16 & 1.02 \\ 0.61 & 1.75 \end{pmatrix}^\top, D_2 = \begin{pmatrix} 1.15 & 0.77 \\ 0.87 & 1.16 \end{pmatrix},$$

$$Y_2 = \begin{pmatrix} -0.70 & 0.20 \\ 0.22 & -0.71 \end{pmatrix}, \Gamma_2 = \begin{pmatrix} 1.09 & 0.11 \\ 0.10 & 0.11 \end{pmatrix}.$$

Choose $\alpha = 2.0, \beta = 1.0, \gamma = 0.5, \mu_2 = 0.02, \lambda_2 = 4.22$, and $\alpha_2 = 1.7$. Moreover, the reference value for the controller gain matrices $K_{1\sigma(t)}^*, K_{2\sigma(t)}^*$, and $K_{3\sigma(t)}^*$ are given as

$$K_{11}^* = \begin{pmatrix} 1.5 & 0.9 & 0.6 \\ 1.6 & 0.7 & 0.8 \end{pmatrix}, K_{21}^* = \begin{pmatrix} 0.9 & 0.9 \\ 0.8 & 1.9 \end{pmatrix}, K_{31}^* = \begin{pmatrix} 1.9 & 0.8 \\ 0.9 & 1.9 \end{pmatrix},$$

and

$$K_{12}^* = \begin{pmatrix} 1.2 & 0.8 & 0.5 \\ 1.5 & 0.7 & 0.9 \end{pmatrix}, K_{22}^* = \begin{pmatrix} 1.1 & 1.2 \\ 1.2 & 1.2 \end{pmatrix}, K_{32}^* = \begin{pmatrix} 1.1 & 0.9 \\ 1.0 & 1.2 \end{pmatrix}.$$

By Theorem 2, we have $\delta_1 = 1211.1, \delta_2 = 1498.8, \delta_3 = 1094.3, \delta_4 = 452.6$, and the observer gain matrices in

$$G_1 = \begin{pmatrix} -4.6090 & 0.0885 & 0.0278 \\ 0.0769 & -4.5825 & 0.0720 \\ 0.0779 & 0.0623 & -4.9515 \end{pmatrix}, F_1 = \begin{pmatrix} 0.0131 & 0.0136 & 0.0134 \\ 0.0133 & 0.0137 & 0.0139 \end{pmatrix},$$

$$M_1 = \begin{pmatrix} 0.2550 & 0.0233 & 0.0923 \\ 0.2847 & 0.1270 & 0.3642 \end{pmatrix}^\top, H_1 = \begin{pmatrix} -0.6955 & 0.2243 \\ 0.2646 & -0.7055 \end{pmatrix},$$

$$G_2 = \begin{pmatrix} -4.5586 & 0.1579 & 0.0229 \\ 0.0107 & -4.4134 & 0.1102 \\ 0.0088 & 0.1704 & -4.9896 \end{pmatrix}, F_2 = \begin{pmatrix} 0.0150 & 0.0145 & 0.0154 \\ 0.0149 & 0.0143 & 0.0156 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 0.2261 & 0.0472 & 0.0919 \\ 0.2817 & 0.1855 & 0.3920 \end{pmatrix}^\top, H_2 = \begin{pmatrix} -0.6847 & 0.2154 \\ 0.2351 & -0.6946 \end{pmatrix}.$$

Figures 4 and 5 show the simulations of the errors $e_x(t)$ and $e_{\tilde{x}}(t)$ under ADT switching, respectively. Figures 6 and 7 present the simulations of the control signals under the bumpless transfer control and bumpy transfer control. It can be found from Figures 6 and 7 that the bumpless transfer controllers have smaller fluctuations than the bumpy transfer controllers. Additionally, a comparison between Examples 1 and 2 shows that without the bumpless transfer controller, the error system has more bumps when the system is switching.

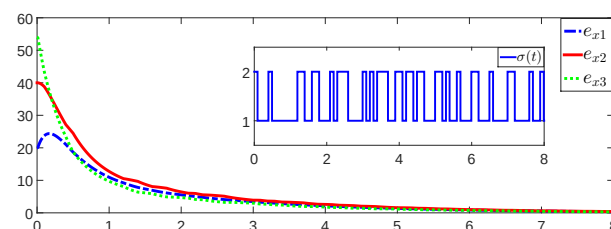


Figure 4. The error $e_x(t)$ under ADT switching.

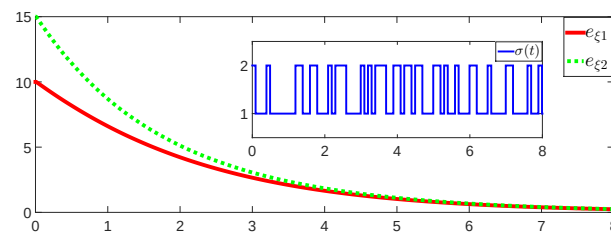


Figure 5. The error $e_{\zeta}(t)$ under ADT switching.

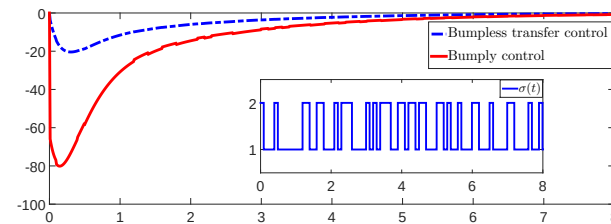


Figure 6. Bumpless and bumpy transfer controllers of the first subsystem.

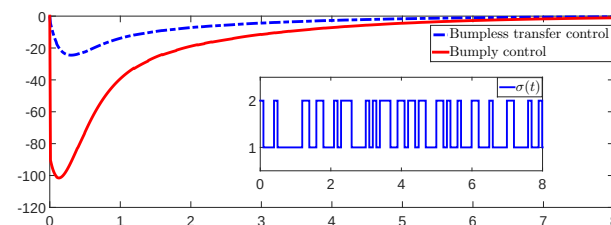


Figure 7. Bumpless and bumpy transfer controllers of the second subsystem.

5. Conclusions

This paper investigates the bumpless transfer control of SPSs by virtue of state and disturbance observers. A novel framework of positive double observers is developed for SPSs. A bumpless transfer controller is proposed based on the double observers. Linear-programming-based conditions are presented, and CLF is used for the stability analysis. In the future, it will be interesting to develop more accurate bumpless transfer conditions to limit bumps in switching systems. Moreover, it is also important to discuss applications of dual observers in different switching forms.

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