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Existence and Hyers–Ulam Stability of Stochastic Delay Systems Governed by the Rosenblatt Process

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Abstract: Under the effect of the Rosenblatt process, time-delay systems of nonlinear stochastic delay differential equations are considered. Utilizing the delayed matrix functions and exact solutions for these systems, the existence and Hyers–Ulam stability results are derived. First, depending on the fixed point theory, the existence and uniqueness of solutions are proven. Next, sufficient criteria for the Hyers–Ulam stability are established. Ultimately, to illustrate the importance of the results, an example is provided.

Keywords: Hyers–Ulam stability; stochastic delay system; Rosenblatt process; delayed matrix function; Krasnoselskii's fixed point theorem

MSC: 37A50; 34K50; 34K20; 47H10

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1. Introduction

Many researchers have paid significant attention to stochastic delay differential equations (SDDEs) and their applications because of their effective modeling in several scientific and engineering fields, such as physics, economics, biology, fluid dynamics, finance, medicine, and so forth (see, for instance, [1–9]). Recently, determining the exact solutions of differential systems has been attempted. Specifically, many new results regarding how to represent solutions for time-delay systems were obtained from the novel study [10,11], which were applied to stability analysis and control problems (see, [12–17] and the references therein).

The Wiener–Ito multiple integral of order q is defined as

$$Z_H^q(\ell) = a(H,q)\int_{R^q}\left(\int_0^\ell \prod_{j=1}^{q}(\varsigma-\Im_j)_+^{-\left(\frac{1}{2}+\frac{1-H}{q}\right)}\mathrm{d}\varsigma\right)\mathrm{d}\mathcal{G}(\Im_1)\ldots\mathrm{d}\mathcal{G}(\Im_q), \tag{1}$$

in terms of the standard Wiener process, $(\mathcal{G}(\Im))_{\Im\in\mathbb{R}}$, where $\mathbf{E}\left(Z_H^q(1)\right)^2 = 1$ and $\Im_+ = \max(\Im, 0)$ are the conditions under which $a(H,q)$ is a normalizing constant. The process $\left(Z_H^q(\ell)\right)_{\ell\geq 0}$, provided by (1), is called the Hermite process. The Hermite process is the fractional Brownian motion (fBm) with a Hurst parameter of $H \in \left(\frac{1}{2}, 1\right)$ for $q = 1$, while it is not Gaussian for $q = 2$. Additionally, the Hermite process, denoted by (1) for $q = 2$, is referred to as the Rosenblatt process. Most of the studies [18–20] involved fBm because of its self-similarity, long-range dependence, and more straightforward calculus of the Gaussian. But, fBm fails in the concrete case of having non-Gaussianity smooth-tongued in the models. In that situation, the Rosenblatt process is applicable. Non-Gaussian processes like the Rosenblatt process have numerous intriguing characteristics such as stationarity of the increments, long-range dependence, and self-similarity (for more details, see [21–29]). Therefore, it seems interesting to study a new class of stochastic differential equations driven by the Rosenblatt process.

On the other hand, studying the stability of (SDDEs) solutions is essential, and Hyers–Ulam stability (HUS) is a crucial topic. In 1940, Ulam [30] created the first proposal that functional equations are stable, during a lecture at Wisconsin University. In 1941, Hyers [31] provided a solution to this problem, after which HUS was established. In addition to providing a solid theoretical foundation for the well-posedness and HUS for SDDEs, the study of HUS for SDDEs also provides a solid theoretical foundation for the approximate solution of SDDEs. When it is rather difficult to acquire a precise solution for the system with HUS, we may substitute an approximate solution for an accurate one, and the HUS can, to a certain extent, ensure the dependability of the estimated solution.

Recently, many researchers have examined the HUS of diverse kinds of stochastic differential equations (see, [32–35] and the references therein).

However, as far as we know, the standard literature has not dealt with the existence and HUS of second-order nonlinear SDDEs driven by the Rosenblatt process. Therefore, in this study, we try, for the first time, to analyze such a topic.

Our study focuses on determining the existence and HUS of the nonlinear SDDEs driven by the Rosenblatt process, taking into account the previous research.

$$\begin{aligned} \aleph''(\ell) + \mathbb{D}\aleph(\ell - \zeta) = \hbar(\ell, \aleph(\ell)) + \Delta(\ell, \aleph(\ell))\frac{dZ_H(\ell)}{d\ell}, \ \ \ell \in \mp := [0, \varpi], \\ \aleph(\ell) \equiv \psi(\ell), \ \aleph'(\ell) \equiv \psi'(\ell), \ \ \ell \in \mp_1 := [-\zeta, 0], \end{aligned} \tag{2}$$

where $\aleph(\ell) \in \mathbb{R}^n$ represents the state vector, $\zeta > 0$ denotes a delay, $\varpi > (m-1)\zeta$, $m = 1, 2, \ldots$, $\psi \in C([-\zeta, 0], \mathbb{R}^n)$, $\mathbb{D} \in \mathbb{R}^{n \times n}$ is any matrix, and $\hbar \in C(\mp \times \mathbb{R}^n, \mathbb{R}^n)$ is a provided function. In the separable Hilbert space $\mathbb{R}^n$, let $\aleph(\cdot)$ have value, and let the norm be $\|\cdot\|$ and the inner product be $\langle \cdot, \cdot \rangle$ with parameter $H \in \left(\frac{1}{2}, 1\right)$, $Z_H(\ell)$ is a Rosenblatt process on an another real separable Hilbert space $(\mathcal{A}, \|\cdot\|_{\mathcal{A}}, \langle \cdot, \cdot \rangle_{\mathcal{A}})$. Furthermore, consider $\Delta \in C(\mp \times \mathbb{R}^n, L_2^0)$, where $L_2^0 = L_2\left(Q^{\frac{1}{2}}\mathcal{A}, \mathbb{R}^n\right)$.

The remaining sections of this paper are structured as follows: In Section 2, we present some notations and necessary preliminaries. In Section 3, by utilizing Krasnoselskii's fixed point theorem, some sufficient conditions are established for the existence and uniqueness of solutions to the system (2). In Section 4, we prove the Hyers–Ulam stability of (2) via Grönwall's inequality lemma approach. Finally, we provide a numerical example to illustrate the effectiveness of the derived results.

2. Preliminaries

During the entire paper, consider $(\Sigma, \eth, \mathbb{P})$ to represent the complete probability space with a probability measure $\mathbb{P}$ on Σ and a filtration $\{\eth_\ell | \, \ell \in \mp\}$ produced by $\{Z_H(s) | \, s \in [0, \ell]\}$. For some $1 < \mu < \infty$, consider the Hilbert space $L^\mu(\Sigma, \eth_\varpi, \mathbb{R}^n)$ to express all $\eth_\varpi$-measurable μth-integrable variables having values in $\mathbb{R}^n$ with norm $\|\aleph\|_{L^\mu}^\mu = \mathbf{E}\|\aleph(\ell)\|^\mu$, where the expectation $\mathbf{E}$ is defined by $\mathbf{E}\aleph = \int_\Sigma \aleph d\mathbb{P}$. Assume that $\mathcal{A}$ and $\mathcal{B}$ are two Banach spaces, $Q \in L_b(\mathcal{A}, \mathcal{A})$ indicates an operator on $\mathcal{A}$ that is self-adjoint trace class and non-negative, and $L_b(\mathcal{A}, \mathcal{B})$ is the space of the bounded linear operators from $\mathcal{A}$ to $\mathcal{B}$. Let $L_2^0 = L_2\left(Q^{\frac{1}{2}}\mathcal{A}, \mathcal{B}\right)$ be the space of all Q-Hilbert–Schmidt operators from $Q^{\frac{1}{2}}\mathcal{A}$ into $\mathcal{B}$, equipped with the norm

$$\|\Xi\|_{L_2^0}^2 = \left\|\Xi Q^{\frac{1}{2}}\right\|^2 = \mathrm{Tr}\left(\Xi Q \Xi^T\right).$$

Provided a norm $\|\Xi\|_{\mathcal{Q}} = \left(\sup_{\ell\in\mp} \mathbf{E}\|\Xi(\ell)\|^{\mu}\right)^{1/\mu}$, let $\mathcal{Q} := C([-\zeta,\varpi], L^{\mu}(\Sigma,\eth_{\varpi},\mathbb{P},\mathbb{R}^n))$ be the Banach space of all μth-integrable and $\eth_{\varpi}$-adapted processes Ξ. A norm $\|\cdot\|$ on $\mathbb{R}^n$ can be represented by the matrix norm

$$\|\mathbb{D}\| = \max\left\{\sum_{i=1}^{n}|d_{i1}|, \sum_{i=1}^{n}|d_{i2}|, \dots, \sum_{i=1}^{n}|d_{in}|\right\},$$

where $\mathbb{D} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$. Furthermore, consider

$$\begin{aligned} &C^1(\mp, L^{\mu}(\Sigma,\eth_{\varpi},\mathbb{P},\mathbb{R}^n)) \\ &= \left\{\aleph \in C(\mp, L^{\mu}(\Sigma,\eth_{\varpi},\mathbb{P},\mathbb{R}^n)) : \aleph' \in C(\mp, L^{\mu}(\Sigma,\eth_{\varpi},\mathbb{P},\mathbb{R}^n))\right\}. \end{aligned}$$

Finally, we assume the initial values

$$\|\psi\|_C^{\mu} = \sup_{s\in\mp_1} \mathbf{E}\|\psi(s)\|^{\mu} \text{ and } \|\psi'\|_C^{\mu} = \sup_{s\in\mp_1} \mathbf{E}\|\psi'(s)\|^{\mu}.$$

Some of the basic definitions and lemmas employed in this study are discussed.

Definition 1 ([13]). *Let the $n \times n$ identity matrix and null matrix be symbolized by $\mathbb{I}$ and Θ, respectively. Then, for $\iota = 0, 1, 2, \dots$, the delayed matrix functions $\mathcal{H}_{\zeta}(\mathbb{D}\ell)$ and $\mathcal{M}_{\zeta}(\mathbb{D}\ell)$ are defined, respectively, by*

$$\mathcal{H}_{\zeta}(\mathbb{D}\ell) := \begin{cases} \Theta, & -\infty < \ell < -\zeta, \\ \mathbb{I}, & -\zeta \le \ell < 0, \\ \mathbb{I} - \mathbb{D}\frac{\ell^2}{2!}, & 0 \le \ell < \zeta, \\ \quad\vdots & \quad\vdots \\ \mathbb{I} - \mathbb{D}\frac{\ell^2}{2!} + \mathbb{D}^2\frac{(\ell-\zeta)^4}{4!} & \\ + \cdots + (-1)^{\iota}\mathbb{D}^{\iota}\frac{(\ell-(\iota-1)\zeta)^{2\iota}}{(2\iota)!}, & (\iota-1)\zeta \le \ell < \iota\zeta, \end{cases} \tag{3}$$

and

$$\mathcal{M}_{\zeta}(\mathbb{D}\ell) := \begin{cases} \Theta, & -\infty < \ell < -\zeta, \\ \mathbb{I}(\ell+\zeta), & -\zeta \le \ell < 0, \\ \mathbb{I}(\ell+\zeta) - \mathbb{D}\frac{\ell^3}{3!}, & 0 \le \ell < \zeta, \\ \quad\vdots & \quad\vdots \\ \mathbb{I}(\ell+\zeta) - \mathbb{D}\frac{\ell^3}{3!} + \mathbb{D}^2\frac{(\ell-\zeta)^5}{5!} & \\ \quad + \cdots + (-1)^{\iota}\mathbb{D}^{\iota}\frac{(\ell-(\iota-1)\zeta)^{2\iota+1}}{(2\iota+1)!}, & (\iota-1)\zeta \le \ell < \iota\zeta, \end{cases} \tag{4}$$

Lemma 1 ([13]). *The solution of (2) can be expressed in the following form:*

$$\begin{aligned} \aleph(\ell) = {} & \mathcal{H}_{\zeta}(\mathbb{D}(\ell-\zeta))\psi(0) + \mathcal{M}_{\zeta}(\mathbb{D}(\ell-\zeta))\psi'(0) \\ & -\mathbb{D}\int_{-\zeta}^{0} \mathcal{M}_{\zeta}(\mathbb{D}(\ell-2\zeta-\varsigma))\psi(\varsigma)\mathrm{d}\varsigma \\ & + \int_0^{\ell} \mathcal{M}_{\zeta}(\mathbb{D}(\ell-\zeta-\varsigma))\hbar(\varsigma,\aleph(\varsigma))\mathrm{d}\varsigma \\ & + \int_0^{\ell} \mathcal{M}_{\zeta}(\mathbb{D}(\ell-\zeta-\varsigma))\Delta(\varsigma,\aleph(\varsigma))dZ_H(\varsigma). \end{aligned}$$

Lemma 2 ([29]). *If $\sigma : \mp \longrightarrow L_2^0$ satisfies*

$$\int_0^{\varpi} \|\sigma(\varsigma)\|^2_{L_2^0}\mathrm{d}\varsigma < \infty,$$

then

$$\mathbf{E}\left\|\int_0^{\ell}\sigma(\varsigma)\mathrm{d}Z_H(\varsigma)\right\|^2 \leq 2H\ell^{2H-1}\int_0^{\ell}\|\sigma(\varsigma)\|^2_{L_2^0}\mathrm{d}\varsigma.$$

Lemma 3 ([36]). *For $\Lambda : \mp \longrightarrow L_2^0$, such that*

$$\int_0^{\ell}\|\Lambda(\varsigma)\|^{\mu}_{L_2^0}\mathrm{d}\varsigma < \infty,$$

and applying Hölder's inequality and the Kahane–Khintchine inequality, there is a constant τ_μ, such that

$$\begin{aligned}
\mathbf{E}\left\|\int_0^{\ell}\Lambda(\varsigma)\mathrm{d}Z_H(\varsigma)\right\|^{\mu} &\leq \tau_\mu\left\{\mathbf{E}\left\|\int_0^{\ell}\Lambda(\varsigma)\mathrm{d}Z_H(\varsigma)\right\|^2\right\}^{\mu/2} \\
&\leq \tau_\mu\left\{2H\ell^{2H-1}\int_0^{\ell}\|\Lambda(\varsigma)\|^2_{L_2^0}\mathrm{d}\varsigma\right\}^{\mu/2} \\
&\leq \tau_\mu\left(2H\ell^{2H-1}\right)^{\mu/2}\left(\int_0^{\ell}\mathrm{d}\varsigma\right)^{\mu/2-1}\int_0^{\ell}\left(\|\Lambda(\varsigma)\|^2_{L_2^0}\right)^{\mu/2}\mathrm{d}\varsigma \\
&= \tau_\mu(2H)^{\mu/2}\ell^{\mu H-1}\int_0^{\ell}\|\Lambda(\varsigma)\|^{\mu}_{L_2^0}\mathrm{d}\varsigma.
\end{aligned}$$

Definition 2 ([37]). *When considering a specific constant $\kappa > 0$, and a function $\Pi \in C(\mp, \mathbb{R}^n)$ fulfilling*

$$\mathbf{E}\left\|\Pi''(\ell) + \mathbb{D}\Pi(\ell-\zeta) - \hbar(\ell,\Pi(\ell)) - \Delta(\ell,\Pi(\ell))\mathrm{d}Z_H(\ell)\right\|^{\mu} \leq \kappa, \quad \ell \in [0,\varpi], \tag{5}$$

implies that there exist a solution $\aleph \in C(\mp, \mathbb{R}^n)$ of (2) and a number $W > 0$ such that

$$\mathbf{E}\|\Pi(\ell) - \aleph(\ell)\|^{\mu} \leq W\kappa, \quad \text{for all } \ell \in [0,\varpi].$$

The system (2) is Hyers–Ulam stable on $[0,\varpi]$.

Remark 1 ([37]). *A function $\Pi \in C(\mp, \mathbb{R}^n)$ is a solution of the inequality (5) if and only if there exists a function $\mathcal{E} \in C(\mp, \mathbb{R}^n)$, such that*

(i) $\mathbf{E}\|\mathcal{E}(\ell)\|^{\mu} \leq \kappa$, $\ell \in \mp$.
(ii) $\Pi''(\ell) = -\mathbb{D}\Pi(\ell-\zeta) + \Delta(\ell,\Pi(\ell))\mathrm{d}Z_H(\ell) + \hbar(\ell,\Pi(\ell)) + \mathcal{E}(\ell)$, $\ell \in \mp$.

Definition 3 ([38]). *The Mittag–Leffler function, containing two parameters, is defined as*

$$\mathbb{E}_{\alpha,\epsilon}(\ell) = \sum_{\iota=0}^{\infty}\frac{\ell^{\iota}}{\Gamma(\alpha\iota+\epsilon)}, \quad \alpha,\ \epsilon > 0,\ \ell \in \mathbb{C}.$$

If $\epsilon = 1$, then

$$\mathbb{E}_{\alpha,1}(\ell) = \mathbb{E}_{\alpha}(\ell) = \sum_{\iota=0}^{\infty}\frac{\ell^{\iota}}{\Gamma(\alpha\iota+1)}, \quad \alpha > 0.$$

Lemma 4 ([15]). *For any $\ell \in [(\iota-1)\zeta, \iota\zeta]$, $\iota = 1, 2, \ldots$, we obtain*

$$\|\mathcal{H}_\zeta(\mathbb{D}(\ell))\| \leq \mathbb{E}_2\left(\|\mathbb{D}\|\ell^2\right),$$

and

$$\|\mathcal{M}_\zeta(\mathbb{D}(\ell))\| \leq (\ell+\zeta)\mathbb{E}_{2,2}\left(\|\mathbb{D}\|(\ell+\zeta)^2\right).$$

Lemma 5. (*Grönwall's inequality, [39]*). *Let $\hbar(\ell)$ and $\wp(\ell)$ be nonnegative, continuous functions on $0 \leq \ell \leq T$, for which the inequality*

$$\hbar(\ell) \leq \eta + \int_0^\ell \wp(s)\hbar(s)\mathrm{d}s, \quad \textit{for } \ell \in [0,T],$$

holds, where $\eta \geq 0$ is a constant. Then,

$$\hbar(\ell) \leq \eta \exp\left(\int_0^\ell \wp(s)\mathrm{d}s\right), \quad \textit{for } \ell \in [0,T].$$

Lemma 6. (*Krasnoselskii's fixed point theorem, [40]*). *Assume that $\mathcal{J}$ is a closed, bounded, and non-empty convex subset of a Banach space $\mathcal{U}$. If O_1 and O_2 are mappings from $\mathcal{J}$ into $\mathcal{U}$, such that*
(i) $O_1\ell + O_2\aleph \in \mathcal{J}$ for every pair $\ell, \aleph \in \mathcal{J}$,
(ii) O_2 is a contraction mapping,
(iii) O_1 is continuous and compact,
then there is $\Im \in \mathcal{J}$, such that $\Im = O_1\Im + O_2\Im$.

3. Main Results

In this section, we present and prove the existence, uniqueness, and Hyers–Ulam stability results of (2). To prove our main results, the assumptions listed below are assumed:

(G1): There exist a function $\varDelta : \mp \times \mathbb{R}^n \longrightarrow L_2^0$ that is continuous, and a constant $U_\varDelta \in L^{r_2}(\mp, \mathbb{R}^+)$ and $r_2 > 1$, such that

$$\mathbf{E}\|\varDelta(\ell,\aleph_1) - \varDelta(\ell,\aleph_2)\|^\mu_{L_2^0} \leq U_\varDelta(\ell)\mathbf{E}\|\aleph_1 - \aleph_2\|^\mu, \quad \text{for all } \ell \in \mp, \aleph_1, \aleph_2 \in \mathbb{R}^n.$$

Let $\mu \in [2,\infty)$ and $\sup_{\ell\in\mp} \mathbf{E}\|\varDelta(\ell,0)\|^\mu_{L_2^0} = W_\varDelta < \infty$.

(G2): There exist a function $\hbar : \mp \times \mathbb{R}^n \longrightarrow L_2^0$ that is continuous, and a constant $U_\hbar \in L^{r_2}(\mp, \mathbb{R}^+)$ and $r_2 > 1$, such that

$$\mathbf{E}\|\hbar(\ell,\aleph_1) - \hbar(\ell,\aleph_2)\|^\mu \leq U_\hbar(\ell)\mathbf{E}\|\aleph_1 - \aleph_2\|^\mu, \; \mathbf{E}\|\hbar(\ell,\aleph)\|^\mu \leq U_\hbar(\ell)\left(1 + \mathbf{E}\|\aleph\|^\mu\right),$$

for all $\ell \in \mp, \aleph_1, \aleph_2 \in \mathbb{R}^n$.

Using Krasnoselskii's fixed point theorem, we now prove the existence and uniqueness results.

Theorem 1. *If* (**G1**)–(**G2**) *holds, then there exists a unique mild solution of the nonlinear stochastic system (2), provided that*

$$2^{\mu-1}W_2 + W_3 < 1, \tag{6}$$

where

$$W_2 := \frac{\tau_\mu (2H)^{\mu/2} \varpi^{\mu(H+1)-\frac{1}{r_2}}}{(\mu r_1 + 1)^{\frac{1}{r_1}}} \left(\mathbb{E}_{2,2}\left(\|\mathbb{D}\|\varpi^2\right)\right)^\mu \|U_\Delta\|_{L^{r_2}(\mp,\mathbb{R}^+)},$$

and

$$W_3 := \frac{\varpi^{\mu+\frac{1}{r_1}}}{(\mu r_1 + 1)^{\frac{1}{r_1}}} \left(\mathbb{E}_{2,2}\left(\|\mathbb{D}\|\varpi^2\right)\right)^\mu \|U_\hbar\|_{L^{r_2}(\mp,\mathbb{R}^+)},$$

for $\frac{1}{r_1} + \frac{1}{r_2} = 1$, $r_1, r_2 > 1$.

Proof. We deal with the set

$$\mathcal{T}_\varrho = \left\{\aleph \in \mathcal{Q} : \|\aleph\|^\mu_{\mathcal{Q}} = \sup_{\ell\in\mp} \mathbf{E}\|\aleph(\ell)\|^\mu \leq \varrho\right\},$$

for each positive number ϱ. Let $\ell \in \mp$. Applying Lemma 1, we then transform problem (2) into a fixed point problem and define an operator $F : \mathcal{Q} \longrightarrow \mathcal{Q}$ by

$$\begin{aligned}(F\aleph)(\ell) = {} & \mathcal{H}_\zeta(\mathbb{D}(\ell-\zeta))\psi(0) + \mathcal{M}_\zeta(\mathbb{D}(\ell-\zeta))\psi'(0) \\ & -\mathbb{D}\int_{-\zeta}^{0} \mathcal{M}_\zeta(\mathbb{D}(\ell-2\zeta-\varsigma))\psi(\varsigma)\mathrm{d}\varsigma \\ & +\int_0^\ell \mathcal{M}_\zeta(\mathbb{D}(\ell-\zeta-\varsigma))\hbar(\varsigma,\aleph(\varsigma))\mathrm{d}\varsigma \\ & +\int_0^\ell \mathcal{M}_\zeta(\mathbb{D}(\ell-\zeta-\varsigma))\Delta(\varsigma,\aleph(\varsigma))\mathrm{d}Z_H(\varsigma),\end{aligned}$$

for $\ell \in \mp$. Decomposing the operator F, the operators $\mathcal{L}_1$ and $\mathcal{L}_2$ can be described on $\mathcal{T}_\varrho$, as provided below:

$$\begin{aligned}(\mathcal{L}_1\aleph)(\ell) = {} & \mathcal{H}_\zeta(\mathbb{D}(\ell-\zeta))\psi(0) + \mathcal{M}_\zeta(\mathbb{D}(\ell-\zeta))\psi'(0) \\ & -\mathbb{D}\int_{-\zeta}^{0} \mathcal{M}_\zeta(\mathbb{D}(\ell-2\zeta-\varsigma))\psi(\varsigma)\mathrm{d}\varsigma \\ & +\int_0^\ell \mathcal{M}_\zeta(\mathbb{D}(\ell-\zeta-\varsigma))\hbar(\varsigma,\aleph(\varsigma))\mathrm{d}\varsigma,\end{aligned} \tag{7}$$

$$(\mathcal{L}_2\aleph)(\ell) = \int_0^\ell \mathcal{M}_\zeta(\mathbb{D}(\ell-\zeta-\varsigma))\Delta(\varsigma,\aleph(\varsigma))\mathrm{d}Z_H(\varsigma). \tag{8}$$

At this point, we observe that $\mathcal{T}_\varrho$ is a convex set, closed and bounded of $\mathcal{Q}$. Consequently, our proof consists of three essential steps:

Step 1. We show the existence of $\varrho > 0$, such that $\mathcal{L}_1\aleph + \mathcal{L}_2\Im \in \mathcal{T}_\varrho$ for all $\aleph, \Im \in \mathcal{T}_\varrho$. For each $\ell \in \mp$ and $\aleph, \Im \in \mathcal{T}_\varrho$, and using (7) and (8), we obtain

$$\begin{aligned}& \|\mathcal{L}_1\aleph + \mathcal{L}_2\Im\|_{\mathcal{Q}}^\mu \\ & = \sup_{\ell\in\mp} \mathbf{E}\|(\mathcal{L}_1\aleph + \mathcal{L}_2\Im)(\ell)\|^\mu \\ & \leq 5^{\mu-1}\Big[\big\|\mathcal{H}_\zeta(\mathbb{D}(\ell-\zeta))\big\|^\mu \mathbf{E}\|\psi(0)\|^\mu + \big\|\mathcal{M}_\zeta(\mathbb{D}(\ell-\zeta))\big\|^\mu \mathbf{E}\|\psi'(0)\|^\mu \\ & \quad +\|\mathbb{D}\|^\mu \mathbf{E}\bigg\|\int_{-\zeta}^{0} \mathcal{M}_\zeta(\mathbb{D}(\ell-2\zeta-\varsigma))\psi(\varsigma)\mathrm{d}\varsigma\bigg\|^\mu \\ & \quad +\mathbf{E}\bigg\|\int_0^\ell \mathcal{M}_\zeta(\mathbb{D}(\ell-\zeta-\varsigma))\hbar(\varsigma,\aleph(\varsigma))\mathrm{d}\varsigma\bigg\|^\mu \\ & \quad +\mathbf{E}\bigg\|\int_0^\ell \mathcal{M}_\zeta(\mathbb{D}(\ell-\zeta-\varsigma))\Delta(\varsigma,\Im(\varsigma))\mathrm{d}Z_H(\varsigma)\bigg\|^\mu\Big] \\ & = \textstyle\sum_{n=1}^5 \mathbb{I}_n.\end{aligned} \tag{9}$$

From Lemma 4, we have

$$\begin{aligned}\mathbf{I}_1 & = 5^{\mu-1}\big\|\mathcal{H}_\zeta(\mathbb{D}(\ell-\zeta))\big\|^\mu \mathbf{E}\|\psi(0)\|^\mu \\ & \leq 5^{\mu-1}\Big(\mathbb{E}_2\Big(\|\mathbb{D}\|(\ell-\zeta)^2\Big)\Big)^\mu \mathbf{E}\|\psi\|_C^\mu,\end{aligned}$$

$$\begin{aligned}\mathbf{I}_2 & = 5^{\mu-1}\big\|\mathcal{M}_\zeta(\mathbb{D}(\ell-\zeta))\big\|^\mu \mathbf{E}\|\psi'(0)\|^\mu \\ & \leq 5^{\mu-1}\Big(\ell\mathbb{E}_{2,2}\Big(\|\mathbb{D}\|\ell^2\Big)\Big)^\mu \mathbf{E}\|\psi'\|_C^\mu,\end{aligned}$$

$$\begin{aligned}\mathbf{I}_3 &= 5^{\mu-1}\|\mathbb{D}\|^{\mu}\mathbf{E}\left\|\int_{-\zeta}^{0}\mathcal{M}_{\zeta}(\mathbb{D}(\ell-2\zeta-\varsigma))\psi(\varsigma)\mathrm{d}\varsigma\right\|^{\mu}\\ &\leq 5^{\mu-1}\|\mathbb{D}\|^{\mu}\zeta^{\mu-1}\mathbf{E}\|\psi\|_C^{\mu}\int_{-\zeta}^{0}\|\mathcal{M}_{\zeta}(\mathbb{D}(\ell-2\zeta-\varsigma))\|^{\mu}\mathrm{d}\varsigma\\ &\leq 5^{\mu-1}\|\mathbb{D}\|^{\mu}\zeta^{\mu}\Big(\ell\mathbb{E}_{2,2}\Big(\|\mathbb{D}\|\ell^2\Big)\Big)^{\mu}\mathbf{E}\|\psi\|_C^{\mu},\end{aligned}$$

$$\begin{aligned}\mathbf{I}_4 &= 5^{\mu-1}\mathbf{E}\left\|\int_0^{\ell}\mathcal{M}_{\zeta}(\mathbb{D}(\ell-\zeta-\varsigma))\Delta(\varsigma,\Im(\varsigma))\mathrm{d}Z_H(\varsigma)\right\|^{\mu}\\ &= 5^{\mu-1}\mathbf{E}\left\{\left\|\int_0^{\ell}\mathcal{M}_{\zeta}(\mathbb{D}(\ell-\zeta-\varsigma))\Delta(\varsigma,\Im(\varsigma))\mathrm{d}Z_H(\varsigma)\right\|^2\right\}^{\mu/2}.\end{aligned}$$

Applying Lemmas 2 and 3, we obtain

$$\begin{aligned}\mathbf{I}_4 &\leq 5^{\mu-1}\tau_{\mu}\left\{\mathbf{E}\left\|\int_0^{\ell}\mathcal{M}_{\zeta}(\mathbb{D}(\ell-\zeta-\varsigma))\Delta(\varsigma,\Im(\varsigma))\mathrm{d}Z_H(\varsigma)\right\|^2\right\}^{\mu/2}\\ &\leq 5^{\mu-1}\tau_{\mu}\left\{2H\ell^{2H-1}\int_0^{\ell}\mathbf{E}\|\mathcal{M}_{\zeta}(\mathbb{D}(\ell-\zeta-\varsigma))\Delta(\varsigma,\Im(\varsigma))\|_{L_2^0}^2\mathrm{d}\varsigma\right\}^{\mu/2}\\ &\leq 5^{\mu-1}\tau_{\mu}\Big(2H\ell^{2H-1}\Big)^{\mu/2}\left\{\int_0^{\ell}\mathbf{E}\|\mathcal{M}_{\zeta}(\mathbb{D}(\ell-\zeta-\varsigma))\Delta(\varsigma,\Im(\varsigma))\|_{L_2^0}^2\mathrm{d}\varsigma\right\}^{\mu/2}\\ &\leq 5^{\mu-1}\tau_{\mu}\Big(2H\ell^{2H-1}\Big)^{\mu/2}\\ &\quad\times\left\{\left(\int_0^{\ell}\Big(\mathbf{E}\|\mathcal{M}_{\zeta}(\mathbb{D}(\ell-\zeta-\varsigma))\Delta(\varsigma,\Im(\varsigma))\|_{L_2^0}^2\Big)^{\mu/2}\mathrm{d}\varsigma\right)^{2/\mu}\left(\int_0^{\ell}\mathrm{d}\varsigma\right)^{\frac{\mu-2}{\mu}}\right\}^{\mu/2}\\ &\leq 5^{\mu-1}\tau_{\mu}(2H)^{\mu/2}\varpi^{\mu H-1}\int_0^{\ell}\mathbf{E}\|\mathcal{M}_{\zeta}(\mathbb{D}(\ell-\zeta-\varsigma))\Delta(\varsigma,\Im(\varsigma))\|_{L_2^0}^{\mu}\mathrm{d}\varsigma.\end{aligned}$$

Using Lemma 4 and (**G1**), we obtain

$$\begin{aligned}\mathbf{I}_4 &\leq 5^{\mu-1}\tau_{\mu}(2H)^{\mu/2}\varpi^{\mu H-1}\int_0^{\ell}\Big((\ell-\varsigma)\mathbb{E}_{2,2}\Big(\|\mathbb{D}\|(\ell-\varsigma)^2\Big)\Big)^{\mu}\mathbf{E}\|\Delta(\varsigma,\Im(\varsigma))\|_{L_2^0}^{\mu}\mathrm{d}\varsigma\\ &\leq 5^{\mu-1}\tau_{\mu}(2H)^{\mu/2}\varpi^{\mu H-1}\\ &\quad\times 2^{\mu-1}\left\{\int_0^{\ell}\Big((\ell-\varsigma)\mathbb{E}_{2,2}\Big(\|\mathbb{D}\|(\ell-\varsigma)^2\Big)\Big)^{\mu}\mathbf{E}\|\Delta(\varsigma,\Im(\varsigma))-\Delta(\varsigma,0)\|_{L_2^0}^{\mu}\mathrm{d}\varsigma\right.\\ &\qquad\left.+\int_0^{\ell}\Big((\ell-\varsigma)\mathbb{E}_{2,2}\Big(\|\mathbb{D}\|(\ell-\varsigma)^2\Big)\Big)^{\mu}\mathbf{E}\|\Delta(\varsigma,0)\|_{L_2^0}^{\mu}\mathrm{d}\varsigma\right\}\\ &\leq (10)^{\mu-1}\tau_{\mu}(2H)^{\mu/2}\varpi^{\mu H-1}\\ &\quad\times\left\{\int_0^{\ell}\Big((\ell-\varsigma)\mathbb{E}_{2,2}\Big(\|\mathbb{D}\|(\ell-\varsigma)^2\Big)\Big)^{\mu}U_{\Delta}(\varsigma)\mathbf{E}\|\Im(\varsigma)\|^{\mu}\mathrm{d}\varsigma\right.\\ &\qquad\left.+W_{\Delta}\int_0^{\ell}\Big((\ell-\varsigma)\mathbb{E}_{2,2}\Big(\|\mathbb{D}\|(\ell-\varsigma)^2\Big)\Big)^{\mu}\mathrm{d}\varsigma\right\}\\ &\leq (10)^{\mu-1}\tau_{\mu}(2H)^{\mu/2}\varpi^{\mu H-1}\left\{\|\Im\|_{\mathcal{Q}}^{\mu}\int_0^{\ell}\Big((\ell-\varsigma)\mathbb{E}_{2,2}\Big(\|\mathbb{D}\|(\ell-\varsigma)^2\Big)\Big)^{\mu}U_{\Delta}(\varsigma)\mathrm{d}\varsigma\right.\\ &\qquad\left.+\frac{\varpi^{\mu+1}W_{\Delta}}{\mu+1}\Big(\mathbb{E}_{2,2}\Big(\|\mathbb{D}\|\varpi^2\Big)\Big)^{\mu}\right\}.\end{aligned} \tag{10}$$

Additionally, using Hölder inequality and (**G1**), we obtain

$$\begin{aligned}
&\int_0^\ell \Big((\ell-\varsigma)\mathbb{E}_{2,2}\Big(\|\mathbb{D}\|(\ell-\varsigma)^2\Big)\Big)^\mu U_\Delta(\varsigma)\mathrm{d}\varsigma \\
&\le \left(\int_0^\ell \Big((\ell-\varsigma)\mathbb{E}_{2,2}\Big(\|\mathbb{D}\|(\ell-\varsigma)^2\Big)\Big)^{\mu r_1}\mathrm{d}\varsigma\right)^{\frac{1}{r_1}}\left(\int_0^\ell U_\Delta^{r_2}(\varsigma)\mathrm{d}\varsigma\right)^{\frac{1}{r_2}} \\
&\le \Big(\mathbb{E}_{2,2}\Big(\|\mathbb{D}\|\varpi^2\Big)\Big)^\mu\left(\int_0^\ell (\ell-\varsigma)^{\mu r_1}\mathrm{d}\varsigma\right)^{\frac{1}{r_1}}\left(\int_0^\ell U_\Delta^{r_2}(\varsigma)\mathrm{d}\varsigma\right)^{\frac{1}{r_2}} \\
&\le \frac{\varpi^{\mu+\frac{1}{r_1}}}{(\mu r_1+1)^{\frac{1}{r_1}}}\Big(\mathbb{E}_{2,2}\Big(\|\mathbb{D}\|\varpi^2\Big)\Big)^\mu\|U_\Delta\|_{L^{r_2}(\mp,\mathbb{R}^+)}.
\end{aligned} \tag{11}$$

Substituting (11) into (10), we obtain

$$\begin{aligned}
\mathbf{I}_4 \le{}& (10)^{\mu-1}\tau_\mu(2H)^{\mu/2}\varpi^{\mu H-1} \\
&\times\left\{\frac{\varrho\varpi^{\mu+\frac{1}{r_1}}}{(\mu r_1+1)^{\frac{1}{r_1}}}\Big(\mathbb{E}_{2,2}\Big(\|\mathbb{D}\|\varpi^2\Big)\Big)^\mu\|U_\Delta\|_{L^{r_2}(\mp,\mathbb{R}^+)}+\frac{\varpi^{\mu+1}W_\Delta}{\mu+1}\Big(\mathbb{E}_{2,2}\Big(\|\mathbb{D}\|\varpi^2\Big)\Big)^\mu\right\} \\
={}& (10)^{\mu-1}W_2\varrho+\frac{(10)^{\mu-1}\tau_\mu(2H)^{\mu/2}\varpi^{\mu(H+1)}W_\Delta}{\mu+1}\Big(\mathbb{E}_{2,2}\Big(\|\mathbb{D}\|\varpi^2\Big)\Big)^\mu.
\end{aligned}$$

Furthermore, using (11) and (**G2**), we obtain

$$\begin{aligned}
\mathbf{I}_5 ={}& 5^{\mu-1}\mathbf{E}\left\|\int_0^\ell \mathcal{M}_\zeta(\mathbb{D}(\ell-\zeta-\varsigma))\hbar(\varsigma,\aleph(\varsigma))\mathrm{d}\varsigma\right\|^\mu \\
\le{}& 5^{\mu-1}\int_0^\ell\Big((\ell-\varsigma)\mathbb{E}_{2,2}\Big(\|\mathbb{D}\|(\ell-\varsigma)^2\Big)\Big)^\mu\mathbf{E}\|\hbar(\varsigma,\aleph(\varsigma))\|^\mu\mathrm{d}\varsigma \\
\le{}& 5^{\mu-1}\int_0^\ell\Big((\ell-\varsigma)\mathbb{E}_{2,2}\Big(\|\mathbb{D}\|(\ell-\varsigma)^2\Big)\Big)^\mu U_\hbar(\varsigma)(1+\mathbf{E}\|\aleph\|^\mu)\mathrm{d}\varsigma \\
\le{}& \frac{5^{\mu-1}(1+\varrho)\varpi^{\mu+\frac{1}{r_1}}}{(\mu r_1+1)^{\frac{1}{r_1}}}\Big(\mathbb{E}_{2,2}\Big(\|\mathbb{D}\|\varpi^2\Big)\Big)^\mu\|U_\hbar\|_{L^{r_2}(\mp,\mathbb{R}^+)} \\
={}& 5^{\mu-1}(1+\varrho)W_3.
\end{aligned}$$

From $\mathbf{I}_1$ to $\mathbf{I}_5$, (9) becomes

$$\begin{aligned}
&\|\mathcal{L}_1\aleph+\mathcal{L}_2\Im\|_{\mathcal{Q}}^\mu \\
&\le 5^{\mu-1}\Big\{\Big(\mathbb{E}_2\Big(\|\mathbb{D}\|(\ell-\zeta)^2\Big)\Big)^\mu\mathbf{E}\|\psi\|_C^\mu \\
&\quad+\Big(\ell\mathbb{E}_{2,2}\Big(\|\mathbb{D}\|\ell^2\Big)\Big)^\mu\mathbf{E}\|\psi'\|_C^\mu \\
&\quad+\|\mathbb{D}\|^\mu\zeta^\mu\Big(\ell\mathbb{E}_{2,2}\Big(\|\mathbb{D}\|\ell^2\Big)\Big)^\mu\mathbf{E}\|\psi\|_C^\mu \\
&\quad+2^{\mu-1}W_2\varrho+\frac{2^{\mu-1}\tau_\mu(2H)^{\mu/2}\varpi^{\mu(H+1)}W_\Delta}{\mu+1}\Big(\mathbb{E}_{2,2}\Big(\|\mathbb{D}\|\varpi^2\Big)\Big)^\mu \\
&\quad+(1+\varrho)W_3\} \\
&\le 5^{\mu-1}\Big\{\theta(\varpi)+\varrho\Big(2^{\mu-1}W_2+W_3\Big)+W_3\Big\},
\end{aligned}$$

where

$$\theta(\ell) = \left(\mathbb{E}_2\left(\|\mathbb{D}\|(\ell-\zeta)^2\right)\right)^{\mu}\mathbf{E}\|\psi\|_C^{\mu} + \left(\ell\mathbb{E}_{2,2}\left(\|\mathbb{D}\|\ell^2\right)\right)^{\mu}\mathbf{E}\|\psi'\|_C^{\mu} + \|\mathbb{D}\|^{\mu}\zeta^{\mu}\left(\ell\mathbb{E}_{2,2}\left(\|\mathbb{D}\|\ell^2\right)\right)^{\mu}\mathbf{E}\|\psi\|_C^{\mu} + \frac{2^{\mu-1}\tau_{\mu}(2H)^{\mu/2}\ell^{\mu(H+1)}W_{\Delta}}{\mu+1}\left(\mathbb{E}_{2,2}\left(\|\mathbb{D}\|\ell^2\right)\right)^{\mu}.$$

As a result, from (6), we obtain $\mathcal{L}_1\aleph + \mathcal{L}_2\Im \in \mathcal{T}_{\varrho}$ for some ϱ sufficiency large.

Step 2. We show that $\mathcal{L}_1 : \mathcal{T}_{\varrho} \longrightarrow \mathcal{Q}$ is a contraction. For each $\ell \in \mp$ and $\aleph, \Im \in \mathcal{T}_{\varrho}$, using (7) and (**G2**), we obtain

$$\begin{aligned}
&\mathbf{E}\|(\mathcal{L}_1\aleph)(\ell) - (\mathcal{L}_1\Im)(\ell)\|^{\mu} \\
&= \mathbf{E}\left\|\int_0^{\ell}\mathcal{M}_{\zeta}(\mathbb{D}(\ell-\zeta-\varsigma))[\hbar(\varsigma,\aleph(\varsigma)) - \hbar(\varsigma,\Im(\varsigma))]\mathrm{d}\varsigma\right\|^{\mu} \\
&\le \mathbf{E}\|\aleph-\Im\|_{\mathcal{Q}}^{\mu}\int_0^{\ell}\left((\ell-\varsigma)\mathbb{E}_{2,2}\left(\|\mathbb{D}\|(\ell-\varsigma)^2\right)\right)^{\mu}U_{\hbar}(\varsigma)\mathrm{d}\varsigma \\
&\le W_3\|\aleph-\Im\|_{\mathcal{Q}}^{\mu}.
\end{aligned}$$

As we can see from (6), noting $W_3 < 1$, that $\mathcal{L}_1$ is a contraction mapping.

Step 3. We show that $\mathcal{L}_2 : \mathcal{T}_{\varrho} \longrightarrow \mathcal{Q}$ is a continuous compact operator. First, we verify the continuity of $\mathcal{L}_2$. Consider $\aleph_n \in \mathcal{T}_{\varrho}$ with $\aleph_n \longrightarrow \aleph$ as $n \longrightarrow \infty$ in $\mathcal{T}_{\varrho}$. Thus, using Lebesgue's dominated convergence theorem and (8), we obtain, for each $\ell \in \mp$,

$$\begin{aligned}
&\mathbf{E}\|(\mathcal{L}_2\aleph_n)(\ell) - (\mathcal{L}_2\aleph)(\ell)\|^{\mu} \\
&\le \tau_{\mu}(2H)^{\mu/2}\varpi^{\mu H-1}\int_0^{\ell}\|\mathcal{M}_{\zeta}(\mathbb{D}(\ell-\zeta-\varsigma))\|^{\mu}\mathbf{E}\|\Delta(\varsigma,\aleph_n(\varsigma)) - \Delta(\varsigma,\aleph(\varsigma))\|_{L_2^0}^{\mu}\mathrm{d}\varsigma \\
&\le \tau_{\mu}(2H)^{\mu/2}\varpi^{\mu H-1}\int_0^{\ell}\left((\ell-\varsigma)\mathbb{E}_{2,2}\left(\|\mathbb{D}\|(\ell-\varsigma)^2\right)\right)^{\mu}U_{\Delta}(\varsigma) \\
&\qquad\times\|\aleph_n-\aleph\|_{\mathcal{Q}}^{\mu}\mathrm{d}\varsigma \longrightarrow 0, \text{ as } n \longrightarrow \infty.
\end{aligned}$$

This proves the continuity of $\mathcal{L}_2 : \mathcal{T}_{\varrho} \longrightarrow \mathcal{Q}$. Thereafter, we show that $\mathcal{L}_2$ is uniformly bounded on $\mathcal{T}_{\varrho}$. For each $\ell \in \mp$, $\aleph \in \mathcal{T}_{\varrho}$, we have

$$\begin{aligned}
\|\mathcal{L}_2\aleph\|_{\mathcal{Q}}^{\mu} &= \sup_{\ell\in\mp}\mathbf{E}\|(\mathcal{L}_2\aleph)(\ell)\|^{\mu} \\
&\le \sup_{\ell\in\mp}\left\{\mathbf{E}\left\|\int_0^{\ell}\mathcal{M}_{\zeta}(\mathbb{D}(\ell-\zeta-\varsigma))\Delta(\varsigma,\aleph(\varsigma))\mathrm{d}Z_H(\varsigma)\right\|^{\mu}\right\} \\
&\le 2^{\mu-1}W_2\varrho + \frac{2^{\mu-1}\tau_{\mu}(2H)^{\mu/2}\varpi^{\mu(H+1)}W_{\Delta}}{\mu+1}\left(\mathbb{E}_{2,2}\left(\|\mathbb{D}\|\varpi^2\right)\right)^{\mu},
\end{aligned}$$

this indicates that, on $\mathcal{T}_{\varrho}$, $\mathcal{L}_2$ is uniformly bounded. Showing that $\mathcal{L}_2$ is equicontinuous is still necessary. For each $\ell_2, \ell_3 \in \mp$, $0 < \ell_2 < \ell_3 \le \varpi$ and $\aleph \in \mathcal{T}_{\varrho}$, using (8), we obtain

$$\begin{aligned}
&(\mathcal{L}_2\aleph)(\ell_3) - (\mathcal{L}_2\aleph)(\ell_2) \\
&= \int_0^{\ell_3}\mathcal{M}_{\zeta}(\mathbb{D}(\ell_3-\zeta-\varsigma))\Delta(\varsigma,\aleph(\varsigma))\mathrm{d}Z_H(\varsigma) \\
&\quad - \int_0^{\ell_2}\mathcal{M}_{\zeta}(\mathbb{D}(\ell_2-\zeta-\varsigma))\Delta(\varsigma,\aleph(\varsigma))\mathrm{d}Z_H(\varsigma) \\
&= \Psi_1 + \Psi_2,
\end{aligned}$$

where

$$\Psi_1 = \int_{\ell_2}^{\ell_3} \mathcal{M}_\zeta(\mathbb{D}(\ell_3 - \zeta - \varsigma))\Delta(\varsigma, \aleph(\varsigma))\mathrm{d}Z_H(\varsigma),$$

and

$$\Psi_2 = \int_0^{\ell_2} \big[\mathcal{M}_\zeta(\mathbb{D}(\ell_3 - \zeta - \varsigma)) - \mathcal{M}_\zeta(\mathbb{D}(\ell_2 - \zeta - \varsigma))\big]\Delta(\varsigma, \aleph(\varsigma))\mathrm{d}Z_H(\varsigma).$$

Thus

$$\begin{aligned} \mathbf{E}\|(\mathcal{L}_2\aleph)(\ell_3) - (\mathcal{L}_2\aleph)(\ell_2)\|^\mu &= \mathbf{E}\|\Psi_1 + \Psi_2\|^\mu \\ &\leq 2^{\mu-1}\big\{\mathbf{E}\|\Psi_1\|^\mu + \mathbf{E}\|\Psi_2\|^\mu\big\}. \end{aligned} \tag{12}$$

Now, we can check $\|\Psi_r\| \longrightarrow 0$ as $\ell_2 \longrightarrow \ell_3$, when $r = 1, 2$. For Ψ_1, we obtain

$$\begin{aligned} \mathbf{E}\|\Psi_1\|^\mu &= \mathbf{E}\bigg\|\int_{\ell_2}^{\ell_3} \mathcal{M}_\zeta(\mathbb{D}(\ell_3 - \zeta - \varsigma))\Delta(\varsigma, \aleph(\varsigma))\mathrm{d}Z_H(\varsigma)\bigg\|^\mu \\ &\leq \tau_\mu(2H)^{\mu/2}(\ell_3 - \ell_2)^{\mu H - 1}\int_{\ell_2}^{\ell_3} \mathbf{E}\|\mathcal{M}_\zeta(\mathbb{D}(\ell - \zeta - \varsigma))\Delta(\varsigma, \aleph(\varsigma))\|^\mu_{L_2^0}\mathrm{d}\varsigma \\ &\leq 2^{\mu-1}\tau_\mu(2H)^{\mu/2}(\ell_3 - \ell_2)^{\mu H - 1} \\ &\quad \times\bigg\{\varrho\int_{\ell_2}^{\ell_3}\Big((\ell - \varsigma)\mathbb{E}_{2,2}\Big(\|\mathbb{D}\|(\ell - \varsigma)^2\Big)\Big)^\mu U_\Delta(\varsigma)\mathrm{d}\varsigma \\ &\qquad + \frac{(\ell_3 - \ell_2)^{\mu+1}W_\Delta}{\mu + 1}\Big(\mathbb{E}_{2,2}\Big(\|\mathbb{D}\|(\ell_3 - \ell_2)^2\Big)\Big)^\mu\bigg\} \longrightarrow 0, \text{ as } \ell_2 \longrightarrow \ell_3. \end{aligned}$$

For Ψ_2, we obtain

$$\begin{aligned} &\mathbf{E}\|\Psi_2\|^\mu \\ &= \mathbf{E}\bigg\|\int_0^{\ell_2}\big[\mathcal{M}_\zeta(\mathbb{D}(\ell_3 - \zeta - \varsigma)) - \mathcal{M}_\zeta(\mathbb{D}(\ell_2 - \zeta - \varsigma))\big]\Delta(\varsigma, \aleph(\varsigma))\mathrm{d}Z_H(\varsigma)\bigg\|^\mu \\ &\leq \tau_\mu(2H)^{\mu/2}\ell_2^{\mu H - 1} \\ &\quad \times\int_0^{\ell_2}\mathbf{E}\|\big[\mathcal{M}_\zeta(\mathbb{D}(\ell_3 - \zeta - \varsigma)) - \mathcal{M}_\zeta(\mathbb{D}(\ell_2 - \zeta - \varsigma))\big]\Delta(\varsigma, \aleph(\varsigma))\|^\mu_{L_2^0}\mathrm{d}\varsigma \\ &\leq 2^{\mu-1}\tau_\mu(2H)^{\mu/2}\ell_2^{\mu H - 1} \\ &\quad \times\bigg\{\varrho\int_0^{\ell_2}\|\mathcal{M}_\zeta(\mathbb{D}(\ell_3 - \zeta - \varsigma)) - \mathcal{M}_\zeta(\mathbb{D}(\ell_2 - \zeta - \varsigma))\|^\mu U_\Delta(\varsigma)\mathrm{d}\varsigma \\ &\qquad + W_\Delta\int_0^{\ell_2}\|\mathcal{M}_\zeta(\mathbb{D}(\ell_3 - \zeta - \varsigma)) - \mathcal{M}_\zeta(\mathbb{D}(\ell_2 - \zeta - \varsigma))\|^\mu\mathrm{d}\varsigma\bigg\} \\ &\leq 2^{\mu-1}\tau_\mu(2H)^{\mu/2}\ell_2^{\mu H - 1} \\ &\quad \times\Big\{\varrho\|U_\Delta\|_{L^{r_2}(\mp, \mathbb{R}^+)} \\ &\qquad \times\bigg(\int_0^{\ell_2}\Big(\|\mathcal{M}_\zeta(\mathbb{D}(\ell_3 - \zeta - \varsigma)) - \mathcal{M}_\zeta(\mathbb{D}(\ell_2 - \zeta - \varsigma))\|^\mu\Big)^{r_1}\bigg)^{1/r_1}\mathrm{d}\varsigma \\ &\qquad + W_\Delta\int_0^{\ell_2}\|\mathcal{M}_\zeta(\mathbb{D}(\ell_3 - \zeta - \varsigma)) - \mathcal{M}_\zeta(\mathbb{D}(\ell_2 - \zeta - \varsigma))\|^\mu\mathrm{d}\varsigma\Big\}. \end{aligned}$$

From (4), knowing that $\mathcal{M}_\zeta(\mathbb{D}\ell)$ is uniformly continuous for $\ell \in \mp$, we obtain

$$\|\mathcal{M}_\zeta(\mathbb{D}(\ell_3 - \zeta - \varsigma)) - \mathcal{M}_\zeta(\mathbb{D}(\ell_2 - \zeta - \varsigma))\| \longrightarrow 0, \text{ as } \ell_2 \longrightarrow \ell_3.$$

Therefore, we have $\|\Psi_r\| \longrightarrow 0$ as $\ell_2 \longrightarrow \ell_3$, when $r = 1, 2$, which leads, via (12), to

$$\mathbf{E}\|(\mathcal{L}_2\aleph)(\ell_3) - (\mathcal{L}_2\aleph)(\ell_2)\|^\mu \longrightarrow 0, \quad \text{as } \ell_2 \longrightarrow \ell_3,$$

for all $\aleph \in \mathcal{T}_\varrho$. Then, $\mathcal{L}_2$ is compact on $\mathcal{T}_\varrho$ via the Arzelà-Ascoli theorem (see [40]). As a result, $F\aleph = \mathcal{L}_1\aleph + \mathcal{L}_2\aleph$ has a fixed point $\aleph$ in $\mathcal{T}_\varrho$, in accordance with Lemma 6. Furthermore, $\aleph$ is also a solution of (2) and $(\mathcal{L}_1\aleph + \mathcal{L}_2\aleph)(\varpi) = \aleph_1$. Therefore, (2) has a mild solution. This completes the proof. □

Next, we verify the Hyers–Ulam stability via Grönwall's inequality lemma approach.

Theorem 2. *If the assumptions of Theorem 1 are satisfied, then the system (2) has Ulam–Hyers stability.*

Proof. Assume that $\aleph$ is the unique solution of (2) and $\Pi \in C(\mp, \mathbb{R}^n)$ is a solution of the inequality (5) with the aid of Theorem 1. Then

$$\begin{aligned}\aleph(\ell) = {} & \mathcal{H}_\zeta(\mathbb{D}(\ell-\zeta))\psi(0) + \mathcal{M}_\zeta(\mathbb{D}(\ell-\zeta))\psi'(0) \\ & -\mathbb{D}\int_{-\zeta}^{0} \mathcal{M}_\zeta(\mathbb{D}(\ell-2\zeta-\varsigma))\psi(\varsigma)\mathrm{d}\varsigma \\ & + \int_0^\ell \mathcal{M}_\zeta(\mathbb{D}(\ell-\zeta-\varsigma))\hbar(\varsigma,\aleph(\varsigma))\mathrm{d}\varsigma \\ & + \int_0^\ell \mathcal{M}_\zeta(\mathbb{D}(\ell-\zeta-\varsigma))\Delta(\varsigma,\aleph(\varsigma))\mathrm{d}Z_H(\varsigma).\end{aligned}$$

Based on Remark 1, then

$$\Pi''(\ell) = -\mathbb{D}\Pi(\ell-\zeta) + \Delta(\ell,\Pi(\ell))\mathrm{d}Z_H(\ell) + \hbar(\ell,\Pi(\ell)) + \mathcal{E}(\ell), \quad \ell \in \mp,$$

can be expressed as

$$\begin{aligned}\Pi(\ell) = {} & \mathcal{H}_\zeta(\mathbb{D}(\ell-\zeta))\psi(0) + \mathcal{M}_\zeta(\mathbb{D}(\ell-\zeta))\psi'(0) \\ & -\mathbb{D}\int_{-\zeta}^{0} \mathcal{M}_\zeta(\mathbb{D}(\ell-2\zeta-\varsigma))\psi(\varsigma)\mathrm{d}\varsigma \\ & + \int_0^\ell \mathcal{M}_\zeta(\mathbb{D}(\ell-\zeta-\varsigma))\Delta(\varsigma,\Pi(\varsigma))\mathrm{d}Z_H(\varsigma) \\ & + \int_0^\ell \mathcal{M}_\zeta(\mathbb{D}(\ell-\zeta-\varsigma))\hbar(\ell,\Pi(\ell))\mathrm{d}\varsigma \\ & + \int_0^\ell \mathcal{M}_\zeta(\mathbb{D}(\ell-\zeta-\varsigma))\mathcal{E}(\varsigma)\mathrm{d}\varsigma.\end{aligned}$$

In the same manner as in the proof of Theorem 1 and, as a consequence of (9), we have

$$\begin{aligned}& \mathbf{E}\|\Pi(\ell) - \aleph(\ell)\|^\mu \\ & \leq 3^{\mu-1}\Bigg\{\mathbf{E}\Bigg\|\int_0^\ell \mathcal{M}_\zeta(\mathbb{D}(\ell-\zeta-\varsigma))[\Delta(\varsigma,\Pi(\varsigma)) - \Delta(\varsigma,\aleph(\varsigma))]\mathrm{d}Z_H(\varsigma)\Bigg\|^\mu \\ & \quad +\mathbf{E}\Bigg\|\int_0^\ell \mathcal{M}_\zeta(\mathbb{D}(\ell-\zeta-\varsigma))[\hbar(\ell,\Pi(\ell)) - \hbar(\ell,\aleph(\ell))]\mathrm{d}\varsigma\Bigg\|^\mu \\ & \quad +\mathbf{E}\Bigg\|\int_0^\ell \mathcal{M}_\zeta(\mathbb{D}(\ell-\zeta-\varsigma))\mathcal{E}(\varsigma)\mathrm{d}\varsigma\Bigg\|^\mu\Bigg\}\end{aligned}$$

$$
\begin{aligned}
&\leq 3^{\mu-1}\Big\{\tau_\mu(2H)^{\mu/2}\varpi^{\mu H-1}\int_0^\ell\Big((\ell-\varsigma)\mathbb{E}_{2,2}\Big(\|\mathbb{D}\|(\ell-\varsigma)^2\Big)\Big)^\mu U_\Delta(\varsigma)\\
&\qquad\qquad\times\mathbf{E}\|\Pi(\varsigma)-\aleph(\varsigma)\|^\mu \mathrm{d}\varsigma\\
&\quad+\int_0^\ell\Big((\ell-\varsigma)\mathbb{E}_{2,2}\Big(\|\mathbb{D}\|^\mu(\ell-\varsigma)^2\Big)\Big)^\mu U_\hbar(\varsigma)\mathbf{E}\|\Pi(\varsigma)-\aleph(\varsigma)\|^\mu \mathrm{d}\varsigma\\
&\quad+\int_0^\ell\Big((\ell-\varsigma)\mathbb{E}_{2,2}\Big(\|\mathbb{D}\|(\ell-\varsigma)^2\Big)\Big)^\mu \mathbf{E}\|\mathcal{E}(\varsigma)\|^\mu \mathrm{d}\varsigma\Big\}\\
&\leq \int_0^\ell\Big((\ell-\varsigma)\mathbb{E}_{2,2}\Big(\|\mathbb{D}\|(\ell-\varsigma)^2\Big)\Big)^\mu\Big(3^{\mu-1}\tau_\mu(2H)^{\mu/2}\varpi^{\mu H-1}U_\Delta(\varsigma)+3^{\mu-1}U_\hbar(\varsigma)\Big)\\
&\qquad\qquad\times\mathbf{E}\|\Pi(\varsigma)-\aleph(\varsigma)\|^\mu \mathrm{d}\varsigma\\
&\quad+\frac{3^{\mu-1}\varpi^{\mu+1}\kappa}{\mu+1}\Big(\mathbb{E}_{2,2}\Big(\|\mathbb{D}\|\varpi^2\Big)\Big)^\mu.
\end{aligned}
$$

Applying Grönwall's inequality (Lemma 5), we obtain

$$
\mathbf{E}\|\Pi(\ell)-\aleph(\ell)\|^\mu \leq \frac{3^{\mu-1}\varpi^{\mu+1}\kappa}{\mu+1}\Big(\mathbb{E}_{2,2}\Big(\|\mathbb{D}\|\varpi^2\Big)\Big)^\mu \exp\Big(3^{\mu-1}(W_2+W_3)\Big),
$$

which implies that

$$
\mathbf{E}\|\Pi(\ell)-\aleph(\ell)\|^\mu \leq W\kappa,
$$

where

$$
W := \frac{3^{\mu-1}\varpi^{\mu+1}}{\mu+1}\Big(\mathbb{E}_{2,2}\Big(\|\mathbb{D}\|\varpi^2\Big)\Big)^\mu \exp\Big(3^{\mu-1}(W_2+W_3)\Big).
$$

Therefore, there exists W, which satisfies Definition 2. This ends the proof. □

4. An Example

Consider the following nonlinear stochastic delay system driven by the Rosenblatt process:

$$
\begin{aligned}
&\aleph''(\ell)+\mathbb{D}\aleph(\ell-0.5)=\hbar(\ell,\aleph(\ell))+\Delta(\ell,\aleph(\ell))\tfrac{\mathrm{d}Z_H(\ell)}{\mathrm{d}\ell}, \quad \text{for } \ell\in\mp:=[0,1],\\
&\aleph(\ell)\equiv\psi(\ell),\ \aleph'(\ell)\equiv\psi'(\ell) \quad \text{for } -0.5\leq\ell\leq 0,
\end{aligned}
\tag{13}
$$

where

$$
\aleph(\ell)=\begin{pmatrix}\aleph_1(\ell)\\ \aleph_2(\ell)\end{pmatrix},\quad \mathbb{D}=\begin{pmatrix}1&0\\0&0\end{pmatrix},
$$

and

$$
\hbar(\ell,\aleph(\ell))=\begin{pmatrix}(\sin\ell)\aleph_1(\ell)\\(\sin\ell)\aleph_2(\ell)\end{pmatrix},\quad \Delta(\ell,\aleph(\ell))=\begin{pmatrix}\frac{\sqrt{\ell}e^{-\ell}}{4}\aleph_1(\ell)\\ \frac{\sqrt{\ell}e^{-\ell}}{4}\aleph_2(\ell)\end{pmatrix}.
$$

Next, by choosing $\mu=r_1=r_2=2$, we obtain

$$
\begin{aligned}
\mathbf{E}\|\Delta(\ell,\aleph)-\Delta(\ell,\Im)\|^2_{L_2^0}&=\left(\frac{\sqrt{\ell}e^{-\ell}}{4}\right)^2\Big[(\aleph_1(\ell)-\Im_1(\ell))^2+(\aleph_2(\ell)-\Im_2(\ell))^2\Big]\\
&=\frac{\ell e^{-2\ell}}{16}\mathbf{E}\|\aleph-\Im\|^2\\
&\leq\frac{1}{16}\mathbf{E}\|\aleph-\Im\|^2
\end{aligned}
$$

for all $\ell\in\mp$, and $\aleph(\ell),\Im(\ell)\in\mathbb{R}^2$. We set $U_\Delta(\ell)=1/16$, such that $U_\Delta\in L^2(\mp,\mathbb{R}^+)$ in (**G1**), we have

$$
\|U_\Delta\|_{L^2(\mp,\mathbb{R}^+)}=\left(\int_0^1\left[\frac{1}{16}\right]^2\mathrm{d}\varsigma\right)^{\frac{1}{2}}=0.0625.
$$

Thus, selecting $H = 0.75$ and $\tau_\mu = 1.15$, we get

$$W_2 = \frac{\tau_\mu (2H)^{\mu/2} \varpi^{\mu(H+1)-\frac{1}{r_2}}}{(\mu r_1 + 1)^{\frac{1}{r_1}}} \left(\mathbb{E}_{2,2}\left(\|\mathbb{D}\|\varpi^2\right)\right)^\mu \|U_\Delta\|_{L^{r_2}(\mp,\mathbb{R}^+)} = 0.065.$$

Furthermore, we have

$$\begin{aligned}\mathbf{E}\|\hbar(\ell,\aleph) - \hbar(\ell,\Im)\|^2 &= \sin^2\ell\left[(\aleph_1(\ell) - \Im_1(\ell))^2 + (\aleph_2(\ell) - \Im_2(\ell))^2\right] \\ &= U_\hbar(\ell)\mathbf{E}\|\aleph - \Im\|^2.\end{aligned}$$

We set $U_\hbar(\ell) = \sin^2\ell$, such that $U_\hbar \in L^2(\mp,\mathbb{R}^+)$ in (**G2**), we have

$$\|U_\hbar\|_{L^2(\mp,\mathbb{R}^+)} = \left(\int_0^1 \sin^4\varsigma d\varsigma\right)^{\frac{1}{2}} = 0.35217.$$

Hence

$$W_3 = \frac{\varpi^{\mu+\frac{1}{r_1}}}{(\mu r_1 + 1)^{\frac{1}{r_1}}} \left(\mathbb{E}_{2,2}\left(\|\mathbb{D}\|\varpi^2\right)\right)^\mu \|U_\hbar\|_{L^{r_2}(\mp,\mathbb{R}^+)} = 0.21752.$$

Finally, we calculate that

$$2^{\mu-1}W_2 + W_3 = 0.3475 < 1,$$

which follows that all the assumptions of Theorems 1 and 2 hold. Therefore, the system (13) has a unique mild solution $\aleph$, and is Hyers–Ulam stable.

5. Conclusions

In this work, based on fixed point theory, we used the solutions of (2) to prove the existence and uniqueness of solutions. After that, we derived the Hyers–Ulam stability results using the delayed matrix functions and Grönwall's inequality. Finally, we verified the theoretical results by providing an example with a numerical simulation, which showed that our results applied to not only all non-singular matrices, but also all singular and arbitrary matrices, not necessarily squares. This is a novel study to prove the well-posedness and Hyers–Ulam stability of (2) using the delayed matrix functions.

In this study, further studies will focus on the obtained results to ascertain the existence and Hyers–Ulam stability of different types of stochastic delay systems, such as fractional or impulsive fractional stochastic delay systems driven by the Rosenblatt process.

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