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The de Rham Cohomology Classes of Hemi-Slant Submanifolds in Locally Product Riemannian Manifolds

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Abstract: This paper aims to discuss the de Rham cohomology of hemi-slant submanifolds in locally product Riemannian manifolds. The integrability and geodesical invariance conditions of the distributions derived from the definition of a hemi-slant submanifold are given. The existence and non-triviality of de Rham cohomology classes of hemi-slant submanifolds are investigated. Finally, an example is presented.

Keywords: almost product structure; locally product Riemannian manifold; de Rham cohomology class; hemi-slant submanifold; Riemannian submanifold

MSC: 14F40; 53C15; 53C25; 53C40

1. Introduction

Anti-slant submanifolds were introduced and searched by Carriazo [1] as a special case of bi-slant submanifolds in almost Hermitian manifolds. Such a class of submanifolds automatically contains both anti-invariant and slant submanifolds. In addition, anti-slant submanifolds are a particular class of generic submanifolds, which was defined by Ronsse [2]. Also, such a class of submanifolds was investigated by different authors under the names of pseudo-slant submanifolds [3] and hemi-slant submanifolds [4] since the phrase "anti-slant" does not refer to slant factors. Throughout the paper, it is preferred to use the phrase "hemi-slant submanifold" instead of such a type of submanifolds.

In [5], Taştan and Özdemir analyzied hemi-slant submanifolds of almost product Riemannian manifolds regarding their fundamental properties and characterization, the integrability of the related distributions, the hemi-slant product, the total umbilicality, the parallelism of the induced canonical structures and the Chen–Ricci inequality. In this context, a similar study to the previously cited one was performed by Hreţcanu and Blaga [6,7] for hemi-slant submanifolds in the setting of metallic Riemannian manifolds. In [8], Al-Solamy et al. gave classification theorems for totally umbilical hemi-slant submanifolds in Kähler manifolds. In [9], Lone et al. studied the geometry of distributions arising from the definition of hemi-slant submanifolds in cosymplectic manifolds. Moreover, the de Rham cohomology of hemi-slant submanifolds was studied in Kähler manifolds [10], nearly Kähler manifolds [11] and metallic Riemannian manifolds [12]. As is known, semi-invariant and CR-submanifolds are hemi-slant submanifolds. There are also noteworthy works on the Rham cohomology of semi-invariant and CR-submanifolds in other various manifolds, such as Kähler manifolds, nearly Kähler manifolds, quasi-Kähler manifolds, paraquaternionic Kähler manifolds, locally product Riemannian manifolds and metallic Riemannian manifolds (for details, see, e.g., [13–18]).

In the light of the above studies, the essential aim of this paper is to examine the de Rham cohomology of hemi-slant submanifolds in locally product Riemannian manifolds. Also, this paper generalizes the results of [18].

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This paper is organized as follows: Section 1 gives a brief information about some investigations on the geometry of hemi-slant submanifolds. Section 2 deals with some preliminary definitions and concepts which will be used in the subsequent sections. In Section 3, we mainly focus on finding the de Rham cohomology classes of hemi-slant submanifolds in locally product Riemannian manifolds.

2. Preliminaries

A Riemannian manifold $(\overline{M}, \overline{g})$ admitting an almost product structure $\overline{\Phi}$ is said to be almost product if the Riemannian metric $\overline{g}$ is $\overline{\Phi}$-compatible, i.e.,

$$\overline{g}(\overline{\Phi}X, Y) = \overline{g}(X, \overline{\Phi}Y) \tag{1}$$

for all $X, Y \in \Gamma(T\overline{M})$, where $\Gamma(T\overline{M})$ is the Lie algebra of vector fields on $\overline{M}$. Particularly, an almost product Riemannian manifold $(\overline{M}, \overline{g}, \overline{\Phi})$ is called a locally product Riemannian manifold, or briefly an LPR-manifold, if $\overline{\nabla}\,\overline{\Phi} = 0$, where $\overline{\nabla}$ stands for the Riemannian connection on $\overline{M}$ [19].

We consider any m-dimensional isometrically immersed submanifold M of an $\overline{m}$-dimensional almost product Riemannian manifold $(\overline{M}, \overline{g}, \overline{\Phi})$. We denote by TM and $TM^{\perp}$ its tangent and normal bundles, respectively. Let $\Gamma(TM)$ and $\Gamma(TM^{\perp})$ be the sets of cross-sections of TM and $TM^{\perp}$, respectively. We also use the same notation $\overline{g}$ for the induced metric on M. We denote by ∇ and $\nabla^{\perp}$ the induced connection on M and the connection on $TM^{\perp}$ induced by $\overline{\nabla}$, respectively. In this case, the formulas of Gauss and Weingarten are given, respectively, by

$$\overline{\nabla}_X Y = \nabla_X Y + \sigma(X, Y) \tag{2}$$

and

$$\overline{\nabla}_X U = -A_U X + \nabla^{\perp}_X U \tag{3}$$

for all $X, Y \in \Gamma(TM)$ and $U \in \Gamma(TM^{\perp})$, where σ is the second fundamental form of M in $\overline{M}$ and A_U is the Weingarten map in regard to U such that

$$\overline{g}(\sigma(X, Y), U) = \overline{g}(A_U X, Y) \tag{4}$$

for all $X, Y \in \Gamma(TM)$ and $U \in \Gamma(TM^{\perp})$.

For any $X \in \Gamma(TM)$, $\overline{\Phi}X$ is worded as follows:

$$\overline{\Phi}X = fX + hX, \tag{5}$$

where $fX \in \Gamma(TM)$ and $hX \in \Gamma(TM^{\perp})$. Similarly, for any $U \in \Gamma(TM^{\perp})$, we have

$$\overline{\Phi}U = tU + sU, \tag{6}$$

where $tU \in \Gamma(TM)$ and $sU \in \Gamma(TM^{\perp})$. It follows that the bundle morphisms f and s are $\overline{g}$-symmetric on TM and $TM^{\perp}$, respectively. Also, from (1), (5) and (6), there exists a connection between h and t such that

$$\overline{g}(hX, U) = \overline{g}(X, tU) \tag{7}$$

for all $X \in \Gamma(TM)$ and $U \in \Gamma(TM^{\perp})$. In addition, we get

$$f^2 + th = I, \tag{8}$$

$$hf + sh = 0, \tag{9}$$

$$ft + ts = 0 \tag{10}$$

and

$$s^2 + ht = I. \tag{11}$$

For more details about LPR-manifolds and their submanifolds, we refer the reader to [20].

An isometrically immersed submanifold M of an almost product Riemannian manifold $(\overline{M}, \overline{g}, \overline{\Phi})$ is called hemi-slant [6] if the tangent bundle TM has the ortogonal direct decomposition $TM = R^{\phi} \oplus R^{\perp}$, where R^{ϕ} is a slant distribution with the Wirtinger angle $\phi \in \left[0, \frac{\pi}{2}\right]$ and $R^{\perp}$ is an anti-invariant distribution.

Moreover, a hemi-slant submanifold M of an almost product Riemannian manifold $(\overline{M}, \overline{g}, \overline{\Phi})$ is said to be proper if $R^{\phi} \neq \{0\}$ and $R^{\perp} \neq \{0\}$, where $\phi \in \left(0, \frac{\pi}{2}\right)$. Otherwise,

(a) M is an invariant submanifold if $\dim R^{\perp} = 0$ and $\phi = 0$;
(b) M is an anti-invariant submanifold if $\dim R^{\phi} = 0$;
(c) M is a slant submanifold with the Wirtinger angle ϕ if $\dim R^{\perp} = 0$;
(d) M is a semi-invariant submanifold if $\phi = 0$.

It can be understood from above that invariant, anti-invariant, slant and semi-invariant submanifolds are non-proper hemi-slant submanifolds.

For a proper hemi-slant submanifold M of an almost product Riemannian manifold $(\overline{M}, \overline{g}, \overline{\Phi})$, by ([5], Lemmas 3.2 and 3.3), we have

$$fR^{\phi} = R^{\phi}, fR^{\perp} = \{0\}, hR^{\perp} = \overline{\Phi}R^{\perp} \text{ and } hR^{\phi} \perp hR^{\perp}.$$

In addition, $TM^{\perp}$ can be expressed in the following form:

$$TM^{\perp} = hR^{\phi} \oplus hR^{\perp} \oplus \nu,$$

where ν stands for the orthogonal complementary distribution of $hR^{\phi} \oplus hR^{\perp}$ in $TM^{\perp}$ and is a $\overline{\Phi}$-invariant subbundle of $TM^{\perp}$. We also recall the following results:

Theorem 1 ([5], Theorem 4.3)**.** *Let M be any hemi-slant submanifold of an LPR-manifold $(\overline{M}, \overline{g}, \overline{\Phi})$. Then, the integrability of the slant distribution R^{ϕ} is equivalent to*

$$A_{hX}Y - A_{hY}X + \nabla^{\perp}_{Y}hX - \nabla^{\perp}_{X}hY \in \Gamma\left(R^{\phi}\right)$$

for all $X, Y \in \Gamma\left(R^{\phi}\right)$.

Corollary 1 ([5], Corollary 4.7)**.** *Let M be any hemi-slant submanifold of an LPR-manifold $(\overline{M}, \overline{g}, \overline{\Phi})$. Then, the integrability of the anti-invariant distribution $R^{\perp}$ is equivalent to*

$$A_{\overline{\Phi}Z}W = 0 \tag{12}$$

for all $Z, W \in \Gamma\left(R^{\perp}\right)$.

Let M be a differentiable manifold. We denote by $Z^n(M)$ and $B^n(M)$ the vector space of all n-cocycles on M and the vector space of all n-coboundaries on M, respectively, i.e.,

$$Z^n(M) = \ker\left(d : \Omega^n(M) \to \Omega^{n+1}(M)\right)$$

and

$$B^n(M) = \mathrm{Im}\left(d : \Omega^{n-1}(M) \to \Omega^n(M)\right),$$

where Ω^n is the vector space of all differential n-forms on M and d is the exterior derivative. The quotient space

$$H^n_{dR}(M) = Z^n(M)/B^n(M)$$

is said to be the n-th de Rham cohomology group of M. The equivalence class of a differential n-form α in $H^n_{dR}(M)$ is called the Rham cohomology class of α, denoted by $[\alpha]$. Furthermore, the dimension of $H^n_{dR}(M)$ is called the n-th Betti number, which is denoted by b_n [21].

In addition, from Hodge theorem ([22], Theorem 8.12), for a closed orientable Riemannian manifold M, there is a canonical isomorphism

$$H^n_{dR}(M) \cong \mathcal{H}^n_{\Delta}(M),$$

where $\mathcal{H}^n_{\Delta}(M)$ denotes the space of all harmonic n-forms on M, i.e., the kernel space of the Hodge–de Rham Laplacian defined by $\Delta = d\delta + \delta d$ on Ω^n, in which δ is the co-differential operator. It is worth noting that the assumption on a hemi-slant submanifold of an LPR-manifold being closed in this paper is crucial to use Hodge theorem. However, Alexandru Rugina [23] obtained L^p harmonic n-forms, which are neither closed nor co-closed on the hyperbolic space $\mathbb{H}^m_{-1}$, $m \geq 3$, where $1 < p \neq 2 < 3$ and $0 \leq n \leq \frac{m}{2}$. On the other hand, by means of condition W, on complete noncompact Riemannian manifolds, Wei [24] showed that a differential n-form is L^p harmonic if and only if it is both closed and co-closed, where $1 < p < 3$.

A differentiable distribution R on a Riemannian manifold M is called geodesically invariant [25] if

$$\nabla_X X \in \Gamma(R),$$

or equivalently

$$(\nabla_X Y + \nabla_Y X) \in \Gamma(R)$$

for all $X, Y \in \Gamma(R)$, where ∇ is the Riemannian connection on M. Moreover, if $\{E_1, \ldots, E_r\}$ is a local orthonormal frame of R, then the mean curvature vector of R is defined by

$$H_R = \frac{1}{r}\sum_{i=1}^{r}\left(\nabla_{E_i}E_i\right)^{\perp},$$

where $\left(\nabla_{E_i}E_i\right)^{\perp}$ stands for the component of $\nabla_{E_i}E_i$ in the orthogonal complementary of R in M. If H_R vanishes identically, i.e., $H_R = 0$, then R is named a minimal distribution [13]. Also, it is not difficult to see that R is minimal if and only if R is geodesically invariant.

3. Main Results

Let M be a hemi-slant submanifold of an almost product Riemannian manifold $\left(\overline{M}, \overline{g}, \overline{\Phi}\right)$ with the slant distribution R^{ϕ} and the anti-invariant distribution $R^{\perp}$ such that $\dim R^{\phi} = p$ and $\dim R^{\perp} = q$, where ϕ is its Wirtinger angle.

We consider two globally well-defined differential forms α and β on M given by $\alpha = \alpha^1 \Lambda \cdots \Lambda \alpha^p$ and $\beta = \beta^{p+1} \Lambda \cdots \Lambda \beta^{p+q}$ such that $\alpha^1, \ldots, \alpha^p, \beta^{p+1}, \ldots, \beta^{p+q}$ are differential one-forms on M with the following properties:

$$\alpha^k(Z) = 0, \alpha^k(E_l) = \delta_{kl}, 1 \leq k, l \leq p \tag{13}$$

and

$$\beta^{p+K}(X) = 0, \beta^{p+K}\left(E_{p+L}\right) = \delta_{KL}, 1 \leq K, L \leq q \tag{14}$$

for all $X \in \Gamma\left(R^{\phi}\right)$ and $Z \in \Gamma\left(R^{\perp}\right)$, where $\mathfrak{B}_{R^{\phi}} = \left\{E_1, \ldots, E_p\right\}$ and $\mathfrak{B}_{R^{\perp}} = \left\{E_{p+1}, \ldots, E_{p+q}\right\}$ are the local orthonormal frames of R^{ϕ} and $R^{\perp}$, respectively. It follows that each of α and β is a volume form. Thus, R^{ϕ} and $R^{\perp}$ are orientable in regard to $\mathfrak{B}_{R^{\phi}}$ and $\mathfrak{B}_{R^{\perp}}$, respectively. In addition, $\alpha \Lambda \beta$ is a globally well-defined differential m-form and M is orientable in regard to $\mathfrak{B}_{R^{\phi}} \cup \mathfrak{B}_{R^{\perp}}$ of TM.

Proposition 1. *Let M be any hemi-slant submanifold of an LPR-manifold $\left(\overline{M}, \overline{g}, \overline{\Phi}\right)$. Then, the geodesical invariance of the slant distribution R^{ϕ} in M is equivalent to*

$$\sigma(X, fX) + \nabla^{\perp}_X hX \in \Gamma\left(hR^{\phi} \oplus \nu\right)$$

for all $X \in \Gamma(R^{\phi})$. *In particular,* R^{ϕ} *is geodesically invariant in* M *if the following expressions hold:*

(a) M *is* R^{ϕ}*-geodesic;*
(b) *For all* $X \in \Gamma(R^{\phi})$,

$$\nabla^{\perp}_{X} hX \in \Gamma(hR^{\phi} \oplus \nu).$$

Proof. Taking into account that the fact $\overline{\nabla}\,\overline{\Phi} = 0$, by virtue of (1), (2), (3) and (5), an elementary calculation gives us

$$\overline{g}(\nabla_X X, Z) = \overline{g}\Big(\sigma(X, fX) + \nabla^{\perp}_{X} hX, \overline{\Phi} Z\Big) \tag{15}$$

for all $X \in \Gamma(R^{\phi})$ and $Z \in \Gamma(R^{\perp})$, which is the desired equality. □

Proposition 2. *Let* M *be any hemi-slant submanifold of an LPR-manifold* $(\overline{M}, \overline{g}, \overline{\Phi})$. *Then, the geodesical invariance of the anti-invariant distribution* $R^{\perp}$ *in* M *is equivalent to*

$$f A_{\overline{\Phi} Z} Z - t \nabla^{\perp}_{Z} hZ \in \Gamma\Big(R^{\perp}\Big)$$

for all $Z \in \Gamma(R^{\perp})$. *Specifically, each leaf of* $R^{\perp}$ *is geodesically invariant in* M *if the following expressions are satisfied:*

(a) *For all* $Z, W \in \Gamma(R^{\perp})$,

$$A_{\overline{\Phi} Z} W = 0;$$

(b) *For all* $Z \in \Gamma(R^{\perp})$,

$$\nabla^{\perp}_{Z} hZ \in \Gamma\Big(hR^{\perp} \oplus \nu\Big).$$

Proof. Because $\overline{M}$ is an LPR-manifold, we deduce from (1), (2), (3) and (5) that

$$\overline{g}(\nabla_Z Z, X) = -\overline{g}\Big(f A_{\overline{\Phi} Z} Z - t \nabla^{\perp}_{Z} hZ, X\Big)$$

for all $X \in \Gamma(R^{\phi})$ and $Z \in \Gamma(R^{\perp})$, which finishes the proof. □

Theorem 2. *Let* M *be any closed proper hemi-slant submanifold of an LPR-manifold* $(\overline{M}, \overline{g}, \overline{\Phi})$. *Then, there exists a canonical de Rham cohomology class* $[\beta]$ *determined by the differential* q*-form* β *in* $H^{q}_{dR}(M)$ *if the following expressions hold:*

(a) *For all* $X, Y \in \Gamma(R^{\phi})$,

$$A_{hX} Y - A_{hY} X + \nabla^{\perp}_{Y} hX - \nabla^{\perp}_{X} hY \in \Gamma(R^{\phi});$$

(b) *For all* $Z \in \Gamma(R^{\perp})$,

$$f A_{\overline{\Phi} Z} Z - t \nabla^{\perp}_{Z} hZ \in \Gamma\Big(R^{\perp}\Big).$$

In addition, $[\beta] \neq 0$ *if the following expressions are verified:*

(c) *For all* $Z, W \in \Gamma(R^{\perp})$,

$$A_{\overline{\Phi} Z} W = 0;$$

(d) *For all* $X \in \Gamma(R^{\phi})$,

$$\sigma(X, fX) + \nabla^{\perp}_{X} hX \in \Gamma(hR^{\phi} \oplus \nu).$$

Proof. By the definition of β, we write

$$d\beta = \sum_{K=1}^{q} (-1)^{p+K} \beta^{p+1} \Lambda \cdots \Lambda d\beta^{p+K} \Lambda \cdots \Lambda \beta^{p+q}, \tag{16}$$

so it is deducible from (14) that β is closed if and only if

$$d\beta(Z, E_{p+1}, \ldots, E_{p+q}) = 0 \tag{17}$$

and

$$d\beta(Z, W, E_{p+1}, \ldots, E_{p+q-1}) = 0 \tag{18}$$

for all $Z, W \in \Gamma(R^{\perp})$. On the other hand, taking account of Theorem 1 and Proposition 2, (a) and (b) imply the integrability of R^{ϕ} and the geodesical invariance of $R^{\perp}$, respectively. Thus, by the two expressions mentioned, it is understood that (17) and (18) hold, that is, $d\beta = 0$. Therefore, we can talk about the existence of a canonical de Rham cohomology class represented by $[\beta]$ in $H^{q}_{dR}(M)$. Assuming (c) and (d) are true, by a similar argument as above, it can be concluded from Corollary 1 and Proposition 1 that α is closed. Also, as can be clearly seen, α is the Hodge dual of β, so β is a co-closed differential q-form. In addition, since M is a closed submanifold, β is an harmonic q-form. As well, M is orientable in regard to $\mathfrak{B}_{R^{\phi}} \cup \mathfrak{B}_{R^{\perp}}$. As a result, Hodge theorem says that $[\beta] \neq 0$ in $H^{q}_{dR}(M)$. □

Theorem 3. *Let M be any closed proper hemi-slant submanifold of an LPR-manifold $(\overline{M}, \overline{g}, \overline{\Phi})$. Then, the differential q-form β represents a canonical de Rham cohomology class denoted by $[\beta]$ in $H^{q}_{dR}(M)$ if the following expressions hold:*

(a) *For all $X, Y \in \Gamma(R^{\phi})$,*

$$A_{hX}Y - A_{hY}X + \nabla^{\perp}_{Y}hX - \nabla^{\perp}_{X}hY \in \Gamma(R^{\phi});$$

(b) *For all $Z, W \in \Gamma(R^{\perp})$,*

$$A_{\overline{\Phi}Z}W = 0;$$

(c) *For all $Z \in \Gamma(R^{\perp})$,*

$$\nabla^{\perp}_{Z}hZ \in \Gamma\Big(hR^{\perp} \oplus \nu\Big).$$

Furthermore, $[\beta] \neq 0$ if the following expressions are verified:

(d) *M is R^{ϕ}-geodesic;*
(e) *For all $X \in \Gamma(R^{\phi})$,*

$$\nabla^{\perp}_{X}hX \in \Gamma(hR^{\phi} \oplus \nu).$$

Proof. By Propositions 1 and 2, the proof follows immediately from Theorem 2. □

Theorem 4. *Let M be any closed totally geodesic proper hemi-slant submanifold of an LPR-manifold $(\overline{M}, \overline{g}, \overline{\Phi})$. Then, the de Rham cohomology class of the differential q-form β in $H^{q}_{dR}(M)$ is non-trivial if the following expressions are valid:*

(a) *For all $Z, W \in \Gamma(R^{\perp})$,*

$$\nabla^{\perp}_{Z}hW \in \Gamma\Big(hR^{\perp} \oplus \nu\Big);$$

(b) *For all $X, Y \in \Gamma(R^{\phi})$,*

$$\nabla^{\perp}_{X}hY \in \Gamma(hR^{\phi} \oplus \nu).$$

Proof. If M is a totally geodesic submanifold with (a) and (b), then all the expressions of Theorem 2 are naturally satisfied, so the proof is complete. □

Theorem 5. *Let M be any closed proper hemi-slant submanifold of an LPR-manifold $(\overline{M}, \overline{g}, \overline{\Phi})$. Then, there is a de Rham cohomology class $[\alpha]$ determined by the differential p-form α in $H^{p}_{dR}(M)$ if the following expressions are satisfied:*

(a) *For all $Z, W \in \Gamma(R^{\perp})$,*

$$A_{\overline{\Phi}Z}W = 0;$$

(b) *For all* $X \in \Gamma(R^{\phi})$,

$$\sigma(X, fX) + \nabla^{\perp}_{X} hX \in \Gamma(hR^{\phi} \oplus \nu).$$

Moreover, $[\alpha] \neq 0$ *if the following expressions are verified:*

(c) *For all* $X, Y \in \Gamma(R^{\phi})$,

$$A_{hX}Y - A_{hY}X + \nabla^{\perp}_{Y} hX - \nabla^{\perp}_{X} hY \in \Gamma(R^{\phi});$$

(d) *For all* $Z \in \Gamma(R^{\perp})$,

$$fA_{\overline{\Phi}Z}Z - t\nabla^{\perp}_{Z} hZ \in \Gamma\left(R^{\perp}\right).$$

Proof. The proof is omitted here because it is analogous to that of Theorem 2. □

Theorem 6. *Let* M *be any closed proper hemi-slant submanifold of an LPR-manifold* $(\overline{M}, \overline{g}, \overline{\Phi})$. *Then, the differential p-form* α *induces a de Rham cohomology class represented by* $[\alpha]$ *in* $H^{p}_{dR}(M)$ *if the following expressions are verified:*

(a) M *is* R^{ϕ}*-geodesic;*
(b) *For all* $X \in \Gamma(R^{\phi})$,

$$\nabla^{\perp}_{X} hX \in \Gamma(hR^{\phi} \oplus \nu);$$

(c) *For all* $Z, W \in \Gamma(R^{\perp})$,

$$A_{\overline{\Phi}Z}W = 0.$$

In addition, $[\alpha] \neq 0$ *if the following expressions are true:*

(d) *For all* $X, Y \in \Gamma(R^{\phi})$,

$$A_{hX}Y - A_{hY}X + \nabla^{\perp}_{Y} hX - \nabla^{\perp}_{X} hY \in \Gamma(R^{\phi});$$

(e) *For all* $Z \in \Gamma(R^{\perp})$,

$$\nabla^{\perp}_{Z} hZ \in \Gamma\left(hR^{\perp} \oplus \nu\right).$$

Proof. Considering Propositions 1 and 2, the proof is easily obtained from Theorem 5. □

Theorem 7. *Let* M *be any closed totally geodesic proper hemi-slant submanifold of an LPR-manifold* $(\overline{M}, \overline{g}, \overline{\Phi})$. *Then, the de Rham cohomology class of the differential p-form* α *in* $H^{p}_{dR}(M)$ *is non-trivial if the following expressions are correct:*

(a) *For all* $Z, W \in \Gamma(R^{\perp})$,

$$\nabla^{\perp}_{Z} hW \in \Gamma\left(hR^{\perp} \oplus \nu\right);$$

(b) *For all* $X, Y \in \Gamma(R^{\phi})$,

$$\nabla^{\perp}_{X} hY \in \Gamma(hR^{\phi} \oplus \nu).$$

Proof. The proof can be achieved similarly to that of Theorem 4. □

Lastly, we present an example of the existence of the Rham cohomology class of a hemi-slant submanifold in the eight-dimensional Euclidean space.

Example 1. *Let* $(\mathbb{R}^8, \langle , \rangle)$ *be the eight-dimensional Euclidean space. We define an almost product Riemannian structure* $(\langle , \rangle, \overline{\Phi})$ *on* $\mathbb{R}^8$ *by*

$$\overline{\Phi}(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = (x_1, -x_2, x_5, x_6, x_3, x_4, -x_7, x_8).$$

Also, it is obvious that $(\mathbb{R}^8, \langle , \rangle, \overline{\Phi})$ *is an LPR-manifold.*

Now, we consider a submanifold M given by the immersion $i : M \longrightarrow \mathbb{R}^8$ as follows:

$$i(u,v,w,z) = \left(u\cos\frac{\phi}{2}, u\sin\frac{\phi}{2}, v\cos\phi, v\sin\phi, w, 0, z, z\right),$$

where $u, v > 0$. Hence, the set $\{E_1, E_2, E_3, E_4\}$ is a local orthonormal frame for TM such that

$$E_1 = \cos\frac{\phi}{2}\frac{\partial}{\partial x_1} + \sin\frac{\phi}{2}\frac{\partial}{\partial x_2}, E_2 = \cos\phi\frac{\partial}{\partial x_3} + \sin\phi\frac{\partial}{\partial x_4},$$

$$E_3 = \frac{\partial}{\partial x_5} \text{ and } E_4 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_7} + \frac{\partial}{\partial x_8}\right).$$

Putting $R^\phi = Span\{E_1, E_2, E_3\}$ and $R^\perp = Span\{E_4\}$, a direct calculation gives us that R^ϕ is a slant distribution with the Wirtinger angle ϕ and $R^\perp$ is the anti-invariant distribution, so M is a four-dimensional hemi-slant submanifold with the Wirtinger angle ϕ of $\left(\mathbb{R}^8, \langle,\rangle, \overline{\Phi}\right)$.

Now, let us consider a differential three-form α and a differential one-form β given by $\alpha = \alpha^1\Lambda\alpha^2\Lambda\alpha^3$ and $\beta = \beta^1$, respectively, where

$$\alpha^1 = \cos\frac{\phi}{2}dx_1 + \sin\frac{\phi}{2}dx_2,$$

$$\alpha^2 = \cos\phi dx_3 + \sin\phi dx_4,$$

$$\alpha^3 = dx_5$$

and

$$\beta^1 = \frac{1}{\sqrt{2}}(dx_7 + dx_8).$$

In this case, it follows that $d\beta = \delta\beta = d\alpha = \delta\alpha = 0$. Thus, we have

$$\Delta\beta = \Delta\alpha = 0,$$

that is, α and β are harmonic forms. For this reason, from Hodge theorem, $H^1_{dR}(M)$ and $H^3_{dR}(M)$ are non-trivial, i.e., $b_1 \neq 0$ and $b_3 \neq 0$.

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