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Oceanic Shallow-Water Investigations on a Variable-Coefficient Davey–Stewartson System

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Abstract: In this paper, a variable-coefficient Davey–Stewartson (vcDS) system is investigated for modeling the evolution of a two-dimensional wave-packet on water of finite depth in inhomogeneous media or nonuniform boundaries, which is where its novelty lies. The Painlevé integrability is tested by the method of Weiss, Tabor, and Carnevale (WTC) with the simplified form of Krustal. The rational solutions are derived by the Hirota bilinear method, where the formulae of the solutions are represented in terms of determinants. Furthermore the fundamental rogue wave solutions are obtained under certain parameter restrains in rational solutions. Finally the physical characteristics of the influences of the coefficient parameters on the solutions are discussed graphically. These rogue wave solutions have comprehensive implications for two-dimensional surface water waves in the ocean.

Keywords: Davey–Stewartson equation; Painlevé property; rational solution; rogue-wave solution; symbolic computation

MSC: 35C08; 35G50; 35Q51



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1. Introduction

Ever since calculus was proposed, the study of differential equations has attracted wide attention. Differential equations have a strong physical background and have important applications in fields such as natural science and technical science. The solutions and properties of differential equations are significant [1,2]. For example, the stability of a differential equation not only reflects the characteristics of the equation itself, it plays a role in practical model analysis [3–6]. Differential equations can be divided into linear equations and nonlinear equations. Nonlinear evolution equations has become a research focus because of its practicability and complexity.

Nonlinear evolution equations (NLEEs) are a class of partial differential equations in mathematical physics that contain the time variable t . They are derived from a large number of nonlinear phenomena in different backgrounds, such as mechanics, physics, and engineering [7,8]. Because of their important position in electromagnetic fluid dynamics, quantum mechanics, and nonlinear optics, they have attracted great attention from the fields of mathematics, physics, and even engineering [9–11]. Methods of solving nonlinear development equations are important components of nonlinear science with strong interdisciplinary aspects and integration [12,13].

Rogue waves (also known as anomalous waves) play an important part in the study of marine science. They are named for the sudden and large amplitude of their appearance [14]. Similarly, scar theory is one of the fundamental pillars in the field of quantum chaos, with scarred functions being superb tool for carry out such studies. Heller coined the term ‘scar’ to describe the increase in the probability density that certain eigenfunctions exhibit along unstable periodic orbits (POs) of a classically chaotic system [15]. By intentionally going to the edge of the chaotic region, Borondo studied the systematics of scar formation at a

very elementary level uncluttered by other irrelevant effects [16]. The presence of rogue waves in the ocean is often extremely destructive, causing many ships to be in distress [17]. Therefore, the phenomenon of rogue waves has been widely studied in recent years and has important applications in practical engineering problems [18–20].

The Davey–Stewartson (DS) equations are an important and nontrivial integrable model originally used to describe a $(2 + 1)$ -dimensional wave packet on the surface of a liquid of finite depth [21–23]. The DS equation is divided into two types, the DS-I and DS-II equations, depending on the strength of surface tension [24,25]. The DS-I equation is written as follows:

$$\begin{cases} iu_t + u_{xx} + u_{yy} + (\varepsilon|u|^2 - 2v)u = 0 \\ v_{xx} - v_{yy} = \varepsilon(|u|^2)_{xx} \end{cases} \quad (1)$$

and the DS-II equation as follows:

$$\begin{cases} iu_t = u_{xx} - u_{yy} + \varepsilon|u|^2u - 2uv \\ v_{xx} + v_{yy} = \varepsilon(|u|^2)_{xx} \end{cases} \quad (2)$$

where the subscripts denote the partial derivatives, x is the scaled horizontal coordinate, y denotes the scaled space coordinate perpendicular to x , t is the scaled time, the complex function $u(x, y, t)$ is the amplitude of a surface wave packet, the real function $v(x, y, t)$ is the velocity potential of the mean flow interacting with the surface wave, the parameter $\varepsilon = 1$ characterizes the focusing case, and the parameter $\varepsilon = -1$ characterizes the defocusing case [24].

When the media are inhomogeneous or the boundaries are nonuniform, variable-coefficient models are able to describe various situations more realistically than their constant-coefficient counterparts [26]. In this paper, we focus our interest on the variable-coefficient Davey–Stewartson (vcDS) equation for ocean waves, ultra-relativistic degenerate dense plasmas, and Bose–Einstein condensates [24]:

$$\begin{cases} iu_t + P_1(t)u_{xx} + P_2(t)u_{yy} - Q_1(t)|u|^2u + Q_2(t)uv = 0, \\ Rv_{xx} - v_{yy} - S(|u|^2)_{xx} = 0, \end{cases} \quad (3)$$

where $u = u(x, y, t)$ represents the wave amplitude, $v = v(x, y, t)$ is a force, the real functions $P_1(t)$ and $P_2(t)$ denote the wave group's dispersion, $Q_1(t)$ is the cubic nonlinear coefficient, $Q_2(t)$ stands for the nonlocal quadratic nonlinearity, and the parameter S equals either 1 or -1 .

In recent years, further research on the DS equation has been emerging. Tajiri and Arai derived the simplest growing-and-decaying mode solution of the DS equation, where the spatial period of homoclinical solutions tend to be infinite, and obtained the simplest rogue wave solution [27]. Based on the Hirota bilinear method, Ohta and Yang proposed a general rogue wave solution of the DS-I equation in determinant form, which has far-reaching significance for two-dimensional water wave dynamics [23]. The DS-I equation with PT symmetry properties was studied in [28]. Recently, three types of soliton interaction solutions have been obtained, extending the understanding of the diversity and generation mechanisms of rogue waves [29]. For the variable-coefficient DS equation Equation (3), its bilinear form, Bäcklund transformation, and Lax pair were obtained via the Bell polynomials, then localized excitations were derived using variable separation [24,30]. When $2P_1(t) = 2P_2(t) = 2Q_1(t) = Q_2(t) = \beta(t)$, $R = S = 1$, with $\beta(t)$ as a real function of the time variable t , the excitation solutions in Equation (3) can be obtained via the variable separation approach [31].

The rest of this paper is organized as follows. In Section 2, the Painlevé test is applied to Equation (3) in order to obtain Painlevé integrable conditions. In Section 3, based on the bilinear forms of Equation (1), the rational solutions of Equation (3) are derived theoretically.

In Section 4, we present rogue wave solutions under certain parameter restraints on the rational solutions. Finally, the conclusions and discussion are provided in Section 5.

2. Painlevé Test

Painlevé integrability plays a very important role in the structure of the solutions of a given NLEE [32,33]. A partial differential equation (PDE) is said to possess the Painlevé property if its solutions are single-valued in the neighborhood of noncharacteristic movable singularity manifolds [34–36].

To apply the Painlevé analysis to Equation (3), we can construct a coupled system

$$\begin{cases} iu_{1,t} + P_1(t)u_{1,xx} + P_2(t)u_{1,yy} - Q_1(t)u_1^2u_2 + Q_2(t)u_1v = 0 \\ -iu_{2,t} + P_1(t)u_{2,xx} + P_2(t)u_{2,yy} - Q_1(t)u_1u_2^2 + Q_2(t)u_2v = 0 \\ Rv_{xx} - v_{yy} - S(u_1 \cdot u_2)_{xx} = 0 \end{cases} \quad (4)$$

where u_2 stands for the complex conjugates of u_1 .

In line with the Painlevé PDE test and simplified Kruskal ansatz [32–34], we assume the solution of Equation (4) with the generalized Laurent expansion form:

$$\begin{cases} u_1(x, y, t) = \phi^{p_1}(x, y, t) \sum_{j=0}^{\infty} u_{1,j}(y, t) \phi^j(x, y, t) \\ u_2(x, y, t) = \phi^{p_2}(x, y, t) \sum_{j=0}^{\infty} u_{2,j}(y, t) \phi^j(x, y, t) \\ v(x, y, t) = \phi^q(x, y, t) \sum_{j=0}^{\infty} v_j(y, t) \phi^j(x, y, t) \end{cases} \quad (5)$$

with $\phi(x, y, t) = x + \psi(y, t)$, where $\psi(y, t)$ is an arbitrary function of y and t and where $u_{1,j}(y, t)$, $u_{2,j}(y, t)$, $v_j(y, t)$ ($j = 0, 1, 2, \dots$) are all analytic functions of y and t , in the neighbourhood of a noncharacteristic movable singularity manifold defined by $\phi(x, y, t) = 0$.

Using leading-order analysis, we can obtain

$$p_1 = -1, \quad p_2 = -1, \quad q = -2, \quad (6)$$

$$v_0 = \frac{2S(P_1(t) + P_2(t)\psi_y^2)}{SQ_2(t) + Q_1(t)(R - \psi_y^2)}, \quad (7)$$

$$u_{1,0}u_{2,0} = \frac{2(R - \psi_y^2)(P_1(t) + P_2(t)\psi_y^2)}{-SQ_2(t) + Q_1(t)(R - \psi_y^2)}. \quad (8)$$

When $P_2(t) = k_1P_1(t)$, $Q_2(t) = -k_1P_1(t)$, $Q_2(t) = (k_1 - 1)P_1(t)$, $R = -1$, $S = 1$, we have the resonances occurring at $j = -1, 0, 2, 3, 3$ and 4 , while $j = -1$ and $j = 0$ correspond to the arbitrariness of the function $\psi(y, t)$ and $u_{2,0}$. Then, v_0 and $u_{1,0}$ can be simplified as

$$v_0 = 2, \quad u_{1,0} = -\frac{2(1 + \psi_y^2)}{u_{2,0}}. \quad (9)$$

Symbolic computation at $j = 1$ yields

$$\begin{aligned} u_{1,1} &= -\frac{2(u_{2,0})_y\psi_y(1 + \psi_y^2) - u_{2,0}[i\phi_t(1 + \psi_y^2) + (3\psi_y^2 - 1)\psi_{yy}]}{u_{2,0}^2(\psi_y^2 - 1)}, \\ u_{2,1} &= -\frac{2(u_{2,0})_y\psi_y - u_{2,0}(i\psi_t + \psi_{yy})}{2(\psi_y^2 - 1)}, \\ v_1 &= 0. \end{aligned}$$

At resonance $j = 2$, v_2 is arbitrary and we have

$$u_{1,2} = \frac{G_1}{H_1}, \quad u_{2,1} = \frac{G_2}{H_2},$$

where

$$\begin{aligned} G_1 &= -4v_2u_{2,0}^2(1+\psi_y^2)(\psi_y^2-1)^2 - 4(u_{2,0})_y^2\psi_y^2(3\psi_y^4+2\psi_y^2-1) \\ &\quad + 2u_{2,0}[i(u_{2,0})_t(1+\psi_y^2)(\psi_y^2-1)^2 + (u_{2,0})_{yy}(3\psi_y^6+\psi_y^4-3\psi_y^2-1) \\ &\quad + 2(u_{2,0})_y\psi_y(i\psi_t(1+\psi_y^2)^2 + (3\psi_y^4-6\psi_y^2-1)\psi_{yy})] \\ &\quad + 2u_{2,0}^2[\psi_t^2(1+\psi_y^2)^2 + 2i\psi_t(1+\psi_y^2)^2 + 3\psi_{yy} + 6\psi_y^2\psi_{yy} + 3\psi_y^4\psi_{yy}^2], \\ H_1 &= 2u_{2,0}^3(1+3\psi_y^2)(\psi_y^2-1)^2, \\ G_2 &= 4v_2u_{2,0}^2(\psi_y^2-1)^2 + 4(u_{2,0})_y^2(1+\psi_y^2) - 2u_{2,0}i[(u_{2,0})_t(\psi_y^2-1)^2 \\ &\quad + i(u_{2,0})_{yy}(3\psi_y^4-\psi_y^2-1) + 2(u_{2,0})_y\psi_y(1+\psi_y^2)^2(\psi_t-3i\psi_{yy})] \\ &\quad + u_{2,0}^2[-\psi_t^2(1+\psi_y^2) + 2i\psi_t(1+5\psi_y^2)\psi_{yy} + \psi_y^2 + 9\psi_y^2\psi_{yy}^2 \\ &\quad + 3\psi_y^3(-4i\psi_{yt}-4\psi_{yyy}) + 4\psi_y(i\psi_{yt}+\psi_{yyy})], \\ H_2 &= u_{2,0}(1+3\psi_y^2)(\psi_y^2-1)^2. \end{aligned}$$

Symbolic computation shows that the compatibility conditions at resonances $j = 3, 3$ and $j = 4$ are satisfied identically, with $u_{2,3}$, v_3 , and $u_{2,4}$ all being arbitrary. Because $u_{1,3}$, $u_{1,4}$, and v_4 are too long and complicated, they are omitted here.

Thus, system (3) has the Painlevé property in the case $P_2(t) = k_1P_1(t)$, $Q_2(t) = -k_1P_1(t)$, $Q_2(t) = (k_1 - 1)P_1(t)$, $R = -1$, $S = 1$.

3. Rational Solutions

In this section, we first present the bilinear equations using the Hirota bilinear method [37]. Through the variable transformation

$$u = \sqrt{2}\frac{f}{g}, \quad v = \lambda - 2(\ln f)_{xx}, \quad (10)$$

where f is a real function and g is a complex one with variables x, y, t , Equation (3) can be converted into the following equations:

$$f^2[(iD_t + P_1(t)D_x^2 + P_2(t)D_y^2)g \cdot f] = 0 \quad (11)$$

$$fg[P_2(t)D_y^2f \cdot f + (P_1(t) + Q_2(t))D_x^2f \cdot f - Q_2(t)\lambda f^2 + 2Q_1(t)|g|^2] = 0 \quad (12)$$

$$RD_x^2f \cdot f - D_y^2f \cdot f + 2S|g|^2 + cf^2 = 0 \quad (13)$$

where c is a real constant and the bilinear operators $D_x^m D_t^n$ are defined as

$$D_x^m D_t^n a \cdot b \equiv \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n a(x, t) b(x', t') \Big|_{x'=x, t'=t}. \quad (14)$$

Then, Equation (3) can be transformed into the following bilinear equations:

$$\begin{cases} (iD_t + P_1(t)D_x^2 + P_2(t)D_y^2)g \cdot f = 0 \\ RD_x^2f \cdot f - D_y^2f \cdot f + 2S|g|^2 + cf^2 = 0 \end{cases} \quad (15)$$

where

$$\frac{P_2(t)}{1} = \frac{P_1(t) + Q_2(t)}{-R} = \frac{-\lambda Q_2(t)}{-c} = \frac{2Q_1(t)}{-2S}. \quad (16)$$

The following Lemma 1 is proposed and used to construct rational solutions of the Davey–Stewartson I equation [23].

Lemma 1. Let $m_{ij}^{(n)}, \varphi_i^{(n)}, \psi_j^{(n)}$ be functions of x_1, x_2, x_{-1}, x_{-2} satisfying the following differential relations:

$$\begin{cases} \partial_{x_1} m_{ij}^{(n)} = \varphi_i^{(n)} \psi_j^{(n)} \\ \partial_{x_2} m_{ij}^{(n)} = \varphi_i^{(n+1)} \psi_j^{(n)} + \varphi_i^{(n)} \psi_j^{(n-1)} \\ \partial_{x_{-1}} m_{ij}^{(n)} = -\varphi_i^{(n-1)} \psi_j^{(n+1)}, \\ \partial_{x_{-2}} m_{ij}^{(n)} = -\varphi_i^{(n-2)} \psi_j^{(n+1)} - \varphi_i^{(n-1)} \psi_j^{(n+2)} \\ m_{ij}^{(n+1)} = m_{ij}^{(n)} + \varphi_i^{(n)} \psi_j^{(n+1)} \\ \partial_{x_\nu} \varphi_i^{(n)} = \varphi_i^{(n+\nu)} \\ \partial_{x_\nu} \psi_j^{(n)} = \psi_j^{(n-\nu)}, \quad \nu = 1, 2, -1, -2. \end{cases} \quad (17)$$

Then, the determination

$$\tau_n = \det_{1 \leq i, j \leq N} (m_{ij}^{(n)}) \quad (18)$$

satisfies the bilinear equations

$$\begin{cases} (D_{x_1} D_{x_{-1}} - 2) \tau_n \cdot \tau_n = -2 \tau_{n+1} \tau_{n-1} \\ (D_{x_1}^2 - D_{x_2}) \tau_{n+1} \cdot \tau_n = 0 \\ (D_{x_{-1}}^2 + D_{x_{-2}}) \tau_{n+1} \cdot \tau_n = 0. \end{cases} \quad (19)$$

The proof of Lemma 1 can be found in [23].

Based on Lemma 1, rational solutions of Equation (3) can be presented by the following theorem.

Theorem 1. The DS Equation (3) has rational solutions with f and g by $N \times N$ determinations

$$f = \tau_0, \quad g = \tau_1, \quad (20)$$

where $\tau_n = \det_{1 \leq i, j \leq N} (m_{ij}^{(n)})$ and the matrix elements are provided by

$$m_{ij}^{(n)} = \sum_{k=0}^{n_i} c_{ik} (p_i \partial_{p_i} + \xi'_i + n)^{n_i-k} \sum_{l=0}^{n_j} \bar{c}_{jl} (\bar{p}_j \partial_{\bar{p}_j} + \bar{\xi}'_j + n)^{n_j-l} \frac{1}{p_i + \bar{p}_j}, \quad (21)$$

$$\xi'_i = \frac{\sqrt{S}(p_i - p_i^{-1})}{2\sqrt{R}} x + \frac{\sqrt{S}(p_i + p_i^{-1})}{2} y + i S (p_i^2 + p_i^{-2}) \int a_1(t) dt, \quad (22)$$

where p_i is the arbitrary complex constant and n_i, n_j are the arbitrary positive integers, $\bar{\xi}'_j$ is the conjugate of ξ'_j , and, according to the bilinear form, the parameter restraints are defined as follows:

$$P_1(t) = R P_2(t), \quad Q_1(t) = -S P_2(t), \quad Q_2(t) = -2 R P_2(t). \quad (23)$$

Then, f and g are the rational solutions.

Proof of Theorem 1. Assuming that

$$\varphi_i^{(n)} = p_i^n e^{p_i x_1 + p_i^2 x_2 + \frac{1}{p_i} x_{-1} + \frac{1}{p_i^2} x_{-2}}, \quad (24)$$

$$\psi_j^{(n)} = (-q_j)^{-n} e^{q_j x_1 - q_j^2 x_2 + \frac{1}{q_j} x_{-1} - \frac{1}{q_j^2} x_{-2}}, \quad (25)$$

it is clear that functions defined above satisfy the differential and difference relations in Equation (20).

Therefore, the matrix element $m_{ij}^{(n)}$ in $\tau_n = \det(m_{ij}^{(n)})$ satisfies the following equation:

$$m_{ij}^{(n)} = \int \varphi_i^{(n)} \psi_j^{(n)} dx_1 = \frac{1}{p_i + q_j} \left(-\frac{p_i}{q_j} \right)^n e^{\xi_i + \eta_j}, \quad (26)$$

where

$$\xi_i = p_i x_1 + p_i^2 x_2 + \frac{1}{p_i} x_{-1} + \frac{1}{p_i^2} x_{-2}, \eta = q_j x_1 - q_j^2 x_2 + \frac{1}{q_j} x_{-1} - \frac{1}{q_j^2} x_{-2}. \quad (27)$$

Furthermore, we can consider the following functions $\varphi_i^{(n)}, \psi_j^{(n)}, m_{ij}^{(n)}$ with respect to p_i, q_j and define A_i, B_j as follows:

$$A_i = \sum_{k=0}^{n_i} c_{ik} (p_i \partial_{p_i})^{n_i-k}, B_j = \sum_{l=0}^{n_j} d_{jl} (q_j \partial_{q_j})^{n_j-l}. \quad (28)$$

Then,

$$\varphi_i^{(n)} = A_i p_i^n e^{\xi_i}, \psi_j^{(n)} = B_j (-q_j)^{-n} e^{\eta_j}, \quad (29)$$

$$m_{ij}^{(n)} = A_i B_j \frac{1}{p_i + q_j} \left(-\frac{p_i}{q_j} \right)^n e^{\xi_i + \eta_j}. \quad (30)$$

Obviously, these functions meet the differential and difference relations in Equation (20).

Based on

$$(Du)\varphi = Du\varphi = u_x \varphi + u\varphi_x = (u_x + uD)\varphi \Rightarrow D_u = u_x + uD, \quad (31)$$

we can obtain

$$(p_i \partial_{p_i}) p_i^n e^{\xi_i} = p_i^n e^{\xi_i} (p_i \partial_{p_i} + \xi'_i + n), \xi'_i = p_i \xi_i^{(1)}, \quad (32)$$

$$(q_j \partial_{q_j}) (-q_j)^{-n} e^{\eta_j} = (-q_j)^{-n} e^{\eta_j} (q_j \partial_{q_j} + \eta'_j - n), \eta'_j = q_j \eta_j^{(1)}. \quad (33)$$

Hence, we have

$$m_{i,j}^{(n)} = \left(-\frac{p_i}{q_j} \right)^n e^{\xi_i + \eta_j} \sum_{k=0}^{n_i} c_{ik} (p_i \partial_{p_i} + \xi'_i + n)^{n_i-k} \sum_{l=0}^{n_j} d_{jl} (q_j \partial_{q_j} + \eta'_j - n)^{n_j-l} \frac{1}{p_i + q_j}, \quad (34)$$

where

$$\xi'_i = -\frac{2}{p_i^2} x_{-2} - \frac{1}{p_i} x_{-1} + p_i x_1 + 2p_i^2 x_2, \quad (35)$$

$$\eta'_j = -\frac{2}{q_j^2} x_{-2} - \frac{1}{q_j} x_{-1} + q_j x_1 - 2q_j^2 x_2. \quad (36)$$

Equation (21) can be presented by using the relations

$$q_j = \bar{p}_j, d_{jl} = \bar{c}_{jl}. \quad (37)$$

By substituting

$$x_1 = A_{11}x + A_{12}y, \quad (38)$$

$$x_{-1} = A_{-11}x + A_{-12}y, \quad (39)$$

$$x_2 = A_2(t), \quad (40)$$

$$x_{-2} = A_{-2}(t), \quad (41)$$

into Equation (19) and comparing the result with Equation (15), we obtain

$$A_{11} = A_{-11} = \frac{\sqrt{S}}{2\sqrt{R}}, \quad (42)$$

$$A_{12} = A_{-12} = \frac{\sqrt{S}}{2}, \quad (43)$$

$$A_2(t) = -A_{-2}(t) = \frac{iS}{2} \int P_2(t), \quad (44)$$

which satisfies the form of ζ'_i in Equation (22) and the parameter restraint in (23). This completes the proof of Theorem 1. $\square$

4. Rogue Wave Solutions

Based on finding rational solutions f and g , we can obtain the rogue wave solutions of Equation (3).

Making use of Equations (10), (21), and (22), the results u and v are called the rogue wave solutions of (3).

When $N = 1$, $n_1 = 1$ in Equation (21), then u, v is the fundamental rational solution.

When $N > 1$, $n_1 = n_2 = \dots = 1$, then u, v is the multi-rogue wave solution.

When $N = 1$, $n_1 > 1$, then u, v is the higher-order rogue wave solution [23].

These are discussed separately in the following.

4.1. Fundamental Rational Solution

When $N = 1$, $n_1 = 1$, then f and g can be written as follows:

$$f = \frac{1}{p_1 + \bar{p}_1} \left[(\zeta'_1 + c_{11} - \frac{p_1}{p_1 + \bar{p}_1})(\bar{\zeta}'_1 + \bar{c}_{11} - \frac{\bar{p}_1}{p_1 + \bar{p}_1}) + \frac{p_1 \bar{p}_1}{(p_1 + \bar{p}_1)^2} \right] \quad (45)$$

$$g = \frac{1}{p_1 + \bar{p}_1} \left[(\zeta'_1 + c_{11} - \frac{p_1}{p_1 + \bar{p}_1} + 1)(\bar{\zeta}'_1 + \bar{c}_{11} - \frac{\bar{p}_1}{p_1 + \bar{p}_1} - 1) + \frac{p_1 \bar{p}_1}{(p_1 + \bar{p}_1)^2} \right] \quad (46)$$

where

$$\zeta'_1 = \frac{\sqrt{S}(p_1 - p_1^{-1})}{2\sqrt{R}}x + \frac{\sqrt{S}(p_1 + p_1^{-1})}{2}y + iS(p_1^2 + p_1^{-2}) \int P_1(t)dt. \quad (47)$$

This solution can be rewritten as

$$f = \frac{1}{p_1 + \bar{p}_1} (\zeta \bar{\zeta} + \Delta), \quad g = \frac{1}{p_1 + \bar{p}_1} [(\zeta + 1)(\bar{\zeta} - 1) + \Delta],$$

where

$$\zeta = ax + by + h(t) + \theta, \quad \Delta = \frac{p_1 \bar{p}_1}{(p_1 + \bar{p}_1)^2}$$

and

$$a = \frac{\sqrt{S}(p_1 - p_1^{-1})}{2\sqrt{R}}, \quad b = \frac{\sqrt{S}(p_1 + p_1^{-1})}{2},$$

$$h(t) = iS(p_1^2 + p_1^{-2}) \int P_1(t)dt, \quad \theta = c_{11} - \frac{p_1}{p_1 + \bar{p}_1}.$$

Graphical illustrations of rogue wave solutions constructed in Mathematica are presented in Figure 1.

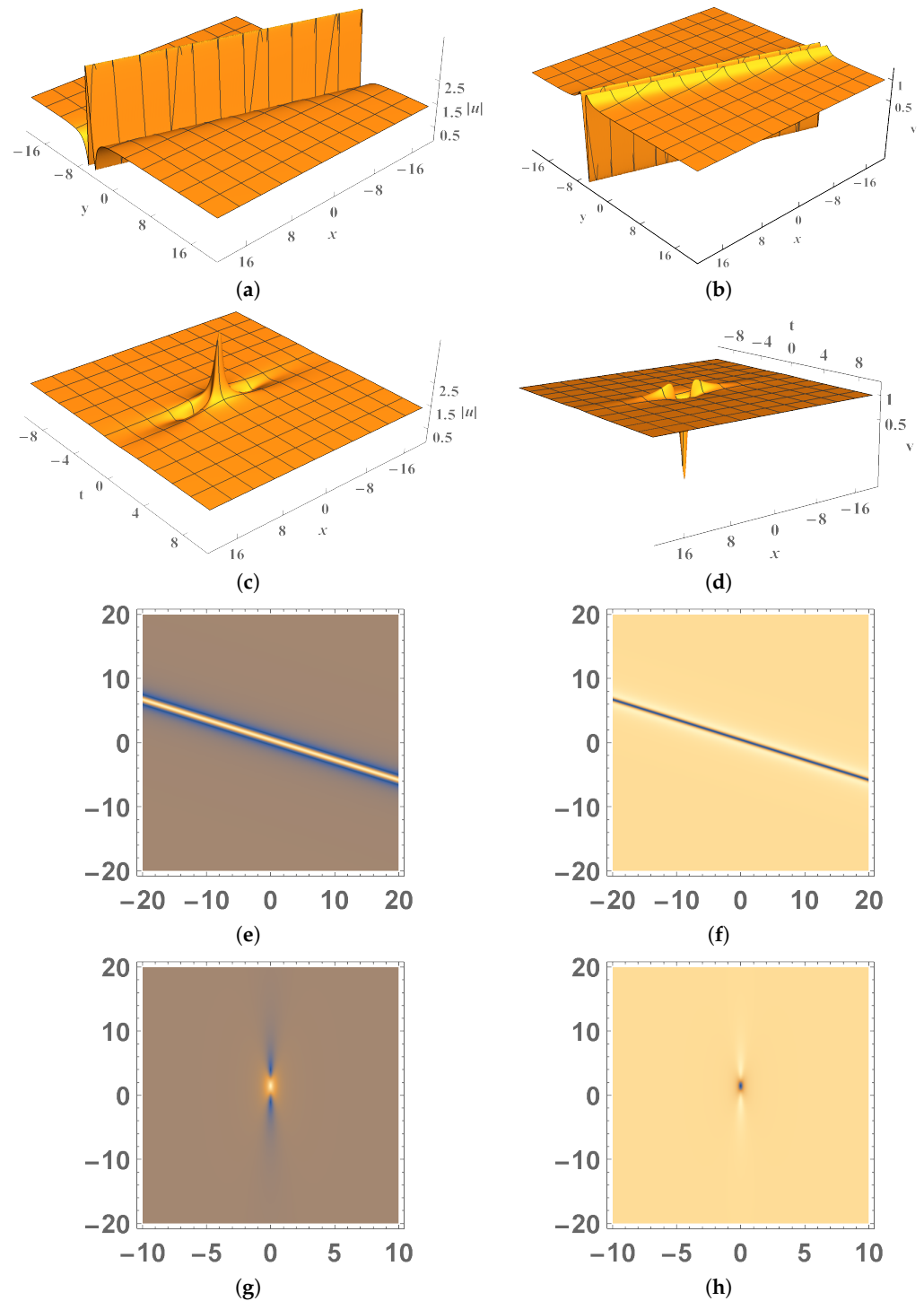


Figure 1. The three-dimensional evolution plot (a–d) and density plot (e–h) of the fundamental rational solution with $N = 1$, $n_1 = 1$, $S = 1$, $R = 1.5$, $p_1 = 1.5$, $c_{11} = 0$, $P_1(t) = 1$: (a,e) $|u|$: $t = 0$; (b,f) v : $t = 0$; (c,g) $|u|$: $y = 0$; (d,h) v : $y = 0$.

From Figure 1a,b, it can be seen that $|u|$ and v both possess abnormal height, satisfying the characteristic of rogue waves. Meanwhile, it is obvious that, as $t \rightarrow \pm\infty$, the solution $|u|$ uniformly approaches the constant background $\sqrt{2}$ and the solution v uniformly approaches the constant background 1 everywhere in the (x, y) plane. It is interesting to note that rogue wave v is a dark rogue wave, which is different from a bright rogue wave u . Because $a_1(t)$ is constant, the figure illustrates that these rogue waves are linear. In

Figure 1c,d, we find that there is only one peak in each figure, which is the characteristic of the fundamental rational solution.

4.2. Multi-Rogue Wave Solution

One subclass of nonfundamental rogue waves consists of multi-rogue waves, which can be obtained when taking $N > 1$, $n_1 = \dots = n_N = 1$ in the rational solution (10) with real values of $(p_1, p_2, \dots, p_N)$. These rogue wave solutions describe the interaction of N individual fundamental rogue waves. In the near field of the (x, y) plane, these line rogue waves intersect and are no longer lines, allowing interesting curvy wave patterns to appear. However, the solution consists of N rogue waves with separate lines in the far field.

To present these multi-rogue waves, we first consider the case with $N = 2$. In this case, the f and g functions of the solutions can be obtained from (10) as follows:

$$f = \begin{vmatrix} m_{11}^0 & m_{02}^0 \\ m_{21}^0 & m_{22}^0 \end{vmatrix}, \quad g = \begin{vmatrix} m_{11}^1 & m_{12}^1 \\ m_{21}^1 & m_{22}^1 \end{vmatrix}, \quad (48)$$

$$m_{ij}^0 = \frac{1}{p_i + \bar{p}_j} \left[\left(-\frac{p_i}{p_i + \bar{p}_j} + \xi_i' + c_{i1} \right) \left(-\frac{\bar{p}_j}{p_i + \bar{p}_j} + \bar{\xi}_i' + \bar{c}_{j1} \right) + \frac{p_i \bar{p}_j}{(p_i + \bar{p}_j)^2} \right], \quad (49)$$

$$m_{ij}^1 = \frac{1}{p_i + \bar{p}_j} \left[\left(-\frac{p_i}{p_i + \bar{p}_j} + \xi_i' + c_{i1} + 1 \right) \left(-\frac{\bar{p}_j}{p_i + \bar{p}_j} + \bar{\xi}_i' + \bar{c}_{j1} - 1 \right) + \frac{p_i \bar{p}_j}{(p_i + \bar{p}_j)^2} \right], \quad (50)$$

$$\xi_i' = \frac{\sqrt{S}(p_i - p_i^{-1})}{2\sqrt{R}}x + \frac{\sqrt{S}(p_i + p_i^{-1})}{2}y + iS(p_i^2 + p_i^{-2}) \int P_1(t)dt. \quad (51)$$

Similarly, with Equation (10), we can select parameters to illustrate graphics, as in Figures 2 and 3.

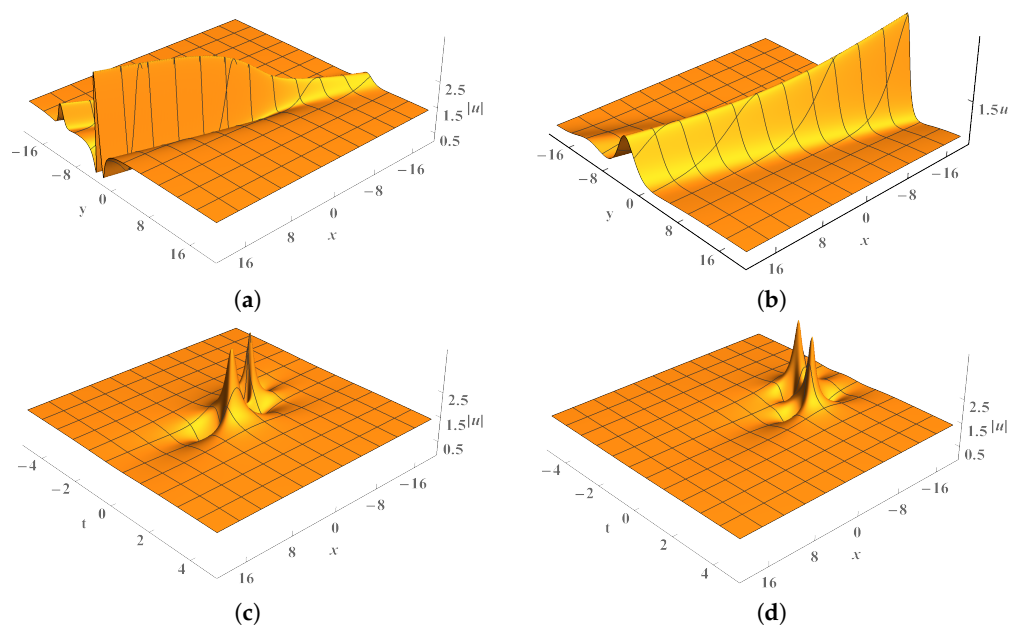


Figure 2. Three-dimensional evolution plot of the multi-rogue rational solution with $N = 2$, $n_1 = n_2 = 1$, $S = 1$, $R = 1.5$, $p_1 = 1.5$, $p_2 = 2$, $c_{11} = 0$, $c_{21} = 1 + i$, $P_1(t) = 1$: (a) $|u| : t = 0$; (b) $|u| : t = 2$; (c) $|u| : y = 0$; (d) $|u| : y = 5$.

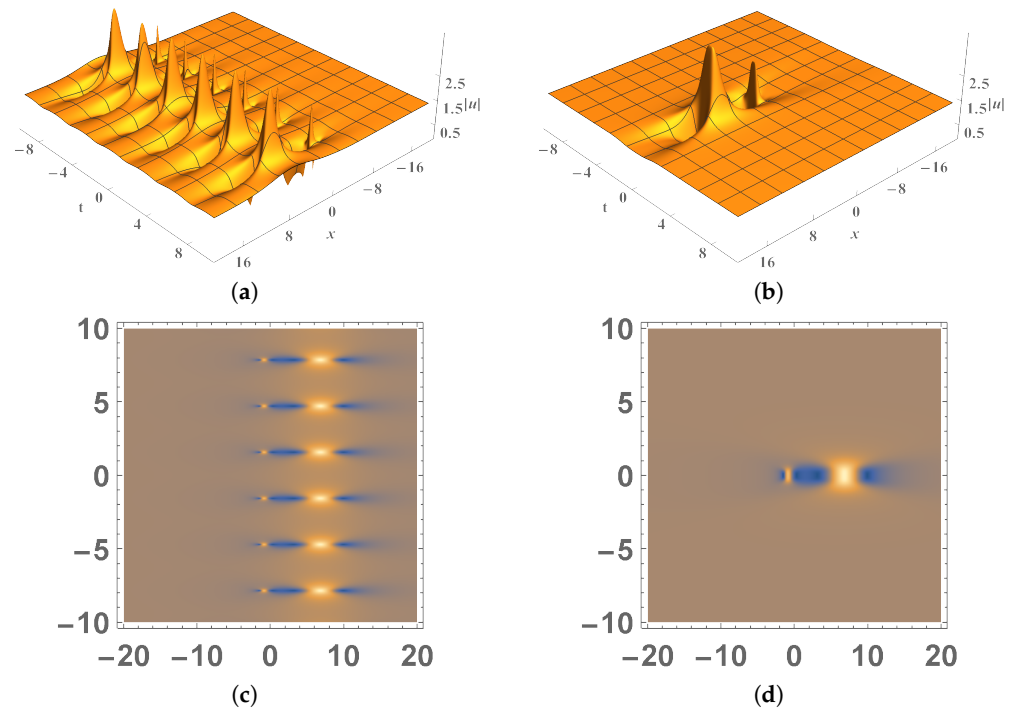


Figure 3. The three-dimensional evolution plot (a,b) and density plot (c,d) of the multi-rogue wave solution with $N = 2$, $n_1 = n_2 = 1$, $S = 1$, $R = 1.5$, $p_1 = 1.5$, $p_2 = 2$, $c_{11} = 0$, $c_{21} = 1 + i$, $y = 0$: (a,c) $|u| : P_1(t) = \sin t$; (c,d) $|u| : P_1(t) = \arctan t$.

From Figure 2a,b, it is clear that as the time t increases, there is a large change in the shape of $|u|$. At the same time, the rogue waves converges and the spatial range involved becomes smaller. The process consists of the rogue waves commencing along a single straight path, gaining speed until they encounter a head-on collision, becoming dispersed and rotated, and ultimately diverging from each other in some direction [29]. From Figure 2c,d, it can be observed that there are two peaks of $|u|$, which meets the characteristic of $N = 2$. Meanwhile, the shape of the peak does not change significantly with increasing y , but the position of the spatiotemporal coordinates changes greatly.

In addition, it can be seen that the amplitude and propagation direction of rogue wave change greatly with the parameter.

4.3. Higher-Order Rogue Wave Solution

Another subclass of nonfundamental rogue waves consists of higher-order rogue waves, which are obtained when we take $N = 1$, $n_1 > 1$ in the rational solution (10) with a real value of p_1 . For instance, if $n_1 = 2$, we obtain a second rogue wave solutions as follows:

$$f = \left[(p_1 \partial_{p_1} + \zeta'_1)^2 + c_{12} \right] \left[(\bar{p}_1 \partial_{\bar{p}_1} + \bar{\zeta}'_1)^2 + \bar{c}_{12} \right] \frac{1}{p_1 + \bar{p}_1}, \quad (52)$$

$$g = \left[(p_1 \partial_{p_1} + \zeta'_1 + 1)^2 + c_{12} \right] \left[(\bar{p}_1 \partial_{\bar{p}_1} + \bar{\zeta}'_1 - 1)^2 + \bar{c}_{12} \right] \frac{1}{p_1 + \bar{p}_1}. \quad (53)$$

The expression for ζ' is shown in Equation (51). Combined with Equation (10), we can now select the parameters and obtain the graphical illustrations in Figures 4 and 5.

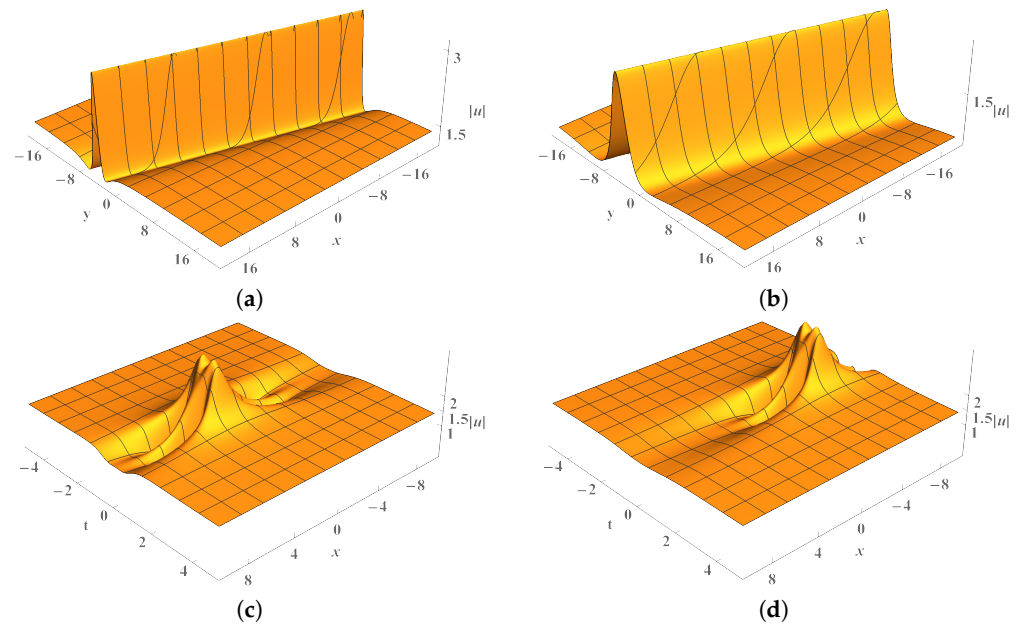


Figure 4. Three-dimensional evolution plot of the higher-order rogue wave solution with $N = 1$, $n_1 = 2$, $S = 1$, $R = 1.5$, $p_1 = 1.5$, $c_{12} = 1$, $P_1(t) = 1$: (a) $|u| : t = 0$; (b) $|u| : t = 1.5$; (c) $|u| : y = 0$; (d) $|u| : y = 2$.

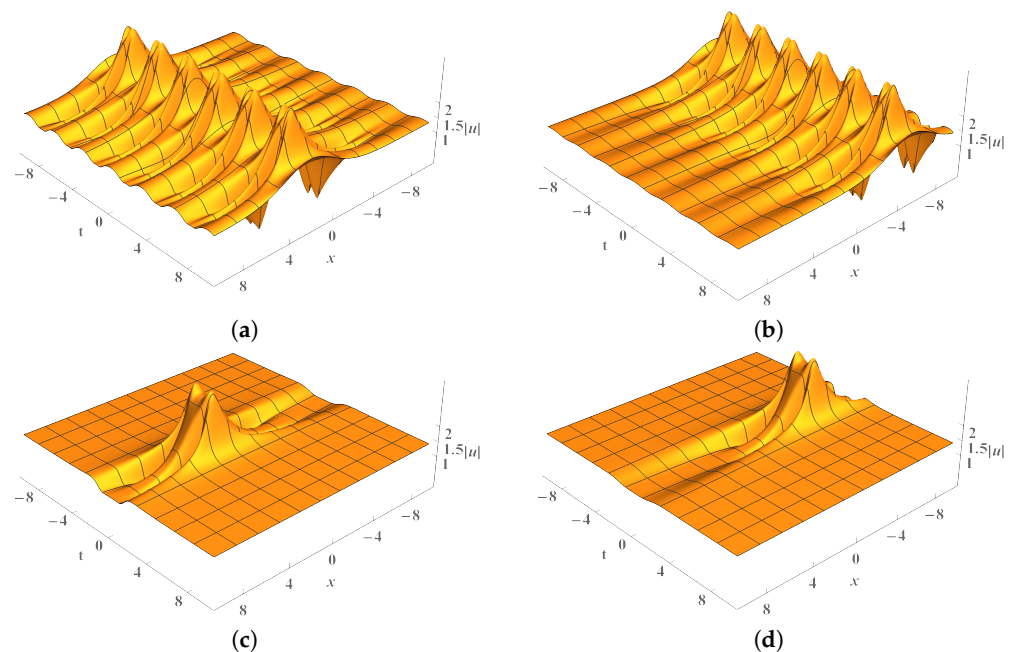


Figure 5. Three-dimensional evolution plot of the higher-order rogue wave solution with $N = 1$, $n_1 = 2$, $S = 1$, $R = 1.5$, $p_1 = 1.5$, $c_{12} = 1$: (a) $|u| : P_1(t) = \sin t : y = 0$; (b) $|u| : P_1(t) = \sin t : y = 2$; (c) $|u| : P_1(t) = \arctan t : y = 0$; (d) $|u| : P_1(t) = \arctan t : y = 2$.

As shown in Figure 4a,b, when the time changes, the rogue waves converge; however, the amplitude of the waves does not change markedly. Observing Figure 4c to Figure 5d and taking different variable coefficients and y values, it can be seen that the shape of the higher-order rogue wave is more complex; indeed, it is the same as that of the multi-rogue wave in that the shape of the peak does not change significantly with changing y , but the spatiotemporal position changes greatly.

Eventually, when $n_1 > 2$, we choose $n_1 = 3$; then,

$$f = \left[(p_1 \partial_{p_1} + \zeta'_1)^3 + c_{13} \right] \left[(\bar{p}_1 \partial_{\bar{p}_1} + \bar{\zeta}'_1)^3 + \bar{c}_{13} \right] \frac{1}{p_1 + \bar{p}_1}, \quad (54)$$

$$g = \left[(p_1 \partial_{p_1} + \zeta'_1 + 1)^3 + c_{13} \right] \left[(\bar{p}_1 \partial_{\bar{p}_1} + \bar{\zeta}'_1 - 1)^3 + \bar{c}_{13} \right] \frac{1}{p_1 + \bar{p}_1}. \quad (55)$$

The corresponding graphics are shown below in Figure 6.

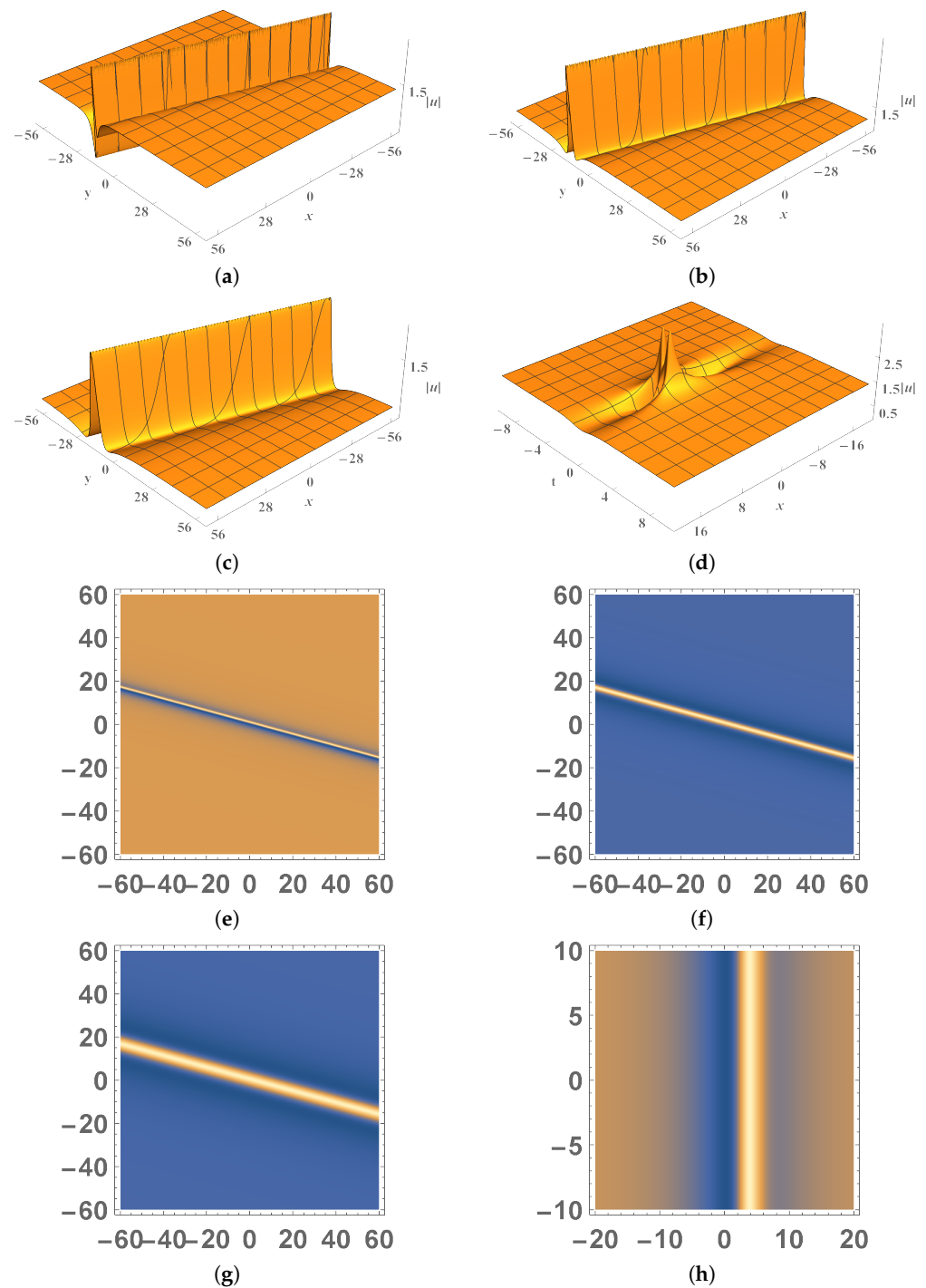


Figure 6. The three-dimensional evolution plot (a–d) and density plot (e–h) of the third-order rogue wave solution with $N = 1$, $n_1 = 3$, $S = 1$, $R = 1.5$, $p_1 = 1.5$, $c_{13} = 1$, $P_1(t) = 1$: (a,e) $|u| : t = 0$; (b,f) $|u| : t = 1$; (c,g) $|u| : t = 2$; (d,h) $|u| : y = 0$.

5. Conclusions

The variable-coefficient DS equation is important in modeling ocean waves as well as ultra-relativistic degenerate dense plasmas Bose–Einstein condensates, and in some other nonlinear contexts where inhomogeneities of the media and nonuniformities of the boundaries need to be taken into consideration. In this paper, we have investigated a variable-coefficient Davey–Stewartson system, namely, a $(2 + 1)$ -dimensional wave packet on the surface of a liquid of finite depth, via symbolic computation.

With respect to $u(x, y, t)$, e.g., the amplitude of a surface wave packet, and $v(x, y, t)$, e.g., the velocity potential of the mean flow interacting with the surface wave, the Painlevé integrability has been tested with the Weiss, Tabor, and Carnevale (WTC) method, proving that System (3) has Painlevé properties with certain parameter restraints. In addition, the bilinear forms of System (3) have been provided by means of variable transformation. Based on the obtained bilinear forms, rational solutions have been obtained by taking certain parameter restraints and deriving the corresponding rogue wave solutions. Furthermore, we have classified different kinds of rogue wave solutions and presented graphical illustrations illustrating their physical properties.

In comparison to other published results, in [23] rogue wave solutions of Equation (1) were derived in terms of the determinant and different types of rogue wave solutions were classified. In this paper, a variable-coefficient Davey–Stewartson (vcDS) Equation (3) is investigated, which can describe situations more realistically than Equation (1) when considering inhomogeneous media and nonuniform boundaries. Furthermore, [29] constructed higher-order lumps, k -order localized rogue waves, and line rogue waves on a $k + 1$ -line soliton background. This direction is more concerned with soliton interactions, which may enlighten us in our following research.

The variable-coefficient constraints for the DS system (3) investigated this paper are strong; weaker coefficient constraints and a wider range of rational and rogue wave solutions are yet to be explored. In the future, we hope to improve the conditions and conclusions of the results presented in this paper in order to obtain rogue wave solutions with a wider form.

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References

1. Alqahtani, R.T.; Kaplan, M. Analyzing Soliton Solutions of the Extended $(3 + 1)$ -Dimensional Sakovich Equation. *Mathematics* **2024**, *12*, 720. [\[CrossRef\]](#)
2. Evans, L.C. *Partial Differential Equations*; American Mathematical Society: Providence, RI, USA, 2010.
3. Pinelas, S.; Selvam, A.; Sabarinathan, S. Ulam–Hyers stability of linear differential equation with general transform. *Symmetry* **2023**, *15*, 2023. [\[CrossRef\]](#)
4. Selvam, A.; Sabarinathan, S.; Nisar, K.S.; Ravichandran, C.; Senthil Kumar, B.V. Results on Ulam-type stability of linear differential equation with integral transform. *Math. Method Appl. Sci.* **2024**, *47*, 2311–2323. [\[CrossRef\]](#)
5. Sivashankar, M.; Sabarinathan, S.; Govindan, V.; Fernandez-Gamiz, U.; Noeiaghdam, S. Stability analysis of COVID-19 outbreak using Caputo-Fabrizio fractional differential equation. *AIMS Math.* **2023**, *8*, 2720–2735. [\[CrossRef\]](#)

6. Selvam, A.; Sabarinathan, S.; Senthil Kumar, B.V.; Byeon, H.; Guedri, K.; Eldin, S.M.; Khan, M.I.; Govindan, V. Ulam-Hyers stability of tuberculosis and COVID-19 co-infection model under Atangana-Baleanu fractal-fractional operator. *Sci. Rep.* **2023**, *13*, 9012. [[CrossRef](#)] [[PubMed](#)]
7. Miah, M.M.; Seadawy, A.R.; Ali, H.S.; Akbar, M. Further investigations to extract abundant new exact traveling wave solutions of some NLEEs. *J. Ocean Eng. Sci.* **2019**, *4*, 387–394. [[CrossRef](#)]
8. Barman, H.K.; Roy, R.; Mahmud, F.; Akbar, M.; Osman, M. Harmonizing wave solutions to the Fokas-Lenells model through the Kudryashov method. *Optik* **2021**, *229*, 166294. [[CrossRef](#)]
9. Gao, X.Y. Two-layer-liquid and lattice considerations through a $(3 + 1)$ -dimensional generalized Yu-Toda-Sasa-Fukuyama system. *Appl. Math. Lett.* **2024**, *152*, 109018. [[CrossRef](#)]
10. Gao, X.Y. Auto-Bäcklund transformation with the solitons and similarity reductions for a generalized nonlinear shallow water wave equation. *Qual. Theory Dyn. Syst.* **2024**, *23*, 181. [[CrossRef](#)]
11. Gao, X.Y. In the shallow water: Auto-Bäcklund, hetero-Bäcklund and scaling transformations via a $(2+1)$ -dimensional generalized Broer-Kaup system. *Qual. Theory Dyn. Syst.* **2024**, *23*, 184. [[CrossRef](#)]
12. Dong, H.N.; Zha, Q.L. Hybrid rogue wave and breather solutions for the nonlinear coupled dispersionless evolution equations. *Wave Motion* **2024**, *125*, 103259. [[CrossRef](#)]
13. Zhang, H.W.; Zong, J.; Tian, G.; Wei, G.M. Analysis of High-Order Bright–Dark Rogue Waves in $(2+1)$ -D Variable-Coefficient Zakharov Equation via Self-Similar and Darboux Transformations. *Mathematics* **2024**, *12*, 1359. [[CrossRef](#)]
14. Smith, R. Giant waves. *Fluid Mech.* **1976**, *77*, 417–431. [[CrossRef](#)]
15. Heller, E.J. Bound-state eigenfunctions of classically chaotic hamiltonian systems: Scars of periodic orbits. *Phys. Rev. Lett.* **1984**, *53*, 1515. [[CrossRef](#)]
16. Arranz, F.J.; Borondo, F.; Benito, R.M. Scar formation at the edge of the chaotic region. *Phys. Rev. Lett.* **1998**, *80*, 944–947. [[CrossRef](#)]
17. Gao, X.Y. Oceanic shallow-water investigations on a generalized Whitham-Broer-Kaup-Boussinesq-Kupershmidt system. *Phys. Fluids* **2023**, *35*, 127106. [[CrossRef](#)]
18. Osborne, A. Rogue waves: Classification, measurement and data analysis, and hyperfast numerical modeling. *Eur. Phys. J. Spec. Top.* **2010**, *185*, 225–245. [[CrossRef](#)]
19. Yan, Z. Vector financial rogue waves. *Phys. Lett. A* **2011**, *375*, 4274–4279. [[CrossRef](#)]
20. Ling, L.M.; Su, H.J. Rogue waves and their patterns for the coupled Fokas–Lenells equations. *Phys. D Nonlinear Phenom.* **2024**, *461*, 134111. [[CrossRef](#)]
21. Davey, A.; Stewartson, K. On three-dimensional packets of surface waves. *Proc. R. Soc. Lond. A* **1974**, *338*, 101–110.
22. Matveev, V.B.; Salle, M.A. *Darboux Transformations and Solitons*; Springer: Berlin, Germany, 1991.
23. Ohta, Y.; Yang, J.K. Rogue waves in the Davey–Stewartson I equation. *Phys. Rev. E* **2012**, *86*, 036604. [[CrossRef](#)] [[PubMed](#)]
24. Zhou, H.P.; Tian, B.; Mo, H.X.; Li, M.; Wang, P. Bäcklund transformation, Lax pair and solitons of the $(2 + 1)$ -dimensional Davey–Stewartsonlike equations with variable coefficients for the electrostatic wave packets. *J. Nonlinear Math. Phys.* **2013**, *20*, 94–105. [[CrossRef](#)]
25. Ablowitz, M.J.; Segur, H. On the transition from two-dimensional to three-dimensional water waves. *Stud. Appl. Math.* **2000**, *104*, 91–127.
26. Zhang, Y.P.; Wang, J.Y.; Wei, G.M. The Painlevé property, Bäcklund transformation, Lax pair and new analytic solutions of a variable-coefficient KdV equation from fluids and plasmas. *Phys. Scr.* **2015**, *90*, 065203.
27. Tajiri, M.; Arai, T. Growing-and-decaying mode solution to the Davey–Stewartson equation. *Phys. Rev. E* **1999**, *60*, 2297. [[CrossRef](#)] [[PubMed](#)]
28. Guo, L.; He, J.; Mihalache, D. Rational and semi-rational solutions to the asymmetric Nizhnik Novikov–Veselov system. *J. Phys. A Math. Theor.* **2021**, *54*. [[CrossRef](#)]
29. Guo, L.J.; Chen, L.; Mihalache, D.; He, J.S. Dynamics of soliton interaction solutions of the Davey–Stewartson I equation. *Phys. Rev. E* **2022**, *105*, 014218. [[CrossRef](#)]
30. Wei, G.M.; Lu, Y.L.; Xie, Y.Q.; Zheng, W.X. Lie symmetry analysis and conservation law of variable-coefficient Davey–Stewartson equation. *Comput. Math. Appl.* **2018**, *75*, 3420–3430. [[CrossRef](#)]
31. Wang, R.J.; Huang, Y.C. Exact solutions and excitations for the Davey–Stewartson equations with nonlinear and gain terms. *Eur. Phys. J. D* **2010**, *57*, 395–401. [[CrossRef](#)]
32. Wei, G.M.; Gao, Y.T.; Xu, T.; Meng, X.H.; Zhang, C.Y. Painlevé property and new analytic solutions for a variable-coefficient Kadomtsev–Petviashvili equation with symbolic computation. *Chin. Phys. Lett.* **2008**, *25*, 1599–1602.
33. Srivastava, S.; Kumar, M. Nonclassical symmetries, optimal classification, and dynamical behavior of similarity solutions of $(3 + 1)$ -dimensional Burgers equation. *Chin. J. Phys.* **2024**, *89*, 404–416. [[CrossRef](#)]
34. Weiss, J.; Tabor, M.; Carnevale, G. Carnevale, The Painlevé property for partial differential equations. *J. Math. Phys.* **1983**, *24*, 522–526. [[CrossRef](#)]
35. Weiss, J.; Tabor, M.; Carnevale, G. The Painlevé property for partial differential equations II: Bäcklund transformation, Lax pairs, and the Schwarzian derivative. *J. Math. Phys.* **1983**, *24*, 1405–1413. [[CrossRef](#)]

-
36. Xu, G.Q. A note on the Painlevé test for nonlinear variable-coefficient PDEs. *Comput. Phys. Commun.* **2009**, *180*, 1137–1144. [[CrossRef](#)]
 37. Hirota, R. *The Direct Method in Soliton Theory*; Cambridge University Press: Cambridge, UK, 2004.

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