mathematics

MDPI

Article

Kamenev-Type Criteria for Testing the Asymptotic Behavior of Solutions of Third-Order Quasi-Linear Neutral Differential Equations

Hail S. Alrashdi [1], Wedad Albalawi [2,*], Ali Muhib [1], Osama Moaaz [3] and Elmetwally M. Elabbasy [1]

1 Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt; hailaldyabai@std.mans.edu.eg (H.S.A.); muhib39@students.mans.edu.eg (A.M.); emelabbasy@mans.edu.eg (E.M.E.)
2 Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia
3 Department of Mathematics, College of Science, Qassim University, P.O. Box 6644, Buraydah 51452, Saudi Arabia; o_moaaz@mans.edu.eg
* Correspondence: wsalbalawi@pnu.edu.sa

Abstract: This paper aims to study the asymptotic properties of nonoscillatory solutions (eventually positive or negative) of a class of third-order canonical neutral differential equations. We use Riccati substitution to reduce the order of the considered equation, and then we use the Philos function class to obtain new criteria of the Kamenev type, which guarantees that all nonoscillatory solutions converge to zero. This approach is characterized by the possibility of applying its conditions to a wider area of equations. This is not the only aspect that distinguishes our results; we also use improved relationships between the solution and the corresponding function, which in turn is reflected in a direct improvement of the criteria. The findings in this article extend and generalize previous findings in the literature and also improve some of these findings.

Keywords: quasi-linear differential equations; asymptotic and oscillatory analysis; third-order; neutral delay arguments

MSC: 34C10; 34K11

check for updates

Citation: Alrashdi, H.S.; Albalawi, W.; Muhib, A.; Moaaz, O.; Elabbasy, E.M. Kamenev-Type Criteria for Testing the Asymptotic Behavior of Solutions of Third-Order Quasi-Linear Neutral Differential Equations. *Mathematics* **2024**, *12*, 1734. https://doi.org/10.3390/math12111734

Academic Editors: Lingju Kong and Min Wang

Received: 7 November 2023
Revised: 21 January 2024
Accepted: 2 February 2024
Published: 3 June 2024

1. Introduction

One type of functional differential equation (FDE) that accounts for the temporal memory of phenomena is the delay differential equation (DDE). Thus, it is simple to understand how these equations are applied in a wide spectrum of fields, including as biological, engineering, and physical models, as well as in other sciences [1,2].

A variety of inquiries concerning oscillatory behavior and asymptotic features of DDE solutions are addressed by oscillation theory, a subfield of qualitative theory. The basic task of oscillation theory is to identify the criteria that eliminate the nonoscillatory solutions. A variety of findings, techniques, and strategies for examining the oscillation of DDEs were gathered in monographs [3–6].

The investigation of oscillation for solutions of ordinary, partial, and fractional FDEs with delay, neutral delay (NDDE), mixed delay, and damping is a recent, significant expansion and enhancement of the oscillation theory. It is known that differential equations with delay have received the most attention, particularly for non-canonical cases. For instance, refer to [7–15] for delay, advanced, and neutral equations, respectively. Furthermore, Refs. [16–21] show how investigations of odd-order equations have evolved. Moreover, one may trace the variation of fractional DDEs in Survey [22]. Whereas [23–25] dealt with damping equations, and [26–29] studied mixed equations. Over the past 20 years, functional dynamic equations have also drawn a lot of attention; see, for instance, [30–32].

In this paper, we present new criteria for the oscillation of quasi-linear third-order neutral DDEs:

$$\left(a(\mathrm{s})\left(\left(x(\mathrm{s})+\eta(\mathrm{s})x(g(\mathrm{s}))\right)''\right)^{r}\right)'+q(\mathrm{s})x^{r}(\tau(\mathrm{s}))=0, \tag{1}$$

where $\mathrm{s}\geq \mathrm{s}_{0,}$ and r is the ratio of any two positive odd integers. Here, in this work, the following assumptions are satisfied:

(I) $a\in C^{1}([\mathrm{s}_0,\infty),(0,\infty))$, $a'(\mathrm{s})\geq 0$, and $\mathcal{T}(\mathrm{s}_0,\infty)=\infty$, where

$$\mathcal{T}(l,\mathrm{s})=\int_{l}^{\mathrm{s}}\frac{1}{a^{1/r}(\theta)}\mathrm{d}\theta; \tag{2}$$

(II) $\eta,q\in C([\mathrm{s}_0,\infty),[0,\infty))$ with $0\leq\eta(\mathrm{s})\leq\eta_0<\infty$ and $q(\mathrm{s})$ does not vanish eventually;

(III) $g,\tau\in C([\mathrm{s}_0,\infty),\mathbb{R})$, $g(\mathrm{s})\leq\mathrm{s}$, $\tau(\mathrm{s})\leq\mathrm{s}$, and $\lim_{\mathrm{s}\to\infty}g(\mathrm{s})=\lim_{\mathrm{s}\to\infty}\tau(\mathrm{s})=\infty$.

For the solution of (1) on $[\mathrm{s}_x,\infty)$, we refer to a real-valued function $x\in C([\mathrm{s}_x,\infty),\mathbb{R})$, $\mathrm{s}_x\geq\mathrm{s}_0$, which satisfies (1) on $[\mathrm{s}_x,\infty)$, and has the properties $(x+\eta\cdot(x\circ g))\in C^{2}([\mathrm{s}_x,\infty),\mathbb{R})$ and $\left(a\cdot\left((x+\eta\cdot(x\circ g))''\right)^{r}\right)\in C([\mathrm{s}_x,\infty),\mathbb{R})$. We only consider those solutions $x(\mathrm{s})$ of (1) satisfying $\sup\{|x(\mathrm{s})|:\mathrm{s}\geq\mathrm{S}\}>0$ for all $\mathrm{S}\geq\mathrm{s}_x$, and we assume that (1) has such solutions. A solution of (1) is said to be oscillatory if it has arbitrarily large zeros in $[\mathrm{s}_0,\infty)$, and is called nonoscillatory otherwise. Equation (1) is said to be oscillatory if all of its solutions are oscillatory.

In the study of neutral equations, the corresponding function z to the solution x, defined as

$$z(\mathrm{s}):=x(\mathrm{s})+\eta(\mathrm{s})x(g(\mathrm{s})), \tag{3}$$

is vital.

Numerous studies have been conducted on third-order functional differential equations and the oscillation behavior of solutions; see [33–40]. There exists a theoretical and applicable interest in the problem of oscillatory properties of neutral DDEs; see Hale [1] for some important applications in various applied sciences.

In what follows, we survey some of the most important research that handles the study of third-order NDDEs using different techniques and some different restrictions to obtain conditions that ensure that the solution is oscillatory or tends to zero to cover the the largest area when applied to special cases.

Baculikova and Dzurina [41] tested the asymptotic features of a pair of third-order NDDEs,

$$\left(a(\mathrm{s})\left(\left(x(\mathrm{s})\pm\eta(\mathrm{s})x(g(\mathrm{s}))\right)''\right)^{r}\right)'+q(\mathrm{s})x^{r}(\tau(\mathrm{s}))=0,\ \mathrm{s}\geq\mathrm{s}_0, \tag{4}$$

where $0\leq\eta(\mathrm{s})\leq\eta<1$. They established novel sufficient conditions that confirm that all nonoscillatory solutions of (4) converge to zero.

Thandapani and Li [42] studied the oscillatory features of the third-order NDDE (1), where $g'(\mathrm{s})\geq g_0>0$, $\tau\circ g=g\circ\tau$ and $0\leq\eta(\mathrm{s})\leq\eta<\infty$. By using the Riccati transformation, they established some sufficient criteria, which confirm that any solution of (1) is oscillatory or tends to zero.

Graef et al. [43] discussed the oscillatory properties of a class of solutions of third-order nonlinear NDDEs:

$$\left(\left(\left(x(\mathrm{s})+\eta(\mathrm{s})x(g(\mathrm{s}))\right)''\right)^{r}\right)'+q(\mathrm{s})x^{r}(\tau(\mathrm{s}))=0$$

where $a=1$ and $\eta(\mathrm{s})\geq 1$. They presented novel sufficient criteria for any solution of the studied equation to be either oscillating or converging to zero.

Kumar and Ganesan [44] discussed the third-order nonlinear NDDE in the form

$$\left(a(\mathrm{s})\varphi(z''(\mathrm{s}))\right)'+q(\mathrm{s})\varphi(x(\tau(\mathrm{s})))=0,\ \mathrm{s}\geq\mathrm{s}_0>0, \tag{5}$$

where $\varphi(u) = |u|^{r-1}u$, $g'(\mathrm{s}) \geq g_0 > 0$ and $\tau \circ g = g \circ \tau$. The third- and first-order equation comparison principles provide the foundation for the obtained results. Below, we present some results obtained in previous studies to facilitate the reader's understanding.

Theorem 1 ([42])**.** *Let $r \geq 1$, $\tau \in C^1([\mathrm{s}_0, \infty))$ and $\tau' > 0$. Assume that*

$$\int_{\mathrm{s}_0}^{\infty} \int_{v}^{\infty} \left(\frac{1}{a(g(u))} \int_{u}^{\infty} Q(\theta) \mathrm{d}\theta \right)^{1/r} \mathrm{d}u \mathrm{d}v = \infty,$$

holds and $\tau(\mathrm{s}) \leq g(\mathrm{s})$. Moreover, assuming there is a function $\rho \in C^1([\mathrm{s}_0, \infty), (0, \infty))$, for all $\mathrm{s}_1 \geq \mathrm{s}_0$ large enough, there exists $\mathrm{s}_1 \geq \mathrm{s}_0$ where

$$\limsup_{\mathrm{s} \to \infty} \int_{\mathrm{s}_2}^{\mathrm{s}} \left(\frac{\rho(l) Q(l)}{2^{r+1}} - \frac{\left(1 + \frac{\eta_0^r}{g_0}\right) \left((\rho'(l))_+ \right)^{r+1}}{(r+1)^{r+1} (\rho(l) \beta_1(\tau(l), \mathrm{s}_1) \tau'(l))^r} \right) \mathrm{d}l = \infty,$$

for Q $= min\{q(\mathrm{s}), q(g(\mathrm{s}))\}$, $(\rho'(\mathrm{s}))_+ := \max\{0, \rho'(\mathrm{s})\}$ and $\beta_1(\mathrm{s}, \mathrm{s}_1) = \int_{\mathrm{s}_1}^{\mathrm{s}} 1 l a^{1/r}(\theta) \mathrm{d}\theta$. Then, (1) is almost oscillatory.

Theorem 2 ([44])**.** *Let $\tau(\mathrm{s}) \leq g(\mathrm{s}) \leq 1$. Assuming that $0 < r \leq 1$,*

$$\int_{\mathrm{s}_1}^{\mathrm{s}} g'(v) \int_{v}^{\infty} \left(\frac{g'(u)}{a(g(u))} \int_{u}^{\infty} Q(t) \mathrm{d}t \right)^{1/r} \mathrm{d}u \mathrm{d}v = \infty$$

and the first-order DDE

$$w'(\mathrm{s}) + \frac{g_0}{g_o + \eta_0^r} Q_1(\mathrm{s}) w\left(g^{-1}(\tau(\mathrm{s})) \right) = 0$$

oscillates, then any positive solution of (5) meets $\lim_{\mathrm{s} \to \infty} x(\mathrm{s}) = 0$, where $g^{-1}(\mathrm{s})$ is an inverse function of $g(\mathrm{s})$, and

$$Q_1(\mathrm{s}) = Q(\mathrm{s}) \left(\int_{\mathrm{s}_1}^{\tau(\mathrm{s})} (\mathcal{T}(l, t) - \mathcal{T}(l, t_1)) \mathrm{d}t \right).$$

Our goal in this study was to examine the asymptotic properties of a class of neutral third-order NDDEs. Based on the improved relationship between x and z that was derived in [45], we obtained new relationships between x and z. The new relationship is characterized by taking into account both cases $\eta \leq 1$ and $\eta > 1$; this was not common in previous third-order studies. We present Kamenev-type criteria that ensure that all solutions of the neutral DDE, (1), either converge to zero or are oscillatory. We begin by deducing some new relationships that help improve the approach. Then, we use the Philos function class to obtain the required conditions. The criteria we obtain improve and extend some results from previous studies. Finally, we employ the results in the special case of our studied equation.

2. Preliminaries

We begin with lemmas, notations that are required throughout this paper. For convenience, we use the symbol $\mathcal{P}$ to state the category of all eventually positive solutions to (1), the symbol $\mathcal{P}_{\downarrow}$ to denote the class of solutions $x \in \mathcal{P}$, whose corresponding function confirms $z'(t) < 0$, and the symbol $\mathcal{P}_{\uparrow}$ to denote the class of solutions $x \in \mathcal{P}$ whose corresponding function confirms $z'(t) > 0$.

Lemma 1 ([41] (Lemma 1))**.** *Assume that $x \in \mathcal{P}$. Then, z meets one of the following possible cases, eventually:*

(i) $z > 0$, $z' > 0$ *and* $z'' > 0$;

(ii) $z > 0,\ z' < 0$ *and* $z'' > 0$.

Lemma 2 ([41] (Lemma 2))**.** *Suppose that* $x \in \mathcal{P}_{\downarrow}$. *If*

$$\int_{s_0}^{\infty}\int_{v}^{\infty}\left(\frac{1}{a(u)}\int_{u}^{\infty}q(\theta)\mathrm{d}\theta\right)^{1/r}\mathrm{d}u\mathrm{d}v=\infty, \tag{6}$$

then $\lim_{s\to\infty} x(s) = \lim_{s\to\infty} z(s) = 0$.

Lemma 3 ([41] (Lemma 3))**.** *Suppose that* $u \in C^2([s_0,\infty),\mathbb{R})$. *Assume that* $u(s) > 0,\ u'(s) \geq 0$ *and* $u''(s) \leq 0$, *on* $[s_0,\infty)$. *Then, there exist a* $s_1 \geq s_0$ *for each* $k_1 \in (0,1)$ *such that*

$$\frac{u(\tau(s))}{u(s)} \geq k_1\frac{\tau(s)}{s},$$

where $s \geq s_1$.

Lemma 4 ([46])**.** *Suppose that* $u \in C^{m+1}([s_0,\infty),\mathbb{R})$, $u^{(j)}(s) > 0$, *for* $j = 0,1,\ldots,m$, *and* $u^{(m+1)}(s) \leq 0$. *Then, there exist a* $s_1 \geq s_0$, *for each* $k_2 \in (0,1)$, *such that*

$$\frac{u(s)}{u'(s)} \geq \frac{k_2}{m}s,$$

where $s \geq s_1$.

Notation 1. *For simplicity, let* $G^{[0]}(s) := s,\ G^{[j]}(s) = G\left(G^{[j-1]}(s)\right),\ G^{[-j]}(s) = G^{-1}\left(G^{[-j+1]}(s)\right)$, *for* $j = 1,2,\ldots$.

Lemma 5 ([45])**.** *Suppose that* $x \in \mathcal{P}_{\uparrow} \cup \mathcal{P}_{\downarrow}$. *Then,*

$$x > \sum_{k=0}^{m}\left(\prod_{n=0}^{2k}\eta\left(g^{[n]}\right)\right)\left(\frac{z\left(g^{[2k]}\right)}{\eta\left(g^{[2k]}\right)} - z\left(g^{[2k+1]}\right)\right), \tag{7}$$

eventually, where $m > 0,\ m \in \mathbb{Z}$.

Let $\Re$ be class of functions, the function $\mathcal{K} \in \Re$, where $\mathcal{K} \in C(H,\mathbb{R})$, $H = \{(s,\theta,\ell) : s_0 \leq \ell \leq \theta \leq s \leq \infty\}$, if $\mathcal{K}$ satisfies the following hypotheses:

(1) $\mathcal{K}(s,s,\ell) = 0,\ \mathcal{K}(s,\ell,\ell) = 0,\ \mathcal{K}(s,\theta,\ell) \neq 0$, for $\ell < \theta < s$;
(2) $\mathcal{K}(s,\theta,\ell)$ possesses the partial derivative $\partial\mathcal{K}/\partial\theta$ on H with the condition that $\partial\mathcal{K}/\partial\theta$ can be integrated locally in terms of θ in H and

$$\frac{\partial\mathcal{K}(s,\theta,\ell)}{\partial\theta} = h(s,\theta,\ell)\mathcal{K}(s,\theta,\ell), \tag{8}$$

for some $h \in C(H,\mathbb{R})$.

This class of functions is defined by Philos [47].

Notation 2. *During the main results, we need to define the following abbreviations:*

$$\psi(s) = \int_{s_0}^{s}\mathcal{T}(s_0,u)\mathrm{d}u,$$

$$\Theta_1(s) = \sum_{k=0}^{m}\left(\prod_{n=0}^{2k}\eta\left(g^{[n]}(\tau(s))\right)\right)\left(\frac{1}{\eta\left(g^{[2k]}(\tau(s))\right)} - 1\right)\frac{\psi\left(g^{[2k]}(\tau(s))\right)}{\psi(\tau(s))},$$

$$\Theta_2(s) = \sum_{k=1}^{m}\left(\prod_{j=1}^{2k-1}\frac{1}{\eta\left(g^{[-j]}(\tau(s))\right)}\right)\left(1 - \frac{1}{\eta\left(g^{[-2k]}(\tau(s))\right)}\frac{\psi\left(g^{[-2k]}(\tau(s))\right)}{\psi\left(g^{[-2k+1]}(\tau(s))\right)}\right),$$

$$\Theta(s) = \begin{cases} 1, & \text{for } \eta = 0 \\ \Theta_1(s), & \text{for } 0 < \eta < 1 \\ \Theta_2(s), & \text{for } \eta > \psi\left(g^{[-2k]}(\tau(s))\right)/\psi\left(g^{[-2k+1]}(\tau(s))\right), \end{cases}$$

and

$$M_0 = \sum_{j=0}^{r+1}\binom{r+1}{r-j+1}(-1)^{r-j+1}\gamma^{r-j+1}\lambda^{j}\frac{\Gamma(\gamma+j-r)\Gamma(\lambda-j+1)}{\Gamma(\gamma+\lambda-r+1)}, \text{ for } r \in \mathbb{Z}^+,$$

where $\gamma, \lambda \in (r, \infty)$,

$$\Gamma(\theta) = \int_0^{+\infty} x^{\theta-1}e^{-x}\mathrm{d}x,\ \theta > 0,$$

and

$$k_0 = \frac{1}{(r+1)^{(r+1)}}.$$

3. Main Results

We present new conditions that guarantee that each solution to DDE (1) oscillates or converges to zero.

Theorem 3. *Suppose that (6) holds and the function* $\mathcal{K} \in \Re$. *In the event that a function,* $\rho \in C^1([s_0, \infty), \mathbb{R}^+)$, *is present and satisfies* $\rho'(s) \geq 0$ *such that*

$$\limsup_{s\to\infty}\int_{\ell}^{s}\mathcal{K}(s,\theta,\ell)\rho(\theta)\left(k_1^r k_2^r\frac{\tau^{2r}(\theta)}{(2\theta)^r}q(\theta)\Theta^r(\theta) - k_0 a(\theta)\left(h(s,\theta,\ell) + \frac{\rho'(\theta)}{\rho(\theta)}\right)^{r+1}\right)\mathrm{d}\theta > 0, \tag{9}$$

for any $k_1, k_2 \in (0,1)$, *then the solution* $x(s)$ *oscillates or tends to zero.*

Proof. Suppose that $x \in \mathcal{P}$. Suppose that there is an $s \geq s_1$ such that $x(s) > 0$, $x(g(s)) > 0$, and $x(\tau(s)) > 0$. Clearly, $z(s) > 0$, $s \geq s_1$. From Lemma 1, we can see that (i) or (ii) is satisfied.

Assume that (ii) is satisfied. Since (6) holds, following from Lemma 2 that $\lim_{s\to\infty} x(s) = 0$.

Now, assume that (i) is satisfied. We have

$$\begin{aligned} z'(s) &\geq \int_{s_0}^{s}\frac{a^{1/r}(u)z''(u)}{a^{1/r}(u)}\mathrm{d}u \geq a^{1/r}(s)z''(s)\int_{s_0}^{s}\frac{1}{a^{1/r}(u)}\mathrm{d}u \\ &\geq a^{1/r}(s)z''(s)\mathcal{T}(s_0,s),\ s \geq s_1 \end{aligned}$$

therefore, we find

$$\left(\frac{z'(s)}{\mathcal{T}(s_0,s)}\right)' = \frac{\mathcal{T}(s_0,s)z''(s) - z'(s)a^{-1/r}(s)}{\mathcal{T}^2(s_0,s)} = \frac{a^{1/r}(s)\mathcal{T}(s_0,s)z''(s) - z'(s)}{a^{1/r}(s)\mathcal{T}^2(s_0,s)} \leq 0,\ s \geq s_1. \tag{10}$$

Since

$$z(s) \geq \int_{s_0}^{s}\frac{\mathcal{T}(s_0,u)z'(u)}{\mathcal{T}(s_0,u)}\mathrm{d}u, \text{ for } s \geq s_1$$

by using (10), we obtain

$$z(s) \geq \frac{z'(s)}{\mathcal{T}(s_0,s)}\int_{s_0}^{s}\mathcal{T}(s_0,u)\mathrm{d}u \geq \frac{z'(s)}{\mathcal{T}(s_0,s)}\psi(s)$$

and so

$$\left(\frac{z(s)}{\psi(s)}\right)' = \frac{\psi(s)z'(s) - z(s)\mathcal{T}(s_0, s)}{\psi^2(s)} = \frac{\mathcal{T}^{-1}(s_0, s)\psi(s)z'(s) - z(s)}{\mathcal{T}^{-1}(s_0, s)\psi^2(s)} \leq 0, \ s \geq s_1. \tag{11}$$

From (3), we have

$$x(s) = z(s) - \eta(s)x(g(s)).$$

Now, assume that $\eta < 1$. Since $z(s)$ satisfies (i), following Lemma 5, that (7) holds. Using $g^{[2k+1]}(s) \leq g^{[2k]}(s) \leq s$, $z'(s) > 0$ and (11), we obtain

$$z\left(g^{[2k+1]}(s)\right) \leq z\left(g^{[2k]}(s)\right) \leq z(s), \ s \geq s_1$$

and

$$z\left(g^{[2k]}(s)\right) \geq \frac{\psi\left(g^{[2k]}(s)\right)z(s)}{\psi(s)}, \text{ for } k = 0, 1, \ldots.$$

Thus, we see that (7) becomes

$$\begin{aligned} x(s) &> \sum_{k=0}^{m}\left(\prod_{n=0}^{2k}\eta\left(g^{[n]}(s)\right)\right)\left(\frac{z\left(g^{[2k]}(s)\right)}{\eta\left(g^{[2k]}(s)\right)} - z\left(g^{[2k]}(s)\right)\right) \\ &> \sum_{k=0}^{m}\left(\prod_{n=0}^{2k}\eta\left(g^{[n]}(s)\right)\right)\left(\frac{1}{\eta\left(g^{[2k]}(s)\right)} - 1\right)z\left(g^{[2k]}(s)\right) \\ &> z(s)\sum_{k=0}^{m}\left(\prod_{n=0}^{2k}\eta\left(g^{[n]}(s)\right)\right)\left(\frac{1}{\eta\left(g^{[2k]}(s)\right)} - 1\right)\frac{\psi\left(g^{[2k]}(s)\right)}{\psi(s)}, \ s \geq s_1. \end{aligned}$$

Using this inequality in (1), we obtain

$$\left(a(s)\left((x(s) + \eta(s)x(g(s)))''\right)^r\right)' \leq -q(s)z^r(\tau(s))\Theta^r(s), \ s \geq s_1. \tag{12}$$

Now, assume that $\eta > 1$. It follows from the (3) that

$$\begin{aligned} x(s) &= \frac{1}{\eta(g^{-1}(s))}\left(z\left(g^{-1}(s)\right) - x\left(g^{-1}(s)\right)\right) \\ &= \frac{z(g^{-1}(s))}{\eta(g^{-1}(s))} - \frac{1}{\eta(g^{-1}(s))\eta(g^{[-2]}(s))}\left(z\left(g^{[-2]}(s)\right) - x\left(g^{[-2]}(s)\right)\right) \\ &= \frac{z(g^{-1}(s))}{\eta(g^{-1}(s))} \\ &\quad - \frac{1}{\eta(g^{-1}(s))\eta(g^{[-2]}(s))}\left(z\left(g^{[-2]}(s)\right) - \frac{1}{\eta(g^{[-3]}(s))}\left(z\left(g^{[-3]}(s)\right) - x\left(g^{[-3]}(s)\right)\right)\right) \\ &= \frac{z(g^{-1}(s))}{\prod_{j=1}^{1}\eta(g^{[-j]}(s))} - \frac{z\left(g^{[-2]}(s)\right)}{\prod_{j=1}^{2}\eta(g^{[-j]}(s))} \\ &\quad + \frac{1}{\prod_{j=1}^{3}\eta(g^{[-j]}(s))}\left(z\left(g^{[-3]}(s)\right) - x\left(g^{[-3]}(s)\right)\right), \ s \geq s_1, \end{aligned}$$

and so on. Thus, we have

$$x(s) > \sum_{k=1}^{m}\left(\prod_{j=1}^{2k-1}\frac{1}{\eta(g^{[-j]}(s))}\right)\left(z\left(g^{[-2k+1]}(s)\right) - \frac{1}{\eta(g^{[-2k]}(s))}z\left(g^{[-2k]}(s)\right)\right), \ s \geq s_1. \tag{13}$$

From the facts that $g^{[-2k]} \geq g^{[-2k+1]} \geq s$, $z' > 0$ and (11), we arrive at

$$z\left(g^{[-2k+1]}(s)\right) \geq z(s),\ s \geq s_1 \tag{14}$$

and

$$z\left(g^{[-2k]}(s)\right) \leq \frac{\psi\left(g^{[-2k]}(s)\right)z\left(g^{[-2k+1]}(s)\right)}{\psi\left(g^{[-2k+1]}(s)\right)},\ s \geq s_1. \tag{15}$$

Using (14) and (15) in (13), we obtain

$$x(s) > z(s)\sum_{k=1}^{m}\left(\prod_{j=1}^{2k-1}\frac{1}{\eta\left(g^{[-j]}(s)\right)}\right)\left(1-\frac{1}{\eta\left(g^{[-2k]}(s)\right)}\frac{\psi\left(g^{[-2k]}(s)\right)}{\psi\left(g^{[-2k+1]}(s)\right)}\right),\ s \geq s_1$$

and so

$$x(\tau(s)) > z(\tau(s))\sum_{k=1}^{m}\left(\prod_{j=1}^{2k-1}\frac{1}{\eta\left(g^{[-j]}(\tau(s))\right)}\right)\left(1-\frac{1}{\eta\left(g^{[-2k]}(\tau(s))\right)}\frac{\psi\left(g^{[-2k]}(\tau(s))\right)}{\psi\left(g^{[-2k+1]}(\tau(s))\right)}\right),\ s \geq s_1.$$

From the above inequality and (1), we obtain (12), therefore,

$$\left(a(s)\left((x(s)+\eta(s)x(g(s)))''\right)^r\right)' \leq 0. \tag{16}$$

Using (16), $a'(s) \geq 0$, and $z''(s) > 0$, we have $z'''(s) \leq 0$. Therefore, there exists an $s_2 \geq s_1$ such that $z(s)$ satisfies

$$z(\tau(s)) > 0,\ z'(s) > 0,\ z''(s) > 0,\ z'''(s) \leq 0,\ s \geq s_2.$$

We define $\omega(s)$ as follow:

$$\omega(s) = \rho(s)\frac{a(s)(z''(s))^r}{(z'(s))^r},\ s \geq s_2. \tag{17}$$

We see that $\omega(s) > 0$ and

$$\begin{aligned}\omega'(s) &= \rho'(s)\frac{a(s)(z''(s))^r}{(z'(s))^r} \\ &\quad + \frac{\rho(s)(z'(s))^r\left(a(s)(z''(s))^r\right)' - r\rho(s)a(s)(z''(s))^r(z'(s))^{r-1}z''(s)}{(z'(s))^{2r}},\ s \geq s_2.\end{aligned}$$

By using (12) and (17), we have

$$\omega'(s) \leq \rho'(s)\frac{\omega(s)}{\rho(s)} - \rho(s)\frac{q(s)z^r(\tau(s))\Theta^r(s)}{(z'(s))^r} - r\frac{\omega^{(r+1)/r}(s)}{a^{1/r}(s)\rho^{1/r}(s)},\ s \geq s_2. \tag{18}$$

By using Lemma 3 with $u(s) = z'(s)$, there exists a $s_3 \geq s_2$ such that

$$\frac{z'(\tau(s))}{z'(s)} \geq k_1\frac{\tau(s)}{s},\ s \geq s_3 \geq s_2. \tag{19}$$

By using Lemma 4, we have

$$\frac{z(s)}{z'(s)} \geq \frac{1}{2}k_2 s,\ s \geq s_3. \tag{20}$$

From (19) and (20), we obtain

$$\frac{1}{z'(s)} \geq k_1 \frac{\tau(s)}{sz'(\tau(s))} \geq k_1 k_2 \frac{\tau^2(s)}{2s} \frac{1}{z(\tau(s))}, \; s \geq s_3. \tag{21}$$

Using (18) and (21), we obtain

$$\omega'(s) \leq \rho'(s)\frac{\omega(s)}{\rho(s)} - k_1^r k_2^r \frac{\tau^{2r}(s)}{(2s)^r}\rho(s)q(s)\Theta^r(s) - r\frac{\omega^{(r+1)/r}(s)}{a^{1/r}(s)\rho^{1/r}(s)}, \; s \geq s_3$$

and so

$$k_1^r k_2^r \frac{\tau^{2r}(s)}{(2s)^r}\rho(s)q(s)\Theta^r(s) \leq -\omega'(s) + \rho'(s)\frac{\omega(s)}{\rho(s)} - r\frac{\omega^{(r+1)/r}(s)}{a^{1/r}(s)\rho^{1/r}(s)}, \; s \geq s_3. \tag{22}$$

Multiplying the above inequality by $\mathcal{K}(s,\theta,\ell)$ and integrating from $\ell \geq s_3$ to s, we obtain

$$\begin{aligned} & \int_\ell^s \mathcal{K}(s,\theta,\ell) k_1^r k_2^r \frac{\tau^{2r}(\theta)}{(2\theta)^r}\rho(\theta)q(\theta)\Theta^r(\theta)d\theta \\ \leq \; & -\int_\ell^s \mathcal{K}(s,\theta,\ell)\omega'(\theta)d\theta + \int_\ell^s \mathcal{K}(s,\theta,\ell)\rho'(\theta)\frac{\omega(\theta)}{\rho(\theta)}d\theta \\ & -r\int_\ell^s \frac{\mathcal{K}(s,\theta,\ell)\omega^{(r+1)/r}(\theta)}{a^{1/r}(\theta)\rho^{1/r}(\theta)}d\theta, \; \ell \geq s_3. \end{aligned} \tag{23}$$

By using (8), for all $s \geq \ell$, we have

$$\begin{aligned} & \int_\ell^s \mathcal{K}(s,\theta,\ell) k_1^r k_2^r \frac{\tau^{2r}(\theta)}{(2\theta)^r}\rho(\theta)q(\theta)\Theta^r(\theta)d\theta \\ \leq \; & \int_\ell^s \mathcal{K}(s,\theta,\ell)\left(\left(h(s,\theta,\ell) + \frac{\rho'(\theta)}{\rho(\theta)}\right)\omega(\theta) - r\frac{\omega^{(r+1)/r}(\theta)}{a^{1/r}(\theta)\rho^{1/r}(\theta)}\right)d\theta, \; s \geq \ell. \end{aligned} \tag{24}$$

Set

$$F(\vartheta) = \left(h(s,\theta,\ell) + \frac{\rho'(\theta)}{\rho(\theta)}\right)\vartheta - r\frac{\vartheta^{(r+1)/r}}{a^{1/r}(\theta)\rho^{1/r}(\theta)}, \; s \geq \ell.$$

A simple calculation implies when

$$\vartheta = k_0\rho(\theta)a(\theta)\left(h(s,\theta,\ell) + \frac{\rho'(\theta)}{\rho(\theta)}\right)^r, \; s \geq \ell$$

$F(\vartheta)$ has the maximum

$$k_0\rho(\theta)a(\theta)\left(h(s,\theta,\ell) + \frac{\rho'(\theta)}{\rho(\theta)}\right)^{r+1}, \; s \geq \ell$$

that is,

$$F(\vartheta) \leq F_{\max} = k_0\rho(\theta)a(\theta)\left(h(s,\theta,\ell) + \frac{\rho'(\theta)}{\rho(\theta)}\right)^{r+1}, \; s \geq \ell. \tag{25}$$

Using (24) and (25), we have

$$\begin{aligned} 0 \; \geq \; & \int_\ell^s \mathcal{K}(s,\theta,\ell) k_1^r k_2^r \frac{\tau^{2r}(\theta)}{(2\theta)^r}\rho(\theta)q(\theta)\Theta^r(\theta)d\theta \\ & -\int_\ell^s \mathcal{K}(s,\theta,\ell)k_0\rho(\theta)a(\theta)\left(h(s,\theta,\ell) + \frac{\rho'(\theta)}{\rho(\theta)}\right)^{r+1}d\theta, \; s \geq \ell \end{aligned}$$

and so

$$\int_{\ell}^{s} \mathcal{K}(s,\theta,\ell)\rho(\theta)\left(k_1^r k_2^r \frac{\tau^{2r}(\theta)}{(2\theta)^r} q(\theta)\Theta^r(\theta) - k_0 a(\theta)\left(h(s,\theta,\ell) + \frac{\rho'(\theta)}{\rho(\theta)}\right)^{r+1}\right) d\theta \le 0,\ s \ge \ell.$$

Taking the super limit, we obtain

$$\limsup_{s\to\infty} \int_{\ell}^{s} \mathcal{K}(s,\theta,\ell)\rho(\theta)\left(k_1^r k_2^r \frac{\tau^{2r}(\theta)}{(2\theta)^r} q(\theta)\Theta^r(\theta) - k_0 a(\theta)\left(h(s,\theta,\ell) + \frac{\rho'(\theta)}{\rho(\theta)}\right)^{r+1}\right) d\theta \le 0,\ s \ge \ell.$$

This contradicts (9) and the proof is complete. □

Theorem 4. *Assume that (6) holds and*

$$\mathcal{K}(s,\theta,\ell) = (s-\theta)^{\sigma}(\theta-\ell)^{\vartheta},$$

where σ, ϑ are constants greater than r. If there is a $\rho \in C^1([s_0,\infty),\mathbb{R}^+)$ satisfying $\rho'(s) \ge 0$ such that

$$\limsup_{s\to\infty} \int_{\ell}^{s} (s-\theta)^{\sigma}(\theta-\ell)^{\vartheta}\rho(\theta)\Phi(s,l,\theta)d\theta > 0 \tag{26}$$

for any $k_1, k_2 \in (0,1)$, then, the solution $x(s)$ is oscillatory or converges to zero, where

$$\Phi(s,l,\theta) := k_1^r k_2^r \frac{\tau^{2r}(\theta)}{(2\theta)^r} q(\theta)\Theta^r(\theta) - k_0 a(\theta)\left(\frac{\vartheta s - (\sigma+\vartheta)\theta + \sigma\ell}{(s-\theta)(\theta-\ell)} + \frac{\rho'(\theta)}{\rho(\theta)}\right)^{r+1}.$$

Proof. Suppose that $x \in \mathcal{P}$. Suppose that there is an $s \ge s_1$ such that $x(s) > 0,\ x(g(s)) > 0$, and $x(\tau(s)) > 0$. Clearly, $z(s) > 0, s \ge s_1$. Since

$$\mathcal{K}(s,\theta,\ell) = (s-\theta)^{\sigma}(\theta-\ell)^{\vartheta},\ s \ge \ell,$$

by using (8), we have

$$h(s,\theta,\ell) = \frac{\vartheta s - (\sigma+\vartheta)\theta + \sigma\ell}{(s-\theta)(\theta-\ell)},\ s \ge \ell.$$

Now, as in the proof of Theorem 3, we arrive at

$$\limsup_{s\to\infty} \int_{\ell}^{s} (s-\theta)^{\sigma}(\theta-\ell)^{\vartheta}\rho(\theta)\Phi(s,l,\theta)d\theta \le 0,\ s \ge \ell.$$

This contradicts (26) and the proof is complete. □

Theorem 5. *Assume that (6) holds and*

$$\mathcal{K}(s,\theta,\ell) = (\mathcal{T}(s_0,s) - \mathcal{T}(s_0,\theta))^{\gamma}(\mathcal{T}(s_0,\theta) - \mathcal{T}(s_0,\ell))^{\lambda},$$

where γ, λ are constants greater than r. If there exists a function $\rho \in C^1([s_0,\infty),\mathbb{R}^+)$ satisfying $\rho'(s) \ge 0$ such that

$$\limsup_{s\to\infty} \int_{\ell}^{s} (\mathcal{T}(s_0,s) - \mathcal{T}(s_0,\theta))^{\gamma}(\mathcal{T}(s_0,\theta) - \mathcal{T}(s_0,\ell))^{\lambda}\rho(\theta)\Psi(s,l,\theta)d\theta > 0 \tag{27}$$

for any $k_1, k_2 \in (0,1)$*, then, the solution* $x(s)$ *is oscillatory or converges to zero, where*

$$\Psi(s,l,\theta) \;:\; = k_1^r k_2^r \frac{\tau^{2r}(\theta)}{(2\theta)^r} q(\theta)\Theta^r(\theta)$$
$$-k_0 a(\theta)\left(\frac{\lambda\mathcal{T}(s_0,s)-(\gamma+\lambda)\mathcal{T}(s_0,\theta)+\gamma\mathcal{T}(s_0,\ell)}{a^{1/r}(\theta)(\mathcal{T}(s_0,s)-\mathcal{T}(s_0,\theta))(\mathcal{T}(s_0,\theta)-\mathcal{T}(s_0,\ell))}+\frac{\rho'(\theta)}{\rho(\theta)}\right)^{r+1}.$$

Proof. Suppose that $x \in \mathcal{P}$. Suppose that there is an $s \geq s_1$ such that $x(s) > 0$, $x(g(s)) > 0$, and $x(\tau(s)) > 0$. Clearly, $z(s) > 0$, $s \geq s_1$. Since

$$\mathcal{K}(s,\theta,\ell) = (\mathcal{T}(s_0,s)-\mathcal{T}(s_0,\theta))^{\gamma}(\mathcal{T}(s_0,\theta)-\mathcal{T}(s_0,\ell))^{\lambda},\ s \geq \ell,$$

by using (8), we have

$$h(s,\theta,\ell) = \frac{\lambda\mathcal{T}(s_0,s)-(\gamma+\lambda)\mathcal{T}(s_0,\theta)+\gamma\mathcal{T}(s_0,\ell)}{a^{1/r}(\theta)(\mathcal{T}(s_0,s)-\mathcal{T}(s_0,\theta))(\mathcal{T}(s_0,\theta)-\mathcal{T}(s_0,\ell))},\ s \geq \ell.$$

Now, as in the proof of Theorem 3, we arrive at

$$\limsup_{s\to\infty}\int_{\ell}^{s}(\mathcal{T}(s_0,s)-\mathcal{T}(s_0,\theta))^{\gamma}(\mathcal{T}(s_0,\theta)-\mathcal{T}(s_0,\ell))^{\lambda}\rho(\theta)\Psi(s,l,\theta)d\theta \leq 0,\ s \geq \ell.$$

This contradicts (27) and the proof is complete. □

Corollary 1. *Suppose that (6) holds,* r *is an odd natural number and* $\rho(s) = 1$*. If there exist two constants* $\gamma, \lambda > r$ *such that*

$$\limsup_{s\to\infty}\frac{\int_{\ell}^{s}(\mathcal{T}(s_0,s)-\mathcal{T}(s_0,\theta))^{\gamma}(\mathcal{T}(s_0,\theta)-\mathcal{T}(s_0,\ell))^{\lambda}k_1^r k_2^r\frac{\tau^{2r}(\theta)}{(2\theta)^r}q(\theta)\Theta^r(\theta)\mathrm{d}\theta}{(\mathcal{T}(s_0,s)-\mathcal{T}(s_0,\ell))^{\gamma+\lambda-r}} > k_0 M_0 \tag{28}$$

for any $k_1, k_2 \in (0,1)$*, then, the solution* $x(s)$ *is oscillatory or converges to zero.*

Proof. As in Theorem 5 with $\rho(s) = 1$, we have to sufficiently prove that (28) leads to (27). From

$$\int_0^1 y^{\gamma-1}(1-y)^{\lambda-1}\mathrm{d}y = \frac{\Gamma(\gamma)\Gamma(\lambda)}{\Gamma(\gamma+\lambda)}.$$

Using $y = \varrho/\delta$, we obtain

$$\begin{aligned}\int_0^{\delta}(\delta-\varrho)^{\gamma+j-r-1}\varrho^{\lambda-j}\mathrm{d}\varrho &= \int_0^1 \delta^{\gamma+\lambda-r}(1-y)^{\gamma+j-r-1}y^{\lambda-j}\mathrm{d}y \\ &= \delta^{\gamma+\lambda-r}\frac{\Gamma(\gamma+j-r)\Gamma(\lambda-j+1)}{\Gamma(\gamma+\lambda-r+1)}.\end{aligned} \tag{29}$$

Let $\varrho = \mathcal{T}(s_0,\theta)-\mathcal{T}(s_0,\ell)$ and $\delta = \mathcal{T}(s_0,s)-\mathcal{T}(s_0,\ell)$. Then, by (9),

$$\begin{aligned}&\int_{\ell}^{s} a(\theta)(\mathcal{T}(s_0,s)-\mathcal{T}(s_0,\theta))^{\gamma}(\mathcal{T}(s_0,\theta)-\mathcal{T}(s_0,\ell))^{\lambda}\mathcal{F}(s,l,\theta)\mathrm{d}\theta \\ =\ &\int_0^{\delta}(\delta-\varrho)^{\gamma-r-1}\varrho^{\lambda-r-1}(\lambda(\delta-\varrho)-\gamma\varrho)^{r+1}\mathrm{d}\varrho,\end{aligned} \tag{30}$$

where

$$\mathcal{F}(s,l,\theta) := \left(\frac{\lambda\mathcal{T}(s_0,s)-(\gamma+\lambda)\mathcal{T}(s_0,\theta)+\gamma\mathcal{T}(s_0,\ell)}{a^{1/r}(\theta)(\mathcal{T}(s_0,s)-\mathcal{T}(s_0,\theta))(\mathcal{T}(s_0,\theta)-\mathcal{T}(s_0,\ell))}\right)^{r+1},\ s \geq \ell,$$

and

$$(\lambda(\delta-\varrho)-\gamma\varrho)^{r+1}=\sum_{j=0}^{r+1}(-1)^{j}\binom{r+1}{j}(\lambda(\delta-\varrho))^{j}(\gamma\varrho)^{r+1-j}. \tag{31}$$

From (30) and (31), we have

$$\begin{aligned}
&\int_{\ell}^{s} a(\theta)(\mathcal{T}(s_0,s)-\mathcal{T}(s_0,\theta))^{\gamma}(\mathcal{T}(s_0,\theta)-\mathcal{T}(s_0,\ell))^{\lambda}F(s,l,\theta)d\theta \\
=\ &\sum_{j=0}^{r+1}\binom{r+1}{r-j+1}(-1)^{r-j+1}\gamma^{r-j+1}\lambda^{j}\int_{0}^{\delta}\varrho^{\lambda-j}(\delta-\varrho)^{\gamma+j-r-1}d\varrho \\
=\ &(\mathcal{T}(s_0,s)-\mathcal{T}(s_0,\ell))^{\gamma+\lambda-r}M_0,\ s\geq\ell.
\end{aligned} \tag{32}$$

Hence, by (28) and (32), (27) holds. The proof is complete. □

Corollary 2. *Suppose that (6) holds, r is an odd natural number, and $\rho(s)=1$. If there exist two constants $\gamma,\lambda>r$ such that*

$$\limsup_{s\to\infty}\frac{\int_{\ell}^{s}(\mathcal{T}(s_0,s)-\mathcal{T}(s_0,\theta))^{\gamma}(\mathcal{T}(s_0,\theta)-\mathcal{T}(s_0,\ell))^{\lambda}\frac{\tau^{2r}(\theta)}{\theta^{r}}q(\theta)\Theta^{r}(\theta)d\theta}{(\mathcal{T}(s_0,s)-\mathcal{T}(s_0,\ell))^{\gamma+\lambda-r}}>2^{r}k_0M_0, \tag{33}$$

then, the solution $x(s)$ is oscillatory or converges to zero.

Proof. We shall show (33) implies (28). Note that (33) implies

$$\left(\frac{k_1k_2}{2}\right)^{r}q(s)\Theta^{r}(\theta)\left(\frac{\tau^{2}(s)}{s}\right)^{r}=\left(\frac{k}{2}\right)^{r}q(s)\Theta^{r}(\theta)\left(\frac{\tau^{2}(s)}{s}\right)^{r}, \tag{34}$$

where $k=k_1k_2$. Conversely, (33) suggests, for $k\in(0,1)$,

$$\limsup_{s\to\infty}\frac{\int_{\ell}^{s}(\mathcal{T}(s_0,s)-\mathcal{T}(s_0,\theta))^{\gamma}(\mathcal{T}(s_0,\theta)-\mathcal{T}(s_0,\ell))^{\lambda}\frac{\tau^{2r}(\theta)}{\theta^{r}}q(\theta)\Theta^{r}(\theta)d\theta}{(\mathcal{T}(s_0,s)-\mathcal{T}(s_0,\ell))^{\gamma+\lambda-r}}>\frac{1}{k^{r}}2^{r}k_0M_0,\ s\geq\ell. \tag{35}$$

Combining (34) and (35), we obtain that (28) holds. Hence, by Corollary 1, we complete the proof. □

Example 1. *For the third-order NDDE*

$$\left(x(s)+\frac{1}{2}x\left(\frac{s}{2}\right)\right)'''+\frac{\kappa}{s^{3}}x\left(\frac{s}{2}\right)=0,\ s>1. \tag{36}$$

Note that $r=1$, $a(s)=1$, $\eta(s)=1/2<1$, $q(s)=\kappa/s^{3}$, $\kappa>0$, $g(s)=s/2$, and $\tau(s)=s/2$. Condition (6) is satisfied, where

$$\int_{s_0}^{\infty}\int_{v}^{\infty}\left(\frac{1}{a(u)}\int_{u}^{\infty}q(\theta)d\theta\right)^{1/r}dudv=\int_{s_0}^{\infty}\int_{v}^{\infty}\int_{u}^{\infty}\frac{\kappa}{\theta^{3}}d\theta dudv=\infty.$$

Note that

$$\mathcal{T}(s_0,s)=\int_{s_0}^{s}\frac{d\theta}{a^{1/r}(\theta)}=\int_{s_0}^{s}d\theta=(s-s_0)=(s-1).$$

We may choose $\gamma = 4,\ \lambda = 5$*, then*

$$\begin{aligned} M_0 &= \sum_{j=0}^{r+1} \binom{r+1}{r-j+1} (-1)^{r-j+1} \gamma^{r-j+1} \lambda^j \frac{\Gamma(\gamma+j-r)\Gamma(\lambda-j+1)}{\Gamma(\gamma+\lambda-r+1)} \\ &= \sum_{j=0}^{1+1} C_{1+1}^{1-j+1} (-1)^{1-j+1} 4^{1-j+1} 5^j \frac{\Gamma(4+j-1)\Gamma(5-j+1)}{\Gamma(4+5-1+1)} = 4.1664 \times 10^{-2} \end{aligned}$$

and so

$$2^r k_0 M_0 = (2)\left(\frac{1}{4}\right)\left(4.1664 \times 10^{-2}\right) = 2.0832 \times 10^{-2}.$$

Now,

$$\psi\left(g^{[2k]}(\tau(\mathrm{s}))\right) = \frac{\mathrm{s}^2}{2^{4k+3}},$$

$$\begin{aligned} \Theta_1(\mathrm{s}) &= \sum_{k=0}^{m} \left(\prod_{n=0}^{2k} \eta\left(g^{[n]}(\tau(\mathrm{s}))\right) \right) \left(\frac{1}{\eta\left(g^{[2k]}(\tau(\mathrm{s}))\right)} - 1 \right) \frac{\psi\left(g^{[2k]}(\tau(\mathrm{s}))\right)}{\psi(\tau(\mathrm{s}))} \\ &= \sum_{k=0}^{20} \left(\frac{1}{2}\right)^{2k+1} (1) \frac{\mathrm{s}^2}{2^{4k+3}} \frac{2^3}{\mathrm{s}^2} = \sum_{k=0}^{20} \left(\frac{1}{2}\right)^{2k+1} \frac{1}{2^{4k}} \\ &\approx 0.50794 := \mu_0, \end{aligned}$$

Moreover, for $\mathrm{s} > \ell > 1$*, the left side of (33) is*

$$\begin{aligned} &\limsup_{\mathrm{s}\to\infty} \frac{\int_\ell^{\mathrm{s}} (\mathcal{T}(\mathrm{s}_0,\mathrm{s}) - \mathcal{T}(\mathrm{s}_0,\theta))^\gamma (\mathcal{T}(\mathrm{s}_0,\theta) - \mathcal{T}(\mathrm{s}_0,\ell))^\lambda \frac{\tau^{2r}(\theta)}{\theta^r} q(\theta) \Theta^r(\theta) \mathrm{d}\theta}{(\mathcal{T}(\mathrm{s}_0,\mathrm{s}) - \mathcal{T}(\mathrm{s}_0,\ell))^{\gamma+\lambda-r}} \\ = &\limsup_{\mathrm{s}\to\infty} \frac{\mu_0 \kappa}{4(\mathrm{s}-\ell)^8} \int_\ell^{\mathrm{s}} \frac{(\mathrm{s}-\theta)^4(\theta-\ell)^5}{\theta^2} \mathrm{d}\theta \\ = &\frac{\mu_0}{1120}\kappa. \end{aligned}$$

Therefore, from Corollary 2, it confirms that every positive solution of (36) approaches zero and that $\kappa \gtrapprox 45.934$.

Example 2. *Consider the third-order NDDE*

$$\left(\mathrm{s}\left(\left(x(\mathrm{s}) + \frac{1}{3} x\left(\frac{\mathrm{s}}{2}\right) \right)''' \right)^3 \right)' + \frac{\kappa}{\mathrm{s}^6} x^3\left(\frac{\mathrm{s}}{2}\right) = 0,\ \mathrm{s} > 1. \tag{37}$$

Note that $r = 3,\ a(\mathrm{s}) = \mathrm{s},\ \eta(\mathrm{s}) = 1/3 < 1,\ q(\mathrm{s}) = \kappa/\mathrm{s}^6,\ \kappa > 0, g(\mathrm{s}) = \mathrm{s}/2,$ *and* $\tau(\mathrm{s}) = \mathrm{s}/2$. *Condition (6) is satisfied, where*

$$\int_{\mathrm{s}_0}^{\infty} \int_v^{\infty} \left(\frac{1}{a(u)} \int_u^{\infty} q(\theta) \mathrm{d}\theta \right)^{1/r} \mathrm{d}u \mathrm{d}v = \int_{\mathrm{s}_0}^{\infty} \int_v^{\infty} \left(\frac{1}{u} \int_u^{\infty} \frac{\kappa}{{}^{\backprime}6} \mathrm{d}\theta \right)^{1/3} \mathrm{d}u \mathrm{d}v = \infty.$$

Note that

$$\mathcal{T}(\mathrm{s}_0,\mathrm{s}) = \int_{\mathrm{s}_0}^{\mathrm{s}} \frac{\mathrm{d}\theta}{a^{1/r}(\theta)} = \frac{3}{2}\left(\mathrm{s}^{2/3} - 1\right).$$

We may choose $\gamma = 4,\ \lambda = 5$, *then*

$$\begin{aligned} M_0 &= \sum_{j=0}^{r+1} \binom{r+1}{r-j+1} (-1)^{r-j+1} \gamma^{r-j+1} \lambda^j \frac{\Gamma(\gamma + j - r)\Gamma(\lambda - j + 1)}{\Gamma(\gamma + \lambda - r + 1)} \\ &= \sum_{j=0}^{3+1} C_{3+1}^{3-j+1} (-1)^{3-j+1} 4^{3-j+1} 5^j \frac{\Gamma(4 + j - 3)\Gamma(5 - j + 1)}{\Gamma(4 + 5 - 3 + 1)} = 27.5 \end{aligned}$$

and so

$$2^r k_0 M_0 = 2^3 \frac{1}{(4)^4} (27.5) = 0.85938.$$

Now,

$$\psi\left(g^{[2k]}(\tau(\mathrm{s}))\right) = \frac{9}{10} \frac{\mathrm{s}^{5/3}}{2^{(10k+5)/3}},$$

$$\begin{aligned} \Theta_1(\mathrm{s}) &= \sum_{k=0}^{m} \left(\prod_{n=0}^{2k} \eta\left(g^{[n]}(\tau(\mathrm{s}))\right) \right) \left(\frac{1}{\eta\left(g^{[2k]}(\tau(\mathrm{s}))\right)} - 1 \right) \frac{\psi\left(g^{[2k]}(\tau(\mathrm{s}))\right)}{\psi(\tau(\mathrm{s}))} \\ &= \sum_{k=0}^{20} \left(\frac{1}{3}\right)^{2k+1} (2) \frac{\mathrm{s}^{5/3}}{2^{(10k+5)/3}} \frac{2^{5/3}}{\mathrm{s}^{5/3}} = \sum_{k=0}^{20} \left(\frac{1}{3}\right)^{2k+1} (2) \frac{1}{2^{10k/3}} \\ &\approx 0.67410 := \mu_0, \end{aligned}$$

and, for $\mathrm{s} > \ell > 1$, *the left side of (33) takes*

$$\begin{aligned} &\limsup_{\mathrm{s}\to\infty} \frac{\int_\ell^{\mathrm{s}} (\mathcal{T}(\mathrm{s}_0, \mathrm{s}) - \mathcal{T}(\mathrm{s}_0, \theta))^\gamma (\mathcal{T}(\mathrm{s}_0, \theta) - \mathcal{T}(\mathrm{s}_0, \ell))^\lambda \frac{\tau^{2r}(\theta)}{\theta^r} q(\theta) \Theta^r(\theta) \mathrm{d}\theta}{(\mathcal{T}(\mathrm{s}_0, \mathrm{s}) - \mathcal{T}(\mathrm{s}_0, \ell))^{\gamma+\lambda-r}} \\ =\ &\limsup_{\mathrm{s}\to\infty} \left(\frac{3}{2}\right)^9 \frac{\mu_0^3 \kappa}{2^6 \left(\mathrm{s}^{2/3} - \ell^{2/3}\right)^6} \int_\ell^{\mathrm{s}} \frac{\left(\mathrm{s}^{2/3} - \theta^{2/3}\right)^4 \left(\theta^{2/3} - \ell^{2/3}\right)^5}{\theta^3} \mathrm{d}\theta \\ =\ &\left(\frac{3}{2}\right)^9 \frac{\mu_0^3 \kappa}{2^6 (20)}. \end{aligned}$$

Hence, by Corollary 2, it confirms that every nonoscillatory solution of (37) converges to zero provided that $\kappa \gtrapprox 93.412$.

Remark 1. *Consider the NDDE*

$$\left(x(\mathrm{s}) + \frac{1}{2} x\left(\frac{\mathrm{s}}{4}\right) \right)''' + \frac{\kappa}{\mathrm{s}^3} x\left(\frac{\mathrm{s}}{2}\right) = 0,\ \mathrm{s} > 1. \tag{38}$$

We find that Theorem 1 in [42] and Theorem 2 in [44] cannot be applied to this equation because $\tau(\mathrm{s}) = \mathrm{s}/2 > g(\mathrm{s}) = \mathrm{s}/4$. *While using the results we obtained, we find that the solutions of (38) are oscillatory or tend to zero. Therefore, our results improve the results in [42,44].*

Remark 2. *We note that additional conditions were mentioned in [42,44], including the composition condition* $(\tau \circ g = g \circ \tau)$, *which is a harsh condition on the delay functions, while we were able to dispense with these conditions in our results. We also note that the results we obtained are considered an expansion and extension of both [41,43], as we find that in [41], (1) was studied when* $0 \le \eta(\mathrm{s}) \le \eta < 1$, *and we find in [43] that Equation (1) was studied when* $a = 1$ *and* $\eta(\mathrm{s}) \ge 1$, *while in our study, Equation (1) was studied when* $0 \le \eta(\mathrm{s}) \le \eta_0 < \infty$.

Remark 3. *From Example 1 in [41], we find that every nonoscillatory solution of (37) converges to zero provided that* $\kappa > 9^3/2$. *However, by using our criterion (33), we find that every nonoscillatory* *solution of (37) converges to zero provided that* $\kappa > 93.412$*. Hence, our findings enhance those presented in [41].*

4. Conclusions

It is known that studying the solution behavior of odd-order differential equations is more difficult than studying even-order equations. This is due to several reasons, one of which is the ability to obtain relationships between the different derivatives of positive solutions, as well as the multiplicity of derivative possibilities for positive solutions. Based on the improved relationship between x and z that was derived in [45], we obtained new relationships between x and z. The new relationship takes into account the cases $\eta \leq 1$ and $\eta > 1$, and this was not usual in previous studies of neutral third-order differential equations. Using the appropriate Riccati substitution, we obtained the Riccati inequality and then applied the Philos approach to obtain new criteria for the asymptotic behavior of the studied equation. The new criteria ensure that all nonoscillatory solutions converge to zero. The results provided in this work improve and extend the well-known results in previous works; for instance, see [41–44]. It would also be of interest to use this approach to study the equation

$$\left(a(\mathrm{s})\left((x(\mathrm{s}) + \eta(\mathrm{s})x(g(\mathrm{s})))^{(n-1)}\right)^{r}\right)' + q(\mathrm{s})x^{r}(\tau(\mathrm{s})) = 0,$$

where $n \geq 3$.

Author Contributions: Conceptualization, H.S.A., W.A., A.M., O.M. and E.M.E.; Methodology, H.S.A., W.A., A.M., O.M. and E.M.E.; Investigation, H.S.A., W.A., A.M., O.M. and E.M.E.; Writing—original draft, H.S.A. and A.M.; Writing—review & editing, W.A., O.M. and E.M.E. All authors have read and agreed to the published version of the manuscript.

Funding: This research is funded by Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2023R157), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Data Availability Statement: Data are contained within the article.

Acknowledgments: The authors gratefully acknowledge the editor and the anonymous reviewers for their comments that improved the final version of the manuscript. Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2023R157), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Hale, J.K. Functional differential equations. In *Oxford Applied Mathematical Sciences*; Springer: New York, NY, USA, 1971; Volume 3.
2. Rihan, F.A. *Delay Differential Equations and Applications to Biology*; Springer Nature Singapore Pte Ltd.: Singapore, 2021.
3. Ladde, G.S.; Lakshmikantham, V.; Zhang, B.G. *Oscillation Theory of Differential Equations with Deviating Arguments*; Marcel Dekker: New York, NY, USA, 1987.
4. Gyori, I.; Ladas, G. *Oscillation Theory of Delay Differential Equations with Applications*; Clarendon Press: Oxford, UK, 1991.
5. Erbe, L.H.; Kong, Q.; Zhong, B.G. *Oscillation Theory for Functional Differential Equations*; Marcel Dekker: New York, NY, USA, 1995.
6. Agarwal, R.P.; Grace, S.R.; O'Regan, D. *Oscillation Theory for Second Order Linear, Half-Linear, Superlinear and Sublinear Dynamic Equations*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 2002.
7. Dzurina, J.; Jadlovska, I. A sharp oscillation result for second-order half-linear noncanonical delay differential equations. *Electron. J. Qual. Theory Differ. Equ.* **2020**, *46*, 1–14. [CrossRef]
8. Dzurina, J.; Jadlovska, I. Kneser-type oscillation criteria for second-order half-linear delay differential equations. *Appl. Math. Comput*. **2020**, *380*, 125289.
9. Jadlovska, I. Oscillation criteria of Kneser-type for second-order half-linear advanced differential equations. *Appl. Math. Lett*. **2020**, *106*, 106354. [CrossRef]
10. Jadlovska, I. New criteria for sharp oscillation of second-order neutral delay differential equations. *Mathematics* **2021**, *9*, 2089. [CrossRef]
11. Dzurina, J.; Grace, S.R.; Jadlovska, I.; Li, T. Oscillation criteria for second-order Emden-Fowler delay differential equations with a sublinear neutral term. *Math. Nachr.* **2020**, *5*, 910–922. [CrossRef]

12. Thandapani, E.; Tamilvanan, S.; Jambulingam, E.S. Oscillation of third order half linear neutral delay differential equations. *Int. J. Pure Appl. Math.* **2012**, *77*, 359–368.
13. Bohner, M.; Grace, S.R.; Jadlovska, I. Oscillation criteria for second-order neutral delay differential equations. *Electron. J. Qual. Theory Differ. Equ.* **2017**, *60*, 1–12. [CrossRef]
14. Bohner, M.; Grace, S.R.; Jadlovská, I. Sharp oscillation criteria for second-order neutral delay differential equations. *Math. Methods Appl. Sci.* **2020**, *17*, 10041–10053. [CrossRef]
15. Moaaz, O.; Ramos, H.; Awrejcewicz, J. Second-order Emden–Fowler neutral differential equations: A new precise criterion for oscillation. *Appl. Math. Lett.* **2021**, *118*, 107172. [CrossRef]
16. Almarri, B.; Moaaz, O.; Anis, M.; Qaraad, B. Third-Order Neutral Differential Equation with a Middle Term and Several Delays: Asymptotic Behavior of Solutions. *Axioms* **2023**, *12*, 166. [CrossRef]
17. Gopal, T.; Ayyappan, G.; Graef, J.R.; Thandapani, E. Oscillatory and asymptotic behavior of solutions of third-order quasi-linear neutral difference equations. *Math. Slovaca* **2022**, *72*, 411–418. [CrossRef]
18. Jadlovska, I.; Chatzarakis, G.E.; Dzurina, J.; Grace, S.R. On sharp oscillation criteria for general third-order delay differential equations. *Mathematics* **2021**, *9*, 1675. [CrossRef]
19. Thandapani, E.; Göktürk, B.; Özdemir, O.; Tunç, E. Oscillatory behavior of semi-canonical nonlinear neutral differential equations of third-order via comparison principles. *Qualit. Theo. Dyn. Syst.* **2023**, *22*, 30. [CrossRef]
20. Dzurina, J.; Thapani, E.; Tamilvanan, S. Oscillation of solutions to third order half-linear neutral differential equations. *Electron. J. Differ. Equ.* **2012**, *2012*, 29.
21. Su, M.; Xu, Z. Oscillation criteria of certain third order neutral differential equations. *Differ. Equ. Appl.* **2012**, *4*, 221–232. [CrossRef]
22. Alzabut, J.; Agarwal, R.P.; Grace, S.R.; Jonnalagadda, J.M.; Selvam, A.G.M.; Wang, C. A survey on the oscillation of solutions for fractional difference equations. *Mathematics* **2022**, *10*, 894. [CrossRef]
23. Graef, J.R.; Özdemir, O.; Kaymaz, A.; Tunc, E. Oscillation of damped second-order linear mixed neutral differential equations. *Monatsh. Math.* **2021**, *194*, 85–104. [CrossRef]
24. Yang, D.; Bai, C. On the oscillation criteria for fourth-order p-Laplacian differential equations with middle term. *J. Funct. Space* **2021**, *2021*, 1–10. [CrossRef]
25. Zeng, Y.; Li, Y.; Luo, L.; Luo, Z. Oscillation of generalized neutral delay differential equations of Emden-Fowler type with with damping. *J. Zhejiang Univ.-Sci. A* **2016**, *43*, 394–400.
26. Santra, S.S.; Scapellato, A. Some conditions for the oscillation of second-order differential equations with several mixed delays. *J. Fix. Point. Theory. A* **2022**, *24*, 18. [CrossRef]
27. Santra, S.S.; El-Nabulsi, R.A.; Khedher, K.M. Oscillation of second-order differential equations with multiple and mixed delays under a canonical operator. *Mathematics* **2021**, *9*, 1323. [CrossRef]
28. Santra, S.S.; Khedher, K.M.; Yao, S.W. New aspects for oscillation of differential systems with mixed delays and impulses. *Symmetry* **2021**, *13*, 780. [CrossRef]
29. Tunç, E.; Özdemir, O. Comparison theorems on the oscillation of even order nonlinear mixed neutral differential equations. *Math. Method. Appl. Sci.* **2023**, *46*, 631–640. [CrossRef]
30. Hassan, T.S.; Sun, Y.; Menaem, A.A. Improved oscillation results for functional nonlinear dynamic equations of second order. *Mathematics* **2020**, *8*, 1897. [CrossRef]
31. O'Regan, D.; Hassan, T.S. Oscillation criteria for solutions to nonlinear dynamic equations of higher order. *Hacet. J. Math. Stat.* **2016**, *45*, 417–427. [CrossRef]
32. Hassan, A.M.; Ramos, H.; Moaaz, O. Second-Order Dynamic Equations with Noncanonical Operator: Oscillatory Behavior. *Fractal Fract.* **2023**, *7*, 134. [CrossRef]
33. Chatzarakis, G.E.; Grace, S.R.; Jadlovská, I.; Li, T.; Tun ç, E. Oscillation criteria for third-order Emden–Fowler differential equations with unbounded neutral coefficients. *Complexity* **2019**, *2019*, 5691758. [CrossRef]
34. Dzurina, J.; Grace, S.R.; Jadlovska, I. On nonexistence of Kneser solutions of third-order neutral delay differential equations. *Appl. Math. Lett.* **2019**, **88**, 193–200. [CrossRef]
35. Moaaz, O.; Alnafisah, Y. An improved approach to investigate the oscillatory properties of third-order neutral differential equations. *Mathematics* **2023**, *11*, 2290. [CrossRef]
36. Moaaz, O.; Awrejcewicz, J.; Muhib, A. Establishing new criteria for oscillation of odd-order nonlinear differential equations. *Mathematics* **2020**, **8**, 937. [CrossRef]
37. Moaaz, O.; Dassios, I.; Muhsin, W.; Muhib, A. Oscillation theory for non-linear neutral delay differential equations of third order. *Appl. Sci.* **2020**, *10*, 4855. [CrossRef]
38. Moaaz, O.; Qaraad, B.; El-Nabulsi, R.A.; Bazighifan, O. New results for kneser solutions of third-order nonlinear neutral differential equations. *Mathematics* **2020**, *8*, 686. [CrossRef]
39. Muhib, A.; Abdeljawad, T.; Moaaz, O.; Elabbasy, E.M. Oscillatory properties of odd-order delay differential equations with distribution deviating arguments. *Appl. Sci.* **2020**, *10*, 5952. [CrossRef]
40. Pátíková, Z.; Fišnarová, S. Use of the modified Riccati technique for neutral half-linear differential equations. *Mathematics* **2021**, *9*, 235. [CrossRef]
41. Baculíková, B.; Džurina, J. Oscillation of third-order neutral differential equations. *Math. Comput. Model.* **2010**, *52*, 215–226. [CrossRef]

42. Thandapani, E.; Li, T. On the oscillation of third-order quasi-linear neutral functional differential equations. *Arch. Math.* **2011**, *47*, 181–199.
43. Graef, J.R.; Tunç, E.; Grace, S. Oscillatory and asymptotic behavior of a third-order nonlinear neutral differential equation. *Opusc. Math.* **2017**, *37*, 839–852. [CrossRef]
44. Kumar, M.S.; Ganesan, V. On the oscillatory behavior of solutions of third order nonlinear neutral differential equations. *Malaya J. Mat.* **2019**, *2019*, 596–599.
45. Moaaz, O.; Cesarano, C.; Almarri, B. An improved relationship between the solution and its corresponding function in neutral fourth-order differential equations and its applications. *Mathematics* **2023**, *11*, 1708. [CrossRef]
46. Kiguradze, I.T.; Chanturia, T.A. *Asymptotic Properties of Solutions of Nonautonomous Ordinary Diferential Equations*; Trans lated from the 1985 Russian original; Kluwer Academic: Dordrecht, The Netherlands, 1993.
47. Philos, C.G. Oscillation theroms for linear differential equations of second order. *Arch. Math.* **1989**, *53*, 482–492. [CrossRef]