mathematics

Article

Fixed Point Theory in Bicomplex Metric Spaces: A New Framework with Applications

Badriah Alamri

Department of Mathematics and Statistics, College of Science, University of Jeddah, Jeddah 23218, Saudi Arabia; baalamri@uj.edu.sa

Abstract: This paper investigates the existence of common fixed points for mappings satisfying generalized rational type contractive conditions in the framework of bicomplex valued metric spaces. Our findings extend well-established results in the existing literature. As an application of our leading result, we explore the existence and uniqueness of solutions of the Volterra integral equation of the second kind.

Keywords: common fixed point; bicomplex valued metric space; generalized contractive condition; Volterra integral equation

MSC: 46S40; 47H10; 54H25

Citation: Alamri, B. Fixed Point Theory in Bicomplex Metric Spaces: A New Framework with Applications. *Mathematics* **2024**, *12*, 1770. https://doi.org/10.3390/math12111770

Academic Editor: Janusz Brzdęk

Received: 29 April 2024
Revised: 29 May 2024
Accepted: 1 June 2024
Published: 6 June 2024

1. Introduction

The concept of complex numbers emerged in the 17th century, with significant contributions from Leonhard Euler, a mathematician who introduced the symbol 'i' to represent the imaginary unit ($\sqrt{-1}$) with the property $i^2 = -1$. While Euler's notation gained widespread adoption, earlier work by mathematicians like Gottfried Wilhelm Leibniz also played a role in the development of complex analysis.

Building upon complex numbers, bicomplex numbers were introduced by Segre [1] and offer a commutative alternative to quaternions and generalize complex numbers. Price [2] and Waugh [3] provide more detailed explorations of bicomplex numbers. Azam et al. [4] introduced the concept of complex valued metric spaces (CVMSs) as a special case of cone metric spaces, enabling the study of rational expressions not definable in the latter. This concept arises from the limitations of cone metric spaces, which rely on underlying Banach spaces that are not division rings. CVMSs allows for generalizations of fixed point theory results involving divisions. In 2017, Choi et al. [5] combined bicomplex numbers with CVMSs, introducing bicomplex valued metric spaces (bi-CVMSs) and establishing common fixed point theorems. Thereafter, Jebril et al. [6] further explored the application of bi-CVMSs by investigating the existence of common fixed points for mappings exhibiting specific contractive inequalities defined by rational expressions. In due course, Beg et al. [7] contributed by establishing fixed point results using an extrapolation technique. Gnanaprakasam et al. [8] further expanded the field by investigating contractive conditions in bi-CVMSs and their application to solving linear equations. Abdou [9] presented a novel approach to solving systems of Fredholm integral equations. Their method leverages common fixed point theorems established within the context of bi-CVMSs. Recent research in bicomplex valued metric spaces has seen a surge in activity, with works by Gu et al. [10], Tassaddiq et al. [11], Albargi et al. [12], Abdou [13], Gnanaprakasam et al. [14], Gurusamy [15], and Ramaswamy et al. [16] exploring fixed and common fixed point results, applications to integral equations, and other related structures. For a deeper understanding in this direction, we recommend consulting references [17–22].

In this research article, we establish common fixed point theorems for mappings satisfying generalized contractive conditions in the framework of bicomplex valued metric spaces. To demonstrate the originality of our main theorem, we present a novel example. Finally, we showcase the applicability of our findings by exploring the existence and uniqueness of solutions for Volterra integral equations of the second kind.

2. Preliminaries

We represent the sets of real numbers, complex numbers, and bicomplex numbers as $\mathbb{C}_0$, $\mathbb{C}_1$, and $\mathbb{C}_2$, respectively. A bicomplex number, as introduced by Segre [1], can be expressed as

$$\varrho = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2$$

where a_1, a_2, a_3, a_4 belong to $\mathbb{C}_0$, i_1 and i_2 are independent imaginary units satisfying $i_1^2 = i_2^2 = -1$ and $i_1 i_2 = i_2 i_1$, and $\mathbb{C}_2$ is defined as

$$\mathbb{C}_2 = \{\varrho : \varrho = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2 : a_1, a_2, a_3, a_4 \in \mathbb{C}_0\}$$

that is,

$$\mathbb{C}_2 = \{\varrho : \varrho = z_1 + i_2 z_2 : z_1, z_2 \in \mathbb{C}_1\}$$

where $z_1 = a_1 + a_2 i_1 \in \mathbb{C}_1$ and $z_2 = a_3 + a_4 i_1 \in \mathbb{C}_1$. If $\varrho = z_1 + i_2 z_2$ and $\hbar = \omega_1 + i_2 \omega_2$, then the sum is

$$\varrho \pm \hbar = (z_1 + i_2 z_2) \pm (\omega_1 + i_2 \omega_2) = (z_1 \pm \omega_1) + i_2 (z_2 \pm \omega_2)$$

and the product is

$$\varrho \cdot \hbar = (z_1 + i_2 z_2) \cdot (\omega_1 + i_2 \omega_2) = (z_1 \omega_1 - z_2 \omega_2) + i_2 (z_1 \omega_2 + z_2 \omega_1).$$

There are four idempotent members in $\mathbb{C}_2$, which are $0, 1$, $e_1 = \frac{1 + i_1 i_2}{2}$, and $e_2 = \frac{1 - i_1 i_2}{2}$. Notably, e_1 and e_2 are non-trivial idempotents satisfying $e_1 + e_2 = 1$ and $e_1 e_2 = 0$. This unique property allows any bicomplex number $z_1 + i_2 z_2$ to be expressed as a specific linear combination of e_1 and e_2, i.e.,

$$\varrho = z_1 + i_2 z_2 = (z_1 - i_1 z_2) e_1 + (z_1 + i_1 z_2) e_2.$$

This decomposition of ϱ into a linear combination of idempotents (e_1 and e_2) with complex coefficients $\varrho_1 = (z_1 - i_1 z_2)$ and $\varrho_2 = (z_1 + i_1 z_2)$ is analogous to the idempotent characterization observed in other mathematical structures. We can refer to ϱ_1 and ϱ_2 as the idempotent components of ϱ. A bicomplex number ϱ denoted by $\varrho = z_1 + i_2 z_2$ is considered invertible within $\mathbb{C}_2$ if there exists another bicomplex number $\hbar$ such that their product $\varrho\hbar$ equals the identity element 1. This element $\hbar$ is then recognized as the multiplicative inverse of ϱ. Also, ϱ is recognized as the multiplicative inverse of $\hbar$.

An element $\varrho = z_1 + i_2 z_2 \in \mathbb{C}_2$ is nonsingular iff $\left|z_1^2 + z_2^2\right| \neq 0$ and singular iff $\left|z_1^2 + z_2^2\right| = 0$. The inverse of ϱ is defined as

$$\varrho^{-1} = \hbar = \frac{z_1 - i_2 z_2}{z_1^2 + z_2^2}.$$

Zero is the at-most member in $\mathbb{C}_0$ that does not possess a multiplicative inverse, and in $\mathbb{C}_1$, $0 = 0 + i0$ is the at-most member that does not possess a multiplicative inverse. We represent the set of singular members of $\mathbb{C}_0$ and $\mathbb{C}_1$ by $\aleph_0$ and $\aleph_1$ in this order. There are many members in $\mathbb{C}_2$ that do not have a multiplicative inverse. We represent this set by $\aleph_2$, and evidently, $\aleph_0 = \aleph_1 \subset \aleph_2$.

A bicomplex number $\varrho = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2 \in \mathbb{C}_2$ is said to be degenerated if the matrix

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}_{2\times 2}$$

is degenerated. In this way ϱ^{-1} exists, and it is degenerated. Define a norm $\|\cdot\| : \mathbb{C}_2 \to \mathbb{C}_0$ as follows:

$$\begin{aligned}
\|\varrho\| &= \|z_1 + i_2 z_2\| = \left\{|z_1|^2 + |z_2|^2\right\}^{\frac{1}{2}} \\
&= \left[\frac{|(z_1 - i_1 z_2)|^2 + |(z_1 + i_1 z_2)|^2}{2}\right]^{\frac{1}{2}} \\
&= \left(a_1^2 + a_2^2 + a_3^2 + a_4^2\right)^{\frac{1}{2}},
\end{aligned}$$

where $\varrho = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2 = z_1 + i_2 z_2 \in \mathbb{C}_2$.

The space $\mathbb{C}_2$ equipped with the norm $\|\cdot\|$ is a Banach space. If $\varrho, \hbar \in \mathbb{C}_2$, then

$$\|\varrho\hbar\| \leq \sqrt{2}\|\varrho\|\|\hbar\|$$

holds instead of

$$\|\varrho\hbar\| \leq \|\varrho\|\|\hbar\|.$$

Therefore, $\mathbb{C}_2$ is not a Banach algebra. Let $\varrho = z_1 + i_2 z_2, \hbar = \omega_1 + i_2 \omega_2 \in \mathbb{C}_2$; then we define

$$\varrho \preceq_{i_2} \hbar \Leftrightarrow \text{ Re}(z_1) \preceq \text{Re}(\omega_1) \text{ and } \text{Im}(z_2) \preceq \text{Im}(\omega_2).$$

If

$$\varrho \preceq_{i_2} \hbar$$

then one of the followings conditions holds:

$$\begin{aligned}
\text{(i) } z_1 &= \omega_1,\ z_2 \prec \omega_2, \\
\text{(ii) } z_1 &\prec \omega_1,\ z_2 = \omega_2, \\
\text{(iii) } z_1 &\prec \omega_1,\ z_2 \prec \omega_2, \\
\text{(iv) } z_1 &= \omega_1,\ z_2 = \omega_2.
\end{aligned}$$

Specifically, $\varrho \precnsim_{i_2} \hbar$ if $\varrho \prec_{i_2} \hbar$ and $\varrho \neq \hbar$; that is, one of (i), (ii), or (iii) holds. Also, $\varrho \prec_{i_2} \hbar$ if only condition (iii) is satisfied. For $\varrho, \hbar \in \mathbb{C}_2$, we can prove the following:

(i) $\varrho \preceq_{i_2} \hbar \Longrightarrow \|\varrho\| \leq \|\hbar\|$;
(ii) $\|\varrho + \hbar\| \leq \|\varrho\| + \|\hbar\|$;
(iii) $\|a\varrho\| \leq a\|\varrho\|$, where a is a non-negative real number;
(iv) $\|\varrho\hbar\| \leq \sqrt{2}\|\varrho\|\|\hbar\|$;
(v) $\|\varrho^{-1}\| = \|\varrho\|^{-1}$ if ϱ is a degenerated bicomplex number with $0 \prec \varrho$;
(vi) $\|\frac{\varrho}{\hbar}\| = \frac{\|\varrho\|}{\|\hbar\|}$ if $\hbar$ is a degenerated bicomplex number.

In their work, Azam et al. [4] introduced the concept of a complex valued metric space (CVMS).

Definition 1 ([4])**.** *Let $\mathcal{A}$ be a non-empty set and $\preceq$ be a partial order defined on $\mathcal{A}$. Additionally, let $d : \mathcal{A} \times \mathcal{A} \to \mathbb{C}_1$ be a mapping satisfying*

(i) $0 \preceq d(\mathfrak{q}, \kappa)$ *and* $d(\mathfrak{q}, \kappa) = 0$ *if and only if* $\mathfrak{q} = \kappa$;
(ii) $d(\mathfrak{q}, \kappa) = d(\kappa, \mathfrak{q})$;
(iii) $d(\mathfrak{q}, \kappa) \preceq d(\mathfrak{q}, \nu) + d(\nu, \kappa)$ *for all* $\mathfrak{q}, \kappa, \nu \in \mathcal{A}$.

Then, $(\mathcal{A}, \preceq, d)$ is a CVMS.

Choi et al. [5] introduced the concept of a bicomplex valued metric space (bi-CVMS) with the following definition:

Definition 2 ([5]). *Let* $\mathcal{A} \neq \emptyset$, $\preceq_{i_2}$ *be a partial order defined on* $\mathcal{A}$, *and* $d : \mathcal{A} \times \mathcal{A} \to \mathbb{C}_2$ *be a mapping satisfying*

- *(i)* $0 \preceq_{i_2} d(\mathfrak{q}, \kappa)$ *and* $d(\mathfrak{q}, \kappa) = 0$ *if and only if* $\mathfrak{q} = \kappa$;
- *(ii)* $d(\mathfrak{q}, \kappa) = d(\kappa, \mathfrak{q})$;
- *(iii)* $d(\mathfrak{q}, \kappa) \preceq_{i_2} d(\mathfrak{q}, \nu) + d(\nu, \kappa)$ *for all* $\mathfrak{q}, \kappa, \nu \in \mathcal{A}$.

Then, $(\mathcal{A}, \preceq_{i_2}, d)$ *is a bi-CVMS.*

Example 1 ([5]). *Let* $\mathcal{A} = \mathbb{C}_2$ *and* $\mathfrak{q}, \kappa \in \mathcal{A}$. *Define* $d : \mathcal{A} \times \mathcal{A} \to \mathbb{C}_2$ *by*

$$d(\mathfrak{q}, \kappa) = |z_1 - \omega_1| + i_2|z_2 - \omega_2|$$

where $\mathfrak{q} = z_1 + i_2 z_2$ *and* $\kappa = \omega_1 + i_2\omega_2 \in \mathbb{C}_2$. *Then* $(\mathcal{A}, \preceq_{i_2}, d)$ *is a bi-CVMS.*

Definition 3 ([5]). *For a bi-CVMS* $(\mathcal{A}, \preceq_{i_2}, d)$:

- *(i)* *A sequence* $\{\mathfrak{q}_r\} \subseteq \mathcal{A}$ *is said to be a convergent sequence and converges to a point* $\mathfrak{q}$ *if for any* $0 \prec_{i_2} \ell \in \mathbb{C}_2$ *there is a natural number* $r_0 \in \mathbb{N}$ *such that* $d(\mathfrak{q}_r, \mathfrak{q}) \prec_{i_2} \ell$ *for all* $r > r_0$, *and we write* $\lim_{r\to\infty} \mathfrak{q}_r = \mathfrak{q}$ *or* $\mathfrak{q}_r \to \mathfrak{q}$ *as* $r \to \infty$.
- *(ii)* *A sequence* $\{\mathfrak{q}_r\} \subseteq \mathcal{A}$ *is said to be a Cauchy sequence in* $(\mathcal{A}, d)$ *if for any* $0 \prec_{i_2} \ell \in \mathbb{C}_2$ *there is a natural number* $r_0 \in \mathbb{N}$ *such that* $d(\mathfrak{q}_r, \mathfrak{q}_{r+m}) \prec_{i_2} \ell$ *for all* $r, m \in \mathbb{N}$ *and* $r > r_0$.
- *(iii)* *If every Cauchy sequence in* $\mathcal{A}$ *is convergent in* $\mathcal{A}$, *then* $(\mathcal{A}, \preceq_{i_2}, d)$ *is said to be a complete bi-CVMS.*

Lemma 1 ([7]). *Let* $(\mathcal{A}, \preceq_{i_2}, d)$ *be a bi-CVMS and let* $\{\mathfrak{q}_r\} \subseteq \mathcal{A}$. *Then* $\{\mathfrak{q}_r\}$ *converges to* $\mathfrak{q}$ *if and only if* $\|d(\mathfrak{q}_r, \mathfrak{q})\| \to 0$ *as* $r \to \infty$.

Lemma 2 ([7]). *Let* $(\mathcal{A}, \preceq_{i_2}, d)$ *be a bi-CVMS and let* $\{\mathfrak{q}_r\} \subseteq \mathcal{A}$. *Then* $\{\mathfrak{q}_r\}$ *is a Cauchy sequence if and only if* $\|d(\mathfrak{q}_r, \mathfrak{q}_{r+m})\| \to 0$ *as* $r \to \infty$, *where* $m \in \mathbb{N}$.

3. Main Result

In the following, we state a Proposition from Tassdique et al. [11], which will be instrumental in our subsequent results.

Proposition 1 ([11]). *Let* $(\mathcal{A}, \preceq_{i_2}, d)$ *be a bi-CVMS and* $\mathcal{V}, \mathcal{W} : (\mathcal{A}, \preceq_{i_2}, d) \to (\mathcal{A}, \preceq_{i_2}, d)$. *Let* $\mathfrak{q}_0 \in \mathcal{A}$. *Define the sequence* $\{\mathfrak{q}_r\}$ *by*

$$\mathfrak{q}_{2r+1} = \mathcal{V}\mathfrak{q}_{2r} \text{ and } \mathfrak{q}_{2r+2} = \mathcal{W}\mathfrak{q}_{2r+1} \tag{1}$$

for all $r = 0, 1, 2, \ldots$

Assume that there exists $\mu_1 : \mathcal{A}\times\mathcal{A}\to[0, 1)$ *satisfying*

$$\mu_1(\mathcal{W}\mathcal{V}\mathfrak{q}, \kappa) \leq \mu_1(\mathfrak{q}, \kappa) \text{ and } \mu_1(\mathfrak{q}, \mathcal{V}\mathcal{W}\kappa) \leq \mu_1(\mathfrak{q}, \kappa)$$

for all $\mathfrak{q}, \kappa \in \mathcal{A}$. *Then*

$$\mu_1(\mathfrak{q}_{2r}, \kappa) \leq \mu_1(\mathfrak{q}_0, \kappa) \text{ and } \mu_1(\mathfrak{q}, \mathfrak{q}_{2r+1}) \leq \mu_1(\mathfrak{q}, \mathfrak{q}_1)$$

for all $\mathfrak{q}, \kappa \in \mathcal{A}$ *and* $r = 0, 1, 2, \ldots$

Theorem 1. *Let* $(\mathcal{A}, \preceq_{i_2}, d)$ *be a complete bi-CVMS and* $\mathcal{V}, \mathcal{W} : \mathcal{A} \to \mathcal{A}$. *If there exist the mappings* $\mu_i : \mathcal{A}\times\mathcal{A}\to[0, 1)$, *where* $i = 1, 2, \ldots 7$, *such that*

- *(a)* $\mu_i(\mathcal{W}\mathcal{V}\mathfrak{q}, \kappa) \leq \mu_i(\mathfrak{q}, \kappa)$ *and* $\mu_i(\mathfrak{q}, \mathcal{V}\mathcal{W}\kappa) \leq \mu_i(\mathfrak{q}, \kappa)$;
- *(b)* $\mu_1(\mathfrak{q}, \kappa) + \sqrt{2}(\mu_2(\mathfrak{q}, \kappa) + \mu_3(\mathfrak{q}, \kappa)) + 2\sqrt{2}(\mu_4(\mathfrak{q}, \kappa) + \mu_5(\mathfrak{q}, \kappa)) + 2(\mu_6(\mathfrak{q}, \kappa) + \mu_7(\mathfrak{q}, \kappa)) < 1$;

(c)

$$d(\mathcal{V}\mathfrak{q},\mathcal{W}\kappa) \precsim_{i_2} \mu_1(\mathfrak{q},\kappa)d(\mathfrak{q},\kappa)+\mu_2(\mathfrak{q},\kappa)\frac{d(\mathfrak{q},\mathcal{V}\mathfrak{q})d(\kappa,\mathcal{W}\kappa)}{1+d(\mathfrak{q},\kappa)}+\mu_3(\mathfrak{q},\kappa)\frac{d(\kappa,\mathcal{V}\mathfrak{q})d(\mathfrak{q},\mathcal{W}\kappa)}{1+d(\mathfrak{q},\kappa)}$$
$$+\mu_4(\mathfrak{q},\kappa)\frac{d(\mathfrak{q},\mathcal{V}\mathfrak{q})d(\mathfrak{q},\mathcal{W}\kappa)}{1+d(\mathfrak{q},\kappa)}+\mu_5(\mathfrak{q},\kappa)\frac{d(\kappa,\mathcal{V}\mathfrak{q})d(\kappa,\mathcal{W}\kappa)}{1+d(\mathfrak{q},\kappa)}$$
$$+\mu_6(\mathfrak{q},\kappa)\frac{d(\mathfrak{q},\mathcal{W}\kappa)[1+d(\mathfrak{q},\mathcal{V}\mathfrak{q})]}{1+d(\mathfrak{q},\kappa)}+\mu_7(\mathfrak{q},\kappa)\frac{d(\kappa,\mathcal{V}\mathfrak{q})[1+d(\kappa,\mathcal{W}\kappa)]}{1+d(\mathfrak{q},\kappa)} \tag{2}$$

for all $\mathfrak{q},\kappa\in\mathcal{A}$*, then* $\mathcal{V}$ *and* $\mathcal{W}$ *possess a unique common fixed point.*

Proof. Consider an arbitrary element $\mathfrak{q}_0$ belonging to the set $\mathcal{A}$. Define a sequence $\{\mathfrak{q}_r\}$ as follows:

$$\mathfrak{q}_{2r+1}=\mathcal{V}\mathfrak{q}_{2r} \text{ and } \mathfrak{q}_{2r+2}=\mathcal{W}\mathfrak{q}_{2r+1}$$

for all $r=0,1,2,\ldots$. By (2), we have

$$d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})=d(\mathcal{V}\mathfrak{q}_{2r},\mathcal{W}\mathfrak{q}_{2r+1}) \precsim_{i_2} \mu_1(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})$$

$$\begin{aligned}
&+\mu_2(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})\frac{d(\mathfrak{q}_{2r},\mathcal{V}\mathfrak{q}_{2r})d(\mathfrak{q}_{2r+1},\mathcal{W}\mathfrak{q}_{2r+1})}{1+d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})}\\
&+\mu_3(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})\frac{d(\mathfrak{q}_{2r+1},\mathcal{V}\mathfrak{q}_{2r})d(\mathfrak{q}_{2r},\mathcal{W}\mathfrak{q}_{2r+1})}{1+d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})}\\
&+\mu_4(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})\frac{d(\mathfrak{q}_{2r},\mathcal{V}\mathfrak{q}_{2r})d(\mathfrak{q}_{2r},\mathcal{W}\mathfrak{q}_{2r+1})}{1+d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})}\\
&+\mu_5(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})\frac{d(\mathfrak{q}_{2r+1},\mathcal{V}\mathfrak{q}_{2r})d(\mathfrak{q}_{2r+1},\mathcal{W}\mathfrak{q}_{2r+1})}{1+d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})}\\
&+\mu_6(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})\frac{d(\mathfrak{q}_{2r},\mathcal{W}\mathfrak{q}_{2r+1})[1+d(\mathfrak{q}_{2r},\mathcal{V}\mathfrak{q}_{2r})]}{1+d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})}\\
&+\mu_7(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})\frac{d(\mathfrak{q}_{2r+1},\mathcal{V}\mathfrak{q}_{2r})[1+d(\mathfrak{q}_{2r+1},\mathcal{W}\mathfrak{q}_{2r+1})]}{1+d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})}\\
=\;&\mu_1(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})\\
&+\mu_2(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})\frac{d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})}{1+d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})}\\
&+\mu_3(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})\frac{d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+1})d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+2})}{1+d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})}\\
&+\mu_4(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})\frac{d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+2})}{1+d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})}\\
&+\mu_5(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})\frac{d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+1})d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})}{1+d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})}\\
&+\mu_6(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})\frac{d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+2})[1+d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})]}{1+d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})}\\
&+\mu_7(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})\frac{d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+1})[1+d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})]}{1+d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})},
\end{aligned}$$

which implies

$$d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2}) \precsim_{i_2} \mu_1(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})$$

$$+\mu_2(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})\frac{d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})}{1+d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})}$$

$$
\begin{aligned}
&+\mu_4(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})\frac{d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+2})}{1+d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})}\\
&+\mu_6(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+2}).
\end{aligned}
$$

This yields

$$
\begin{aligned}
\|d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})\| \leq\ & \mu_1(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})\|d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})\|\\
&+\mu_2(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})\left\|\frac{d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})}{1+d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})}\right\|\\
&+\mu_4(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})\left\|\frac{d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+2})}{1+d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})}\right\|\\
&+\mu_6(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})\|d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})+d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})\|
\end{aligned}
$$

which implies

$$
\begin{aligned}
\|d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})\| \leq\ & \mu_1(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})\|d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})\|\\
&+\sqrt{2}\mu_2(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})\left\|\frac{d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})}{1+d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})}\right\|\|d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})\|\\
&+\sqrt{2}\mu_4(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})\left\|\frac{d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})}{1+d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})}\right\|\|d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})+d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})\|\\
&+\mu_6(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})\|d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})+d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})\|\\
\leq\ & \mu_1(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})\|d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})\|\\
&+\sqrt{2}\mu_2(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})\|d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})\|\\
&+\sqrt{2}\mu_4(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})\|d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})+d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})\|\\
&+\mu_6(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})\|d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})+d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})\|.
\end{aligned}
$$

This implies

$$
\begin{aligned}
\|d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})\| \leq\ & \mu_1(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})\|d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})\|\\
&+\sqrt{2}\mu_2(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})\|d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})\|\\
&+\sqrt{2}\mu_4(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})\|d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})\|\\
&+\sqrt{2}\mu_4(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})\|d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})\|\\
&+\mu_6(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})\|d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})\|\\
&+\mu_6(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})\|d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})\|
\end{aligned}
$$

By Proposition 1, we have

$$
\begin{aligned}
\|d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})\| \leq\ & \mu_1(\mathfrak{q}_0,\mathfrak{q}_1)\|d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})\|\\
&+\sqrt{2}\mu_2(\mathfrak{q}_0,\mathfrak{q}_1)\|d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})\|\\
&+\sqrt{2}\mu_4(\mathfrak{q}_0,\mathfrak{q}_1)\|d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})\|\\
&+\sqrt{2}\mu_4(\mathfrak{q}_0,\mathfrak{q}_1)\|d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})\|\\
&+\mu_6(\mathfrak{q}_0,\mathfrak{q}_1)\|d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})\|\\
&+\mu_6(\mathfrak{q}_0,\mathfrak{q}_1)\|d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})\|
\end{aligned}
$$

which implies

$$
\|d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})\| \leq \frac{\mu_1(\mathfrak{q}_0,\mathfrak{q}_1)+\sqrt{2}\mu_4(\mathfrak{q}_0,\mathfrak{q}_1)+\mu_6(\mathfrak{q}_0,\mathfrak{q}_1)}{1-\sqrt{2}\mu_2(\mathfrak{q}_0,\mathfrak{q}_1)-\sqrt{2}\mu_4(\mathfrak{q}_0,\mathfrak{q}_1)-\mu_6(\mathfrak{q}_0,\mathfrak{q}_1)}\|d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})\|. \tag{3}
$$

Similarly, by (2), we have

$$
\begin{aligned}
d(\mathfrak{q}_{2r+3},\mathfrak{q}_{2r+2}) = d(\mathcal{V}\mathcal{W}\mathfrak{q}_{2r+1},\mathcal{W}\mathfrak{q}_{2r+1}) &\preceq_{i_2} \mu_1(\mathcal{W}\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+1})d(\mathcal{W}\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+1}) \\
&+\mu_2(\mathcal{W}\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+1})\frac{d(\mathcal{W}\mathfrak{q}_{2r+1},\mathcal{V}\mathcal{W}\mathfrak{q}_{2r+1})d(\mathfrak{q}_{2r+1},\mathcal{W}\mathfrak{q}_{2r+1})}{1+d(\mathcal{W}\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+1})} \\
&+\mu_3(\mathcal{W}\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+1})\frac{d(\mathfrak{q}_{2r+1},\mathcal{V}\mathcal{W}\mathfrak{q}_{2r+1})d(\mathcal{W}\mathfrak{q}_{2r+1},\mathcal{W}\mathfrak{q}_{2r+1})}{1+d(\mathcal{W}\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+1})} \\
&+\mu_4(\mathcal{W}\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+1})\frac{d(\mathcal{W}\mathfrak{q}_{2r+1},\mathcal{V}\mathcal{W}\mathfrak{q}_{2r+1})d(\mathcal{W}\mathfrak{q}_{2r+1},\mathcal{W}\mathfrak{q}_{2r+1})}{1+d(\mathcal{W}\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+1})} \\
&+\mu_5(\mathcal{W}\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+1})\frac{d(\mathfrak{q}_{2r+1},\mathcal{V}\mathcal{W}\mathfrak{q}_{2r+1})d(\mathfrak{q}_{2r+1},\mathcal{W}\mathfrak{q}_{2r+1})}{1+d(\mathcal{W}\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+1})} \\
&+\mu_6(\mathcal{W}\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+1})\frac{d(\mathcal{W}\mathfrak{q}_{2r+1},\mathcal{W}\mathfrak{q}_{2r+1})[1+d(\mathcal{W}\mathfrak{q}_{2r+1},\mathcal{V}\mathcal{W}\mathfrak{q}_{2r+1})]}{1+d(\mathcal{W}\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+1})} \\
&+\mu_7(\mathcal{W}\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+1})\frac{d(\mathfrak{q}_{2r+1},\mathcal{V}\mathcal{W}\mathfrak{q}_{2r+1})[1+d(\mathfrak{q}_{2r+1},\mathcal{W}\mathfrak{q}_{2r+1})]}{1+d(\mathcal{W}\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+1})} \\
&= \mu_1(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})d(\mathfrak{q}_{2r+2},\mathfrak{q}_{2r+1}) \\
&+\mu_2(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})\frac{d(\mathfrak{q}_{2r+2},\mathfrak{q}_{2r+3})d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})}{1+d(\mathfrak{q}_{2r+2},\mathfrak{q}_{2r+1})} \\
&+\mu_3(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})\frac{d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+3})d(\mathfrak{q}_{2r+2},\mathfrak{q}_{2r+2})}{1+d(\mathfrak{q}_{2r+2},\mathfrak{q}_{2r+1})} \\
&+\mu_4(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})\frac{d(\mathfrak{q}_{2r+2},\mathfrak{q}_{2r+3})d(\mathfrak{q}_{2r+2},\mathfrak{q}_{2r+2})}{1+d(\mathfrak{q}_{2r+2},\mathfrak{q}_{2r+1})} \\
&+\mu_5(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})\frac{d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+3})d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})}{1+d(\mathfrak{q}_{2r+2},\mathfrak{q}_{2r+1})} \\
&+\mu_6(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})\frac{d(\mathfrak{q}_{2r+2},\mathfrak{q}_{2r+2})[1+d(\mathfrak{q}_{2r+2},\mathfrak{q}_{2r+3})]}{1+d(\mathfrak{q}_{2r+2},\mathfrak{q}_{2r+1})} \\
&+\mu_7(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})\frac{d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+3})[1+d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})]}{1+d(\mathfrak{q}_{2r+2},\mathfrak{q}_{2r+1})} \\
&= \mu_1(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})d(\mathfrak{q}_{2r+2},\mathfrak{q}_{2r+1}) \\
&+\mu_2(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})\frac{d(\mathfrak{q}_{2r+2},\mathfrak{q}_{2r+3})d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})}{1+d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})} \\
&+\mu_5(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})\frac{d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+3})d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})}{1+d(\mathfrak{q}_{2r+2},\mathfrak{q}_{2r+1})} \\
&+\mu_7(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+3}).
\end{aligned}
$$

This implies

$$
\begin{aligned}
\|d(\mathfrak{q}_{2r+2},\mathfrak{q}_{2r+3})\| \leq{}& \mu_1(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})\|d(\mathfrak{q}_{2r+2},\mathfrak{q}_{2r+1})\| \\
&+\mu_2(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})\left\|\frac{d(\mathfrak{q}_{2r+2},\mathfrak{q}_{2r+3})d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})}{1+d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})}\right\| \\
&+\mu_5(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})\left\|\frac{d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+3})d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})}{1+d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})}\right\| \\
&+\mu_7(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})\|d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+3})\|
\end{aligned}
$$

that is,

$$
\begin{aligned}
\|d(\mathfrak{q}_{2r+2},\mathfrak{q}_{2r+3})\| \leq{}& \mu_1(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})\|d(\mathfrak{q}_{2r+2},\mathfrak{q}_{2r+1})\| \\
&+\sqrt{2}\mu_2(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})\|d(\mathfrak{q}_{2r+2},\mathfrak{q}_{2r+3})\|\left\|\frac{d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})}{1+d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})}\right\|
\end{aligned}
$$

$$
\begin{aligned}
&+\sqrt{2}\mu_5(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})\|d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+3})\|\left\|\frac{d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})}{1+d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})}\right\| \\
&+\mu_7(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})\|d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+3})\|
\end{aligned}
$$

which further yields

$$
\begin{aligned}
\|d(\mathfrak{q}_{2r+2},\mathfrak{q}_{2r+3})\| \leq\; & \mu_1(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})\|d(\mathfrak{q}_{2r+2},\mathfrak{q}_{2r+1})\| \\
&+\sqrt{2}\mu_2(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})\|d(\mathfrak{q}_{2r+2},\mathfrak{q}_{2r+3})\| \\
&+\sqrt{2}\mu_5(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})\|d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+3})\| \\
&+\mu_7(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})\|d(\mathfrak{q}_{2r+2},\mathfrak{q}_{2r+3})\| \\
\leq\; & \mu_1(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})\|d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})\| \\
&+\sqrt{2}\mu_2(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})\|d(\mathfrak{q}_{2r+2},\mathfrak{q}_{2r+3})\| \\
&+\sqrt{2}\mu_5(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})(\|d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})\|+\|d(\mathfrak{q}_{2r+2},\mathfrak{q}_{2r+3})\|) \\
&+\mu_7(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})(\|d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})\|+\|d(\mathfrak{q}_{2r+2},\mathfrak{q}_{2r+3})\|).
\end{aligned}
$$

By Proposition 1, we have

$$
\begin{aligned}
\|d(\mathfrak{q}_{2r+2},\mathfrak{q}_{2r+3})\| \leq\; & \mu_1(\mathfrak{q}_0,\mathfrak{q}_1)\|d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})\| \\
&+\sqrt{2}\mu_2(\mathfrak{q}_0,\mathfrak{q}_1)\|d(\mathfrak{q}_{2r+2},\mathfrak{q}_{2r+3})\| \\
&+\sqrt{2}\mu_5(\mathfrak{q}_0,\mathfrak{q}_1)\|d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})\| \\
&+\sqrt{2}\mu_5(\mathfrak{q}_0,\mathfrak{q}_1)\|d(\mathfrak{q}_{2r+2},\mathfrak{q}_{2r+3})\| \\
&+\mu_7(\mathfrak{q}_0,\mathfrak{q}_1)\|d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})\| \\
&+\mu_7(\mathfrak{q}_0,\mathfrak{q}_1)\|d(\mathfrak{q}_{2r+2},\mathfrak{q}_{2r+3})\|
\end{aligned}
$$

which implies

$$
\|d(\mathfrak{q}_{2r+2},\mathfrak{q}_{2r+3})\| \leq \frac{\mu_1(\mathfrak{q}_0,\mathfrak{q}_1)+\sqrt{2}\mu_5(\mathfrak{q}_0,\mathfrak{q}_1)+\mu_7(\mathfrak{q}_0,\mathfrak{q}_1)}{1-\sqrt{2}\mu_2(\mathfrak{q}_0,\mathfrak{q}_1)-\sqrt{2}\mu_5(\mathfrak{q}_0,\mathfrak{q}_1)-\mu_7(\mathfrak{q}_0,\mathfrak{q}_1)}\|d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})\|. \tag{4}
$$

For all $r = 0, 1, 2, \ldots$, let

$$
\lambda = \max\left\{\begin{array}{c}\frac{\mu_1(\mathfrak{q}_0,\mathfrak{q}_1)+\sqrt{2}\mu_4(\mathfrak{q}_0,\mathfrak{q}_1)+\mu_6(\mathfrak{q}_0,\mathfrak{q}_1)}{1-\sqrt{2}\mu_2(\mathfrak{q}_0,\mathfrak{q}_1)-\sqrt{2}\mu_4(\mathfrak{q}_0,\mathfrak{q}_1)-\mu_6(\mathfrak{q}_0,\mathfrak{q}_1)}, \\ \frac{\mu_1(\mathfrak{q}_0,\mathfrak{q}_1)+\sqrt{2}\mu_5(\mathfrak{q}_0,\mathfrak{q}_1)+\mu_7(\mathfrak{q}_0,\mathfrak{q}_1)}{1-\sqrt{2}\mu_2(\mathfrak{q}_0,\mathfrak{q}_1)-\sqrt{2}\mu_5(\mathfrak{q}_0,\mathfrak{q}_1)-\mu_7(\mathfrak{q}_0,\mathfrak{q}_1)}\end{array}\right\} < 1.
$$

Then from (3) and (4), we conclude that

$$
\|d(\mathfrak{q}_r,\mathfrak{q}_{r+1})\| \leq \lambda\|d(\mathfrak{q}_{r-1},\mathfrak{q}_r)\|
$$

for all $r \in \mathbb{N}$. Utilizing the principle of mathematical induction, we can construct a sequence $\{\mathfrak{q}_r\}$ in $\mathcal{A}$ satisfying the following

$$
\|d(\mathfrak{q}_r,\mathfrak{q}_{r+1})\| \leq \lambda\|d(\mathfrak{q}_{r-1},\mathfrak{q}_r)\| \leq \lambda^2\|d(\mathfrak{q}_{r-2},\mathfrak{q}_{r-1})\| \leq \ldots \leq \lambda^r\|d(\mathfrak{q}_0,\mathfrak{q}_1)\|
$$

for all $r \in \mathbb{N}$. Now, for $m > r$, we get

$$
\begin{aligned}
\|d(\mathfrak{q}_r,\mathfrak{q}_m)\| &\leq \|d(\mathfrak{q}_r,\mathfrak{q}_{r+1})\| + \|d(\mathfrak{q}_{r+1},\mathfrak{q}_{r+2})\| + \ldots + \|d(\mathfrak{q}_{m-1},\mathfrak{q}_m)\| \\
&\leq \lambda^r\|d(\mathfrak{q}_0,\mathfrak{q}_1)\| + \lambda^{r+1}\|d(\mathfrak{q}_0,\mathfrak{q}_1)\| + \cdots + \lambda^{m-1}\|d(\mathfrak{q}_0,\mathfrak{q}_1)\| \\
&= \left[\lambda^r + \lambda^{r+1} + \ldots + \lambda^{m-1}\right]\|d(\mathfrak{q}_0,\mathfrak{q}_1)\| \\
&= \lambda^r\left[1 + \lambda + \lambda^2 \ldots + \lambda^{m-n-1}\right]\|d(\mathfrak{q}_0,\mathfrak{q}_1)\|
\end{aligned}
$$

$$\leq \quad \frac{\lambda^r}{1-\lambda}\|d(\mathfrak{q}_0,\mathfrak{q}_1)\|.$$

Now, by taking $r, m \to \infty$, we get

$$\|d(\mathfrak{q}_r,\mathfrak{q}_m)\| \to 0.$$

By Lemma 2, $\{\mathfrak{q}_r\}$ is a Cauchy sequence in $\mathcal{A}$. As $(\mathcal{A}, \preceq_{i_2}, d)$ is complete, so there exists $\mathfrak{q}^* \in \mathcal{A}$ such that $\mathfrak{q}_r \to \mathfrak{q}^*$ as $r \to \infty$. Now, we show that $\mathfrak{q}^*$ is a fixed point of $\mathcal{V}$. From (2), we have

$$\begin{aligned}
d(\mathfrak{q}^*,\mathcal{V}\mathfrak{q}^*) &\preceq_{i_2} d(\mathfrak{q}^*,\mathcal{W}\mathfrak{q}_{2r+1}) + d(\mathcal{W}\mathfrak{q}_{2r+1},\mathcal{V}\mathfrak{q}^*)\\
&= d(\mathfrak{q}^*,\mathcal{W}\mathfrak{q}_{2r+1}) + d(\mathcal{V}\mathfrak{q}^*,\mathcal{W}\mathfrak{q}_{2r+1})\\
&\preceq_{i_2} \begin{pmatrix}
d(\mathfrak{q}^*,\mathfrak{q}_{2r+2}) + \mu_1(\mathfrak{q}^*,\mathfrak{q}_{2r+1})d(\mathfrak{q}^*,\mathfrak{q}_{2r+1})\\
+\mu_2(\mathfrak{q}^*,\mathfrak{q}_{2r+1})\frac{d(\mathfrak{q}^*,\mathcal{V}\mathfrak{q}^*)d(\mathfrak{q}_{2r+1},\mathcal{W}\mathfrak{q}_{2r+1})}{1+d(\mathfrak{q}^*,\mathfrak{q}_{2r+1})}\\
+\mu_3(\mathfrak{q}^*,\mathfrak{q}_{2r+1})\frac{d(\mathfrak{q}_{2r+1},\mathcal{V}\mathfrak{q}^*)d(\mathfrak{q}^*,\mathcal{W}\mathfrak{q}_{2r+1})}{1+d(\mathfrak{q}^*,\mathfrak{q}_{2r+1})}\\
+\mu_4(\mathfrak{q}^*,\mathfrak{q}_{2r+1})\frac{d(\mathfrak{q}^*,\mathcal{V}\mathfrak{q}^*)d(\mathfrak{q}^*,\mathcal{W}\mathfrak{q}_{2r+1})}{1+d(\mathfrak{q}^*,\mathfrak{q}_{2r+1})}\\
+\mu_5(\mathfrak{q}^*,\mathfrak{q}_{2r+1})\frac{d(\mathfrak{q}_{2r+1},\mathcal{V}\mathfrak{q}^*)d(\mathfrak{q}_{2r+1},\mathcal{W}\mathfrak{q}_{2r+1})}{1+d(\mathfrak{q}^*,\mathfrak{q}_{2r+1})}\\
+\mu_6(\mathfrak{q}^*,\mathfrak{q}_{2r+1})\frac{d(\mathfrak{q}^*,\mathcal{W}\mathfrak{q}_{2r+1})[1+d(\mathfrak{q}^*,\mathcal{V}\mathfrak{q}^*)]}{1+d(\mathfrak{q}^*,\mathfrak{q}_{2r+1})}\\
+\mu_7(\mathfrak{q}^*,\mathfrak{q}_{2r+1})\frac{d(\mathfrak{q}_{2r+1},\mathcal{V}\mathfrak{q}^*)[1+d(\mathfrak{q}_{2r+1},\mathcal{W}\mathfrak{q}_{2r+1})]}{1+d(\mathfrak{q}^*,\mathfrak{q}_{2r+1})}
\end{pmatrix}\\
&= \begin{pmatrix}
d(\mathfrak{q}^*,\mathfrak{q}_{2r+2}) + \mu_1(\mathfrak{q}^*,\mathfrak{q}_{2r+1})d(\mathfrak{q}^*,\mathfrak{q}_{2r+1})\\
+\mu_2(\mathfrak{q}^*,\mathfrak{q}_{2r+1})\frac{d(\mathfrak{q}^*,\mathcal{V}\mathfrak{q}^*)d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})}{1+d(\mathfrak{q}^*,\mathfrak{q}_{2r+1})}\\
+\mu_3(\mathfrak{q}^*,\mathfrak{q}_{2r+1})\frac{d(\mathfrak{q}_{2r+1},\mathcal{V}\mathfrak{q}^*)d(\mathfrak{q}^*,\mathfrak{q}_{2r+2})}{1+d(\mathfrak{q}^*,\mathfrak{q}_{2r+1})}\\
+\mu_4(\mathfrak{q}^*,\mathfrak{q}_{2r+1})\frac{d(\mathfrak{q}^*,\mathcal{V}\mathfrak{q}^*)d(\mathfrak{q}^*,\mathfrak{q}_{2r+2})}{1+d(\mathfrak{q}^*,\mathfrak{q}_{2r+1})}\\
+\mu_5(\mathfrak{q}^*,\mathfrak{q}_{2r+1})\frac{d(\mathfrak{q}_{2r+1},\mathcal{V}\mathfrak{q}^*)d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})}{1+d(\mathfrak{q}^*,\mathfrak{q}_{2r+1})}\\
+\mu_6(\mathfrak{q}^*,\mathfrak{q}_{2r+1})\frac{d(\mathfrak{q}^*,\mathfrak{q}_{2r+2})[1+d(\mathfrak{q}^*,\mathcal{V}\mathfrak{q}^*)]}{1+d(\mathfrak{q}^*,\mathfrak{q}_{2r+1})}\\
+\mu_7(\mathfrak{q}^*,\mathfrak{q}_{2r+1})\frac{d(\mathfrak{q}_{2r+1},\mathcal{V}\mathfrak{q}^*)[1+d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})]}{1+d(\mathfrak{q}^*,\mathfrak{q}_{2r+1})}
\end{pmatrix}.
\end{aligned}$$

By Proposition 1, we have

$$d(\mathfrak{q}^*,\mathcal{V}\mathfrak{q}^*) \preceq_{i_2} \begin{pmatrix}
d(\mathfrak{q}^*,\mathfrak{q}_{2r+2}) + \mu_1(\mathfrak{q}^*,\mathfrak{q}_1)d(\mathfrak{q}^*,\mathfrak{q}_{2r+1})\\
+\mu_2(\mathfrak{q}^*,\mathfrak{q}_1)\frac{d(\mathfrak{q}^*,\mathcal{V}\mathfrak{q}^*)d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})}{1+d(\mathfrak{q}^*,\mathfrak{q}_{2r+1})}\\
+\mu_3(\mathfrak{q}^*,\mathfrak{q}_1)\frac{d(\mathfrak{q}_{2r+1},\mathcal{V}\mathfrak{q}^*)d(\mathfrak{q}^*,\mathfrak{q}_{2r+2})}{1+d(\mathfrak{q}^*,\mathfrak{q}_{2r+1})}\\
+\mu_4(\mathfrak{q}^*,\mathfrak{q}_1)\frac{d(\mathfrak{q}^*,\mathcal{V}\mathfrak{q}^*)d(\mathfrak{q}^*,\mathfrak{q}_{2r+2})}{1+d(\mathfrak{q}^*,\mathfrak{q}_{2r+1})}\\
+\mu_5(\mathfrak{q}^*,\mathfrak{q}_1)\frac{d(\mathfrak{q}_{2r+1},\mathcal{V}\mathfrak{q}^*)d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})}{1+d(\mathfrak{q}^*,\mathfrak{q}_{2r+1})}\\
+\mu_6(\mathfrak{q}^*,\mathfrak{q}_1)\frac{d(\mathfrak{q}^*,\mathfrak{q}_{2r+2})[1+d(\mathfrak{q}^*,\mathcal{V}\mathfrak{q}^*)]}{1+d(\mathfrak{q}^*,\mathfrak{q}_{2r+1})}\\
+\mu_7(\mathfrak{q}^*,\mathfrak{q}_1)\frac{d(\mathfrak{q}_{2r+1},\mathcal{V}\mathfrak{q}^*)[1+d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})]}{1+d(\mathfrak{q}^*,\mathfrak{q}_{2r+1})}
\end{pmatrix}.$$

This implies that

$$\|d(\mathfrak{q}^*,\mathcal{V}\mathfrak{q}^*)\| \quad \leq \quad \begin{pmatrix}
\|d(\mathfrak{q}^*,\mathfrak{q}_{2r+2})\| + \mu_1(\mathfrak{q}^*,\mathfrak{q}_1)\|d(\mathfrak{q}^*,\mathfrak{q}_{2r+1})\|\\
+\mu_2(\mathfrak{q}^*,\mathfrak{q}_1)\left\|\frac{d(\mathfrak{q}^*,\mathcal{V}\mathfrak{q}^*)d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})}{1+d(\mathfrak{q}^*,\mathfrak{q}_{2r+1})}\right\|\\
+\mu_3(\mathfrak{q}^*,\mathfrak{q}_1)\left\|\frac{d(\mathfrak{q}_{2r+1},\mathcal{V}\mathfrak{q}^*)d(\mathfrak{q}^*,\mathfrak{q}_{2r+2})}{1+d(\mathfrak{q}^*,\mathfrak{q}_{2r+1})}\right\|\\
+\mu_4(\mathfrak{q}^*,\mathfrak{q}_1)\left\|\frac{d(\mathfrak{q}^*,\mathcal{V}\mathfrak{q}^*)d(\mathfrak{q}^*,\mathfrak{q}_{2r+2})}{1+d(\mathfrak{q}^*,\mathfrak{q}_{2r+1})}\right\|\\
+\mu_5(\mathfrak{q}^*,\mathfrak{q}_1)\left\|\frac{d(\mathfrak{q}_{2r+1},\mathcal{V}\mathfrak{q}^*)d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})}{1+d(\mathfrak{q}^*,\mathfrak{q}_{2r+1})}\right\|\\
+\mu_6(\mathfrak{q}^*,\mathfrak{q}_1)\left\|\frac{d(\mathfrak{q}^*,\mathfrak{q}_{2r+2})[1+d(\mathfrak{q}^*,\mathcal{V}\mathfrak{q}^*)]}{1+d(\mathfrak{q}^*,\mathfrak{q}_{2r+1})}\right\|\\
+\mu_7(\mathfrak{q}^*,\mathfrak{q}_1)\left\|\frac{d(\mathfrak{q}_{2r+1},\mathcal{V}\mathfrak{q}^*)[1+d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})]}{1+d(\mathfrak{q}^*,\mathfrak{q}_{2r+1})}\right\|
\end{pmatrix}$$

$$\leq \left(\begin{array}{c} \|d(\mathfrak{q}^*,\mathfrak{q}_{2r+2})\| + \mu_1(\mathfrak{q}^*,\mathfrak{q}_1)\|d(\mathfrak{q}^*,\mathfrak{q}_{2r+1})\| \\ +\mu_2(\mathfrak{q}^*,\mathfrak{q}_1)\left\|\frac{d(\mathfrak{q}^*,\mathcal{V}\mathfrak{q}^*)d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})}{1+d(\mathfrak{q}^*,\mathfrak{q}_{2r+1})}\right\| \\ +\mu_3(\mathfrak{q}^*,\mathfrak{q}_1)\left\|\frac{d(\mathfrak{q}_{2r+1},\mathcal{V}\mathfrak{q}^*)d(\mathfrak{q}^*,\mathfrak{q}_{2r+2})}{1+d(\mathfrak{q}^*,\mathfrak{q}_{2r+1})}\right\| \\ +\mu_4(\mathfrak{q}^*,\mathfrak{q}_1)\left\|\frac{d(\mathfrak{q}^*,\mathcal{V}\mathfrak{q}^*)d(\mathfrak{q}^*,\mathfrak{q}_{2r+2})}{1+d(\mathfrak{q}^*,\mathfrak{q}_{2r+1})}\right\| \\ +\mu_5(\mathfrak{q}^*,\mathfrak{q}_1)\left\|\frac{d(\mathfrak{q}_{2r+1},\mathcal{V}\mathfrak{q}^*)d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})}{1+d(\mathfrak{q}^*,\mathfrak{q}_{2r+1})}\right\| \\ +\mu_6(\mathfrak{q}^*,\mathfrak{q}_1)\left\|\frac{d(\mathfrak{q}^*,\mathfrak{q}_{2r+2})[1+d(\mathfrak{q}^*,\mathcal{V}\mathfrak{q}^*)]}{1+d(\mathfrak{q}^*,\mathfrak{q}_{2r+1})}\right\| \\ +\sqrt{2}\mu_7(\mathfrak{q}^*,\mathfrak{q}_1)\|d(\mathfrak{q}_{2r+1},\mathcal{V}\mathfrak{q}^*)\|\left\|\frac{[1+d(\mathfrak{q}_{2r+1},\mathfrak{q}_{2r+2})]}{1+d(\mathfrak{q}^*,\mathfrak{q}_{2r+1})}\right\| \end{array}\right).$$

Letting $r \to \infty$, we have $\|d(\mathfrak{q}^*,\mathcal{V}\mathfrak{q}^*)\| \leq \sqrt{2}\mu_7(\mathfrak{q}^*,\mathfrak{q}^*)\|d(\mathfrak{q}^*,\mathcal{V}\mathfrak{q}^*)\|$. By condition $\sqrt{2}\mu_7(\mathfrak{q}^*,\mathfrak{q}^*) < 1$. Thus, $\|d(\mathfrak{q}^*,\mathcal{V}\mathfrak{q}^*)\| = 0$ and $\mathfrak{q}^* = \mathcal{V}\mathfrak{q}^*$. Now we prove that $\mathfrak{q}^*$ is a fixed point of $\mathcal{W}$. By (2), we have

$$d(\mathfrak{q}^*,\mathcal{W}\mathfrak{q}^*) \preceq_{i_2} (d(\mathfrak{q}^*,\mathcal{V}\mathfrak{q}_{2r}) + d(\mathcal{V}\mathfrak{q}_{2r},\mathcal{W}\mathfrak{q}^*))$$

$$\preceq_{i_2} \left(\begin{array}{c} d(\mathfrak{q}^*,\mathcal{V}\mathfrak{q}_{2r}) + \mu_1(\mathfrak{q}_{2r},\mathfrak{q}^*)d(\mathfrak{q}_{2r},\mathfrak{q}^*) \\ +\mu_2(\mathfrak{q}_{2r},\mathfrak{q}^*)\frac{d(\mathfrak{q}_{2r},\mathcal{V}\mathfrak{q}_{2r})d(\mathfrak{q}^*,\mathcal{W}\mathfrak{q}^*)}{1+d(\mathfrak{q}_{2r},\mathfrak{q}^*)} \\ +\mu_3(\mathfrak{q}_{2r},\mathfrak{q}^*)\frac{d(\mathfrak{q}^*,\mathcal{V}\mathfrak{q}_{2r})d(\mathfrak{q}_{2r},\mathcal{W}\mathfrak{q}^*)}{1+d(\mathfrak{q}_{2r},\mathfrak{q}^*)} \\ +\mu_4(\mathfrak{q}_{2r},\mathfrak{q}^*)\frac{d(\mathfrak{q}_{2r},\mathcal{V}\mathfrak{q}_{2r})d(\mathfrak{q}_{2r},\mathcal{W}\mathfrak{q}^*)}{1+d(\mathfrak{q}_{2r},\mathfrak{q}^*)} + \mu_5(\mathfrak{q}_{2r},\mathfrak{q}^*)\frac{d(\mathfrak{q}^*,\mathcal{V}\mathfrak{q}_{2r})d(\mathfrak{q}^*,\mathcal{W}\mathfrak{q}^*)}{1+d(\mathfrak{q}_{2r},\mathfrak{q}^*)} \\ +\mu_6(\mathfrak{q}_{2r},\mathfrak{q}^*)\frac{d(\mathfrak{q}_{2r},\mathcal{W}\mathfrak{q}^*)[1+d(\mathfrak{q}_{2r},\mathcal{V}\mathfrak{q}_{2r})]}{1+d(\mathfrak{q}_{2r},\mathfrak{q}^*)} + \mu_7(\mathfrak{q}_{2r},\mathfrak{q}^*)\frac{d(\mathfrak{q}^*,\mathcal{V}\mathfrak{q}_{2r})[1+d(\mathfrak{q}^*,\mathcal{W}\mathfrak{q}^*)]}{1+d(\mathfrak{q}_{2r},\mathfrak{q}^*)} \end{array}\right)$$

$$= \left(\begin{array}{c} d(\mathfrak{q}^*,\mathfrak{q}_{2r+1}) + \mu_1(\mathfrak{q}_{2r},\mathfrak{q}^*)d(\mathfrak{q}_{2r},\mathfrak{q}^*) \\ +\mu_2(\mathfrak{q}_{2r},\mathfrak{q}^*)\frac{d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})d(\mathfrak{q}^*,\mathcal{W}\mathfrak{q}^*)}{1+d(\mathfrak{q}_{2r},\mathfrak{q}^*)} \\ +\mu_3(\mathfrak{q}_{2r},\mathfrak{q}^*)\frac{d(\mathfrak{q}^*,\mathfrak{q}_{2r+1})d(\mathfrak{q}_{2r},\mathcal{W}\mathfrak{q}^*)}{1+d(\mathfrak{q}_{2r},\mathfrak{q}^*)} \\ +\mu_4(\mathfrak{q}_{2r},\mathfrak{q}^*)\frac{d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})d(\mathfrak{q}_{2r},\mathcal{W}\mathfrak{q}^*)}{1+d(\mathfrak{q}_{2r},\mathfrak{q}^*)} + \mu_5(\mathfrak{q}_{2r},\mathfrak{q}^*)\frac{d(\mathfrak{q}^*,\mathfrak{q}_{2r+1})d(\mathfrak{q}^*,\mathcal{W}\mathfrak{q}^*)}{1+d(\mathfrak{q}_{2r},\mathfrak{q}^*)} \\ +\mu_6(\mathfrak{q}_{2r},\mathfrak{q}^*)\frac{d(\mathfrak{q}_{2r},\mathcal{W}\mathfrak{q}^*)[1+d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})]}{1+d(\mathfrak{q}_{2r},\mathfrak{q}^*)} + \mu_7(\mathfrak{q}_{2r},\mathfrak{q}^*)\frac{d(\mathfrak{q}^*,\mathfrak{q}_{2r+1})[1+d(\mathfrak{q}^*,\mathcal{W}\mathfrak{q}^*)]}{1+d(\mathfrak{q}_{2r},\mathfrak{q}^*)} \end{array}\right).$$

By Proposition 1, we have

$$d(\mathfrak{q}^*,\mathcal{W}\mathfrak{q}^*) \preceq_{i_2} \left(\begin{array}{c} d(\mathfrak{q}^*,\mathfrak{q}_{2r+1}) + \mu_1(\mathfrak{q}_0,\mathfrak{q}^*)d(\mathfrak{q}_{2r},\mathfrak{q}^*) \\ +\mu_2(\mathfrak{q}_0,\mathfrak{q}^*)\frac{d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})d(\mathfrak{q}^*,\mathcal{W}\mathfrak{q}^*)}{1+d(\mathfrak{q}_{2r},\mathfrak{q}^*)} \\ +\mu_3(\mathfrak{q}_0,\mathfrak{q}^*)\frac{d(\mathfrak{q}^*,\mathfrak{q}_{2r+1})d(\mathfrak{q}_{2r},\mathcal{W}\mathfrak{q}^*)}{1+d(\mathfrak{q}_{2r},\mathfrak{q}^*)} \\ +\mu_4(\mathfrak{q}_0,\mathfrak{q}^*)\frac{d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})d(\mathfrak{q}_{2r},\mathcal{W}\mathfrak{q}^*)}{1+d(\mathfrak{q}_{2r},\mathfrak{q}^*)} + \mu_5(\mathfrak{q}_0,\mathfrak{q}^*)\frac{d(\mathfrak{q}^*,\mathfrak{q}_{2r+1})d(\mathfrak{q}^*,\mathcal{W}\mathfrak{q}^*)}{1+d(\mathfrak{q}_{2r},\mathfrak{q}^*)} \\ +\mu_6(\mathfrak{q}_0,\mathfrak{q}^*)\frac{d(\mathfrak{q}_{2r},\mathcal{W}\mathfrak{q}^*)[1+d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})]}{1+d(\mathfrak{q}_{2r},\mathfrak{q}^*)} + \mu_7(\mathfrak{q}_0,\mathfrak{q}^*)\frac{d(\mathfrak{q}^*,\mathfrak{q}_{2r+1})[1+d(\mathfrak{q}^*,\mathcal{W}\mathfrak{q}^*)]}{1+d(\mathfrak{q}_{2r},\mathfrak{q}^*)} \end{array}\right).$$

This implies that

$$\|d(\mathfrak{q}^*,\mathcal{W}\mathfrak{q}^*)\| \leq \left(\begin{array}{c} \|d(\mathfrak{q}^*,\mathfrak{q}_{2r+1})\| + \mu_1(\mathfrak{q}_0,\mathfrak{q}^*)\|d(\mathfrak{q}_{2r},\mathfrak{q}^*)\| \\ +\mu_2(\mathfrak{q}_0,\mathfrak{q}^*)\left\|\frac{d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})d(\mathfrak{q}^*,\mathcal{W}\mathfrak{q}^*)}{1+d(\mathfrak{q}_{2r},\mathfrak{q}^*)}\right\| \\ +\mu_3(\mathfrak{q}_0,\mathfrak{q}^*)\left\|\frac{d(\mathfrak{q}^*,\mathfrak{q}_{2r+1})d(\mathfrak{q}_{2r},\mathcal{W}\mathfrak{q}^*)}{1+d(\mathfrak{q}_{2r},\mathfrak{q}^*)}\right\| \\ +\mu_4(\mathfrak{q}_0,\mathfrak{q}^*)\left\|\frac{d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})d(\mathfrak{q}_{2r},\mathcal{W}\mathfrak{q}^*)}{1+d(\mathfrak{q}_{2r},\mathfrak{q}^*)}\right\| \\ +\mu_5(\mathfrak{q}_0,\mathfrak{q}^*)\left\|\frac{d(\mathfrak{q}^*,\mathfrak{q}_{2r+1})d(\mathfrak{q}^*,\mathcal{W}\mathfrak{q}^*)}{1+d(\mathfrak{q}_{2r},\mathfrak{q}^*)}\right\| \\ +\mu_6(\mathfrak{q}_0,\mathfrak{q}^*)\left\|\frac{d(\mathfrak{q}_{2r},\mathcal{W}\mathfrak{q}^*)[1+d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})]}{1+d(\mathfrak{q}_{2r},\mathfrak{q}^*)}\right\| \\ +\mu_7(\mathfrak{q}_0,\mathfrak{q}^*)\left\|\frac{d(\mathfrak{q}^*,\mathfrak{q}_{2r+1})[1+d(\mathfrak{q}^*,\mathcal{W}\mathfrak{q}^*)]}{1+d(\mathfrak{q}_{2r},\mathfrak{q}^*)}\right\| \end{array}\right)$$

$$\leq \left(\begin{array}{c} \|d(\mathfrak{q}^*,\mathfrak{q}_{2r+1})\| + \mu_1(\mathfrak{q}_0,\mathfrak{q}^*)\|d(\mathfrak{q}_{2r},\mathfrak{q}^*)\| \\ +\mu_2(\mathfrak{q}_0,\mathfrak{q}^*)\left\|\frac{d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})d(\mathfrak{q}^*,\mathcal{W}\mathfrak{q}^*)}{1+d(\mathfrak{q}_{2r},\mathfrak{q}^*)}\right\| \\ +\mu_3(\mathfrak{q}_0,\mathfrak{q}^*)\left\|\frac{d(\mathfrak{q}^*,\mathfrak{q}_{2r+1})d(\mathfrak{q}_{2r},\mathcal{W}\mathfrak{q}^*)}{1+d(\mathfrak{q}_{2r},\mathfrak{q}^*)}\right\| \\ +\mu_4(\mathfrak{q}_0,\mathfrak{q}^*)\left\|\frac{d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})d(\mathfrak{q}_{2r},\mathcal{W}\mathfrak{q}^*)}{1+d(\mathfrak{q}_{2r},\mathfrak{q}^*)}\right\| \\ +\mu_5(\mathfrak{q}_0,\mathfrak{q}^*)\left\|\frac{d(\mathfrak{q}^*,\mathfrak{q}_{2r+1})d(\mathfrak{q}^*,\mathcal{W}\mathfrak{q}^*)}{1+d(\mathfrak{q}_{2r},\mathfrak{q}^*)}\right\| \\ +\sqrt{2}\mu_6(\mathfrak{q}_0,\mathfrak{q}^*)\|d(\mathfrak{q}_{2r},\mathcal{W}\mathfrak{q}^*)\|\left\|\frac{[1+d(\mathfrak{q}_{2r},\mathfrak{q}_{2r+1})]}{1+d(\mathfrak{q}_{2r},\mathfrak{q}^*)}\right\| \\ +\mu_7(\mathfrak{q}_0,\mathfrak{q}^*)\left\|\frac{d(\mathfrak{q}^*,\mathfrak{q}_{2r+1})[1+d(\mathfrak{q}^*,\mathcal{W}\mathfrak{q}^*)]}{1+d(\mathfrak{q}_{2r},\mathfrak{q}^*)}\right\| \end{array}\right)$$

Letting $r \to \infty$, we have $\frac{1}{1-\sqrt{2}\mu_6(\mathfrak{q}_0,\mathfrak{q}^*)}\|d(\mathfrak{q}^*,\mathcal{W}\mathfrak{q}^*)\| = 0$. Thus, $\mathfrak{q}^* = \mathcal{W}\mathfrak{q}^*$. Thus, $\mathfrak{q}^*$ is a common fixed point of $\mathcal{V}$ and $\mathcal{W}$. Now we prove that $\mathfrak{q}^*$ is unique. We suppose that

$$\mathfrak{q}^{/} = \mathcal{V}\mathfrak{q}^{/} = \mathcal{W}\mathfrak{q}^{/}$$

but $\mathfrak{q}^* \neq \mathfrak{q}^{/}$. Now from (2), we have

$$d\left(\mathfrak{q}^*,\mathfrak{q}^{/}\right) = d\left(\mathcal{V}\mathfrak{q}^*,\mathcal{W}\mathfrak{q}^{/}\right)$$

$$\preceq_{i_2} \mu_1\left(\mathfrak{q}^*,\mathfrak{q}^{/}\right)d\left(\mathfrak{q}^*,\mathfrak{q}^{/}\right) + \mu_2\left(\mathfrak{q}^*,\mathfrak{q}^{/}\right)\frac{d(\mathfrak{q}^*,\mathcal{V}\mathfrak{q}^*)d\left(\mathfrak{q}^{/},\mathcal{W}\mathfrak{q}^{/}\right)}{1+d\left(\mathfrak{q}^*,\mathfrak{q}^{/}\right)}$$

$$+\mu_3\left(\mathfrak{q}^*,\mathfrak{q}^{/}\right)\frac{d\left(\mathfrak{q}^{/},\mathcal{V}\mathfrak{q}^*\right)d\left(\mathfrak{q}^*,\mathcal{W}\mathfrak{q}^{/}\right)}{1+d\left(\mathfrak{q}^*,\mathfrak{q}^{/}\right)}$$

$$+\mu_4\left(\mathfrak{q}^*,\mathfrak{q}^{/}\right)\frac{d(\mathfrak{q}^*,\mathcal{V}\mathfrak{q}^*)d\left(\mathfrak{q}^*,\mathcal{W}\mathfrak{q}^{/}\right)}{1+d\left(\mathfrak{q}^*,\mathfrak{q}^{/}\right)} + \mu_5\left(\mathfrak{q}^*,\mathfrak{q}^{/}\right)\frac{d\left(\mathfrak{q}^{/},\mathcal{V}\mathfrak{q}^*\right)d\left(\mathfrak{q}^{/},\mathcal{W}\mathfrak{q}^{/}\right)}{1+d\left(\mathfrak{q}^*,\mathfrak{q}^{/}\right)}$$

$$+\mu_6\left(\mathfrak{q}^*,\mathfrak{q}^{/}\right)\frac{d\left(\mathfrak{q}^*,\mathcal{W}\mathfrak{q}^{/}\right)[1+d(\mathfrak{q}^*,\mathcal{V}\mathfrak{q}^*)]}{1+d\left(\mathfrak{q}^*,\mathfrak{q}^{/}\right)} + \mu_7\left(\mathfrak{q}^*,\mathfrak{q}^{/}\right)\frac{d\left(\mathfrak{q}^{/},\mathcal{V}\mathfrak{q}^*\right)\left[1+d\left(\mathfrak{q}^{/},\mathcal{W}\mathfrak{q}^{/}\right)\right]}{1+d\left(\mathfrak{q}^*,\mathfrak{q}^{/}\right)}$$

$$\begin{aligned} = \;& \mu_1\left(\mathfrak{q}^*,\mathfrak{q}^{/}\right)d\left(\mathfrak{q}^*,\mathfrak{q}^{/}\right) + \mu_2\left(\mathfrak{q}^*,\mathfrak{q}^{/}\right)\frac{d(\mathfrak{q}^*,\mathfrak{q}^*)d\left(\mathfrak{q}^{/},\mathfrak{q}^{/}\right)}{1+d\left(\mathfrak{q}^*,\mathfrak{q}^{/}\right)} + \mu_3\left(\mathfrak{q}^*,\mathfrak{q}^{/}\right)\frac{d\left(\mathfrak{q}^{/},\mathfrak{q}^*\right)d\left(\mathfrak{q}^*,\mathfrak{q}^{/}\right)}{1+d\left(\mathfrak{q}^*,\mathfrak{q}^{/}\right)} \\ & +\mu_4\left(\mathfrak{q}^*,\mathfrak{q}^{/}\right)\frac{d(\mathfrak{q}^*,\mathfrak{q}^*)d\left(\mathfrak{q}^*,\mathfrak{q}^{/}\right)}{1+d\left(\mathfrak{q}^*,\mathfrak{q}^{/}\right)} + \mu_5\left(\mathfrak{q}^*,\mathfrak{q}^{/}\right)\frac{d\left(\mathfrak{q}^{/},\mathfrak{q}^*\right)d\left(\mathfrak{q}^{/},\mathfrak{q}^{/}\right)}{1+d\left(\mathfrak{q}^*,\mathfrak{q}^{/}\right)} \\ & +\mu_6\left(\mathfrak{q}^*,\mathfrak{q}^{/}\right)\frac{d\left(\mathfrak{q}^*,\mathfrak{q}^{/}\right)[1+d(\mathfrak{q}^*,\mathfrak{q}^*)]}{1+d\left(\mathfrak{q}^*,\mathfrak{q}^{/}\right)} + \mu_7\left(\mathfrak{q}^*,\mathfrak{q}^{/}\right)\frac{d\left(\mathfrak{q}^{/},\mathfrak{q}^*\right)\left[1+d\left(\mathfrak{q}^{/},\mathfrak{q}^{/}\right)\right]}{1+d\left(\mathfrak{q}^*,\mathfrak{q}^{/}\right)} \\ = \;& \mu_1\left(\mathfrak{q}^*,\mathfrak{q}^{/}\right)d\left(\mathfrak{q}^*,\mathfrak{q}^{/}\right) + \mu_3\left(\mathfrak{q}^*,\mathfrak{q}^{/}\right)\frac{d\left(\mathfrak{q}^{/},\mathfrak{q}^*\right)d\left(\mathfrak{q}^*,\mathfrak{q}^{/}\right)}{1+d\left(\mathfrak{q}^*,\mathfrak{q}^{/}\right)} + \mu_6\left(\mathfrak{q}^*,\mathfrak{q}^{/}\right)\frac{d\left(\mathfrak{q}^*,\mathfrak{q}^{/}\right)}{1+d\left(\mathfrak{q}^*,\mathfrak{q}^{/}\right)} \\ & +\mu_7\left(\mathfrak{q}^*,\mathfrak{q}^{/}\right)\frac{d\left(\mathfrak{q}^{/},\mathfrak{q}^*\right)}{1+d\left(\mathfrak{q}^*,\mathfrak{q}^{/}\right)}. \end{aligned}$$

This implies that

$$\begin{aligned} \left\|d\left(\mathfrak{q}^*,\mathfrak{q}^{/}\right)\right\| \;\leq\; & \mu_1\left(\mathfrak{q}^*,\mathfrak{q}^{/}\right)\left\|d\left(\mathfrak{q}^*,\mathfrak{q}^{/}\right)\right\| \\ & +\sqrt{2}\mu_3\left(\mathfrak{q}^*,\mathfrak{q}^{/}\right)\left\|d\left(\mathfrak{q}^*,\mathfrak{q}^{/}\right)\right\|\frac{\left\|d\left(\mathfrak{q}^*,\mathfrak{q}^{/}\right)\right\|}{\left\|1+d\left(\mathfrak{q}^*,\mathfrak{q}^{/}\right)\right\|} \end{aligned}$$

$$+\mu_6\left(\mathfrak{q}^*,\mathfrak{q}^/\right)\frac{\left\|d\left(\mathfrak{q}^*,\mathfrak{q}^/\right)\right\|}{\left\|1+d\left(\mathfrak{q}^*,\mathfrak{q}^/\right)\right\|}+\mu_7\left(\mathfrak{q}^*,\mathfrak{q}^/\right)\frac{\left\|d\left(\mathfrak{q}^/,\mathfrak{q}^*\right)\right\|}{\left\|1+d\left(\mathfrak{q}^*,\mathfrak{q}^/\right)\right\|}.$$

Since $\left\|d\left(\mathfrak{q}^*,\mathfrak{q}^/\right)\right\| \leq \left\|1+d\left(\mathfrak{q}^*,\mathfrak{q}^/\right)\right\|$, so $\frac{\left\|d\left(\mathfrak{q}^*,\mathfrak{q}^/\right)\right\|}{\left\|1+d\left(\mathfrak{q}^*,\mathfrak{q}^/\right)\right\|} \leq 1$ and $\frac{1}{\left\|1+d\left(\mathfrak{q}^*,\mathfrak{q}^/\right)\right\|} \leq 1$. Thus,

$$\begin{aligned}\left\|d\left(\mathfrak{q}^*,\mathfrak{q}^/\right)\right\| &\leq \mu_1\left(\mathfrak{q}^*,\mathfrak{q}^/\right)\left\|d\left(\mathfrak{q}^*,\mathfrak{q}^/\right)\right\|+\sqrt{2}\mu_3\left(\mathfrak{q}^*,\mathfrak{q}^/\right)\left\|d\left(\mathfrak{q}^*,\mathfrak{q}^/\right)\right\| \\ &\quad+\mu_6\left(\mathfrak{q}^*,\mathfrak{q}^/\right)\left\|d\left(\mathfrak{q}^*,\mathfrak{q}^/\right)\right\|+\mu_7\left(\mathfrak{q}^*,\mathfrak{q}^/\right)\left\|d\left(\mathfrak{q}^*,\mathfrak{q}^/\right)\right\| \\ &= \left(\mu_1\left(\mathfrak{q}^*,\mathfrak{q}^/\right)+\sqrt{2}\mu_3\left(\mathfrak{q}^*,\mathfrak{q}^/\right)+\mu_6\left(\mathfrak{q}^*,\mathfrak{q}^/\right)+\mu_7\left(\mathfrak{q}^*,\mathfrak{q}^/\right)\right)\left\|d\left(\mathfrak{q}^*,\mathfrak{q}^/\right)\right\|.\end{aligned}$$

As $\left(\mu_1\left(\mathfrak{q}^*,\mathfrak{q}^/\right)+\sqrt{2}\mu_3\left(\mathfrak{q}^*,\mathfrak{q}^/\right)+\mu_6\left(\mathfrak{q}^*,\mathfrak{q}^/\right)+\mu_7\left(\mathfrak{q}^*,\mathfrak{q}^/\right)\right) < 1$, we have

$$\left\|d\left(\mathfrak{q}^*,\mathfrak{q}^/\right)\right\| = 0.$$

Thus, $\mathfrak{q}^* = \mathfrak{q}^/$. □

Corollary 1. *Let $(\mathcal{A},\preceq_{i_2},d)$ be a complete bi-CVMS and $\mathcal{V}:\mathcal{A}\to\mathcal{A}$. If there exist the mappings $\mu_i:\mathcal{A}\times\mathcal{A}\to[0,1)$, where $i=1,2,\ldots 7$ such that*

(a) $\mu_i(\mathcal{V}\mathfrak{q},\kappa) \leq \mu_i(\mathfrak{q},\kappa)$ *and* $\mu_i(\mathfrak{q},\mathcal{V}\kappa) \leq \mu_i(\mathfrak{q},\kappa)$;

(b) $\mu_1(\mathfrak{q},\kappa)+\sqrt{2}(\mu_2(\mathfrak{q},\kappa)+\mu_3(\mathfrak{q},\kappa))+2\sqrt{2}(\mu_4(\mathfrak{q},\kappa)+\mu_5(\mathfrak{q},\kappa))+2(\mu_6(\mathfrak{q},\kappa)+\mu_7(\mathfrak{q},\kappa))<1$;

(c)

$$\begin{aligned}d(\mathcal{V}\mathfrak{q},\mathcal{V}\kappa) \preceq_{i_2} &\mu_1(\mathfrak{q},\kappa)d(\mathfrak{q},\kappa)+\mu_2(\mathfrak{q},\kappa)\frac{d(\mathfrak{q},\mathcal{V}\mathfrak{q})d(\kappa,\mathcal{V}\kappa)}{1+d(\mathfrak{q},\kappa)}+\mu_3(\mathfrak{q},\kappa)\frac{d(\kappa,\mathcal{V}\mathfrak{q})d(\mathfrak{q},\mathcal{V}\kappa)}{1+d(\mathfrak{q},\kappa)} \\ &+\mu_4(\mathfrak{q},\kappa)\frac{d(\mathfrak{q},\mathcal{V}\mathfrak{q})d(\mathfrak{q},\mathcal{V}\kappa)}{1+d(\mathfrak{q},\kappa)}+\mu_5(\mathfrak{q},\kappa)\frac{d(\kappa,\mathcal{V}\mathfrak{q})d(\kappa,\mathcal{V}\kappa)}{1+d(\mathfrak{q},\kappa)} \\ &+\mu_6(\mathfrak{q},\kappa)\frac{d(\mathfrak{q},\mathcal{V}\kappa)[1+d(\mathfrak{q},\mathcal{V}\mathfrak{q})]}{1+d(\mathfrak{q},\kappa)}+\mu_7(\mathfrak{q},\kappa)\frac{d(\kappa,\mathcal{V}\mathfrak{q})[1+d(\kappa,\mathcal{V}\kappa)]}{1+d(\mathfrak{q},\kappa)}\end{aligned}$$

for all $\mathfrak{q},\kappa\in\mathcal{A}$, then $\mathcal{V}$ has a unique fixed point.

Proof. Take $\mathcal{W}=\mathcal{V}$ in Theorem 1. □

Corollary 2. *Let $(\mathcal{A},\preceq_{i_2},d)$ be a complete bi-CVMS and $\mathcal{V},\mathcal{W}:\mathcal{A}\to\mathcal{A}$. If there exist the mappings $\mu_i:\mathcal{A}\times\mathcal{A}\to[0,1)$, where $i=1,2,\ldots 6$, such that*

(a) $\mu_i(\mathcal{W}\mathcal{V}\mathfrak{q},\kappa) \leq \mu_i(\mathfrak{q},\kappa)$ *and* $\mu_i(\mathfrak{q},\mathcal{V}\mathcal{W}\kappa) \leq \mu_i(\mathfrak{q},\kappa)$;

(b) $\mu_1(\mathfrak{q},\kappa)+\sqrt{2}(\mu_2(\mathfrak{q},\kappa)+\mu_3(\mathfrak{q},\kappa))+2\sqrt{2}(\mu_4(\mathfrak{q},\kappa)+\mu_5(\mathfrak{q},\kappa))+2\mu_6(\mathfrak{q},\kappa)<1$;

(c)

$$\begin{aligned}d(\mathcal{V}\mathfrak{q},\mathcal{W}\kappa) \preceq_{i_2} &\mu_1(\mathfrak{q},\kappa)d(\mathfrak{q},\kappa)+\mu_2(\mathfrak{q},\kappa)\frac{d(\mathfrak{q},\mathcal{V}\mathfrak{q})d(\kappa,\mathcal{W}\kappa)}{1+d(\mathfrak{q},\kappa)}+\mu_3(\mathfrak{q},\kappa)\frac{d(\kappa,\mathcal{V}\mathfrak{q})d(\mathfrak{q},\mathcal{W}\kappa)}{1+d(\mathfrak{q},\kappa)} \\ &+\mu_4(\mathfrak{q},\kappa)\frac{d(\mathfrak{q},\mathcal{V}\mathfrak{q})d(\mathfrak{q},\mathcal{W}\kappa)}{1+d(\mathfrak{q},\kappa)}+\mu_5(\mathfrak{q},\kappa)\frac{d(\kappa,\mathcal{V}\mathfrak{q})d(\kappa,\mathcal{W}\kappa)}{1+d(\mathfrak{q},\kappa)}+\mu_6(\mathfrak{q},\kappa)\frac{d(\mathfrak{q},\mathcal{W}\kappa)[1+d(\mathfrak{q},\mathcal{V}\mathfrak{q})]}{1+d(\mathfrak{q},\kappa)}\end{aligned}$$

for all $\mathfrak{q},\kappa\in\mathcal{A}$, then $\mathcal{V}$ and $\mathcal{W}$ have a unique common fixed point.

Proof. Setting $\mu_7:\mathcal{A}\times\mathcal{A}\to[0,1)$ by $\mu_7(\mathfrak{q},\kappa)=0$ in Theorem 1. □

Corollary 3. *Let $(\mathcal{A},\preceq_{i_2},d)$ be a complete bi-CVMS and $\mathcal{V},\mathcal{W}:\mathcal{A}\to\mathcal{A}$. If there exist the mappings $\mu_i:\mathcal{A}\times\mathcal{A}\to[0,1)$, where $i=1,2,\ldots 5$, such that*

(a) $\mu_i(\mathcal{W}\mathcal{V}\mathfrak{q},\kappa) \leq \mu_i(\mathfrak{q},\kappa)$ *and* $\mu_i(\mathfrak{q},\mathcal{V}\mathcal{W}\kappa) \leq \mu_i(\mathfrak{q},\kappa)$;

(b) $\mu_1(\mathfrak{q},\kappa)+\sqrt{2}(\mu_2(\mathfrak{q},\kappa)+\mu_3(\mathfrak{q},\kappa))+2\sqrt{2}(\mu_4(\mathfrak{q},\kappa)+\mu_5(\mathfrak{q},\kappa))<1$;

(c)

$$d(\mathcal{V}\mathfrak{q},\mathcal{W}\kappa)\preceq_{i_2}\mu_1(\mathfrak{q},\kappa)d(\mathfrak{q},\kappa)+\mu_2(\mathfrak{q},\kappa)\frac{d(\mathfrak{q},\mathcal{V}\mathfrak{q})d(\kappa,\mathcal{W}\kappa)}{1+d(\mathfrak{q},\kappa)}+\mu_3(\mathfrak{q},\kappa)\frac{d(\kappa,\mathcal{V}\mathfrak{q})d(\mathfrak{q},\mathcal{W}\kappa)}{1+d(\mathfrak{q},\kappa)}$$

$$+\mu_4(\mathfrak{q},\kappa)\frac{d(\mathfrak{q},\mathcal{V}\mathfrak{q})d(\mathfrak{q},\mathcal{W}\kappa)}{1+d(\mathfrak{q},\kappa)}+\mu_5(\mathfrak{q},\kappa)\frac{d(\kappa,\mathcal{V}\mathfrak{q})d(\kappa,\mathcal{W}\kappa)}{1+d(\mathfrak{q},\kappa)}$$

for all $\mathfrak{q},\kappa\in\mathcal{A}$, *then* $\mathcal{V}$ *and* $\mathcal{W}$ *have a unique common fixed point.*

Proof. Setting $\mu_6,\mu_7:\mathcal{A}\times\mathcal{A}\to[0,1)$ by $\mu_6(\mathfrak{q},\kappa)=\mu_7(\mathfrak{q},\kappa)=0$ in Theorem 1. ☐

Corollary 4. *Let* $(\mathcal{A},\preceq_{i_2},d)$ *be a complete bi-CVMS and* $\mathcal{V},\mathcal{W}:\mathcal{A}\to\mathcal{A}$. *If there exist the mappings* $\mu_i:\mathcal{A}\times\mathcal{A}\to[0,1)$, *where* $i=1,2,\dots 4$, *such that*

(a) $\mu_i(\mathcal{W}\mathcal{V}\mathfrak{q},\kappa)\leq\mu_i(\mathfrak{q},\kappa)$ *and* $\mu_i(\mathfrak{q},\mathcal{V}\mathcal{W}\kappa)\leq\mu_i(\mathfrak{q},\kappa)$;

(b) $\mu_1(\mathfrak{q},\kappa)+\sqrt{2}(\mu_2(\mathfrak{q},\kappa)+\mu_3(\mathfrak{q},\kappa))+2\sqrt{2}\mu_4(\mathfrak{q},\kappa)<1$;

(c)

$$d(\mathcal{V}\mathfrak{q},\mathcal{W}\kappa)\preceq_{i_2}\mu_1(\mathfrak{q},\kappa)d(\mathfrak{q},\kappa)+\mu_2(\mathfrak{q},\kappa)\frac{d(\mathfrak{q},\mathcal{V}\mathfrak{q})d(\kappa,\mathcal{W}\kappa)}{1+d(\mathfrak{q},\kappa)}$$

$$+\mu_3(\mathfrak{q},\kappa)\frac{d(\kappa,\mathcal{V}\mathfrak{q})d(\mathfrak{q},\mathcal{W}\kappa)}{1+d(\mathfrak{q},\kappa)}+\mu_4(\mathfrak{q},\kappa)\frac{d(\mathfrak{q},\mathcal{V}\mathfrak{q})d(\mathfrak{q},\mathcal{W}\kappa)}{1+d(\mathfrak{q},\kappa)}$$

for all $\mathfrak{q},\kappa\in\mathcal{A}$, *then* $\mathcal{V}$ *and* $\mathcal{W}$ *have a unique common fixed point.*

Proof. Setting $\mu_5,\mu_6,\mu_7:\mathcal{A}\times\mathcal{A}\to[0,1)$ by $\mu_6(\mathfrak{q},\kappa)=\mu_7(\mathfrak{q},\kappa)=0$ in Theorem 1. ☐

Our main result generalizes a key finding by Tassaddiq et al. [11] in this way.

Corollary 5 ([11])**.** *Let* $(\mathcal{A},\preceq_{i_2},d)$ *be a complete bi-CVMS and* $\mathcal{V},\mathcal{W}:\mathcal{A}\to\mathcal{A}$. *If there exist the mappings* $\mu_i:\mathcal{A}\times\mathcal{A}\to[0,1)$, *where* $i=1,2,3$, *such that:*

(a) $\mu_i(\mathcal{W}\mathcal{V}\mathfrak{q},\kappa)\leq\mu_i(\mathfrak{q},\kappa)$ *and* $\mu_i(\mathfrak{q},\mathcal{V}\mathcal{W}\kappa)\leq\mu_i(\mathfrak{q},\kappa)$;

(b) $\mu_1(\mathfrak{q},\kappa)+\sqrt{2}(\mu_2(\mathfrak{q},\kappa)+\mu_3(\mathfrak{q},\kappa))<1$;

(c)

$$d(\mathcal{V}\mathfrak{q},\mathcal{W}\kappa)\preceq_{i_2}\mu_1(\mathfrak{q},\kappa)d(\mathfrak{q},\kappa)+\mu_2(\mathfrak{q},\kappa)\frac{d(\mathfrak{q},\mathcal{V}\mathfrak{q})d(\kappa,\mathcal{W}\kappa)}{1+d(\mathfrak{q},\kappa)}+\mu_3(\mathfrak{q},\kappa)\frac{d(\kappa,\mathcal{V}\mathfrak{q})d(\mathfrak{q},\mathcal{W}\kappa)}{1+d(\mathfrak{q},\kappa)}$$

for all $\mathfrak{q},\kappa\in\mathcal{A}$, *then* $\mathcal{V}$ *and* $\mathcal{W}$ *have a unique common fixed point.*

Proof. Setting $\mu_4,\mu_5,\mu_6,\mu_7:\mathcal{A}\times\mathcal{A}\to[0,1)$ by $\mu_4(\mathfrak{q},\kappa)=\mu_5(\mathfrak{q},\kappa)=\mu_6(\mathfrak{q},\kappa)=\mu_7(\mathfrak{q},\kappa)=0$ in Theorem 1. ☐

Corollary 6. *Let* $(\mathcal{A},\preceq_{i_2},d)$ *be a complete bi-CVMS and* $\mathcal{V},\mathcal{W}:\mathcal{A}\to\mathcal{A}$. *If there exist the mappings* $\mu_i:\mathcal{A}\times\mathcal{A}\to[0,1)$, *where* $i=1,2$, *such that*

(a) $\mu_i(\mathcal{W}\mathcal{V}\mathfrak{q},\kappa)\leq\mu_i(\mathfrak{q},\kappa)$ *and* $\mu_i(\mathfrak{q},\mathcal{V}\mathcal{W}\kappa)\leq\mu_i(\mathfrak{q},\kappa)$;

(b) $\mu_1(\mathfrak{q},\kappa)+\sqrt{2}\mu_2(\mathfrak{q},\kappa)<1$;

(c)

$$d(\mathcal{V}\mathfrak{q},\mathcal{W}\kappa)\preceq_{i_2}\mu_1(\mathfrak{q},\kappa)d(\mathfrak{q},\kappa)+\mu_2(\mathfrak{q},\kappa)\frac{d(\mathfrak{q},\mathcal{V}\mathfrak{q})d(\kappa,\mathcal{W}\kappa)}{1+d(\mathfrak{q},\kappa)}$$

for all $\mathfrak{q},\kappa\in\mathcal{A}$, *then* $\mathcal{V}$ *and* $\mathcal{W}$ *have a unique common fixed point.*

Proof. Setting $\mu_3,\mu_4,\mu_5,\mu_6,\mu_7:\mathcal{A}\times\mathcal{A}\to[0,1)$ by $\mu_3(\mathfrak{q},\kappa)=\mu_4(\mathfrak{q},\kappa)=\mu_5(\mathfrak{q},\kappa)=\mu_6(\mathfrak{q},\kappa)=\mu_7(\mathfrak{q},\kappa)=0$ in Theorem 1. ☐

Example 2. *Let* $\mathcal{A} = [0,1]$ *and* $d : \mathcal{A} \times \mathcal{A} \to \mathbb{C}$ *be defined by*

$$d(\mathfrak{q}, \kappa) = |\mathfrak{q} - \kappa| + i_2|\mathfrak{q} - \kappa|$$

for all $\mathfrak{q}, \kappa \in \mathcal{A}$.

Then $(\mathcal{A}, \preceq_{i_2}, d)$ *is a complete bi-CVMS. Define* $\mathcal{V}, \mathcal{W} : \mathcal{A} \to \mathcal{A}$ *by*

$$\mathcal{V}\mathfrak{q} = \frac{\mathfrak{q}}{3} \text{ and } \mathcal{W}\mathfrak{q} = \frac{\mathfrak{q}}{2}$$

Consider

$$\mu_i : \mathcal{A} \times \mathcal{A} \to [0,1)$$

for $i = 1, 2, \ldots, 5$ *by*

$$\mu_1(\mathfrak{q}, \kappa) = \frac{\mathfrak{q}}{5} + \frac{\kappa}{6}$$

$$\mu_2(\mathfrak{q}, \kappa) = \frac{\mathfrak{q}}{8} + \frac{\kappa}{9}$$

$$\mu_3(\mathfrak{q}, \kappa) = \frac{\mathfrak{q}^2\kappa^2}{32}$$

$$\mu_4(\mathfrak{q}, \kappa) = \frac{\mathfrak{q}^2}{16} + \frac{\kappa^2}{25}.$$

Then, evidently,

$$\mu_1(\mathfrak{q}, \kappa) + \sqrt{2}(\mu_2(\mathfrak{q}, \kappa) + \mu_3(\mathfrak{q}, \kappa)) + 2\sqrt{2}\mu_4(\mathfrak{q}, \kappa) < 1.$$

Now

$$\mu_1(\mathcal{W}\mathcal{V}\mathfrak{q}, \kappa) = \mu_1\Big(\mathcal{W}(\frac{\mathfrak{q}}{3}), \kappa\Big) = \mu_1\Big(\frac{\mathfrak{q}}{6}, \kappa\Big) = \frac{\mathfrak{q}}{30} + \frac{\kappa}{6} \leq \frac{\mathfrak{q}}{5} + \frac{\kappa}{6} = \mu_1(\mathfrak{q}, \kappa)$$

and

$$\mu_1(\mathfrak{q}, \mathcal{V}\mathcal{W}\kappa) = \mu_1\Big(\mathfrak{q}, \mathcal{V}(\frac{\kappa}{2})\Big) = \mu_1\Big(\mathfrak{q}, \frac{\kappa}{6}\Big) = \frac{\mathfrak{q}}{5} + \frac{\kappa}{36} \leq \frac{\mathfrak{q}}{5} + \frac{\kappa}{6} = \mu_1(\mathfrak{q}, \kappa).$$

Also,

$$\mu_2(\mathcal{W}\mathcal{V}\mathfrak{q}, \kappa) = \mu_2\Big(\mathcal{W}(\frac{\mathfrak{q}}{3}), \kappa\Big) = \mu_2\Big(\frac{\mathfrak{q}}{6}, \kappa\Big) = \frac{\mathfrak{q}}{48} + \frac{\kappa}{9} \leq \frac{\mathfrak{q}}{8} + \frac{\kappa}{9} = \mu_2(\mathfrak{q}, \kappa)$$

and

$$\mu_2(\mathfrak{q}, \mathcal{V}\mathcal{W}\kappa) = \mu_2\Big(\mathfrak{q}, \mathcal{V}(\frac{\kappa}{2})\Big) = \mu_2\Big(\mathfrak{q}, \frac{\kappa}{6}\Big) = \frac{\mathfrak{q}}{8} + \frac{\kappa}{54} \leq \frac{\mathfrak{q}}{8} + \frac{\kappa}{9} = \mu_2(\mathfrak{q}, \kappa)$$

and

$$\mu_3(\mathcal{W}\mathcal{V}\mathfrak{q}, \kappa) = \mu_3\Big(\mathcal{W}(\frac{\mathfrak{q}}{3}), \kappa\Big) = \mu_3\Big(\frac{\mathfrak{q}}{6}, \kappa\Big) = \frac{\mathfrak{q}^2\kappa^2}{192} \leq \frac{\mathfrak{q}^2\kappa^2}{32} = \mu_3(\mathfrak{q}, \kappa)$$

$$\mu_3(\mathfrak{q}, \mathcal{V}\mathcal{W}\kappa) = \mu_3\Big(\mathfrak{q}, \mathcal{V}(\frac{\kappa}{2})\Big) = \mu_3\Big(\mathfrak{q}, \frac{\kappa}{6}\Big) = \frac{\mathfrak{q}^2\kappa^2}{192} \leq \frac{\mathfrak{q}^2\kappa^2}{32} = \mu_3(\mathfrak{q}, \kappa)$$

and

$$\mu_4(\mathcal{W}\mathcal{V}\mathfrak{q}, \kappa) = \mu_4\Big(\mathcal{W}(\frac{\mathfrak{q}}{3}), \kappa\Big) = \mu_4\Big(\frac{\mathfrak{q}}{6}, \kappa\Big) = \frac{\mathfrak{q}^2}{96} + \frac{\kappa^2}{25} \leq \frac{\mathfrak{q}^2}{16} + \frac{\kappa^2}{25} = \mu_4(\mathfrak{q}, \kappa)$$

$$\mu_4(\mathfrak{q}, \mathcal{V}\mathcal{W}\kappa) = \mu_4\Big(\mathfrak{q}, \mathcal{V}(\frac{\kappa}{2})\Big) = \mu_4\Big(\mathfrak{q}, \frac{\kappa}{6}\Big) = \frac{\mathfrak{q}^2}{96} + \frac{\kappa^2}{150} \leq \frac{\mathfrak{q}^2}{16} + \frac{\kappa^2}{25} = \mu_4(\mathfrak{q}, \kappa).$$

Now,

$$\begin{aligned} d(\mathcal{V}\mathfrak{q}, \mathcal{W}\kappa) &= d(\frac{\mathfrak{q}}{3}, \frac{\kappa}{2}) = \Big|\frac{\mathfrak{q}}{3} - \frac{\kappa}{2}\Big| + i_2\Big|\frac{\mathfrak{q}}{3} - \frac{\kappa}{2}\Big| \\ &= \Big|\frac{2\mathfrak{q} - 3\kappa}{6}\Big| + i_2\Big|\frac{2\mathfrak{q} - 3\kappa}{6}\Big| \end{aligned}$$

$$
\begin{aligned}
&\preceq_{i_2} \left|\frac{2\mathfrak{q}-2\kappa}{6}\right| + i_2\left|\frac{2\mathfrak{q}-2\kappa}{3}\right| \\
&= \frac{1}{3}(|\mathfrak{q}-\kappa| + i_2|\mathfrak{q}-\kappa|) \\
&\preceq_{i_2} \frac{11}{30}(|\mathfrak{q}-\kappa| + i_2|\mathfrak{q}-\kappa|) \\
&\preceq_{i_2} \mu_1(\mathfrak{q},\kappa)d(\mathfrak{q},\kappa) + \mu_2(\mathfrak{q},\kappa)\frac{d(\mathfrak{q},\mathcal{V}\mathfrak{q})d(\kappa,\mathcal{W}\kappa)}{1+d(\mathfrak{q},\kappa)} \\
&+ \mu_3(\mathfrak{q},\kappa)\frac{d(\kappa,\mathcal{V}\mathfrak{q})d(\mathfrak{q},\mathcal{W}\kappa)}{1+d(\mathfrak{q},\kappa)} + \mu_4(\mathfrak{q},\kappa)\frac{d(\mathfrak{q},\mathcal{V}\mathfrak{q})d(\mathfrak{q},\mathcal{W}\kappa)}{1+d(\mathfrak{q},\kappa)}.
\end{aligned}
$$

Then it is very simple to prove that all the conditions of Corollary 4 are satisfied and 0 *is a common fixed point of mappings* $\mathcal{V}$ *and* $\mathcal{W}$.

Corollary 7. *Let* $(\mathcal{A},\preceq_{i_2},d)$ *be a complete bi-CVMS and* $\mathcal{V}:\mathcal{A}\to\mathcal{A}$. *If there exist the mappings* $\mu_i:\mathcal{A}\times\mathcal{A}\to[0,1)$, *where* $i=1,2,\ldots 7$, *such that*

(a) $\mu_i(\mathcal{V}\mathfrak{q},\kappa)\le\mu_i(\mathfrak{q},\kappa)$ *and* $\mu_i(\mathfrak{q},\mathcal{V}\kappa)\le\mu_i(\mathfrak{q},\kappa)$;

(b) $\mu_1(\mathfrak{q},\kappa)+\sqrt{2}(\mu_2(\mathfrak{q},\kappa)+\mu_3(\mathfrak{q},\kappa))+2\sqrt{2}(\mu_4(\mathfrak{q},\kappa)+\mu_5(\mathfrak{q},\kappa))+2(\mu_6(\mathfrak{q},\kappa)+\mu_7(\mathfrak{q},\kappa))< 1,$;

(c)

$$
\begin{aligned}
d(\mathcal{V}^r\mathfrak{q},\mathcal{V}^r\kappa) \preceq_{i_2} &\ \mu_1(\mathfrak{q},\kappa)d(\mathfrak{q},\kappa) + \mu_2(\mathfrak{q},\kappa)\frac{d(\mathfrak{q},\mathcal{V}^r\mathfrak{q})d(\kappa,\mathcal{V}^r\kappa)}{1+d(\mathfrak{q},\kappa)} + \mu_3(\mathfrak{q},\kappa)\frac{d(\kappa,\mathcal{V}^r\mathfrak{q})d(\mathfrak{q},\mathcal{V}^r\kappa)}{1+d(\mathfrak{q},\kappa)} \\
&+\mu_4(\mathfrak{q},\kappa)\frac{d(\mathfrak{q},\mathcal{V}^r\mathfrak{q})d(\mathfrak{q},\mathcal{V}^r\kappa)}{1+d(\mathfrak{q},\kappa)} + \mu_5(\mathfrak{q},\kappa)\frac{d(\kappa,\mathcal{V}^r\mathfrak{q})d(\kappa,\mathcal{V}^r\kappa)}{1+d(\mathfrak{q},\kappa)} \\
&+\mu_6(\mathfrak{q},\kappa)\frac{d(\mathfrak{q},\mathcal{V}^r\kappa)[1+d(\mathfrak{q},\mathcal{V}^r\mathfrak{q})]}{1+d(\mathfrak{q},\kappa)} + \mu_7(\mathfrak{q},\kappa)\frac{d(\kappa,\mathcal{V}^r\mathfrak{q})[1+d(\kappa,\mathcal{V}^r\kappa)]}{1+d(\mathfrak{q},\kappa)}
\end{aligned}
$$

for all $\mathfrak{q},\kappa\in\mathcal{A}$, *then* $\mathcal{V}$ *has a unique fixed point.*

Proof. From Corollary 1, we have $\mathfrak{q}\in\mathcal{A}$ such that $\mathcal{V}^r\mathfrak{q}=\mathfrak{q}$. Now, from

$$
\begin{aligned}
d(\mathcal{V}\mathfrak{q},\mathfrak{q}) &= d(\mathcal{V}\mathcal{V}^r\mathfrak{q},\mathcal{V}^r\mathfrak{q}) \\
&= d(\mathcal{V}^r\mathcal{V}\mathfrak{q},\mathcal{V}^r\mathfrak{q}) \preceq_{i_2} \mu_1(\mathcal{V}\mathfrak{q},\mathfrak{q})d(\mathcal{V}\mathfrak{q},\mathfrak{q}) + \mu_2(\mathcal{V}\mathfrak{q},\mathfrak{q})\frac{d(\mathcal{V}\mathfrak{q},\mathcal{V}^r\mathcal{V}\mathfrak{q})d(\mathfrak{q},\mathfrak{q})}{1+d(\mathcal{V}\mathfrak{q},\mathfrak{q})} \\
&\quad+ \mu_3(\mathcal{V}\mathfrak{q},\mathfrak{q})\frac{d(\mathfrak{q},\mathcal{V}^r\mathcal{V}\mathfrak{q})d(\mathcal{V}\mathfrak{q},\mathcal{V}^r\mathfrak{q})}{1+d(\mathcal{V}\mathfrak{q},\mathfrak{q})} \\
&\quad+\mu_4(\mathcal{V}\mathfrak{q},\mathfrak{q})\frac{d(\mathcal{V}\mathfrak{q},\mathcal{V}^r\mathcal{V}\mathfrak{q})d(\mathcal{V}\mathfrak{q},\mathcal{V}^r\mathfrak{q})}{1+d(\mathcal{V}\mathfrak{q},\mathfrak{q})} + \mu_5(\mathcal{V}\mathfrak{q},\mathfrak{q})\frac{d(\mathfrak{q},\mathcal{V}^r\mathcal{V}\mathfrak{q})d(\mathfrak{q},\mathcal{V}^r\mathfrak{q})}{1+d(\mathcal{V}\mathfrak{q},\mathfrak{q})} \\
&\quad+\mu_6(\mathcal{V}\mathfrak{q},\mathfrak{q})\frac{d(\mathcal{V}\mathfrak{q},\mathcal{V}^r\mathfrak{q})[1+d(\mathcal{V}\mathfrak{q},\mathcal{V}^r\mathcal{V}\mathfrak{q})]}{1+d(\mathcal{V}\mathfrak{q},\mathfrak{q})} + \mu_7(\mathcal{V}\mathfrak{q},\mathfrak{q})\frac{d(\mathfrak{q},\mathcal{V}^r\mathcal{V}\mathfrak{q})[1+d(\mathfrak{q},\mathcal{V}^r\mathfrak{q})]}{1+d(\mathcal{V}\mathfrak{q},\mathfrak{q})} \\
&= \mu_1(\mathcal{V}\mathfrak{q},\mathfrak{q})d(\mathcal{V}\mathfrak{q},\mathfrak{q}) + \mu_2(\mathcal{V}\mathfrak{q},\mathfrak{q})\frac{d(\mathcal{V}\mathfrak{q},\mathcal{V}\mathfrak{q})d(\mathfrak{q},\mathfrak{q})}{1+d(\mathcal{V}\mathfrak{q},\mathfrak{q})} \\
&\quad+\mu_3(\mathcal{V}\mathfrak{q},\mathfrak{q})\frac{d(\mathfrak{q},\mathcal{V}\mathfrak{q})d(\mathcal{V}\mathfrak{q},\mathfrak{q})}{1+d(\mathcal{V}\mathfrak{q},\mathfrak{q})} \\
&\quad+\mu_4(\mathcal{V}\mathfrak{q},\mathfrak{q})\frac{d(\mathcal{V}\mathfrak{q},\mathcal{V}\mathfrak{q})d(\mathcal{V}\mathfrak{q},\mathfrak{q})}{1+d(\mathcal{V}\mathfrak{q},\mathfrak{q})} + \mu_5(\mathcal{V}\mathfrak{q},\mathfrak{q})\frac{d(\mathfrak{q},\mathcal{V}\mathfrak{q})d(\mathfrak{q},\mathfrak{q})}{1+d(\mathcal{V}\mathfrak{q},\mathfrak{q})} \\
&\quad+\mu_6(\mathcal{V}\mathfrak{q},\mathfrak{q})\frac{d(\mathcal{V}\mathfrak{q},\mathfrak{q})[1+d(\mathcal{V}\mathfrak{q},\mathcal{V}\mathfrak{q})]}{1+d(\mathcal{V}\mathfrak{q},\mathfrak{q})} + \mu_7(\mathcal{V}\mathfrak{q},\mathfrak{q})\frac{d(\mathfrak{q},\mathcal{V}\mathfrak{q})[1+d(\mathfrak{q},\mathfrak{q})]}{1+d(\mathcal{V}\mathfrak{q},\mathfrak{q})} \\
&= \mu_1(\mathcal{V}\mathfrak{q},\mathfrak{q})d(\mathcal{V}\mathfrak{q},\mathfrak{q}) + \mu_3(\mathcal{V}\mathfrak{q},\mathfrak{q})\frac{d(\mathfrak{q},\mathcal{V}\mathfrak{q})d(\mathcal{V}\mathfrak{q},\mathfrak{q})}{1+d(\mathcal{V}\mathfrak{q},\mathfrak{q})} + \mu_6(\mathcal{V}\mathfrak{q},\mathfrak{q})\frac{d(\mathcal{V}\mathfrak{q},\mathfrak{q})}{1+d(\mathcal{V}\mathfrak{q},\mathfrak{q})} \\
&\quad+\mu_7(\mathcal{V}\mathfrak{q},\mathfrak{q})\frac{d(\mathfrak{q},\mathcal{V}\mathfrak{q})}{1+d(\mathcal{V}\mathfrak{q},\mathfrak{q})}
\end{aligned}
$$

which implies that

$$\begin{aligned}\|d(\mathcal{V}\mathfrak{q},\mathfrak{q})\| &\leq \mu_1(\mathcal{V}\mathfrak{q},\mathfrak{q})\|d(\mathcal{V}\mathfrak{q},\mathfrak{q})\| + \sqrt{2}\mu_3(\mathcal{V}\mathfrak{q},\mathfrak{q})\|d(\mathfrak{q},\mathcal{V}\mathfrak{q})\|\frac{\|d(\mathcal{V}\mathfrak{q},\mathfrak{q})\|}{\|1+d(\mathcal{V}\mathfrak{q},\mathfrak{q})\|} \\ &\quad +\mu_6(\mathcal{V}\mathfrak{q},\mathfrak{q})\frac{\|d(\mathcal{V}\mathfrak{q},\mathfrak{q})\|}{\|1+d(\mathcal{V}\mathfrak{q},\mathfrak{q})\|} + \mu_7(\mathcal{V}\mathfrak{q},\mathfrak{q})\frac{\|d(\mathfrak{q},\mathcal{V}\mathfrak{q})\|}{\|1+d(\mathcal{V}\mathfrak{q},\mathfrak{q})\|}.\end{aligned}$$

Since $\|d(\mathcal{V}\mathfrak{q},\mathfrak{q})\| \leq \|1+d(\mathcal{V}\mathfrak{q},\mathfrak{q})\|$ and $1 \leq \|1+d(\mathcal{V}\mathfrak{q},\mathfrak{q})\|$, so $\frac{\|d(\mathcal{V}\mathfrak{q},\mathfrak{q})\|}{\|1+d(\mathcal{V}\mathfrak{q},\mathfrak{q})\|} \leq 1$ and $\frac{1}{\|1+d(\mathcal{V}\mathfrak{q},\mathfrak{q})\|} \leq 1$. Thus, we have

$$\begin{aligned}\|d(\mathcal{V}\mathfrak{q},\mathfrak{q})\| &\leq \mu_1(\mathcal{V}\mathfrak{q},\mathfrak{q})\|d(\mathcal{V}\mathfrak{q},\mathfrak{q})\| + \sqrt{2}\mu_3(\mathcal{V}\mathfrak{q},\mathfrak{q})\|d(\mathfrak{q},\mathcal{V}\mathfrak{q})\| \\ &\quad +\mu_6(\mathcal{V}\mathfrak{q},\mathfrak{q})\|d(\mathcal{V}\mathfrak{q},\mathfrak{q})\| + \mu_7(\mathcal{V}\mathfrak{q},\mathfrak{q})\|d(\mathfrak{q},\mathcal{V}\mathfrak{q})\| \\ &= \Big(\mu_1(\mathcal{V}\mathfrak{q},\mathfrak{q}) + \sqrt{2}\mu_3(\mathcal{V}\mathfrak{q},\mathfrak{q}) + \mu_6(\mathcal{V}\mathfrak{q},\mathfrak{q}) + \mu_7(\mathcal{V}\mathfrak{q},\mathfrak{q})\Big)\|d(\mathfrak{q},\mathcal{V}\mathfrak{q})\|.\end{aligned}$$

As $\Big(\mu_1(\mathcal{V}\mathfrak{q},\mathfrak{q}) + \sqrt{2}\mu_3(\mathcal{V}\mathfrak{q},\mathfrak{q}) + \mu_6(\mathcal{V}\mathfrak{q},\mathfrak{q}) + \mu_7(\mathcal{V}\mathfrak{q},\mathfrak{q})\Big) < 1$, so it is possible only if $\|d(\mathcal{V}\mathfrak{q},\mathfrak{q})\| = 0$. Thus, $\mathcal{V}\mathfrak{q} = \mathfrak{q}$. □

4. Inferred Findings

Corollary 8. *Let $(\mathcal{A},\preceq_{i_2},d)$ be a complete bi-CVMS and $\mathcal{V},\mathcal{W} : \mathcal{A} \to \mathcal{A}$. If there exist the mappings $\mu_i : \mathcal{A}\to[0,1)$, where $i = 1,2,\dots 7$, such that*

(a) $\mu_i(\mathcal{W}\mathcal{V}\mathfrak{q}) \leq \mu_i(\mathfrak{q})$ *and* $\mu_i(\mathcal{V}\mathcal{W}\kappa) \leq \mu_i(\kappa)$;

(b) $\mu_1(\mathfrak{q})+\sqrt{2}(\mu_2(\mathfrak{q}) + \mu_3(\mathfrak{q}))+2\sqrt{2}(\mu_4(\mathfrak{q}) + \mu_5(\mathfrak{q}))+2(\mu_6(\mathfrak{q}) + \mu_7(\mathfrak{q}))< 1$;

(c)

$$\begin{aligned}d(\mathcal{V}\mathfrak{q},\mathcal{W}\kappa) \preceq_{i_2} \mu_1(\mathfrak{q})d(\mathfrak{q},\kappa) &+ \mu_2(\mathfrak{q})\frac{d(\mathfrak{q},\mathcal{V}\mathfrak{q})d(\kappa,\mathcal{W}\kappa)}{1+d(\mathfrak{q},\kappa)} + \mu_3(\mathfrak{q})\frac{d(\kappa,\mathcal{V}\mathfrak{q})d(\mathfrak{q},\mathcal{W}\kappa)}{1+d(\mathfrak{q},\kappa)} \\ &+\mu_4(\mathfrak{q})\frac{d(\mathfrak{q},\mathcal{V}\mathfrak{q})d(\mathfrak{q},\mathcal{W}\kappa)}{1+d(\mathfrak{q},\kappa)} + \mu_5(\mathfrak{q})\frac{d(\kappa,\mathcal{V}\mathfrak{q})d(\kappa,\mathcal{W}\kappa)}{1+d(\mathfrak{q},\kappa)} \\ &+\mu_6(\mathfrak{q})\frac{d(\mathfrak{q},\mathcal{W}\kappa)[1+d(\mathfrak{q},\mathcal{V}\mathfrak{q})]}{1+d(\mathfrak{q},\kappa)} + \mu_7(\mathfrak{q})\frac{d(\kappa,\mathcal{V}\mathfrak{q})[1+d(\kappa,\mathcal{W}\kappa)]}{1+d(\mathfrak{q},\kappa)},\end{aligned}$$

for all $\mathfrak{q},\kappa \in \mathcal{A}$, then $\mathcal{V}$ and $\mathcal{W}$ have a unique common fixed point.

Proof. Define $\mu_i : \mathcal{A}\times\mathcal{A}\to[0,1)$ by

$$\mu_i(\mathfrak{q},\kappa) = \mu_i(\mathfrak{q})$$

for all $\mathfrak{q},\kappa \in \mathcal{A}$. Then for all $\mathfrak{q},\kappa \in \mathcal{A}$, we have

(a) $\mu_i(\mathcal{W}\mathcal{V}\mathfrak{q},\kappa) = \mu_i(\mathcal{W}\mathcal{V}\mathfrak{q}) \leq \mu_i(\mathfrak{q}) = \mu_i(\mathfrak{q},\kappa)$ and $\mu_i(\mathfrak{q},\mathcal{V}\mathcal{W}\kappa) = \mu_i(\mathfrak{q}) = \mu_i(\mathfrak{q},\kappa)$;

(b)

$$\begin{aligned}&\mu_1(\mathfrak{q},\kappa) + \sqrt{2}(\mu_2(\mathfrak{q},\kappa) + \mu_3(\mathfrak{q},\kappa)) + 2\sqrt{2}(\mu_4(\mathfrak{q},\kappa) + \mu_5(\mathfrak{q},\kappa)) + 2(\mu_6(\mathfrak{q},\kappa) + \mu_7(\mathfrak{q},\kappa)) \\ &= \mu_1(\mathfrak{q}) + \sqrt{2}(\mu_2(\mathfrak{q}) + \mu_3(\mathfrak{q})) + 2\sqrt{2}(\mu_4(\mathfrak{q}) + \mu_5(\mathfrak{q})) + 2(\mu_6(\mathfrak{q}) + \mu_7(\mathfrak{q})) < 1;\end{aligned}$$

(c)

$$\begin{aligned}d(\mathcal{V}\mathfrak{q},\mathcal{W}\kappa) &\preceq {}_{i_2}\mu_1(\mathfrak{q})d(\mathfrak{q},\kappa) + \mu_2(\mathfrak{q})\frac{d(\mathfrak{q},\mathcal{V}\mathfrak{q})d(\kappa,\mathcal{W}\kappa)}{1+d(\mathfrak{q},\kappa)} + \mu_3(\mathfrak{q})\frac{d(\kappa,\mathcal{V}\mathfrak{q})d(\mathfrak{q},\mathcal{W}\kappa)}{1+d(\mathfrak{q},\kappa)} \\ &\quad +\mu_4(\mathfrak{q})\frac{d(\mathfrak{q},\mathcal{V}\mathfrak{q})d(\mathfrak{q},\mathcal{W}\kappa)}{1+d(\mathfrak{q},\kappa)} + \mu_5(\mathfrak{q})\frac{d(\kappa,\mathcal{V}\mathfrak{q})d(\kappa,\mathcal{W}\kappa)}{1+d(\mathfrak{q},\kappa)} \\ &\quad +\mu_6(\mathfrak{q})\frac{d(\mathfrak{q},\mathcal{W}\kappa)[1+d(\mathfrak{q},\mathcal{V}\mathfrak{q})]}{1+d(\mathfrak{q},\kappa)} + \mu_7(\mathfrak{q})\frac{d(\kappa,\mathcal{V}\mathfrak{q})[1+d(\kappa,\mathcal{W}\kappa)]}{1+d(\mathfrak{q},\kappa)}\end{aligned}$$

$$\begin{aligned}
&= \mu_1(\mathfrak{q},\kappa)d(\mathfrak{q},\kappa)+\mu_2(\mathfrak{q},\kappa)\frac{d(\mathfrak{q},\mathcal{V}\mathfrak{q})d(\kappa,\mathcal{W}\kappa)}{1+d(\mathfrak{q},\kappa)}+\mu_3(\mathfrak{q},\kappa)\frac{d(\kappa,\mathcal{V}\mathfrak{q})d(\mathfrak{q},\mathcal{W}\kappa)}{1+d(\mathfrak{q},\kappa)}\\
&\quad+\mu_4(\mathfrak{q},\kappa)\frac{d(\mathfrak{q},\mathcal{V}\mathfrak{q})d(\mathfrak{q},\mathcal{W}\kappa)}{1+d(\mathfrak{q},\kappa)}+\mu_5(\mathfrak{q},\kappa)\frac{d(\kappa,\mathcal{V}\mathfrak{q})d(\kappa,\mathcal{W}\kappa)}{1+d(\mathfrak{q},\kappa)}\\
&\quad+\mu_6(\mathfrak{q},\kappa)\frac{d(\mathfrak{q},\mathcal{W}\kappa)[1+d(\mathfrak{q},\mathcal{V}\mathfrak{q})]}{1+d(\mathfrak{q},\kappa)}+\mu_7(\mathfrak{q},\kappa)\frac{d(\kappa,\mathcal{V}\mathfrak{q})[1+d(\kappa,\mathcal{W}\kappa)]}{1+d(\mathfrak{q},\kappa)}.
\end{aligned}$$

By Theorem 1, $\mathcal{V}$ and $\mathcal{W}$ have a unique common fixed point. □

Corollary 9. *Let $(\mathcal{A},\preceq_{i_2},d)$ be a complete bi-CVMS, and let $\mathcal{V},\mathcal{W}:\mathcal{A}\to\mathcal{A}$. If there exist the constants $\mu_i\in[0,1)$ for $i=1,2,\ldots,7$ with $\mu_1+\sqrt{2}(\mu_2+\mu_3)+2\sqrt{2}(\mu_4+\mu_5)+2(\mu_6+\mu_7)<1$, such that*

$$\begin{aligned}
d(\mathcal{V}\mathfrak{q},\mathcal{W}\kappa) \preceq_{i_2} \mu_1 d(\mathfrak{q},\kappa)+\mu_2\frac{d(\mathfrak{q},\mathcal{V}\mathfrak{q})d(\kappa,\mathcal{W}\kappa)}{1+d(\mathfrak{q},\kappa)}+\mu_3\frac{d(\kappa,\mathcal{V}\mathfrak{q})d(\mathfrak{q},\mathcal{W}\kappa)}{1+d(\mathfrak{q},\kappa)}\\
+\mu_4\frac{d(\mathfrak{q},\mathcal{V}\mathfrak{q})d(\mathfrak{q},\mathcal{W}\kappa)}{1+d(\mathfrak{q},\kappa)}+\mu_5\frac{d(\kappa,\mathcal{V}\mathfrak{q})d(\kappa,\mathcal{W}\kappa)}{1+d(\mathfrak{q},\kappa)}\\
+\mu_6\frac{d(\mathfrak{q},\mathcal{W}\kappa)[1+d(\mathfrak{q},\mathcal{V}\mathfrak{q})]}{1+d(\mathfrak{q},\kappa)}+\mu_7\frac{d(\kappa,\mathcal{V}\mathfrak{q})[1+d(\kappa,\mathcal{W}\kappa)]}{1+d(\mathfrak{q},\kappa)},
\end{aligned}$$

for all $\mathfrak{q},\kappa\in\mathcal{A}$, then $\mathcal{V}$ and $\mathcal{W}$ have a unique common fixed point.

Proof. Take $\mu_i(\cdot)=\mu_i$, where $i=1,2,\ldots,7$, in Corollary 8. □

Now we present more general results, which, subsequently, recover some results from Gnanaprakasam et al. [8] as a special case.

Corollary 10 ([8])**.** *Let $(\mathcal{A},\preceq_{i_2},d)$ be a complete bi-CVMS, and let $\mathcal{V},\mathcal{W}:\mathcal{A}\to\mathcal{A}$. If there exist the constants $\mu_1,\mu_2,\mu_3\in[0,1)$ with $\mu_1+\sqrt{2}\mu_2+\sqrt{2}\mu_3<1$ such that*

$$d(\mathcal{V}\mathfrak{q},\mathcal{W}\kappa) \preceq_{i_2} \mu_1 d(\mathfrak{q},\kappa)+\mu_2\frac{d(\mathfrak{q},\mathcal{V}\mathfrak{q})d(\kappa,\mathcal{W}\kappa)}{1+d(\mathfrak{q},\kappa)}+\mu_3\frac{d(\kappa,\mathcal{V}\mathfrak{q})d(\mathfrak{q},\mathcal{W}\kappa)}{1+d(\mathfrak{q},\kappa)}$$

for all $\mathfrak{q},\kappa\in\mathcal{A}$, then $\mathcal{V}$ and $\mathcal{W}$ have a unique common fixed point.

Proof. Take $\mu_i=0$, where $i=4,5,6,7$, in Corollary 9. □

Corollary 11 ([8])**.** *Let $(\mathcal{A},\preceq_{i_2},d)$ be a complete bi-CVMS, and let $\mathcal{V}:\mathcal{A}\to\mathcal{A}$. If there exist the constants $\mu_1,\mu_2,\mu_3\in[0,1)$, with $\mu_1+\sqrt{2}\mu_2+\sqrt{2}\mu_3<1$, such that*

$$d(\mathcal{V}\mathfrak{q},\mathcal{V}\kappa) \preceq_{i_2} \mu_1 d(\mathfrak{q},\kappa)+\mu_2\frac{d(\mathfrak{q},\mathcal{V}\mathfrak{q})d(\kappa,\mathcal{V}\kappa)}{1+d(\mathfrak{q},\kappa)}+\mu_3\frac{d(\kappa,\mathcal{V}\mathfrak{q})d(\mathfrak{q},\mathcal{V}\kappa)}{1+d(\mathfrak{q},\kappa)}$$

for all $\mathfrak{q},\kappa\in\mathcal{A}$, then $\mathcal{V}$ has a unique fixed point.

Proof. Set $\mathcal{W}=\mathcal{V}$ in Corollary 10. □

In the following way, we derive the principal result established by Beg et al. [7].

Corollary 12 ([7])**.** *Let $(\mathcal{A},\preceq_{i_2},d)$ be a complete bi-CVMS, and let $\mathcal{V},\mathcal{W}:\mathcal{A}\to\mathcal{A}$. If there exist $\mu_1,\mu_2\in[0,1)$, with $\mu_1+\sqrt{2}\mu_2<1$, such that*

$$d(\mathcal{V}\mathfrak{q},\mathcal{W}\kappa) \preceq_{i_2} \mu_1 d(\mathfrak{q},\kappa)+\mu_2\frac{d(\mathfrak{q},\mathcal{V}\mathfrak{q})d(\kappa,\mathcal{W}\kappa)}{1+d(\mathfrak{q},\kappa)}$$

for all $\mathfrak{q},\kappa\in\mathcal{A}$, then $\mathcal{V}$ and $\mathcal{W}$ have a unique common fixed point.

Proof. Take $\mu_i = 0$, where $i = 3, 4, 5, 6, 7$, in Corollary 9. $\square$

5. Applications

The powerful tools of fixed point theory have proven immensely valuable for tackling various problems across diverse mathematical disciplines. In this section, we demonstrate the applicability of the established fixed point results to the realm of integral equations. These integral equations play a crucial role in numerous scientific and engineering fields, including population dynamics, heat transfer, finance, and many more. These equations often lack closed-form solutions, necessitating the use of alternative methods to establish their existence and uniqueness. Here, we will use the framework developed in the previous section (Corollary 4) to address a specific type of integral equation named the Volterra integral equation of the second kind. By translating the integral equation into a fixed point problem, we will demonstrate how our fixed point theorems can be utilized to guarantee the existence and, potentially, the uniqueness of solutions for this equation.

We consider the space $\mathcal{A}$ consisting of all real-valued continuous functions defined on the closed interval $[0,1]$, denoted by $(C[0,1], \mathbb{R})$. Furthermore, define a distance function $d : \mathcal{A} \times \mathcal{A} \to \mathbb{C}_2$ as follows:

$$d(\mathfrak{q}, \kappa) = \max_{t \in [0,1]} (1+i)(|\mathfrak{q}(t) - \kappa(t)|)$$

for all $\mathfrak{q}, \kappa \in \mathcal{A}$ and $t \in [0,1]$. Then $(\mathcal{A}, \preceq_{i_2}, d)$ qualifies as a complete bi-CVMS. Consider

$$\mathfrak{q}(t) = \int_0^1 K_1(t, s, \mathfrak{q}(s))ds + g(t), \tag{5}$$

$$\mathfrak{q}(t) = \int_0^1 K_2(t, s, \mathfrak{q}(s))ds + g(t), \tag{6}$$

where $g : [0,1] \to \mathbb{R}$ and $K_1, K_2 : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ are continuous for $t \in [0,1]$. Within the set of bicomplex numbers $\mathbb{C}_2$, we introduce a partial order denoted by $\preceq_{i_2}$. This relation holds between two elements $\mathfrak{q}(t)$ and $\kappa(t)$ if and only if $\mathfrak{q} \leq \kappa$.

Theorem 2. *Suppose the following condition*

$$|K_1(t, s, \mathfrak{q}(s)) - K_2(t, s, \kappa(s))| \leq \mu_1(\mathfrak{q}, \kappa)|\mathfrak{q}(s) - \kappa(s)|$$

holds for all $\mathfrak{q}, \kappa \in \mathcal{A}$, with $\mathfrak{q} \neq \kappa$ and for some control function $\mu_1 : \mathcal{A} \times \mathcal{A} \to [0,1)$; then the integral operators defined by (5) and (6) possess a unique solution that is common to both operators.

Proof. Define the continuous mappings $\mathcal{V}, \mathcal{W} : \mathcal{A} \to \mathcal{A}$ by

$$\mathcal{V}\mathfrak{q}(t) = \int_0^1 K_1(t, s, \mathfrak{q}(s))ds + g(t),$$

$$\mathcal{W}\mathfrak{q}(t) = \int_0^1 K_2(t, s, \mathfrak{q}(s))ds + g(t),$$

for all $t \in [0,1]$. Consider

$$\begin{aligned} d(\mathcal{V}\mathfrak{q}, \mathcal{W}\kappa) &= \max_{t \in [0,1]} (1+i_2)|\mathcal{V}\mathfrak{q}(t) - T\kappa(t)| \\ &= \max_{t \in [0,1]} (1+i_2)\left(\left|\int_0^1 K_1(t, s, \mathfrak{q}(s))ds - \int_0^1 K_2(t, s, \kappa(s))ds\right|\right) \\ &\preceq_{i_2} \max_{t \in [0,1]} (1+i_2)\left(\int_0^1 |K_1(t, s, \mathfrak{q}(s)) - K_2(t, s, \kappa(s))|ds\right) \end{aligned}$$

$$
\begin{aligned}
&\precsim_{i_2} \max_{t\in[0,1]} (1+i_2)\left(\mu_1(\mathfrak{q},\kappa)\int_0^1 |\mathfrak{q}(s)-\kappa(s)|ds\right)\\
&\precsim_{i_2} \mu_1(\mathfrak{q},\kappa)d(\mathfrak{q},\kappa)+\mu_2(\mathfrak{q},\kappa)\frac{d(\mathfrak{q},\mathcal{V}\mathfrak{q})d(\kappa,\mathcal{W}\kappa)}{1+d(\mathfrak{q},\kappa)}\\
&+\mu_3(\mathfrak{q},\kappa)\frac{d(\kappa,\mathcal{V}\mathfrak{q})d(\mathfrak{q},\mathcal{W}\kappa)}{1+d(\mathfrak{q},\kappa)}+\mu_4(\mathfrak{q},\kappa)\frac{d(\mathfrak{q},\mathcal{V}\mathfrak{q})d(\mathfrak{q},\mathcal{W}\kappa)}{1+d(\mathfrak{q},\kappa)}\\
&+\mu_5(\mathfrak{q},\kappa)\frac{d(\kappa,\mathcal{V}\mathfrak{q})d(\kappa,\mathcal{W}\kappa)}{1+d(\mathfrak{q},\kappa)}+\mu_6(\mathfrak{q},\kappa)\frac{d(\mathfrak{q},\mathcal{W}\kappa)[1+d(\mathfrak{q},\mathcal{V}\mathfrak{q})]}{1+d(\mathfrak{q},\kappa)}\\
&+\mu_7(\mathfrak{q},\kappa)\frac{d(\kappa,\mathcal{V}\mathfrak{q})[1+d(\kappa,\mathcal{W}\kappa)]}{1+d(\mathfrak{q},\kappa)}
\end{aligned}
$$

for any control functions $\mu_2,\mu_3,\mu_4,\mu_5,\mu_6,\mu_7 : \mathcal{A}\times\mathcal{A}\to[0,1)$. Hence, all the conditions stipulated in Theorem 1 are met; the integral equations defined by (5) and (6) possess a unique common solution. □

6. Conclusions

In the present research article, we have obtained common fixed point results for rational contractions involving control functions of two variables in the framework of bi-CVMS. In this way, we have derived the leading results of Beg et al. [7], Gnanaprakasam et al. [8], and Tassaddiq et al. [11]. Furthermore, to substantiate the validity of our findings, we have presented a non-trivial example that demonstrates the effectiveness of the obtained results. To illustrate the applicability of our main result, we further investigated the existence and uniqueness of solutions for Volterra integral equations of the second kind.

Building upon the recent work of Özgür et al. [23,24] on the fixed-circle problem in metric spaces, a compelling avenue for future research would be to investigate this problem within the context of complete bi-CVMS. Even more generally, exploring suitable contractive conditions to establish existence and uniqueness theorems for fixed circles of self-mappings in these spaces holds significant promise for uncovering geometric interpretations and expanding the theory.

Funding: This research was funded by the University of Jeddah, grant number UJ-23-DR-31.

Data Availability Statement: All data required for this research are included within this paper.

Acknowledgments: This work was funded by the University of Jeddah, Jeddah, Saudi Arabia, under grant (No. UJ-23-DR-31). The author, therefore, thanks the University of Jeddah for its technical and financial support.

Conflicts of Interest: The author declare no conflicts of interest.

References

1. Segre, C. Le Rappresentazioni Reali delle Forme Complesse a Gli Enti Iperalgebrici. *Math. Ann.* **1892**, *40*, 413–467. [CrossRef]
2. Price, G.B. *An Introduction to Multicomplex Spaces and Functions*; Marcel Dekker: New York, NY, USA, 1991.
3. Waugh, M.A. *Introduction to Bicomplex Numbers*; University of Delhi: New Delhi, India, 2018.
4. Azam, A.; Fisher, B.; Khan, M. Common fixed point theorems in complex valued metric spaces. *Num. Funct. Anal. Optim.* **2011**, *32*, 243–253. [CrossRef]
5. Choi, J.; Datta, S.K.; Biswas, T.; Islam, N. Some fixed point theorems in connection with two weakly compatible mappings in bicomplex valued metric spaces. *Honam. Math. J.* **2017**, *39*, 115–126. [CrossRef]
6. Jebril, I.H.; Datta, S.K.; Sarkar, R.; Biswas, N. Common fixed point theorems under rational contractions for a pair of mappings in bicomplex valued metric spaces. *J. Interdiscip. Math.* **2020**, *22*, 1071–1082 [CrossRef]
7. Beg, I.; Datta, S.K.; Pal, D. Fixed point in bicomplex valued metric spaces. *Int. J. Nonlinear Anal. Appl.* **2021**, *12*, 717–727.
8. Gnanaprakasam, A.J.; Boulaaras, S.M.; Mani, G.; Cherif, B.; Idris, S.A. Solving system of linear equations via bicomplex valued metric space. *Demonstr. Math.* **2021**, *54*, 474–487. [CrossRef]
9. Abdou, A.A.N. Solving the Fredholm integral equation by common fixed point results in bicomplex valued metric spaces. *Mathematics* **2023**, *11*, 3249. [CrossRef]

10. Gu, Z.; Mani, G.; Gnanaprakasam, A.J.; Li, Y. Solving a system of nonlinear integral equations via common fixed point theorems on bicomplex partial metric space. *Mathematics* **2021**, *9*, 1584. [CrossRef]
11. Tassaddiq, A.; Ahmad, J.; Al-Mazrooei, A.E.; Lateef, D.; Lakhani, F. On common fixed point results in bicomplex valued metric spaces with application. *AIMS Math.* **2023**, *8*, 5522–5539. [CrossRef]
12. Albargi, A.H.; Shammaky, A.E.; Ahmad, J. Common fixed point results in bicomplex valued metric spaces with application. *Mathematics* **2023**, *11*, 1207. [CrossRef]
13. Abdou, A.A.N. Common fixed point theorems for multi-valued mappings in bicomplex valued metric spaces with application. *AIMS Math.* **2023**, *8*, 20154–20168. [CrossRef]
14. Mani, G.; Haque, S.; Gnanaprakasam, A.J.; Ege, O.; Mlaiki, N. The study of bicomplex-valued controlled metric spaces with applications to fractional differential equations. *Mathematics* **2023**, *11*, 2742. [CrossRef]
15. Gurusamy, S. Bicomplex valued bipolar metric spaces and fixed point theorems. *Math. Anal. Its Contemp. Appl.* **2022**, *4*, 29–43.
16. Ramaswamy, R.; Mani, G.; Gnanaprakasam, A.J.; Gnanaprakasam, A.J.; Abdelnaby, O.A.A.; Radenović, S. An application to fixed-point results in tricomplex-valued metric spaces using control functions. *Mathematics* **2022**, *10*, 3344. [CrossRef]
17. Abdullahi, M.S.; Azam, A. Multivalued fixed points results via rational type contractive conditions in complex valued metric spaces. *J. Int. Math. Virtual Inst.* **2017**, *7*, 119–146.
18. Arshad, M.; Fahimuddin; Shoaib, A.; Hussain, A. Fixed point results for α-ψ-locally graphic contraction in dislocated qusai metric spaces. *Math. Sci.* **2014**, *8*, 79–85.
19. Azam, A.; Ahmad, J.; Kumam, P. Common fixed point theorems for multi-valued mappings in complex-valued metric spaces. *J. Inequal. Appl.* **2013**, *2013*, 578. [CrossRef]
20. Mukheimer, A.A. Some Common fixed point theorems in complex valued b-metric spaces. *Sci. World J.* **2014**, *2014*, 587825. [CrossRef]
21. Ullah, N.; Shagari, M.S.; Azam, A. Fixed point theorems in complex valued extended b-metric spaces. *Moroc. J. Pure Appl. Anal.* **2019**, *5*, 140–163. [CrossRef]
22. Carmel Pushpa Raj, J.; Arul Xavier, A.; Maria Joseph, J.; Marudai, M. Common fixed point theorems under rational contractions in complex valued extended b-metric spaces. *Int. J. Nonlinear Anal. Appl.* **2022**, *13*, 3479–3490.
23. Özgür, N.Y.; Tas, N. Some fixed-circle theorems on metric spaces. *Bull. Malays. Math. Sci. Soc.* **2019**, *42*, 1433–1449. [CrossRef]
24. Özgür, N.Y.; Tas, N.; Çelik, U. New fixed-circle results on S-metric spaces. *Bull. Math. Anal. Appl.* **2017**, *9*, 10–23.