Properties of Convex Lattice Sets under the Discrete Legendre Transform

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Abstract: The discrete Legendre transform is a powerful tool for analyzing the properties of convex lattice sets. In this paper, for \( t > 0 \), we study a class of convex lattice sets and establish a relationship between vertices of the polar of convex lattice sets and vertices of the polar of its \( t \)-dilation. Subsequently, we show that there exists a class of convex lattice sets such that its polar is itself. In addition, we calculate upper and lower bounds for the discrete Mahler product of a class of convex lattice sets.

Keywords: convex lattice sets; polar; discrete Legendre transform; discrete Mahler product

MSC: 52C07; 11H06; 52B20

1. Introduction

We denote by \( \mathbb{E}^n \) the \( n \)-dimensional Euclidean space. A convex body \( C \) in \( \mathbb{E}^n \) is a compact convex subset with a nonempty interior. If a convex body \( C \) contains the origin in its interior, then the polar body \( C^* \) is defined by

\[
C^* = \{ x \in \mathbb{E}^n : \langle x, y \rangle \leq 1 \text{ for all } y \in C \},
\]

where \( \langle x, y \rangle \) denotes the standard inner product of \( x \) and \( y \) in \( \mathbb{E}^n \). For instance, the polar body of a origin-symmetric circle with a radius of 2 is the origin-symmetric circle with a radius of \( \frac{1}{2} \) in \( \mathbb{E}^2 \). See Figure 1.

![Figure 1. Origin-symmetric circle with radius of 2 and its polar.](image)

The famous Blaschke–Santaló inequality in convex geometry provides a relationship between \( V(K) \) and \( V(K^*) \). It asserts that

\[
V(K)V(K^*) \leq \frac{\pi^n}{\Gamma^2(1 + \frac{n}{2})},
\]

with equality if and only if \( K \) is an ellipsoid [1]. Here, \( V(\cdot) \) denotes the volume in \( \mathbb{E}^n \).

For \( n \leq 3 \), the inequality was proven by Blaschke [2], and later, it was generalized to higher dimension by Santaló [3]. The case of equality was characterized by Petty [4]. The Blaschke–Santaló inequality has been extensively studied, see, e.g., [5–7].
Blaschke–Santaló inequality considers the upper bound for $V(K)V(K^*)$ in $\mathbb{E}^n$. For the discrete setting, we consider the finite sets of integer lattice points which are necessarily full-dimensional unless indicated otherwise. A fundamental problem in the discrete setting is to establish the lower and upper bounds for the number of integer lattice points, see, e.g., [8,9]. More information about integer lattice points can be found in [10–14].

Let $\mathbb{Z}^n$ be the set of integer lattice in $\mathbb{E}^n$, i.e., the lattice of all points with integer coordinates. Let $K$ be a finite set in $\mathbb{Z}^n$. We say that $K$ is a convex lattice set if

$$K = \text{conv}(K) \cap \mathbb{Z}^n,$$

where $\text{conv}(K)$ denotes the convex hull of $K$. We denote by $\#(\cdot)$ the lattice point enumerator for convex lattice sets. Let $\#(K)'(K^*_{2\mathbb{Z}})$ be the discrete Mahler product of convex lattice set $K$, where $K^*_{2\mathbb{Z}}$ denotes the polar of convex lattice set $K$ in $\mathbb{Z}^n$ (see Definition 1).

The Legendre transform is a classical and powerful tool in mathematics. In the discrete case, Murota considered the discrete Legendre transform in $\mathbb{Z}^n$ [15,16]. Let $f : \mathbb{Z}^n \rightarrow \mathbb{Z}$ be proper and convex, then the discrete Legendre transform $f^* : \mathbb{Z}^n \rightarrow \mathbb{Z}$ is defined by

$$f^*(p) = \sup\{\langle x, p \rangle - f(x) : x \in \mathbb{Z}^n\}, p \in \mathbb{Z}^n, \quad (1)$$

where $\mathbb{Z} = \mathbb{Z} \cup \{+\infty\}$ and $\sup(\cdot)$ denotes supremum.

The paper is organized as follows. In Section 2, for $t > 0$, we study a class of convex lattice sets and establish a relationship between the vertices of $K^*_{2\mathbb{Z}}$ and the vertices of $(tK)^*_{2\mathbb{Z}}$. Furthermore, we consider a specific class of convex lattice sets that satisfy $K^*_{2\mathbb{Z}} = K$. Additionally, we derive the upper and lower bounds for the discrete Mahler product of a class of convex lattice sets.

2. Properties of Convex Lattice Sets

Denote by $\mathcal{K}_{\mathbb{Z}}^n$ the set of all convex lattice sets containing the origin. For $K \in \mathcal{K}_{\mathbb{Z}}^n$, let $K^0 = \text{vert}(\text{conv}(K))$, where $\text{vert}$ means vertices.

Polytopes can be characterized through two distinct methods. If a polytope is given as the bounded intersection of finitely many closed halfspaces, then it is called an $H$-polytope [1,17]. Assume that the hyperplane $h_i$ can be written in the form

$$h_i(x) = h_{i1}x_1 + \cdots + h_{i(n-1)}x_{n-1} + \beta_i, \quad i = 1, \ldots, m, \quad (2)$$

with $h_{ij}, \beta_i \in \mathbb{Z}$ for $j = 1, \ldots, n-1$. We turn to a subclass of $\mathcal{K}_{\mathbb{Z}}^n$, consisting of all convex lattice sets in $\mathbb{Z}^n$, generated by the hyperplane $h_i$, that include the origin, which will be denoted by $\mathcal{D}_{\mathbb{Z}}^n$. For $K \in \mathcal{D}_{\mathbb{Z}}^n$, it is generated by

$$K = \left( \bigcap_{i=1}^m h_i^- \right) \cap \mathbb{Z}^n,$$

where $h_i^-$ denotes the closed halfspace bounded by $h_i$.

Clearly, $h_i$ represents the boundary of $\text{conv}(K)$. In special cases, points in the convex lattice set $K$ may lie on $h_i(x)$. Any point in the convex lattice set $K$ is a lattice point with integer coordinates. Based on coefficients of $h_i$, when $x_1, \ldots, x_{n-1}$ are integers, we can deduce that value of $h_i$, i.e., the $x_n$ is an integer.

A polytope given as the convex hull of finitely many points is called a $V$-polytope [1,17]. Let $K \in \mathcal{D}_{\mathbb{Z}}^n$. Assume that $K^0 = \{A_1, \ldots, A_k\}$, then

$$K = \text{conv}(K^0) \cap \mathbb{Z}^n = \text{conv}\{A_1, \ldots, A_k\} \cap \mathbb{Z}^n,$$

where $A_r = (a_{r1}, \ldots, a_{rn}) \in \mathbb{Z}^n$ with $A_r \neq (0, \ldots, 0)$ for $1 \leq r \leq k$.

For (2) and $p \in \mathbb{Z}^{n-1}$, by the discrete Legendre transform (1), we obtain that

$$h_i^*(p) = \sup\{\langle x, p \rangle - h_i(x) : x \in \mathbb{Z}^{n-1}\}$$
\[
\sup\\{ (x, p) - (b_1x_1 + \cdots + b_{i(n-1)}x_{n-1} + \beta_i) : x \in \mathbb{Z}^{n-1} \},
\]
for \(i = 1, \ldots, m\).

For \(h_i(x)\), the discrete Legendre transform \(h_i^*(p) : \mathbb{Z}^{n-1} \to \mathbb{Z}\) is given by
\[
h_i^*(p) = \begin{cases} 
-\beta_i, & p = (b_1, \cdots, b_{i(n-1)}), \\
+\infty, & \text{otherwise}, 
\end{cases}
\]
for \(i = 1, \ldots, m\).

The process can be found in [18]. Next, we introduce a lemma for the convex function.

**Lemma 1 ([19]).** Let \(f\) be a finite convex function on a convex set \(S \subset \mathbb{Z}^n\). Then, the function
\[
F(x) = \begin{cases} 
f(x), & \text{if } x \in S, \\
+\infty, & \text{if } x \notin S,
\end{cases}
\]
is a proper convex function in \(\mathbb{Z}^n\).

Obviously, by Lemma 1, \(h_i^*(p)\) is a proper convex function in \(\mathbb{Z}^{n-1}\).

Let \(p = p_i = (p_{i1}, \cdots, p_{i(n-1)})\). Then, for \(p = p_i = (b_1, \cdots, b_{i(n-1)})\), we obtain that \(h_i^*(p_i) = -\beta_i\). Assume that
\[
B_i = (p_{i1}, \cdots, p_{i(n-1)}, h_i^*(p_i)) = (b_1, \cdots, b_{i(n-1)}, -\beta_i)
\]
with \(b_{ij}, \beta_i \in \mathbb{Z}\) for \(i = 1, \ldots, m, j = 1, \cdots, n-1\).

Next, we can propose a definition of polar body of convex lattice sets \(K\) in \(\mathbb{Z}^n\).

**Definition 1.** The polar of convex lattice set \(K\) in \(\mathbb{Z}^n\) is defined by
\[
K^*_2 = \operatorname{conv}(\bigcup_{i=1}^m B_i) \cap \mathbb{Z}^2.
\]

In fact, Definition 1 is equivalent to the definition of polar of convex lattice sets in [18]. In [18], He and Si provide the definition of the polar of a convex lattice set \(K\) in \(\mathbb{Q}^n\) and \(\mathbb{Z}^n\), where \(\mathbb{Q}^n\) denotes the set of \(n\)-tuple arrays of rational numbers, that is, the set of points with rational coordinates.

We give a simple example of the definition of the polar of a convex lattice set \(K\).

**Example 1.** Let \(K \in \mathbb{R}^2_2\), \(K^c = \{(1,2), (-1,0), (0,-1)\}\). Clearly, we obtain that \(K\) is bounded by the following hyperplanes
\[
\begin{align*}
h_1(x) &= x + 1, \\
h_2(x) &= -x - 1, \\
h_3(x) &= 3x - 1.
\end{align*}
\]

Then, by (1) and Definition 1, we obtain that
\[
K^*_2 = \operatorname{conv}\{(1, -1), (-1, 1), (3, 1)\} \cap \mathbb{Z}^2.
\]

See Figure 2.
If $K$ satisfies the property $(tC)^* = t^{-1}C^*$. In the subsequent discussion, we consider a similar property of convex lattice sets, i.e., the relation between $K_{2n}^*$ and $(tK)^{2n}_*$. 

**Theorem 1.** If $K \in \mathcal{P}_Z^{2n}$, $t \in \mathbb{Z}^+$, then $(tK)^{2n}_*$ can be obtained by multiplying the $n$-th coordinate component of each point in $(K^*)^{2n}_*$ by a factor of $t$, where $\mathbb{Z}^+$ denotes the set of positive integers.

**Proof.** According to the sign characteristics of coordinates, we divide $\mathbb{Z}^n$ into the following $2^n$ blocks:

$$I_1 = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n : x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, \ldots, x_{n-1} \geq 0, x_n \geq 0\},$$

$$I_2 = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n : x_1 \leq 0, x_2 \geq 0, x_3 \geq 0, \ldots, x_{n-1} \geq 0, x_n \geq 0\},$$

$$I_3 = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n : x_1 \leq 0, x_2 \leq 0, x_3 \leq 0, \ldots, x_{n-1} \geq 0, x_n \geq 0\},$$

$$\vdots$$

$$I_{2n-2+1} = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n : x_1 \leq 0, x_2 \leq 0, x_3 \leq 0, \ldots, x_{n-1} \leq 0, x_n \geq 0\},$$

$$I_{2n-2+2} = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n : x_1 \geq 0, x_2 \leq 0, x_3 \geq 0, \ldots, x_{n-1} \geq 0, x_n \geq 0\},$$

$$I_{2n-2+3} = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n : x_1 \geq 0, x_2 \geq 0, x_3 \leq 0, \ldots, x_{n-1} \geq 0, x_n \geq 0\},$$

$$\vdots$$

$$I_{2^n-2+1} = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n : x_1 \leq 0, x_2 \leq 0, x_3 \leq 0, \ldots, x_{n-1} \leq 0, x_n \leq 0\},$$

$$I_{2^n-2+2} = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n : x_1 \geq 0, x_2 \leq 0, x_3 \geq 0, \ldots, x_{n-1} \geq 0, x_n \leq 0\},$$

$$I_{2^n-2+3} = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n : x_1 \geq 0, x_2 \geq 0, x_3 \leq 0, \ldots, x_{n-1} \geq 0, x_n \leq 0\},$$

$$\vdots$$

$$I_{2^n} = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n : x_1 \geq 0, x_2 \leq 0, x_3 \leq 0, \ldots, x_{n-1} \leq 0, x_n \leq 0\}.$$

Let the vertices of convex lattice set $K$ be $K^0 = \{A_1, \ldots, A_k\}$, where $A_r = (a_{r1}, \ldots, a_{rn}) \in \mathbb{Z}^n$ with $A_r \neq (0, \ldots, 0)$ for $1 \leq r \leq k$.

Assume that $K$ is bounded by the hyperplanes as follows

$$h_i(x) = b_{i1}x_1 + \cdots + b_{i(n-1)}x_{n-1} + b_i, \quad (3)$$

with $b_{ij}, b_i \in \mathbb{Z}$ for $i = 1, \ldots, m$, $j = 1, \ldots, n-1$. 

**Figure 2.** $K$, $K^*_{2n}$ (from left).
By (3) and (1), the discrete Legendre transform \( h^*_i(p) : \mathbb{Z}^{n-1} \to \mathbb{Z} \) is given by
\[
h^*_i(p) = \begin{cases} -\beta_i, & p = (b_{i1}, \cdots, b_{i(n-1)}), \\ +\infty, & \text{otherwise}, \end{cases}
\]
where \( b_{ij}, \beta_i \in \mathbb{Z} \) for \( i = 1, \cdots, m, j = 1, \cdots, n - 1 \).

Then, we have
\[
K_{\mathbb{Z}^s}^* = \text{conv}\left\{ \bigcup_{i=1}^m (b_{i1}, \cdots, b_{i(n-1)}, -\beta_i) \right\} \cap \mathbb{Z}^n.
\]

Let
\[
K_0 = \{(b_{11}, \cdots, b_{1(n-1)}, -\beta_1), \cdots, (b_{m1}, \cdots, b_{m(n-1)}, -\beta_m)\}.
\]
Obviously, \((K_{\mathbb{Z}^s}^*)^p \subseteq K_0\).

For \( tK \), it is easy to obtain that \((tK)^p = \{tA_1, \cdots, tA_k\}\). Then, \((tA_{r1}, \cdots, tA_{rn}) \in (tK)^p\).

For the blocks \( t\mathbb{Z}_s \) and \( t\mathbb{Z}_{s-1} \), where \( 1 \leq s \leq 2^{n-1} \), we know that the hyperplanes of the boundary enclosing \( tK \) can be obtained by translation of the hyperplanes of the boundary enclosing \( K \). In other words, in \( t\mathbb{Z}_s \) and \( t\mathbb{Z}_{s-1} \), where \( 1 \leq s \leq 2^{n-1} \), the coefficients of the hyperplanes of the boundary enclosing \( tK \) is equal to the coefficients of the hyperplanes of the boundary enclosing \( K \).

Therefore, we assume that \( tK \) is bounded by the hyperplanes as follows:
\[
g_i(x) = b_{i1}x_1 + \cdots + b_{i(n-1)}x_{n-1} + a_i,
\]
where \( b_{ij}, \beta_i \in \mathbb{Z} \) for \( i = 1, \cdots, m, j = 1, \cdots, n - 1 \).

Let \( A_r \in h_i(x) \), and then \( tA_r \in g_i(x) \) for \( r = 1, \cdots, k, i = 1, \cdots, m \). Consequently, we have
\[
b_{i1}ta_{r1} + \cdots + b_{i(n-1)}ta_{rn} + a_i = ta_{rn},
\]
and then, we obtain that
\[
a_i = t(a_{rn} - b_{i1} - \cdots - b_{i(n-1)}).
\]

It follows from \( tA_r \in g_i(x) \) that
\[
b_{i1}a_{r1} + \cdots + b_{i(n-1)}a_{rn} + \beta_i = a_{rn}.
\]

Then, we have
\[
a_i = t\beta_i.
\]

Therefore, we obtain that
\[
g_i(x) = b_{i1}x_1 + \cdots + b_{i(n-1)}x_{n-1} + t\beta_i
\]
where \( b_{ij}, \beta_i \in \mathbb{Z} \) for \( i = 1, \cdots, m, j = 1, \cdots, n - 1 \).

By (5) and (1), the discrete Legendre transform \( g^*_i(p) : \mathbb{Z}^{n-1} \to \mathbb{Z} \) is given by
\[
g^*_i(p) = \begin{cases} -t\beta_i, & p = (b_{i1}, \cdots, b_{i(n-1)}), \\ +\infty, & \text{otherwise}, \end{cases}
\]
where \( b_{ij}, \beta_i \in \mathbb{Z} \) for \( i = 1, \cdots, m, j = 1, \cdots, n - 1 \).
Similar to (4), we have
\[(tK)_{2n}^* = \text{conv} \left\{ \bigcup_{i=1}^{m} (b_{i1}, \cdots, b_{i(n-1)}, -t\beta_i) \right\} \cap \mathbb{Z}^n. \tag{6} \]

Let
\[K_1 = \{(b_{i1}, \cdots, b_{i(n-1)}, -t\beta_1), \cdots, (b_{m1}, \cdots, b_{m(n-1)}, -t\beta_m)\}.\]

Obviously, \(((tK)_{2n}^*)^0 \subseteq K_1.\]

By (4) and (6), \(((tK)_{2n}^*)^0\) can multiply the \(n\)-th coordinate component of each point in \((K_{2n}^*)^0\) by a factor of \(t. \]

**Remark 1.** Let \(t\) be a positive rational number denoted by \(\frac{p}{q}\) with \(\gcd(p, q) = 1\) for \(p, q \in \mathbb{Z}\), where \(\gcd\) denotes the greatest common divisor. If the common divisor of
\[a_{11}, \cdots, a_{1n}, a_{21}, \cdots, a_{2n}, \cdots, a_{k1}, \cdots, a_{kn}\]
is \(q\), then the relation between \((K_{2n}^*)^0\) and \(((tK)_{2n}^*)^0\) satisfies Theorem 1.

We illustrate Theorem 1 with an example in \(\mathbb{Z}^2\).

**Example 2.** Let \(K \in \mathcal{P}_{2n}^2, t = 2.\) The vertices of \(K\) are \(K^0 = \{(1, 2), (-1, 0), (0, -1)\}.\) Then, we can obtain that \((K_{22}^*)^0 = \{(1, -1), (-1, 1), (3, 1)\},\)
\[2K = \text{conv}\{(2, 4), (-2, 0), (0, -2)\} \cap \mathbb{Z}^2,\]
\(((2K)_{22}^*)^0 = \{(1, 2), (-1, 2), (3, 2)\}.\) Therefore, we obtain that \(((2K)_{22}^*)^0\) can be obtained by multiplying the second coordinate component of each point in \((K_{22}^*)^0\) by a factor of 2. See Figure 3.

![Figure 3](image-url)

**Figure 3.** \(K\) (top left), \(K_{22}^*\) (top right), \(2K\) (bottom left), \((2K)_{22}^*\) (bottom right).

Next, in order to prove the relation between \(K_{2n}^*\) and \(K\), we introduce a special class convex lattice sets.
Assume that $K$ is bounded by the following hyperplanes
\[
\begin{align*}
h_i(x) &= \pm \sqrt{2\beta_1}x_1 \pm \sqrt{2\beta_1}x_2 \pm \cdots \pm \sqrt{2\beta_1}x_{n-1} + \beta_1, \\
h_{i+1}(x) &= -\beta_1,
\end{align*}
\]
(7)
or
\[
\begin{align*}
h_i(x) &= \pm \sqrt{2\beta_1}x_1 \pm \sqrt{2\beta_1}x_2 \pm \cdots \pm \sqrt{2\beta_1}x_{n-1} - \beta_1, \\
h_{i+1}(x) &= \beta_1,
\end{align*}
\]
(8)
with $\sqrt{2\beta_1} \in \mathbb{Z}$, $\beta_1 \neq 0$, $i = 1, \cdots, m$.

For $K$, it is generated by the hyperplanes (7) or (8), and then, we have $K_{2n}^* = K$. The result is summarized in the following theorem.

**Theorem 2.** If $K \in \mathcal{P}_{2n}^*$ is bounded by the hyperplanes (7) or (8), then $K_{2n}^* = K$.

**Proof.** Without loss of generality, assume that $K$ is bounded by the following hyperplanes
\[
\begin{align*}
h_i(x) &= \pm b_1x_1 \pm b_2x_2 \pm \cdots \pm b_{n-1}x_{n-1} + \beta_1, \\
h_{i+1}(x) &= -\beta_1,
\end{align*}
\]
(9)
where $b_j, \beta_1 \in \mathbb{Z}$, $\beta_1 \neq 0$, $i = 1, \cdots, m$, $j = 1, \cdots, n - 1$.

Let $b_j > 0$ for $j = 1, \cdots, n - 1$. Summarizing the above arguments, we obtain Table 1.

It follows from (1), Definition 1 and (9) that
\[
K_{2n}^* = \text{conv}\{(b_1, \cdots, b_{n-1}), -\beta_1), \cdots, (-b_1, \cdots, -b_{n-1}), -\beta_1), (0, \cdots, 0, \beta_1)\} \cap \mathbb{Z}^n.
\]
(10)

Let
\[
K_1 = \{(b_1, \cdots, b_{n-1}), -\beta_1), \cdots, (-b_1, \cdots, -b_{n-1}), -\beta_1), (0, \cdots, 0, \beta_1)\},
\]
and then, $(K_{2n}^*)^c \subseteq K_1$. By Table 1 and (10), we obtain that $K_1 = (K_{2n}^*)^c$.

Without loss of generality, we take $(-b_1, \cdots, -b_{n-1}, -\beta_1) \in K_{2n}^*$. According to sign characteristics of coordinates, we obtain that $(-b_1, \cdots, -b_{n-1}, -\beta_1) \in l_{2n-2} + 1$.

Indeed, if $K$ is bounded by the hyperplanes (9), then $K_{2n}^* = K$, which is equivalent to $(K_{2n}^*)^c = K$. For $(-b_1, \cdots, -b_{n-1}, -\beta_1) \in (K_{2n}^*)^c$, we have $(-b_1, \cdots, -b_{n-1}, -\beta_1) \in K^c$.

Therefore, by Table 1, in $x_1 \leq 0$, we obtain that
\[
\begin{align*}
b_1 \cdot (-b_1) + \cdots + b_{n-1} \cdot (-b_{n-1}) + \beta_1 &= -\beta_1, \\
h_1 \cdot (-b_1) + \cdots + b_{n-1} \cdot (-b_{n-1}) + \beta_1 &= -\beta_1.
\end{align*}
\]
(11)

By summing over (11), we have
\[
b_1^2 = 2\beta_1.
\]
(12)

Analogously, for $(K_{2n}^*)^c$, we obtain that
\[
b_1^2 = 2\beta_1, \cdots, b_{n-1}^2 = 2\beta_1.
\]
(13)

Combining (12) and (13), we have
\[
b_1 = \cdots = b_{n-1} = \pm \sqrt{2\beta_1},
\]
where $b_j, \beta_1 \in \mathbb{Z}$, $\beta_1 \neq 0$ for $j = 1, \cdots, n - 1$. 
Therefore, it is easy to obtain that $K$ with
\[ K = \{ x \in \mathbb{R}^n : h_i(x) \leq 0, \quad i = 1, \cdots, m \} \]

where $h_i(x) = \sum_{j=1}^{n} a_{ij} x_j + b_i$.

Similarly, if $K \in \mathcal{D}_2^m$ is bounded by the hyperplanes (11), then $K^*_2 = K$.

We illustrate Theorem 2 with an example in $\mathbb{Z}^2$.

Example 3. Let $K \in \mathcal{D}_2^2$, $\beta_1 = 2$. By (11), we see that $K$ is bounded by the following hyperplanes
\[
\begin{align*}
  h_1(x) &= 2x + 2, \\
  h_2(x) &= -2x + 2, \\
  h_3(x) &= -2.
\end{align*}
\]

Therefore, it is easy to obtain that $K^* = \{ (2, -2), (-2, -2), (0, 2) \}$,
\[ K^*_2 = \text{conv} \{ (2, -2), (-2, -2), (0, 2) \} \cap \mathbb{Z}^2. \]

Obviously, $K^*_2 = K$. See Figure 4.
Remark 2. In Theorem 2, we demonstrated the situation where $K$ is bounded by the hyperplanes (7). To illustrate the case where $K$ is bounded by the hyperplanes (8), we now provide a simple example in $\mathbb{Z}^2$.

Let $K \in \mathcal{K}^2$, $\beta_1 = 2$. By (8), assume that $K$ is bounded by the following hyperplanes

$$
\begin{align*}
    h_1(x) &= -2x - 2, \\
    h_2(x) &= 2x - 2, \\
    h_3(x) &= 2.
\end{align*}
$$

Then, we obtain that $K^0 = \{(0, -2), (-2, 2), (2, 2)\}$,

$$
K^*_\mathbb{Z}^2 = \text{conv}\{(-2, 2), (0, -2), (2, 2)\} \cap \mathbb{Z}^2.
$$

Clearly, $K^*_\mathbb{Z}^2 = K$. See Figure 5.

Indeed, the definition of $K$ indicates that $K$ is bounded by the intersection of finitely many closed halfspaces. Assume that

$$
\begin{align*}
    h_{\beta_{ij}}(x) &= b_{ij}x_1 + \cdots + b_{i(n-1)}x_{n-1} + \beta_{ij}, \text{ for } \beta_{1} > 0, i = 1, \cdots, m_1, \\
    h_{\beta_{il}}(x) &= d_{il}x_1 + \cdots + d_{i(n-1)}x_{n-1} + \beta_{il}, \text{ for } \beta_{2} < 0, l = 1, \cdots, m_2,
\end{align*}
$$

with $b_{ij}, d_{il}, \beta_1, \beta_2 \in \mathbb{Z}$ for $j = 1, \cdots, n - 1$.

In the following, we focus on a specific subclass of $\mathcal{K}^n$, which comprises all convex lattice sets in $\mathbb{Z}^n$ formed by the intersection of hyperplanes that include the origin as its
interior. Specifically, this subclass is defined as the intersection of $\mathbb{Z}^n$ with the union of two sets,
\[
\left\{ \left( \bigcap_{i=1}^{m_1} h_{\beta_{1i}} \cap \{ x_n \geq 0 \} \right) \cup \left( \bigcap_{i=1}^{m_2} h_{\beta_{2i}} \cap \{ x_n < 0 \} \right) \right\} \cap \mathbb{Z}^n,
\]
and will be denoted by $\mathcal{C}_Z^n$.

In order to provide the upper and lower bounds for the discrete Mahler product of convex lattice sets, we need the following lemmas.

**Lemma 2** ([20]). For any origin-symmetric convex body $K$, we have
\[
\frac{1}{6^n} \leq \frac{\#(K)}{\#(K^*) V(K)} \leq \frac{6^n \cdot V(B_n)^2}{c^n},
\]
where $c > 0$, and $B_n$ denotes the $n$-dimensional Euclidean unit ball.

**Lemma 3** ([18]). If $K \in \mathcal{C}_Z^n$ and $K$ is origin-symmetric, then $K$ is a cross-polytope that holds if and only if the discrete Mahler product is the smallest.

We now prove the upper and lower bounds for the discrete Mahler product of convex lattice sets. (The discrete Mahler product considers the relationship between $\#(K)$ and $\#((K^*)_Z^n)$ in $\mathbb{Z}^n$. For convex body $K$, $V(K) V(K^*)$ is referred to as the Mahler product. The Mahler conjecture [21] is an open question that proposes the lower bound for $V(K) V(K^*)$.)

**Theorem 3.** If $K \in \mathcal{C}_Z^n$ is origin-symmetric, then the upper and lower bounds of the discrete Mahler product of $K$ are
\[
(2n + 1) \cdot 3^n \leq \#(K) \#(K^*_Z) \leq \frac{(2n + 1) \cdot 36^n \cdot (2t + 1)^n}{c^n \cdot V(B_n)^2}.
\]

**Proof.** Let $K_1 \in \mathcal{C}_Z^n$ be origin-symmetric. Then, we assume that $K_1$ is bounded by the following hyperplanes
\[
h_{\nu,1}(x) = b_{ij} x_1 + \cdots + b_{i(n-1)} x_{n-1} + \beta_1, \text{ for } \beta_1 > 0, i = 1, \cdots, m_1,
\]
\[
h_{\nu,2}(x) = d_{ij} x_1 + \cdots + d_{i(n-1)} x_{n-1} + \beta_2, \text{ for } \beta_2 < 0, i = 1, \cdots, m_2,
\]
with $b_{ij}, d_{ij}, \beta_1, \beta_2 \in \mathbb{Z}$ for $j = 1, \cdots, n-1$.

Therefore, for $K_1$, we have
\[
\beta_1 = -\beta_2 \geq 1, |b_{ij}| = |d_{ij}| \geq 1, m_1 = m_2,
\]
and then, we let $t = \beta_1 = -\beta_2 \geq 1$ and $|b_{ij}| = |d_{ij}| \geq 1$, where $b_{ij}, d_{ij}, \beta_1, \beta_2 \in \mathbb{Z}$ for $i = 1, \cdots, m_1, j = 1, \cdots, n - 1$.

Let $K_2$ be the cross-polytope. Obviously, $K_2 \in \mathcal{C}_Z^n$. Then, we have
\[
\beta_1 = -\beta_2 = 1, |b_{ij}| = |d_{ij}| = 1, \#(K_2) = 2n + 1.
\]

A cross-polytope is a specific type of polytope, and its polar body is indeed a cube. By Lemma 3, the smallest discrete Mahler product can be calculated by
\[
\#(K_2) \#((K_2)^*_Z) = (2n + 1) \cdot \#([-1,1] \cap \mathbb{Z}^n) = (2n + 1) \cdot 3^n.
\]

In particular, when $t = 1$ and $|b_{ij}| = |d_{ij}| = 1$, it is easy to obtain $K_1 = K_2$. We now separate the situation into three cases.

**Case 1:** $|b_{ij}| = |d_{ij}| \geq 2, t = \beta_1 = -\beta_2 \geq 2$. 
Obviously, for \( t \geq 2 \), \( |b_{ij}| = |d_{ij}| \geq 2 \), we obtain that \( K_2 \subset K_1 \).

**Case 2:** \( |b_{ij}| = |d_{ij}| \geq 2, t = \beta_1 = -\beta_2 = 1 \).

Fix \( t = \beta_1 = -\beta_2 = 1 \). For \( K_1 \in \mathcal{C}_n \) is origin-symmetric, we see that \( |b_{ij}| = |d_{ij}| \leq 1 \) is a contradiction.

**Case 3:** \( |b_{ij}| = |d_{ij}| = 1, t = \beta_1 = -\beta_2 \geq 2 \).

For \( |b_{ij}| = |d_{ij}| = 1, t = \beta_1 = -\beta_2 \geq 2 \), it is easy to obtain \( K_2 \subset K_1 \).

By the above arguments, for \( |b_{ij}| = |d_{ij}| \geq 1, t = \beta_1 = -\beta_2 \geq 2 \), we have \( K_2 \subset K_1 \).

Therefore, by the definition of polar body of convex body, we obtain that

\[
(\text{conv}(K_1))^* \subset (\text{conv}(K_2))^*,
\]

and then, we have

\[
\#((\text{conv}(K_1))^*) \leq \#((\text{conv}(K_2))^*). \tag{16}
\]

By Lemma 2 and (16), we obtain that

\[
\#(K_1) \leq \frac{6^n \cdot \#((\text{conv}(K_1))^*) \cdot V(\text{conv}(K_1))}{c^n \cdot V(B_n)^2}
\]

\[
\leq \frac{6^n \cdot \#((\text{conv}(K_2))^*) \cdot V(\text{conv}(K_1))}{c^n \cdot V(B_n)^2}
\]

\[
\leq \frac{6^n \cdot \#(K_2) \cdot 6^n}{c^n \cdot V(B_n)^2}
\]

\[
\leq \frac{36^n \cdot (2n + 1)}{c^n \cdot V(B_n)^2}.
\]

By the definition of \( K_1 \), the upper bound of \( (K_1)^*_{Z^n} \) is an \( n \)-dimensional cube with edge length \( 2t \), that is, when \( t = |b_{ij}| = |d_{ij}| \geq 2 \),

\[
(K_1)^*_{Z^n} = \text{conv}\{(\pm t, \pm t, \cdots, \pm t), \cdots, (\pm t, \pm t, \cdots, \pm t)\} \cap \mathbb{Z}^n.
\]

Then, we can see that

\[
\#((K_1)^*_{Z^n}) \leq \#([-t, t]^n \cap \mathbb{Z}^n) = (2t + 1)^n.
\]

Consequently, we have

\[
\#(K_1)\#((K_1)^*_{Z^n}) \leq \frac{36^n \cdot (2n + 1)}{c^n \cdot V(B_n)^2} \cdot (2t + 1)^n
\]

\[
= \frac{36^n \cdot (2n + 1) \cdot (2t + 1)^n}{c^n \cdot V(B_n)^2}.
\]

Therefore, we have

\[
(2n + 1) \cdot 3^n \leq \#(K_1)\#((K_1)^*_{Z^n}) \leq \frac{(2n + 1) \cdot 36^n \cdot (2t + 1)^n}{c^n \cdot V(B_n)^2},
\]

and then, the upper and lower bounds for \( \#(K)\#((K)^*_{Z^n}) \) are given by

\[
(2n + 1) \cdot 3^n \leq \#(K)\#((K)^*_{Z^n}) \leq \frac{(2n + 1) \cdot 36^n \cdot (2t + 1)^n}{c^n \cdot V(B_n)^2}.
\]

\[\square\]
3. Conclusions

The discrete Legendre transform is a mathematical operation in various fields such as signal processing, economics, and game theory. The discrete Legendre transform is an active field of research, especially in economics and game theory; see, e.g., [22], or discrete convex analysis, see, e.g., [15,23]. It is worth noting that the discrete Legendre transform is relevant to image processing, such as in image compression [24] and image encoding [25]. Moreover, it also serves as a critical method for analyzing the properties of convex lattice sets. Within the context of this study, for \( t > 0 \), we study a class of convex lattice sets and establish a relationship between \((K_{2n}^*)^t\) and \(((tK)^{2n})^t\). More precisely, we obtain that the vertices of \((tK)^{2n}\) derived by scaling the \( n \)-th coordinate component of each vertex in \(K_{2n}^*\) by a factor of \( t \). This discovery provides a deeper understanding of how the scaling of a lattice set impacts the polar of convex lattice set. Subsequently, we turn our attention to a specific class of convex lattice sets that is distinguished by the property \( K_{2n}^* = K \). The class of convex lattice sets is characterized by the presence of a bounding hyperplane \( x_n = a \), where \( a \in \mathbb{Z}\setminus\{0\} \), and the fact that all other bounding hyperplanes intersect the \( x_n \)-axis at the same point. The study of the number of lattice points has always been a focus of public attention. Next, we consider the upper and lower bounds of the discrete Mahler product for convex lattice sets. When the discrete Mahler product reaches its upper bound, the convex lattice set \( K \) is a cross-polytope. The lower bound of the discrete Mahler product is a fraction related to the constant \( c \) and the volume of the unit sphere.

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